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Weak Solutions of a Stochastic Landau–Lifshitz–Gilbert Equation

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We consider a Landau-Lifshitz-Gilber equation perturbed by a multiplicative space-dependent noise for a ferromagnet filling a bounded three-dimensional domain. We show the existence of weak martingale solutions taking values in a sphere $\mathbb{S}^2$. The regularity of weak solutions is also discussed. Our research is in response to the paper by Kohn *et al.* "Magnetic elements at finite temperature and large deviation theory." *Journal of Nonlinear Science* 15 (2005): 223–53, which calls for the study of stochastic equations with spatially varying magnetization.

1 Introduction

The study of the theory of ferromagnetism was initiated by Weiss, see [6] and references therein, and further developed by Landau and Lifshitz [28] and Gilbert [18]. According to this theory, the magnetization u of a ferromagnetic material occupying a region $D \subset \mathbb{R}^3$ at temperatures below the critical (so-called Curie) temperature satisfies, for $t > 0$ and $x \in D$, the following Landau–Lifshitz–Gilbert (LLG) equation:

$$\frac{\partial u}{\partial t}(t,x) = \lambda_1 u(t,x) \times H(t,x) - \lambda_2 u(t,x) \times (u(t,x) \times H(t,x)), \tag{1.1}$$

where $\times$ is the vector cross product in $\mathbb{R}^3$ and H is the so-called effective field, which is the negative of the gradient (with respect to u) of the total magnetic energy functional, $\mathcal{E}$. In the physical situation, the total magnetic energy consists of the anisotropy

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energy, exchange energy and the stray energy. Visintin [38] studied the deterministic Equation (1.1) taking into account all three contributions to the total magnetic energy. In this paper, we prove the existence of weak solutions to a stochastic LLG equation taking into account the exchange energy only. Under this simplifying assumption the LLG equation, being a mixture of the Schrödinger equation and the equation for the heat flow of harmonic maps (see below) is still difficult and its stochastic version has never been studied before. The same simplifying assumptions are often made in the deterministic case, see Alouges and Soyeur [2] or Bertsch *et al.* [4]. The idea to ignore all terms in the micromagnetic energy but the exchange energy goes back to Stoner and Wohlfarth [34]. Let us note that in the deterministic case the model with the exchange energy only has been studied by Gioia and James [19] and DeSimone [13] as a limiting case of the models including other types of energy.

In the simplified situation where the energy functional consists of the exchange energy only, $\mathcal{E}(u) = \frac{1}{2}\int_D |\nabla u(x)|^2\,\mathrm{d}x$, we have $H = \Delta u$ and obtain the following version of the LLG equation:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \lambda_1 u \times \Delta u - \lambda_2 u \times (u \times \Delta u), \quad t > 0, \ \ x \in D, \\ \frac{\partial u}{\partial n}(t,x) &= 0 \quad t > 0, \ \ x \in \partial D, \\ u(0,x) &= u_0(x) \quad x \in D; \end{aligned} \tag{1.2}$$

here $\lambda_1 \in \mathbb{R}$, $\lambda_2 > 0$, n is the outer unit normal vector at the boundary ∂D and we assume that at time $t = 0$ the material is saturated, that is

$$|u_0(x)| = 1 \quad \text{for all } x \in D. \tag{1.3}$$

Let us recall that the stationary solutions of Equation (1.2) correspond to the equilibrium states of the ferromagnet and are not unique in general. An important problem in the theory of ferromagnetism is to describe phase transitions between different equilibrium states induced by thermal fluctuations of the field H. Therefore, the LLG equation needs to be modified in order to incorporate random fluctuations of the field H into the dynamics of the magnetization u and to describe noise-induced transitions between equilibrium states of the ferromagnet. The program to analyze noise-induced transitions was initiated by Néel [30] and further developed in [5, 24] and others. A simple way to incorporate the noise into the LLG equation is to perturb the effective field by a Gaussian noise, that is, to replace H in (1.1) by $H + \xi$, where informally speaking, ξ is the white noise with respect to time variable while in general it can be colored with respect to the space variables, that is

$$\mathbb{E}\xi(s,x)\xi(t,y) = C(x,y)\delta(t-s), \quad s,t \geq 0, \quad x,y \in D.$$

The case when $C(x,y) = \delta(x-y)$ corresponds to the space–time Gaussian white noise. It is well known [12] in the theory of stochastic partial differential equations (PDEs) that a rigorous interpretation of ξ is via the relationship $\xi = \dot{W}$, where W is a Wiener process on $L^2(D,\mathbb{R}^3)$ (say). By a Wiener process on $L^2(D,\mathbb{R}^3)$, we mean a process

$$W(t) = \sum_{k=1}^{\infty} a_k W_k(t) e_k, \quad t \geq 0, \tag{1.4}$$

where $\{W_k : k \geq 1\}$ is a family of independent real-valued Wiener processes, $\{a_k : k \geq 1\}$ is a sequence of real numbers and $\{e_k : k \geq 1\}$ is a complete orthonormal basis of $L^2(D,\mathbb{R}^3)$. Informally, in the case when all $a_k = 1$ we find that $\xi = \dot{W}$ is a space–time white noise. Therefore, a stochastic version of the simplified LLG equation (1.2) takes the form

$$\frac{\partial u}{\partial t} = \lambda_1 u \times (\Delta u + \mathrm{d}W) - \lambda_2 u \times (u \times (\Delta u + \mathrm{d}W)). \tag{1.5}$$

However, the questions of how to understand the stochastic term in this equation and which particular Wiener process (1.4) should be chosen are not obvious. There is strong evidence, see, for example [17, 24], and it was already argued in [5], that the stochastic terms in (1.5) should be understood in the Stratonovich sense and we adopt this approach throughout the paper (see also [7, 10] for a corresponding model of stochastic harmonic maps flow). Moreover, following [17, 27], we will assume that the smallness of λ_2 in physical problems justifies the neglect of noise in the second term on the right-hand side of (1.5). Then a stochastic version of the simplified LLG equation takes the form

$$\mathrm{d}u(t) = (\lambda_1 u(t) \times \Delta u(t) - \lambda_2 u(t) \times (u(t) \times \Delta u(t)))\,\mathrm{d}t + \sum (a_k u(t) \times e_k) \circ \mathrm{d}W_k(t), \tag{1.6}$$

where $\circ\,\mathrm{d}W(t)$ stands for the Stratonovich differential.

The question of how to choose a Wiener process modeling the thermal fluctuations is more delicate. The choice of W such that $\dot{W}$ is a space–time white noise is put forward in Sections 3.2 and 3.3 of an interesting paper [3], where it is argued that it leads to correct physical conclusions. However, it seems unlikely that Equation (1.6) with $a_k = 1$ for all k has an integrable solution. It is well known that stochastic wave equations, stochastic Schrödinger equations and other dispersive equations are well posed only if the driving Wiener process is sufficiently smooth in space (i.e., a_k converge to zero fast enough). We leave this issue for future investigation and in this paper assume that the

Wiener process driving Equation (1.6) is sufficiently smooth in space. Then, for simplicity of presentation, we assume that the driving noise is one-dimensional.

Finally, the stochastic version of the LLG equation we are going to study in this paper has the form:

$$
\begin{aligned}
&\mathrm{d}u(t) = (\lambda_1 u(t) \times \Delta u(t) - \lambda_2 u(t) \times (u(t) \times \Delta u(t)))\,\mathrm{d}t + (u(t) \times h) \circ \mathrm{d}W(t),\\
&\frac{\partial u}{\partial n}(t,x) = 0 \quad \text{on } (0,\infty) \times \partial D,\\
&u(0,x) = u_0(x) \quad \text{on } D,
\end{aligned}
\tag{1.7}
$$

where W is a real-valued Wiener process and $h : D \to \mathbb{R}^3$ is a given bounded function. We remark that the smallness of λ_2 in physical problems gave us an excuse to neglect adding noise to the field in the second term on the right-hand side of (1.5), thus simplifying the form of Equation (1.7) we study in this paper; however, the smallness of the parameter λ_2 is not used in our proof of existence of a weak solution of (1.7) in the following sections.

While stochastic PDEs and their applications in physics is now a well-developed area, see, for example, [9, 12] and references therein, to the best of our knowledge fully nonlinear stochastic dispersive equations such as Equation (1.7) have not been studied. The only exception is a recent paper [21] made available to the authors after this work has been completed. In [21] an equation similar to ours is studied in the whole space $\mathbb{R}^d$ using difference method. The noise considered in [21] corresponds to a choice of a function h in our Equation (1.7) to be constant across the domain D. It is also not clear how a solution to the stochastic Landau–Lifshitz equation is defined. Our work is strongly motivated by a series of papers by Kohn and collaborators [26, 27]. In these works, the thin film approximation is considered in the case of uniform magnetization, when Equation (1.7) reduces to an ordinary stochastic differential equation. The question how to extend their results to the case of nonuniform (dependent on location) magnetization has been formulated as an open problem in [27, Sections 8.3 and 11]. Our paper opens the way to address this issue providing a result on the existence of an appropriately defined weak solution u satisfying the saturation condition

$$|u(t,x)| = 1, \quad x \in D, \ \ t \geq 0$$

for all times. The existence and uniqueness of smooth solutions and the analysis of phase transitions will be the subject of forthcoming papers. Finally, we note that the above saturation condition is not satisfied in room temperature, where the Landau–Lifshitz–Bloch equation should be used instead, see [25] for details. Analysis of the Landau–Lifshitz–Bloch equation driven by noise is an open problem at present.

In current applications of ferromagnetic materials such as magnetic memory elements, the element thickness may be much smaller than the other dimensions and a two-dimensional approximation may be adequate. However, in hard disks the horizontal dimensions of individual single domain grains in the ferromagnetic film are comparable with the film thickness, thus the domain is effectively three-dimensional; see, for example, [1, 16, 32]. This provides a motivation to study Equation (1.7) in its full three-dimensional version. It is also interesting that (1.7) is a stochastic PDE closely related to other important PDEs and its mathematical analysis shares with them some difficulties. If $h=0$, $\lambda_2=1$ and Equation (1.7) has a smooth solution u such that $|u(t,x)|=1$, then Guo and Hong [20] have shown that the equation reduces to the following perturbation of the harmonic maps equation:

$$\frac{\partial u}{\partial t}=\Delta u+|\nabla u|^2 u+\lambda_1 u\times\Delta u.$$

If $h=0$, $\lambda_2=0$, then, using the Hashimoto transform, (1.7) can be transformed into the nonlinear Schrödinger equation, see [14].

The paper is structured as follows. In Section 2, Equation (1.7) is reformulated as an evolution equation in the space $L^2(D,\mathbb{R}^3)$ and the notion of a weak martingale solution is made precise. Our weak martingale solution of Equation (1.7) takes values in $W^{1,2}(D,\mathbb{R}^3)$ so we need to interpret the symbol $u\times\Delta u$ and the Neumann boundary condition in Equation (1.7) in a weak sense via the Stokes theorem. Section 2 also contains the main result of this paper on the existence of a weak martingale solution, formulated as Theorem 2.7. Sections 3–5 are devoted to the proof of Theorem 2.7. In Section 3, we introduce the Faedo–Galerkin approximations and prove for them some uniform bounds in various norms. In Section 4, we use the method of compactness and the Skorohod theorem to show the existence of a probability space on which a weak martingale solution is identified in Section 5. We remark that in Section 3 and parts of Sections 4 and 5, our approach is similar to that of Flandoli and Gatarek [15], where the two-dimensional stochastic Navier–Stokes equation is studied. However, the stochastic LLG equation is fully nonlinear and therefore the regularizing properties of the heat semigroup can be not directly used. In Sections 4 and 5, we also make greater use of the Skorohod theorem and avoid using a martingale representation theorem, instead using more direct analytic arguments. Finally, in the appendices we collected, for the reader's convenience, some facts scattered in the literature that are used in the course of the proof.

Notation. The bounded domain D is fixed throughout the paper. We assume that the boundary ∂D is C^1; this is sufficient to ensure that the Stokes theorem and standard theorems concerning Sobolev spaces, which we use below, are valid. The domain D is omitted in the notation of relevant functional spaces. We will use the notation $\mathbb{L}^p$ for the space $L^p(D,\mathbb{R}^3)$, $\mathbb{W}^{k,p}$ for the Sobolev space $W^{k,p}(D,\mathbb{R}^3)$ and so on. We write $\mathbb{H}^k$ instead of $\mathbb{W}^{k,2}$ and we sometimes denote by $\mathbb{H}$ the Hilbert space $\mathbb{L}^2$ and by V the Hilbert space $\mathbb{H}^1$. The dual space of V is denoted by V′. Occasionally, we use the same notation for the corresponding spaces of functions which take matrix values such as $L^2(D,\mathbb{R}^3\otimes\mathbb{R}^3)$ and $H^1(D,\mathbb{R}^3\otimes\mathbb{R}^3)$.

We use the notation $\langle\cdot,\cdot\rangle$ for an inner product and also for a pairing of vectors from a space and its dual; a norm is denoted by $|\cdot|$. Subscripts denote the appropriate spaces, except that a subscript may be omitted if the space is a Euclidean space.

2 Definition of a Solution and the Main Results

Assumption 2.1. We assume that we have a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$, where $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$ is the filtration, and this probability space satisfies the so-called usual conditions, that is

(i) $\mathbb{P}$ is complete on $(\Omega,\mathcal{F})$;
(ii) for each $t\geq 0$, $\mathcal{F}_t$ contains all $(\mathcal{F},\mathbb{P})$-null sets;
(iii) the filtration $\mathbb{F}$ is right-continuous.

We also assume that defined on this space is a real-valued, $(\mathcal{F}_t)$-adapted Wiener process $(W(t))_{t\geq 0}$. □

Let us observe that since we assume that $h\in\mathbb{L}^\infty$ the map G defined by

$$G:\mathbb{H}\ni u\mapsto u\times h\in\mathbb{H} \tag{2.1}$$

is well defined and, see for instance [8], we have, at least on an informal level, the following equality relating the Stratonovich and Itô differentials:

$$(Gu)\circ\,\mathrm{d}W(t)=\tfrac{1}{2}G'(u)[Gu]\,\mathrm{d}t+G(u)\,\mathrm{d}W(t),\quad u\in\mathbb{H}. \tag{2.2}$$

Since G is a linear map, we infer that $G'(u)[G(u)]=G^2u=(u\times h)\times h$. Thus, we have

$$(Gu)\circ\,\mathrm{d}W(t)=Gu\,\mathrm{d}W(t)+\tfrac{1}{2}(u\times h)\times h\,\mathrm{d}t,\quad u\in\mathbb{H}. \tag{2.3}$$

Denote by A the Laplacian with the Neumann boundary conditions acting on $\mathbb{R}^3$-valued functions, that is

$$D(A) := \left\{ u \in \mathbb{H}^2 : \frac{\partial u}{\partial n} = 0 \text{ on } \partial D \right\}, \qquad Au := -\Delta u, \quad u \in \mathbb{D}(A), \tag{2.4}$$

where $n = (n_1, n_2, n_3)$ is the unit outward normal vector field on ∂D and $\frac{\partial u}{\partial n}$ is the directional derivative of u in the direction n.

It is well known that A is a self-adjoint operator in $\mathbb{H}$ and that $(I + A)^{-1}$ is compact. Hence there exists an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of $\mathbb{H}$ consisting of eigenvectors of A. Define $A_1 := I + A$. It is also known that

$$D(A_1^{1/2}) = \mathbb{H}^1. \tag{2.5}$$

Suppose that $u \in D(A)$ and $v \in \mathbb{H}^1$. Then, by the Stokes theorem, we obtain

$$\langle Au, v\rangle_{\mathbb{L}^2} = \int_D \langle \nabla u(x), \nabla v(x)\rangle \, dx - \int_{\partial D} \left\langle \frac{\partial u}{\partial n}(y), v(y) \right\rangle d\sigma(y) = \int_D \langle \nabla u(x), \nabla v(x)\rangle \, dx, \tag{2.6}$$

where σ is the surface measure on ∂D.

The definition of weak solution will be preceded by some identities, mostly following Visintin's paper [38].

Proposition 2.2. If $v \in \mathrm{V}$ and $u \in D(A)$, then

$$\int_D \langle u(x) \times Au(x), v(x)\rangle \, dx = \sum_i \int_D \left\langle \frac{\partial u}{\partial x_i}(x), \frac{\partial (v \times u)}{\partial x_i}(x) \right\rangle dx. \tag{2.7}$$

$\square$

Proof of formula (2.7). Because $u \in \mathbb{H}^2$, $Au \in \mathbb{L}^2$ and by the Sobolev–Gagliardo inequalities $u \in \mathbb{L}^\infty$. Hence, $u \times Au$ belongs to $\mathbb{L}^2$ and the left-hand side of (2.7) is well defined. Invoking again the Sobolev–Gagliardo inequalities, we obtain $v \in \mathbb{L}^6$, $\frac{\partial v}{\partial x_i} \in \mathbb{L}^2$, $i = 1, 2, 3$, and $u \in \mathbb{L}^\infty$ and $\frac{\partial u}{\partial x_i} \in \mathbb{L}^6$, $j = 1, 2, 3$. Therefore, the right-hand side of equality (2.7) is well defined as well. Since

$$\langle a \times (b \times c), d\rangle = \langle c, (d \times a) \times b\rangle, \quad ab, c, d \in \mathbb{R}^3,$$

we have

$$\int_D \langle u(x) \times Au(x), v(x)\rangle \, dx = \int_D \langle Au(x), v(x) \times u(x)\rangle \, dx = -\sum_i \int_D \left\langle \frac{\partial^2 u}{\partial x_i^2}(x), v(x) \times u(x) \right\rangle dx.$$

Applying the Stokes theorem to the last integral, see, for example [36, Theorem 1.2, p.7], we infer that

$$\begin{aligned}
&\int_D \langle u(x) \times Au(x), v(x)\rangle \,\mathrm{d}x \\
&\quad = \int_D \sum_i \left\langle \frac{\partial u}{\partial x_i}(x), \frac{\partial (v(x) \times u(x))}{\partial x_i} \right\rangle \mathrm{d}x - \int_{\partial D} \sum_i \nu_i \left\langle \frac{\partial u}{\partial x_i}(x), v(x) \times u(x) \right\rangle \mathrm{d}\sigma_x \\
&\quad = \int_D \sum_i \left\langle \frac{\partial u}{\partial x_i}(x), \frac{\partial (v \times u)}{\partial x_i}(x) \right\rangle \mathrm{d}x - \int_{\partial D} \left\langle \frac{\partial u}{\partial \nu}(x), v(x) \times u(x) \right\rangle \mathrm{d}\sigma(x),
\end{aligned}$$

and (2.7) follows. ■

Lemma 2.3. If $v \in \mathrm{V}$ and $u \in D(A)$, then

$$\sum_i \int_D \left\langle \frac{\partial u}{\partial x_i}(x), v(x) \times \frac{\partial u}{\partial x_i}(x) \right\rangle \mathrm{d}x = 0. \tag{2.8}$$

□

Proof. Taking into account that $\langle a \times b, b\rangle = 0$ it is enough to show that the integral in (2.8) is well defined but this follows immediately from the Gagliardo–Nirenberg inequality ■

Thus, we obtain the following result as a direct consequence of Proposition 2.2 and Lemma 2.3.

Corollary 2.4. If $v \in \mathrm{V}$ and $u \in D(A)$, then

$$\int_D \langle u(x) \times Au(x), v(x)\rangle \,\mathrm{d}x = \sum_i \int_D \left\langle \frac{\partial u}{\partial x_i}(x), \frac{\partial v}{\partial x_i}(x) \times u(x) \right\rangle \mathrm{d}x. \tag{2.9}$$

□

Now we are ready to formulate the definition of a weak solution.

Definition 2.5. Given $T \in (0, \infty)$, a weak martingale solution $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}, W, u)$ to Equation (1.7), for the time interval $[0, T]$, consists of a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, with the filtration satisfying the usual conditions, a real-valued $(\mathcal{F}_t)$-adapted Wiener process $W = (W_t)_{t\in[0,T]}$, and a progressively measurable process $u : [0, T] \times \Omega \to \mathbb{H}^1$ such that:

(a) for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$u(\cdot,\omega) \in C([0,T];V') \quad \text{and} \quad \sup_{t\in[0,T]} |u(t,\omega)|_{\mathbb{H}^1} < \infty; \tag{2.10}$$

(b) we have

$$\begin{aligned}&|u(t,x)| = 1, \text{ for Lebesgue—a.e. } x \in D \quad \text{for all } t \in [0,T],\\ &\mathbb{P} \text{ a.e.};\end{aligned} \tag{2.11}$$

(c) for each $\varphi \in \mathbb{W}^{1,4}$, we have:

$$\begin{aligned}\langle u(t),\varphi\rangle_{\mathbb{L}^2} - \langle u_0,\varphi\rangle_{\mathbb{L}^2} = &-\lambda_1 \int_0^t \sum_i \int_D \left\langle \frac{\partial u}{\partial x_i}(s,x), \frac{\partial \varphi}{\partial x_i}(x) \times u(s,x)\right\rangle \mathrm{d}x\,\mathrm{d}s\\ &- \lambda_2 \int_0^t \sum_i \int_D \left\langle \frac{\partial u}{\partial x_i}(s,x), \frac{\partial (u\times\varphi)}{\partial x_i}(s,x) \times u(s,x)\right\rangle \mathrm{d}x\,\mathrm{d}s\\ &+ \int_0^t \int_D \langle u(s,x)\times h(x), \varphi(x)\rangle \mathrm{d}x \circ \mathrm{d}W(s)\end{aligned} \tag{2.12}$$

for all $t \in [0,T]$, $\mathbb{P}$ a.e. □

The significance of conditions (a)–(c) in Definition 2.5 is as follows. Condition (b) requires the magnetization to remain saturated at all times, while condition (a) requires that the magnetic energy stays bounded on the time interval $[0,T]$. If there exists a measurable process u which satisfies conditions (a) and (b) and also takes values in $D(A)$, then, in view of Corollary 2.4, it seems reasonable to say that u solves Equation (1.7) only if condition (c) holds. However, condition (c) makes sense even for processes which satisfy (a) and (b) without taking values in $D(A)$. In the following sections, we will prove the existence of a weak solution which in fact solves Equation (1.7) in a stronger sense than in Definition 2.5. In order to formulate our main result we need to explain precisely how Corollary 2.4 motivates our notation $u \times \Delta u$, where u is our weak martingale solution.

Notation 2.6. We wish to use the notation $u \times \Delta u$ for our weak martingale solution u even when we do not know that u has weak second-order derivatives. By $u \times \Delta u : [0,T] \times \Omega \to \mathbb{H}$, we mean a measurable process in $\mathbb{H}$ such that for almost every $(t,\omega) \in [0,T] \times \Omega$ the following identity is satisfied for all ϕ in $\mathbb{W}^{1,4}$:

$$\langle (u\times\Delta u)(t,\omega),\phi\rangle_{\mathbb{L}^2} = \sum_{i=1}^3 \left\langle \frac{\partial u(t,\omega)}{\partial x_i}, u(t,\omega) \times \frac{\partial \phi}{\partial x_i}\right\rangle_{\mathbb{L}^2}. \tag{2.13}$$

□

Clearly for a process u in $\mathbb{H}^1$ the right-hand side of (2.13) makes sense and since the orthonormal basis $\{e_n : n \in \mathbb{N}\}$ is contained in $\mathbb{W}^{1,4}$, when there exists a process $u \times \Delta u$, u and property (2.13) determines $u \times \Delta u$ uniquely up to a set of measure zero.

Now we can formulate the main result of this paper.

Theorem 2.7. Assume that $u_0 \in \mathbb{H}^1$ satisfies (1.3) and $h \in \mathbb{L}^\infty \cap \mathbb{W}^{1,3}$. For $T \in (0, \infty)$, there exists a weak martingale solution $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}, W, u)$ to Equation (1.7) such that

(a) we have

$$\mathbb{E} \int_0^T |(u \times \Delta u)(t)|^2_{\mathbb{L}^2} \, \mathrm{d}t < \infty \tag{2.14}$$

and

$$\mathbb{E} \left[\sup_{t\in[0,T]} |u(t)|^4_{\mathbb{H}^1} \right] < \infty, \tag{2.15}$$

(b) for every $t \in [0, T]$, $\mathbb{P}$-a.s.,

$$\begin{aligned} u(t) = u_0 + \lambda_1 \int_0^t (u \times \Delta u)(s) \, \mathrm{d}s - \lambda_2 \int_0^t u(s) \times (u \times \Delta u)(s) \, \mathrm{d}s \\ + \int_0^t (u(s) \times h) \circ \mathrm{d}W(s), \end{aligned} \tag{2.16}$$

where the first two integrals are the Bochner integrals in $\mathbb{L}^2$ and the Stratonovich integral is well defined in $\mathbb{L}^2$,

(c) for every $\alpha \in (0, \frac{1}{2})$, $\mathbb{P}$-a.s.,

$$u(\cdot) \in C^\alpha([0, T], \mathbb{L}^2). \tag{2.17}$$

□

This theorem can be considered as an extension of similar result obtained in [2] for the deterministic Landau–Lifschitz–Gilbert equation. Let us note that the approximations used in [2] are different from ours. Part (b) of the theorem shows that a weak solution we constructed has in fact more regularity than required by its definition. This result was also proved in [2] for deterministic equation but was not explicitly stated.

The remaining part of the paper is devoted to the proof of this theorem.

3 Faedo–Galerkin Approximation

Let π_n denote the orthogonal projection from $\mathbb{H}$ onto $\mathbb{H}_n := \mathrm{linspan}\{e_1, \dots, e_n\}$. Let us note that $A(\mathbb{H}_n) \subset \mathbb{H}_n \subset \mathbb{L}^\infty$ so that we have the following.

Lemma 3.1. Define the maps

$$F_n^1 : \mathbb{H}_n \ni u \mapsto \pi_n(u \times \Delta u) \in \mathbb{H}_n, \tag{3.1}$$

$$F_n^2 : \mathbb{H}_n \ni u \mapsto \pi_n(u \times (u \times \Delta u)) \in \mathbb{H}_n, \tag{3.2}$$

$$G_n : \mathbb{H}_n \ni u \mapsto \pi_n(u \times h) \in \mathbb{H}_n. \tag{3.3}$$

The maps F_n^1 and F_n^2 are Lipschitz on balls, the map G_n is linear and

$$|G_n u|_{\mathbb{H}_n} \leq |h|_{\mathbb{L}^\infty} |u|_{\mathbb{H}}, \quad u \in \mathbb{H}_n. \tag{3.4}$$

□

We have the following simple consequences of the identity $\langle a \times b, b\rangle = 0$ and the fact that π_n is self-adjoint.

Lemma 3.2. Assume that $h \in \mathbb{L}^\infty$. Then for all $u \in \mathbb{H}_n$, and $i = 1, 2$,

$$\langle F_n^i(u), u\rangle_{\mathbb{H}} = 0 \quad \text{and} \quad \langle G_n u, u\rangle_{\mathbb{H}} = 0.$$

□

The following simple facts will be useful for estimating the $\mathbb{H}^1$ norm of the Faedo-Galerkin approximants:

Lemma 3.3. For all $u \in \mathbb{H}_n$, and $i = 1, 2$,

$$\langle F_n^1(u), \Delta u\rangle_{\mathbb{H}} = 0, \tag{3.5}$$

$$\langle F_n^2(u), \Delta u\rangle_{\mathbb{H}} = -|u \times \Delta u|_{\mathbb{H}}^2. \tag{3.6}$$

□

Proof of Lemma 3.3. Let us fix $u \in \mathbb{H}_n$. Because $\Delta u \in \mathbb{H}_n$ for $u \in \mathbb{H}_n$, we have the following sequences of equalities:

$$\begin{aligned}
\langle F_n^1(u), \Delta u\rangle_{\mathbb{H}} &= \langle \pi_n(u \times \Delta u), \Delta u\rangle_{\mathbb{H}} = \langle u \times \Delta u, \pi_n(\Delta u)\rangle_{\mathbb{H}} \\
&= \langle u \times \Delta u, \Delta u\rangle_{\mathbb{H}} = 0, \\
\langle F_n^2(u), \Delta u\rangle_{\mathbb{H}} &= \langle \pi_n(u \times (u \times \Delta u)), \Delta u\rangle_{\mathbb{H}} = \langle u \times (u \times \Delta u), \pi_n(\Delta u)\rangle_{\mathbb{H}} \\
&= \langle u \times (u \times \Delta u), \Delta u\rangle_{\mathbb{H}} = -|u \times \Delta u|_{\mathbb{H}}^2,
\end{aligned}$$

where the last equality follows from the identity

$$\langle a \times (a \times b), b\rangle = -|a \times b|^2, \quad a, b \in \mathbb{R}^3.$$

This concludes the proof of the Lemma. ■

Let $F_n = \lambda_1 F_n^1 - \lambda_2 F_n^2$. In view of Lemmas 3.1 and 3.2 the following stochastic differential equation on $\mathbb{H}_n$

$$\begin{aligned} \mathrm{d}u_n(t) &= F_n(u_n(t))\,\mathrm{d}t + G_n u_n(t) \circ \mathrm{d}W(t), \\ u_n(0) &= \pi_n u_0 \end{aligned} \tag{3.7}$$

has a unique strong global solution $u_n = (u_n(t))$, $t \geq 0$ (see, e.g., [11, Theorem 10.6]). Note that, since the map G_n is linear, by putting

$$\hat{F}_n(u) := F_n(u) + \tfrac{1}{2} G_n'(u)(G_n u) = F_n(u) + \tfrac{1}{2} G_n^2 u = F_n(u) + \tfrac{1}{2} \pi_n[(\pi_n(u \times h)) \times h] \tag{3.8}$$

the Stratonovich stochastic differential Equation (3.7) can be written in the following Itô form, see for instance [8],

$$\mathrm{d}u_n(t) = \hat{F}_n(u_n(t))\,\mathrm{d}t + G_n(u_n(t))\,\mathrm{d}W(t).$$

Note also that

$$|u_n(0)|_{\mathbb{H}} = |\pi_n u_0|_{\mathbb{H}} \leq |u_0|_{\mathbb{H}}, \quad n \in \mathbb{N}. \tag{3.9}$$

Definition 3.4 (Fractional power spaces of $A_1 = I + A$). For any nonnegative real number β we define the Hilbert space $X^\beta := D(A_1^\beta)$, which is the domain of the fractional power operator A_1^β with the graph norm $|\cdot|_{X^\beta} := |\mathrm{A}_1^\beta \cdot|_{\mathbb{L}^2}$. The space $X^0 = \mathbb{L}^2$ is identified with its dual. For positive real β, the dual of X^β is denoted by $X^{-\beta}$ and the norm $|\cdot|_{X^{-\beta}}$ of $X^{-\beta}$ satisfies $|x|_{X^{-\beta}} = |A_1^{-\beta} x|_{\mathbb{L}^2}$ when x is in $\mathbb{L}^2$. $\square$

Our aim is to prove the following a priori estimates.

Theorem 3.5. Assume that $h \in \mathbb{L}^\infty \cap \mathbb{W}^{1,3}$ and $T \in (0, \infty)$.

For each $n = 1, 2, \ldots$ and every $t \in [0, T]$

$$|u_n(t)|_{\mathbb{L}^2} = |u_n(0)|_{\mathbb{L}^2} \quad \mathbb{P} \text{ a.s.}, \tag{3.10}$$

moreover, given $2 \le p < \infty$ and $\beta > \frac{1}{4}$ there exists a constant C, which does not depend on n but which may depend on u_0, h, T, p, and β such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\nabla u_n(t)|_{\mathbb{L}^2}^{2p}\right] \le C, \tag{3.11}$$

$$\mathbb{E}\left[\left(\int_0^T |u_n(t) \times \Delta u_n(t)|_{\mathbb{L}^2}^2 \, \mathrm{d}t\right)^p\right] \le C, \tag{3.12}$$

$$\mathbb{E}\left[\left(\int_0^T |u_n(t) \times (u_n(t) \times \Delta u_n(t))|_{\mathbb{L}^{\frac{3}{2}}}^2 \, \mathrm{d}t\right)^{\frac{p}{2}}\right] \le C, \tag{3.13}$$

$$\mathbb{E}\left[\int_0^T |\pi_n[u_n(t) \times (u_n(t) \times \Delta u_n(t))]|_{X^{-\beta}}^2 \, \mathrm{d}t\right] \le C, \tag{3.14}$$

for all $n \in \mathbb{N}$. □

The rest of this section is devoted to the proof of Theorem 3.5.

Proof of inequality (3.10). We will apply the Itô Lemma to a process $\varphi(u_n)$, where $\varphi : \mathbb{H}_n \ni u \mapsto \frac{1}{2}|u|_{\mathbb{H}}^2 \in \mathbb{R}$. Since $G_n^* = -G_n$, in view of Lemmas 3.1 and 3.2, we obtain

$$\begin{aligned}\tfrac{1}{2}\mathrm{d}|u_n|_{\mathbb{H}}^2 &= \langle \hat{F}_n(u_n), u_n\rangle_{\mathbb{H}} \, \mathrm{d}t + \langle G_n u_n, u_n\rangle_{\mathbb{H}} \, \mathrm{d}W + \tfrac{1}{2}|G_n u_n|_{\mathbb{H}}^2 \, \mathrm{d}t \\ &= \tfrac{1}{2}\langle G_n^2 u_n, u_n\rangle_{\mathbb{H}} \, \mathrm{d}t - \tfrac{1}{2}\langle G_n^2 u_n, u_n\rangle_{\mathbb{H}} \, \mathrm{d}t = 0.\end{aligned}$$

Therefore,

$$\tfrac{1}{2}|u_n(t)|_{\mathbb{H}}^2 = \tfrac{1}{2}|u_n(0)|_{\mathbb{H}}^2, \quad \text{for all } t \ge 0, \ \mathbb{P} \text{ a.s.}$$

as required. ■

Proof of inequality (3.11). Let us introduce a function ϕ defined by

$$\phi(u) = \tfrac{1}{2}|\nabla u|_{\mathbb{H}}^2, \quad u \in \mathbb{H}_n.$$

Then

$$\phi'(u)g = \langle \nabla u, \nabla g\rangle_{\mathbb{H}} \quad \text{and} \quad \phi''(u)(g,k) = \langle \nabla g, \nabla k\rangle_{\mathbb{H}}, \ \text{for all } u, g, k \in \mathbb{H}_n.$$

Hence, by the Stokes formula (2.6), we have

$$\phi'(u)g = -\langle \Delta u, g\rangle_{\mathbb{H}}, \quad u, g \in \mathbb{H}_n. \tag{3.15}$$

Therefore, the Itô formula from [31] yields for all $t \geq 0$, $\mathbb{P}$-a.s.

$$\phi(u_n(t)) - \phi(u_n(0)) = \int_0^t \langle \nabla u_n(s), \nabla[\hat{F}_n(u_n)](s)\rangle_{\mathbb{H}}\, \mathrm{d}s + \int_0^t \langle \nabla u_n(s), \nabla[G_n u_n](s)\rangle_{\mathbb{H}}\, \mathrm{d}W(s) + \frac{1}{2}\int_0^t |\nabla[G_n u_n](s)|_{\mathbb{H}}^2\, \mathrm{d}s.$$

Since, by Lemma 3.3 and formulae (3.8)–(3.15), we have

$$\begin{aligned}\langle \nabla u_n(s), \nabla[\hat{F}_n(u_n)](s)\rangle_{\mathbb{H}} &= \langle \nabla u_n(s), \nabla[F_n(u_n)](s)\rangle_{\mathbb{H}} + \tfrac{1}{2}\langle \nabla u_n(s), \nabla[G_n^2(u_n)](s)\rangle_{\mathbb{H}}\\ &= -\langle \Delta u_n(s), F_n(u_n(s))\rangle_{\mathbb{H}} + \tfrac{1}{2}\langle \nabla u_n(s), \nabla[G_n^2(u_n)](s)\rangle_{\mathbb{H}}\\ &= -\lambda_2 |u_n(s) \times \Delta u_n(s)|_{\mathbb{H}}^2 + \tfrac{1}{2}\langle \nabla u_n(s), \nabla[G_n^2(u_n)](s)\rangle_{\mathbb{H}},\end{aligned}$$

we obtain for $t \geq 0$, $\mathbb{P}$-a.s.

$$\begin{aligned}&\frac{1}{2}|\nabla u_n(t)|_{\mathbb{H}}^2 + \lambda_2 \int_0^t |u_n(s) \times \Delta u_n(s)|_{\mathbb{H}}^2 ds\\ &= \frac{1}{2}|\nabla u_n(0)|_{\mathbb{H}}^2 + \int_0^t \langle \nabla u_n(s), \nabla[G_n u_n](s)\rangle_{\mathbb{H}}\, \mathrm{d}W(s)\\ &\quad + \frac{1}{2}\int_0^t |\nabla[G_n u_n](s)|_{\mathbb{H}}^2\, \mathrm{d}s + \frac{1}{2}\int_0^t \langle \nabla u_n(s), \nabla[G_n^2(u_n)](s)\rangle_{\mathbb{H}}\, \mathrm{d}s.\end{aligned} \tag{3.16}$$

Recall that we defined $A_1 = I + A$, thus we obtain the estimate

$$\begin{aligned}|\nabla G_n u_n(s)|_{\mathbb{L}^2}^2 &\leq |A_1^{\frac{1}{2}} \pi_n(u_n(s) \times h)|_{\mathbb{L}^2}^2 = |G_n u_n(s)|_{\mathbb{H}^1}^2\\ &\leq |A_1^{\frac{1}{2}}(u_n(s) \times h)|_{\mathbb{L}^2}^2 = |u_n(s) \times h|_{\mathbb{H}^1}^2\\ &= |\nabla(u_n(s) \times h)|_{\mathbb{L}^2}^2 + |u_n(s) \times h|_{\mathbb{L}^2}^2\\ &\leq 2[|\nabla u_n(s) \times h|_{\mathbb{L}^2}^2 + |u_n(s) \times \nabla h|_{\mathbb{L}^2}^2] + |u_n(s) \times h|_{\mathbb{L}^2}^2.\end{aligned} \tag{3.17}$$

From Hölder's inequality and the Sobolev imbedding of $\mathbb{H}^1$ into $\mathbb{L}^6$, we have

$$\begin{aligned}|u_n(s) \times \nabla h|_{\mathbb{L}^2}^2 &\leq |u_n(s)|_{\mathbb{L}^6}^2 |\nabla h|_{\mathbb{L}^3}^2\\ &\leq c(|\nabla u_n(s)|_{\mathbb{L}^2}^2 + |u_n(s)|_{\mathbb{L}^2}^2)|\nabla h|_{\mathbb{L}^3}^2.\end{aligned} \tag{3.18}$$

We obtain from (3.17), (3.18), and (3.10)

$$\begin{aligned}|\nabla G_n u_n(s)|_{\mathbb{L}^2}^2 &\leq 2[|h|_{\mathbb{L}^\infty}^2 |\nabla u_n(s)|_{\mathbb{L}^2}^2 + c(|\nabla u_n(s)|_{\mathbb{L}^2}^2 + |u_n(s)|_{\mathbb{L}^2}^2)|\nabla h|_{\mathbb{L}^3}^2] + |h|_{\mathbb{L}^\infty}^2 |u_n(s)|_{\mathbb{L}^2}^2\\ &\leq a|\nabla u_n(s)|_{\mathbb{L}^2}^2 + b, \quad \text{for all } s \geq 0, \ \mathbb{P}\text{-a.s.}\end{aligned} \tag{3.19}$$

for some constants a and b depending only on h and u_0.

The same reasoning that leads to inequality (3.19) yields the following estimate.

$$\begin{aligned}|\nabla G_n^2 u_n(s)|_{\mathbb{L}^2}^2 &\le a|\nabla(G_n u_n(s))|_{\mathbb{L}^2}^2 + b\\ &\le a(a|\nabla u_n(s)|_{\mathbb{L}^2}^2 + b) + b\\ &= a_1|\nabla u_n(s)|_{\mathbb{L}^2}^2 + b_1 \quad \text{for all } s\ge 0,\ \mathbb{P}\text{-a.s..}\end{aligned} \tag{3.20}$$

Thanks to the estimates in (3.19) and (3.20), we have, from Equation (3.16),

$$\begin{aligned}\sup_{r\in[0,t]}|\nabla u_n(r)|_{\mathbb{L}^2}^2 \le |\nabla u_n(0)|_{\mathbb{L}^2}^2 &+ 2\sup_{r\in[0,t]}\left|\int_0^r \langle\nabla u_n(s),\nabla G_n u_n(s)\rangle_{\mathbb{L}^2}\,\mathrm{d}W(s)\right|\\ &+\int_0^t (a_2|\nabla u_n(s)|_{\mathbb{L}^2}^2 + b_2)\,\mathrm{d}s \quad \text{for all } t\ge 0,\ \mathbb{P}\text{-a.s.,}\end{aligned} \tag{3.21}$$

where a_2 and b_2 depend on h and u_0 but not on n.

Take $p\in[2,\infty)$ and recall that $T\in(0,\infty)$. We now raise both sides of inequality (3.21) to the power p, take expectations and invoke the Burkholder–Davis–Gundy (BDG) inequality (see Appendix 3) and Jensen's inequality to obtain for any $t\in[0,T]$:

$$\begin{aligned}\mathbb{E}\left[\sup_{r\in[0,t]}|\nabla u_n(r)|_{\mathbb{L}^2}^{2p}\right] &\le c\mathbb{E}\left[\left(\int_0^t \langle\nabla u_n(s),\nabla G_n u_n(s)\rangle_{\mathbb{L}^2}^2\,\mathrm{d}s\right)^{\frac{p}{2}}\right] + \int_0^t (a_3\mathbb{E}[|\nabla u_n(s)|_{\mathbb{L}^2}^{2p}] + b_3)\,\mathrm{d}s\\ &\le a_T + b_T\int_0^t \mathbb{E}\left[\sup_{r\in[0,s]}|\nabla u_n(r)|_{\mathbb{L}^2}^{2p}\right] ds,\end{aligned} \tag{3.22}$$

where c, a_3, b_3, a_T, and b_T depend on h, u_0, p, and T but not on n. Finally, using the Gronwall inequality we obtain (3.11). ■

Proof of inequality (3.12). Since $\lambda_2>0$, we can proceed from identity (3.16) and write down inequalities like (3.21) and (3.22) but with $\int_0^t |u_n(s)\times\Delta u_n(s)|_{\mathrm{H}}^2\,ds$ on the left-hand side in place of $\sup_{r\in[0,t]}|\nabla u_n(r)|_{\mathbb{L}^2}^2$. Inequality (3.12) then follows immediately by using inequality (3.11) on the right-hand side. ■

Proof of inequality (3.13). By Hölder's inequality and the Sobolev imbedding of $\mathbb{H}^1$ into $\mathbb{L}^6$ we have

$$\begin{aligned}|u_n(t)\times(u_n(t)\times\Delta u_n(t))|_{\mathbb{L}^{\frac{3}{2}}} &\le |u_n(t)|_{\mathbb{L}^6}|u_n(t)\times\Delta u_n(t)|_{\mathbb{L}^2}\\ &\le c|u_n(t)|_{\mathbb{H}^1}|u_n(t)\times\Delta u_n(t)|_{\mathbb{L}^2}.\end{aligned} \tag{3.23}$$

We use the inequality (3.23) to obtain the estimate in (3.13) (in the following, c_1 and c_2 are constants not depending on n):

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T |u_n(t)\times(u_n(t)\times\Delta u_n(t))|^2_{\mathbb{L}^{\frac{3}{2}}}\,\mathrm{d}t\right)^{\frac{p}{2}}\right]\\
&\le c_1\mathbb{E}\left[\sup_{r\in[0,T]}|u_n(r)|^p_{\mathbb{H}^1}\left(\int_0^T |u_n(t)\times\Delta u_n(t)|^2_{\mathbb{L}^2}\,\mathrm{d}t\right)^{\frac{p}{2}}\right]\\
&\le c_1\left(\mathbb{E}\left[\sup_{t\in[0,T]}|u_n(t)|^{2p}_{\mathbb{H}^1}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\left(\int_0^T |u_n(t)\times\Delta u_n(t)|^2_{\mathbb{L}^2}\,\mathrm{d}t\right)^{p}\right]\right)^{\frac{1}{2}}\\
&\le c_2,
\end{aligned}$$

thanks to inequalities (3.10)–(3.12). ■

Proof of inequality (3.14). Let $\beta>\frac{1}{4}$. The fractional power space X^{β} is continuously imbedded in $\mathbb{L}^3$ (see, e.g., [22, Theorem 1.6.1]). Consequently, $\mathbb{L}^{\frac{3}{2}}$ is continuously imbedded in $X^{-\beta}$. Thus, we have

$$\begin{aligned}
&\mathbb{E}\int_0^T |\pi_n[u_n(t)\times(u_n(t)\times\Delta u_n(t))]|^2_{X^{-\beta}}\,\mathrm{d}t\\
&\le \mathbb{E}\int_0^T |u_n(t)\times(u_n(t)\times\Delta u_n(t))|^2_{X^{-\beta}}\,\mathrm{d}t\\
&\le c\mathbb{E}\int_0^T |u_n(t)\times(u_n(t)\times\Delta u_n(t))|^2_{\mathbb{L}^{\frac{3}{2}}}\,\mathrm{d}t,
\end{aligned}$$

and we invoke inequality (3.13) to complete the proof. ■

4 Tightness, Continuing the Proof of Theorem 2.7

In this section, we show that the set of laws $\{\mathcal{L}(u_n): n\in\mathbb{N}\}$, on a suitable path space, is tight; we then use Skorohod's theorem to obtain a probability space and a pointwise convergent sequence defined on this space whose limit is a weak martingale solution of Equation (1.7). We leave for Section 5 the verification of property (c) of Definition 2.5 and the proofs of Theorem 2.7(b) and (c).

Equation (3.7) can be written in the following way:

$$\begin{aligned}
u_n(t) &= u_{0,n}+\lambda_1\int_0^t F_n^1(u_n(s))\,\mathrm{d}s-\lambda_2\int_0^t F_n^2(u_n(s))\,\mathrm{d}s+\frac{1}{2}\int_0^t G_n^2u_n(s)\,\mathrm{d}s+\int_0^t G_nu_n(s)\,\mathrm{d}W(s),\\
&=: u_{0,n}+\sum_{i=1}^4 u_n^i(t)\quad t\ge 0.
\end{aligned}$$

Our first aim is to prove the following:

Lemma 4.1. The terms u_n^i as path-valued random variables have the following uniform bounds.

(1) There exists $C\in(0,\infty)$ such that, for all $n\in\mathbb{N}$,

$$\mathbb{E}[|u_n^1|^2_{W^{1,2}(0,T;\mathbb{L}^2)}]\leq C.$$

(2) For each $\beta>\frac{1}{4}$ there exists $C\in(0,\infty)$ such that, for all $n\in\mathbb{N}$,

$$\mathbb{E}[|u_n^2|^2_{W^{1,2}(0,T;X^{-\beta})}]\leq C.$$

(3) There exists $C\in(0,\infty)$ such that for all $n\in\mathbb{N}$

$$|u_n^3|^2_{W^{1,2}(0,T;\mathbb{L}^2)}\leq C,\quad \mathbb{P}\text{-a.s.}$$

(4) For each $p\in[2,\infty)$ and $\alpha\in(0,\frac{1}{2})$ there exists $C>0$ such that, for all $n\in\mathbb{N}$,

$$\mathbb{E}[|u_n^4|^p_{W^{\alpha,p}(0,T;\mathbb{L}^2)}]\leq C. \tag{4.1}$$

It follows that for each $\beta>\frac{1}{4}$, $\alpha\in(0,\frac{1}{2})$, and $p\in[2,\infty)$ we have the uniform bound for u_n:

$$\sup_{n\in\mathbb{N}}\mathbb{E}[|u_n|^2_{W^{\alpha,p}(0,T;X^{-\beta})}]<\infty. \tag{4.2}$$

□

Proof. The first three of the above inequalities follow from Theorem 3.5.

In order to prove (4.1) let us first observe that in view of the first a priori estimate (3.10) in Theorem 3.5, for each $p\in[2,\infty)$, we have

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left[\sup_{t\in[0,T]}|u_n(t)|^p_{\mathbb{L}^2}\right]<\infty.$$

Thus, inequality (4.1) is a direct consequence of inequality (3.4) and Lemma A.1.

The inequality (4.2) follows from the other inequalities thanks to the continuous imbeddings

(i) $\mathbb{L}^2\hookrightarrow X^{-\beta}$, and,

(ii) by Simon [33, Corollary 18, p. 138], $W^{1,2}(0,T;X^{-\beta})\hookrightarrow W^{\alpha,p}(0,T;X^{-\beta})$ if $\frac{1}{2}>\alpha-\frac{1}{p}$. ■

Lemma 4.2. For any $p \in [2,\infty)$ and $q \in [2,6)$ and $\beta > \frac{1}{4}$ the set of laws $\{\mathcal{L}(u_n) : n \in \mathbb{N}\}$ on the Banach space

$$L^p(0,T;\mathbb{L}^q) \cap C([0,T];X^{-\beta})$$

is tight. □

Proof. We use some quite deep compactness results in path spaces by Flandoli and Gatarek [15] and also listed in Appendix 2, together with the fact that X^{ν} is compactly imbedded in $X^{\nu'}$ whenever ν and ν' are real numbers with $\nu > \nu'$ (see, e.g., [22, Theorem 1.4.8]). The idea is to show that the laws $\mathcal{L}(u_n)$ are concentrated on a ball in a space of paths which is compactly imbedded in another space of paths.

Now to be more precise. Take $p \in [2,\infty)$ and $\beta' > \frac{1}{4}$ and $\alpha \in (0,\frac{1}{2})$.

For $\gamma \in [-\beta',\frac{1}{2})$ the space $L^p(0,T;\mathbb{H}^1) \cap W^{\alpha,p}(0,T;X^{-\beta'})$ is compactly imbedded in $L^p(0,T;X^{\gamma})$, by Lemma A.2 and [15, Theorem 2.1]. We note that for any positive real number r

$$\begin{aligned}
&\mathbb{P}\{|u_n|_{L^p(0,T;\mathbb{H}^1)\cap W^{\alpha,p}(0,T;X^{-\beta'})} > r\}\\
&\quad \le \mathbb{P}\left\{|u_n|_{L^p(0,T;\mathbb{H}^1)} > \frac{r}{2}\right\} + \mathbb{P}\left\{|u_n|_{W^{\alpha,p}(0,T;X^{-\beta'})} > \frac{r}{2}\right\}\\
&\quad \le \frac{4}{r^2}\mathbb{E}[|u_n|^2_{L^p(0,T;\mathbb{H}^1)} + |u_n|^2_{W^{\alpha,p}(0,T;X^{-\beta'})}]
\end{aligned}$$

and the expected value on the right-hand side of the last inequality is uniformly bounded in n, by the estimates in (4.2), (3.10), and (3.11). This implies that the family of laws $\{\mathcal{L}(u_n) : n \in \mathbb{N}\}$ is tight on $L^p(0,T;X^{\gamma})$. Given $q \in [2,6)$ we can find $\gamma < \frac{1}{2}$ such that X^{γ} is continuously imbedded in $\mathbb{L}^q$ (see [22, Theorem 1.6.1]) and hence $L^p(0,T;X^{\gamma})$ is continuously imbedded in $L^p(0,T;\mathbb{L}^q)$. Since tightness of laws is preserved by continuous maps, $\{\mathcal{L}(u_n) : n \in \mathbb{N}\}$ is also tight on $L^p(0,T;\mathbb{L}^q)$.

By Flandoli and Gatarek [15, Theorem 2.2], if $\beta > \beta'$ and $\alpha p > 1$, then $W^{\alpha,p}(0,T;X^{-\beta'})$ is compactly imbedded in $C([0,T];X^{-\beta})$. This allows us to conclude that the family of laws $\{\mathcal{L}(u_n) : n \in \mathbb{N}\}$ is tight on $C([0,T];X^{-\beta})$.

The lemma is proved by combining these two tightness results. ■

4.1 Proof of the existence of a solution

By Lemma 4.2, we can find a subsequence of (u_n), denoted in the same way as the full sequence, such that the laws $\mathcal{L}(u_n, W)$ converge weakly to a certain probability measure μ on $L^p(0,T;\mathbb{L}^q) \cap C([0,T];X^{-\beta}) \times C([0,T];\mathbb{R})$, where $p \in [2,\infty)$, $q \in [2,6)$ and $\beta > \frac{1}{4}$ are chosen real numbers. In what follows, we choose $p = 4$, $q = 4$, and $\beta = \frac{1}{2}$.

Proposition 4.3. There exists a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and there exists a sequence (u_n', W_n') of $L^4(0,T;\mathbb{L}^4)\cap C([0,T];X^{-\beta})\times C([0,T];\mathbb{R})$-valued random variables defined on $(\Omega', \mathcal{F}', \mathbb{P}')$ such that the laws of (u_n, W) and (u_n', W_n') on $L^4(0,T;\mathbb{L}^4)\cap C([0,T];X^{-\beta})\times C([0,T];\mathbb{R})$ are equal for each $n\in\mathbb{N}$ and there exists an $L^4(0,T;\mathbb{L}^4)\cap C([0,T];X^{-\beta})\times C([0,T];\mathbb{R})$-valued random variable (u', W') defined on $(\Omega', \mathcal{F}', \mathbb{P}')$ such that the law of (u', W') on $L^4(0,T;\mathbb{L}^4)\cap C([0,T];X^{-\beta})\times C([0,T];\mathbb{R})$ is equal to μ and

$$u_n' \to u' \text{ in } L^4(0,T;\mathbb{L}^4)\cap C([0,T];X^{-\beta})\ \mathbb{P}' \text{ a.s.} \tag{4.3}$$

and

$$W_n' \to W' \text{ in } C([0,T];\mathbb{R})\ \mathbb{P}' \text{ a.s.} \tag{4.4}$$

□

Proof. This result follows from the Skorohod theorem, see, for example [23, Theorem 4.30], once we observe that $L^4(0,T;\mathbb{L}^4)\cap C([0,T];X^{-\beta})\times C([0,T];\mathbb{R})$ is a separable metric space. ■

Remark 4.4. The Borel subsets of $C([0,T];\mathbb{H}_n)$ are Borel subsets of $L^4(0,T;\mathbb{L}^4)\cap C([0,T];X^{-\beta})$ (see [37, Theorem 1.1, Chapter 1]) and $\mathbb{P}\{u_n\in C([0,T];\mathbb{H}_n)\}=1$. Hence, we may assume that u_n' takes values in $\mathbb{H}_n$ and that the laws on $C([0,T];\mathbb{H}_n)$ of u_n and u_n' are equal. □

Thanks to Remark 4.4, it is straightforward to show that the sequence (u_n') satisfies the same estimates as the original sequence (u_n). In particular, from Theorem 3.5

$$\sup_{t\in[0,T]} |u_n'(t)|_{\mathbb{L}^2} \le |u_0|_{\mathbb{L}^2},\ \mathbb{P}'\text{-a.s.}, \tag{4.5}$$

$$\sup_{n\in\mathbb{N}} \mathbb{E}'\left[\sup_{t\in[0,T]} |u_n'(t)|_{\mathbb{H}^1}^{2r}\right] < \infty \quad \text{for any real } r\ge 1, \tag{4.6}$$

$$\sup_{n\in\mathbb{N}} \mathbb{E}'\left[\left(\int_0^T |u_n'(t)\times \Delta u_n'(t)|_{\mathbb{L}^2}^2\, dt\right)^r\right] < \infty \quad \text{for any real } r\ge 1, \tag{4.7}$$

$$\sup_{n\in\mathbb{N}} \mathbb{E}'\int_0^T |u_n'(t)\times (u_n'(t)\times \Delta u_n'(t))|_{\mathbb{L}^{3/2}}^2\, dt < \infty, \tag{4.8}$$

$$\sup_{n\in\mathbb{N}} \mathbb{E}'\int_0^T \left|\pi_n[u_n'(t)\times (u_n'(t)\times \Delta u_n'(t))]\right|_{X^{-\beta}}^2\, dt < \infty. \tag{4.9}$$

Estimates (4.5) and (4.6) for u'_n lead to corresponding estimates for u'. Extend the definitions of $|\cdot|_{\mathbb{H}}$ and $|\cdot|_{\mathbb{H}^1}$ to the domain $X^{-\frac{1}{2}}$ as follows:

$$|v|_{\mathbb{H}} := \infty \text{ if } v \in X^{-\frac{1}{2}} \backslash \mathbb{H}$$

and

$$|v|_{\mathbb{H}^1} := \infty \text{ if } v \in X^{-\frac{1}{2}} \backslash \mathbb{H}^1;$$

these extended maps are lower semicontinuous. Similarly, the extended maps

$$v \in C([0,T]; X^{-\frac{1}{2}}) \mapsto \sup_{t\in[0,T]} |v(t)|_{\mathbb{H}} \quad \text{and} \quad v \in C([0,T]; X^{-\frac{1}{2}}) \mapsto \sup_{t\in[0,T]} |v(t)|_{\mathbb{H}^1}$$

are lower semicontinuous. Therefore, pointwise convergence of u'_n to u' in $C([0,T]; X^{-\frac{1}{2}})$ and (4.5) imply that

$$\sup_{t\in[0,T]} |u'(t)|_{\mathbb{H}} \le |u_0|_{\mathbb{H}} \tag{4.10}$$

and

$$\sup_{t\in[0,T]} |u'(t)|_{X^{-\beta}} \le c|u_0|_{\mathbb{H}} \tag{4.11}$$

$\mathbb{P}'$ a.s., where c in (4.11) is a positive real constant. Pointwise convergence of u'_n to u' in $C([0,T]; X^{-\frac{1}{2}})$ and inequality (4.6) and Fatou's lemma imply that u' enjoys the following property:

$$\mathbb{E}'\left[\sup_{t\in[0,T]} |u'(t)|^4_{\mathbb{H}^1}\right] < \infty. \tag{4.12}$$

Pointwise convergence of u'_n to u' in $L^4(0,T;\mathbb{L}^4)$ and uniform integrability from (4.6) and (4.12) yield

$$\mathbb{E}'\int_0^T |u'_n(t) - u'(t)|^4_{\mathbb{L}^4}\, \mathrm{d}t \to 0 \tag{4.13}$$

as n goes to infinity.

By inequalities (4.7)–(4.9), we can also assume that, given real $r \ge 1$, there exist measurable processes on $[0,T]\times\Omega'$, $Y \in L^{2r}(\Omega';\ L^2(0,T;\mathbb{L}^2))$ and $Z \in L^2(\Omega';\ L^2(0,T;\mathbb{L}^{\frac{3}{2}}))$ such that

$$u'_n \times \Delta u'_n \to Y \text{ weakly in } L^{2r}(\Omega'; L^2(0,T;\mathbb{L}^2)), \tag{4.14}$$

$$u'_n \times (u'_n \times \Delta u'_n) \to Z \text{ weakly in } L^2(\Omega'; L^2(0,T;\mathbb{L}^{\frac{3}{2}})), \tag{4.15}$$

and

$$\pi_n(u'_n \times (u'_n \times \Delta u'_n)) \to Z \text{ weakly in } L^2(\Omega'; L^2(0,T; X^{-\beta})). \tag{4.16}$$

Inequality (4.6) implies that $\sup_{n\in\mathbb{N}} \mathbb{E}' \int_0^T |u'_n(t)|^2_{\mathbb{H}^1}\, dt < \infty$ and this implies that we can assume there is weak convergence of weak derivatives

$$\frac{\partial u'_n}{\partial x_i} \to \frac{\partial u'}{\partial x_i} \text{ weakly in } L^2(\Omega'; L^2(0,T;\mathbb{L}^2)), \quad i = 1,2,3. \tag{4.17}$$

We use this fact to prove Lemma 4.5. This lemma shows that the process Y from (4.14) is $u' \times \Delta u'$ in the weak sense described in Notation 2.6.

Lemma 4.5. For any measurable process $\varphi \in L^4(\Omega'; L^4(0,T;\mathbb{W}^{1,4}))$ in the Sobolev space $\mathbb{W}^{1,4}$, we have the equality

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}' \int_0^T \langle u'_n(s) \times \Delta u'_n(s), \varphi(s)\rangle_{\mathbb{L}^2}\, ds \\ = \mathbb{E}' \int_0^T \langle Y(s), \varphi(s)\rangle_{\mathbb{L}^2}\, ds \\ = \mathbb{E}' \int_0^T \sum_{i=1}^3 \left\langle \frac{\partial u'(s)}{\partial x_i}, u'(s) \times \frac{\partial \varphi(s)}{\partial x_i}\right\rangle_{\mathbb{L}^2} ds. \end{aligned}$$

□

Remark 4.6. It follows by considering processes of the form $\varphi(t,\omega) := \chi_B(t,\omega)\psi$, where χ_B is the indicator of a measurable subset B of $[0,T] \times \Omega'$ and ψ is a fixed element of $\mathbb{W}^{1,4}$, that for each $\psi \in \mathbb{W}^{1,4}$

$$\langle Y(t,\omega), \psi\rangle_{\mathbb{L}^2} = \sum_{i=1}^3 \left\langle \frac{\partial u'(t,\omega)}{\partial x_i}, u'(t,\omega) \times \frac{\partial \psi}{\partial x_i}\right\rangle_{\mathbb{L}^2} \tag{4.18}$$

for almost every $(t,\omega) \in [0,T] \times \Omega'$. Since $\mathbb{W}^{1,4}$ is separable, for (t,ω) outside a set of measure zero, equality (4.18) holds for all $\psi \in \mathbb{W}^{1,4}$. □

Proof. We now prove Lemma 4.5. By Corollary 2.4, for each $n \in \mathbb{N}$, we have

$$\langle u'_n(t) \times \Delta u'_n(t), \varphi\rangle_{\mathbb{L}^2} = \sum_{i=1}^3 \left\langle \frac{\partial u'_n(t)}{\partial x_i}, u'_n(t) \times \frac{\partial \varphi(t)}{\partial x_i}\right\rangle_{\mathbb{L}^2} \tag{4.19}$$

for almost every $t \in [0,T]$ $\mathbb{P}'$ a.s. Indeed, by Remark 4.4 the law of u'_n is supported by $C([0,T];\mathbb{H}_n)$ and $\mathbb{H}_n \subset D(A)$. For each $i \in \{1,2,3\}$, we may write

$$\left\langle \frac{\partial u'_n}{\partial x_i}, u'_n \times \frac{\partial \varphi}{\partial x_i}\right\rangle_{\mathbb{L}^2} - \left\langle \frac{\partial u'}{\partial x_i}, u' \times \frac{\partial \varphi}{\partial x_i}\right\rangle_{\mathbb{L}^2} = \left\langle \frac{\partial u'_n}{\partial x_i} - \frac{\partial u'}{\partial x_i}, u' \times \frac{\partial \varphi}{\partial x_i}\right\rangle_{\mathbb{L}^2} + \left\langle \frac{\partial u'_n}{\partial x_i}, (u'_n - u') \times \frac{\partial \varphi}{\partial x_i}\right\rangle_{\mathbb{L}^2}. \tag{4.20}$$

Because $\varphi \in L^4(\Omega'; L^4(0,T; \mathbb{W}^{1,4}))$, in view of (4.13) and (4.6) and Hölder's inequality, we infer that

$$\mathbb{E}' \int_0^T \left| \left\langle \frac{\partial u_n'(t)}{\partial x_i}, (u_n'(t) - u'(t)) \times \frac{\partial \varphi(t)}{\partial x_i} \right\rangle_{\mathbb{L}^2} \right| \mathrm{d}t$$
$$\leq C \left(\mathbb{E}' \left[\sup_{t\in[0,T]} |u_n'(t)|^2_{\mathbb{H}^1} \right] \right)^{\frac{1}{2}} \times \left(\mathbb{E}' \int_0^T |u_n'(t) - u'(t)|^4_{\mathbb{L}^4} \, \mathrm{d}t \right)^{\frac{1}{4}} \times \left(\mathbb{E}' \int_0^T |\varphi(t)|^4_{\mathbb{W}^{1,4}} \, \mathrm{d}t \right)^{\frac{1}{4}} \to 0. \tag{4.21}$$

Since the process $u' \times \frac{\partial \varphi}{\partial x_i}$ is in $L^2(\Omega'; L^2(0,T); \mathbb{L}^2)$, the weak convergence of weak derivatives in (4.17) yields

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T \left\langle \frac{\partial u_n'(t)}{\partial x_i} - \frac{\partial u'(t)}{\partial x_i}, u'(t) \times \frac{\partial \varphi(t)}{\partial x_i} \right\rangle_{\mathbb{L}^2} \mathrm{d}t = 0. \tag{4.22}$$

Hence we have proved that

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T \left\langle \frac{\partial u_n'(t)}{\partial x_i}, u_n'(t) \times \frac{\partial \varphi(t)}{\partial x_i} \right\rangle_{\mathbb{L}^2} \mathrm{d}t = \mathbb{E}' \int_0^T \left\langle \frac{\partial u'(t)}{\partial x_i}, u'(t) \times \frac{\partial \varphi(t)}{\partial x_i} \right\rangle_{\mathbb{L}^2} \mathrm{d}t. \tag{4.23}$$

We also have, using (4.14):

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T \langle u_n'(t) \times \Delta u_n'(t), \varphi \rangle_{\mathbb{L}^2} \, \mathrm{d}t = \mathbb{E}' \int_0^T \langle Y(t), \varphi \rangle_{\mathbb{L}^2} \, \mathrm{d}t. \tag{4.24}$$

We equate the expressions from the right-hand sides of (4.23) and (4.24) to prove the lemma. ■

We will need the next lemma when we come to apply Itô's formula to show that $|u'(t,\omega)| = 1$ a.e. on D.

Lemma 4.7. For any bounded measurable function $\phi : D \to \mathbb{R}$ we have $\langle Y(s,\omega), \phi u'(s,\omega) \rangle_{\mathbb{L}^2} = 0$ for almost every $(s,\omega) \in [0,T] \times \Omega'$. □

Proof. Let $B \subset [0,T] \times \Omega'$ be a measurable set and χ_B be the indicator function of B. Since u_n' converges to u' in $L^2(\Omega'; L^2(0,T; \mathbb{L}^2))$ (see (4.13)), we also have that $\chi_B \phi u_n'$ converges to $\chi_B \phi u'$ in $L^2(\Omega'; L^2(0,T; \mathbb{L}^2))$. This fact and the fact that $u_n' \times \Delta u_n'$ converges to Y weakly in $L^2(\Omega'; L^2(0,T; \mathbb{L}^2))$ implies that

$$0 = \mathbb{E}' \int_0^T \chi_B(s) \langle u_n'(s) \times \Delta u_n'(s), \phi u_n'(s) \rangle_{\mathbb{L}^2} \, \mathrm{d}s \to \mathbb{E}' \int_0^T \chi_B(s) \langle Y(s), \phi u'(s) \rangle_{\mathbb{H}} \, \mathrm{d}s$$

as n goes to infinity. This proves the lemma. ■

Now we show that the process Z from (4.15) is $u' \times Y$.

Lemma 4.8. For any process $\psi \in L^4(\Omega'; L^4(0,T;\mathbb{L}^4))$, we have

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T \langle u_n'(s) \times (u_n'(s) \times \Delta u_n'(s)), \psi(s)\rangle_{\mathbb{L}^2}\, ds$$

$$= \mathbb{E}' \int_0^T {}_{\mathbb{L}^{\frac{3}{2}}}\langle Z(s), \psi(s)\rangle_{\mathbb{L}^3}\, ds \tag{4.25}$$

$$= \mathbb{E}' \int_0^T {}_{\mathbb{L}^{\frac{3}{2}}}\langle u'(s) \times Y(s), \psi(s)\rangle_{\mathbb{L}^3}\, ds. \tag{4.26}$$

□

Remark 4.9. Since $L^4(\Omega'; L^4(0,T;\mathbb{L}^4))$ is dense in $L^2(\Omega'; L^2(0,T;\mathbb{L}^3))$, we may conclude that $Z = u' \times Y$ as elements of $L^2(\Omega'; L^2(0,T;\mathbb{L}^{\frac{3}{2}}))$. □

Proof. Put $Y_n := u_n' \times \Delta u_n'$ for each $n \in \mathbb{N}$.

Since $\psi \in L^4(\Omega'; L^4(0,T;\mathbb{L}^4))$ defines an element of the dual space of $L^2(\Omega'; L^2(0,T;\mathbb{L}^{\frac{3}{2}}))$, (4.15) implies that the first equality in the statement of the lemma holds.

Now to prove equality (4.26). We may write

$$\begin{aligned}\langle u_n' \times Y_n, \psi\rangle_{\mathbb{L}^2} - {}_{\mathbb{L}^{\frac{3}{2}}}\langle u' \times Y, \psi\rangle_{\mathbb{L}^3} &= \langle Y_n, \psi \times u_n'\rangle_{\mathbb{L}^2} - \langle Y, \psi \times u'\rangle_{\mathbb{L}^2}\\ &= \langle Y_n - Y, \psi \times u'\rangle_{\mathbb{L}^2} + \langle Y_n, \psi \times (u_n' - u')\rangle_{\mathbb{L}^2}.\end{aligned}$$

Since by (4.13) u' belongs to $L^4(\Omega'; L^4(0,T;\mathbb{L}^4))$, the process $\psi \times u'$ belongs to $L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$. Hence the weak convergence in (4.14) ensures the integral $\mathbb{E}' \int_0^T \langle Y_n(t) - Y(t), \psi(t) \times u'(t)\rangle_{\mathbb{L}^2}\, dt$ converges to zero as n goes to infinity. Since the sequence (Y_n) is bounded in $L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$ (see (4.7)), we may apply Hölder's inequality and (4.13) to infer that $\mathbb{E}' \int_0^T \langle Y_n(t), \psi(t) \times (u_n'(t) - u'(t))\rangle_{\mathbb{L}^2}\, dt$ also converges to zero as n goes to infinity. This completes the proof of the lemma. ■

5 End of the Proof of Theorem 2.7

The process u' we constructed is a candidate for a solution to the LLG equation. However, in order to define a weak martingale solution we also need to construct a driving Wiener process.

We define a sequence of $\mathbb{H}$-valued processes $(M_n(t))_{t\in[0,T]}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$M_n(t) := u_n(t) - u_n(0) - \lambda_1 \int_0^t \pi_n(u_n(s) \times \Delta u_n(s))\,ds \\ + \lambda_2 \int_0^t \pi_n(u_n(s) \times (u_n(s) \times \Delta u_n(s)))\,ds - \frac{1}{2}\int_0^t \pi_n[(\pi_n(u_n(s) \times h)) \times h]\,ds.$$

By the definition of $u_n(t)$, we have $M_n(t) = \int_0^t \pi_n(u_n(s) \times h)\,dW(s)$ for each $t \in [0,T]$. We also define a sequence of $\mathbb{H}$-valued processes $(M'_n(t))_{t\in[0,T]}$ on the probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ by

$$M'_n(t) := u'_n(t) - u_n(0) - \lambda_1 \int_0^t \pi_n(u'_n(s) \times \Delta u'_n(s))\,ds \\ + \lambda_2 \int_0^t \pi_n(u'_n(s) \times (u'_n(s) \times \Delta u'_n(s)))\,ds - \frac{1}{2}\int_0^t \pi_n[(\pi_n(u'_n(s) \times h)) \times h]\,ds.$$

Lemma 5.1. For each $t \in (0,T]$ the sequence of random variables $M'_n(t)$ converges weakly in $L^2(\Omega'; X^{-\beta})$ to the limit

$$M'(t) := u'(t) - u_0 - \lambda_1 \int_0^t (u' \times \Delta u')(s)\,ds \\ + \lambda_2 \int_0^t u'(s) \times (u' \times \Delta u')(s)\,ds - \frac{1}{2}\int_0^t (u'(s) \times h) \times h\,ds$$

as n goes to infinity. □

Proof. Let $t \in (0,T]$. Let $U \in L^2(\Omega'; X^\beta)$. We have

$$\mathbb{E}'[{}_{X^{-\beta}}\langle M'_n(t), U\rangle_{X^\beta}] = \mathbb{E}'\Big[{}_{X^{-\beta}}\langle u'_n(t), U\rangle_{X^\beta} - {}_{X^{-\beta}}\langle u_n(0), U\rangle_{X^\beta} \\ - \lambda_1 \int_0^t \langle u'_n(s) \times \Delta u'_n(s), \pi_n U\rangle_{\mathbb{L}^2}\,ds \\ + \lambda_2 \int_0^t {}_{X^{-\beta}}\langle \pi_n(u'_n(s) \times (u'_n(s) \times \Delta u'_n(s))), U\rangle_{X^\beta}\,ds \\ - \frac{1}{2}\int_0^t \langle \pi_n(u'_n(s) \times h) \times h, \pi_n U\rangle_{\mathbb{L}^2}\,ds\Big].$$

Pointwise convergence of u'_n to u' in $C([0,T]; X^{-\beta})$ as well as the convergence in (4.14), (4.16), and (4.13) imply that we can take the limit as n goes to infinity. We obtain

$$\lim_{n\to\infty} \mathbb{E}'[{}_{X^{-\beta}}\langle M'_n(t), U\rangle_{X^\beta}] = \mathbb{E}'\Big[{}_{X^{-\beta}}\langle u'(t), U\rangle_{X^\beta} - {}_{X^{-\beta}}\langle u_0, U\rangle_{X^\beta} - \lambda_1 \int_0^t \langle Y(s), U\rangle_{\mathbb{L}^2}\,ds \\ + \lambda_2 \int_0^t {}_{X^{-\beta}}\langle Z(s), U\rangle_{X^\beta}\,ds - \frac{1}{2}\int_0^t \langle (u'(s) \times h) \times h, U\rangle_{\mathbb{L}^2}\,ds\Big].$$

By definition of $(u' \times \Delta u')$, the right-hand side of the last equality is $\mathbb{E}'[{}_{X^{-\beta}}\langle M'(t), U\rangle_{X^{\beta}}]$. ■

Lemma 5.2. We have the following:

(1) the process $(W'(t))_{t\in[0,T]}$ is a real-valued Brownian motion on $(\Omega', \mathcal{F}', \mathbb{P}')$ and if $0 \le s < t \le T$, then the increment $W'(t) - W'(s)$ is independent of the σ-algebra generated by $u'(r)$ and $W'(r)$ for $r \in [0, s]$;

(2) for each $t \in [0, T]$, we have $M'(t) = \int_0^t (u'(s) \times h)\, \mathrm{d}W'(s)$. □

Proof. Proof of 1. Recall from Proposition 4.3 that the distributions of (u_n, W) and (u'_n, W'_n) on $L^4(0, T; \mathbb{L}^4) \cap C([0, T]; X^{-\beta}) \times C([0, T]; \mathbb{R})$ are equal for each n in $\mathbb{N}$ and u'_n converges to u' in $L^4(0, T; \mathbb{L}^4) \cap C([0, T]; X^{-\beta})$ and W'_n converges to W' in $C([0, T]; \mathbb{R})$ $\mathbb{P}'$ a.e. It is straightforward to show that, for each $n \in \mathbb{N}$, the process $(W'_n(t))_{t\in[0,T]}$ is a real-valued Brownian motion on the probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and that, for $0 \le s < t \le T$, the increment $W'_n(t) - W'_n(s)$ is independent of the σ-algebra generated by $u'_n(r)$ and $W'_n(r)$ for $r \in [0, s]$.

To see that the process $(W'(t))_{t\in[0,T]}$ has the right finite-dimensional distributions to be a Brownian motion on $(\Omega', \mathcal{F}', \mathbb{P}')$, we consider characteristic functions. Let $k \in \mathbb{N}$ and let $s_0 = 0 < s_1 < \cdots < s_k \le T$. For $(t_1, \ldots, t_k) \in \mathbb{R}^k$ we have, for each $n \in \mathbb{N}$,

$$\mathbb{E}'[\mathrm{e}^{i\sum_{j=1}^{k} t_j (W'_n(s_j) - W'_n(s_{j-1}))}] = \mathrm{e}^{-\frac{1}{2}\sum_{j=1}^{k} t_j^2 (s_j - s_{j-1})}.$$

By Lebesgue's dominated convergence theorem, the same equality holds with W' in place of W'_n. Thus $(W'(t))_{t\in[0,T]}$ is a Brownian motion on $(\Omega', \mathcal{F}', \mathbb{P}')$.

Now let $0 \le r_1 < \cdots < r_k \le s < t \le T$ and let $\phi_1, \ldots, \phi_k$ be continuous and bounded real-valued functions on $X^{-\beta}$ and let $\psi_1, \ldots, \psi_k, \psi$ be continuous and bounded real-valued functions on $\mathbb{R}$. We have, for each $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}'\left[\left(\prod_{i=1}^{k} \phi_i(u'_n(r_i)) \prod_{j=1}^{k} \psi_j(W'_n(r_j))\right) \psi(W'_n(t) - W'_n(s))\right] \\
&\quad = \mathbb{E}'\left[\prod_{i=1}^{k} \phi_i(u'_n(r_i)) \prod_{j=1}^{k} \psi_j(W'_n(r_j))\right] \mathbb{E}'[\psi(W'_n(t) - W'_n(s))].
\end{aligned}$$

By Lebesgue's dominated convergence theorem, the same equality holds with u' in place of u'_n and W' in place of W'_n. This proves that $W'(t) - W'(s)$ is independent of the σ-algebra generated by $u'(r)$ and $W'(r)$ for $r \in [0, s]$. ■

Proof of 2. *Step* (i). We show that $M_n'(t) = \int_0^t \pi_n(u_n'(s) \times h)\, \mathrm{d}W_n'(s)$ $\mathbb{P}'$ a.e. for each $t \in [0, T]$ and each $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$ and $t \in (0, T]$. For each $m \in \mathbb{N}$ define the partition $\{s_j^m := \frac{jT}{m}, j = 0, \ldots, m\}$ of $[0, T]$. For each m the random variables in $\mathbb{H}$:

$$M_n(t) - \sum_{j=0}^{m-1} \pi_n(u_n(s_j^m) \times h)(W(t \wedge s_{j+1}^m) - W(t \wedge s_j^m))$$

and

$$M_n'(t) - \sum_{j=0}^{m-1} \pi_n(u_n'(s_j^m) \times h)(W_n'(t \wedge s_{j+1}^m) - W_n'(t \wedge s_j^m))$$

have the same distribution. Since $M_n(t) - \sum_{j=0}^{m-1} \pi_n(u_n(s_j^m) \times h)(W(t \wedge s_{j+1}^m) - W(t \wedge s_j^m))$ converges in $L^2(\Omega; H)$ to $M_n(t) - \int_0^t \pi_n(u_n(s) \times h)\, \mathrm{d}W(s) = 0$ as m goes to infinity, we have $M_n'(t) - \int_0^t \pi_n(u_n'(s) \times h)\, \mathrm{d}W_n'(s) = 0$ $\mathbb{P}'$ a.e.

Step (ii). We show that $M_n'(t)$ converges in $L^2(\Omega'; X^{-\beta})$ to $\int_0^t (u'(s) \times h)\, \mathrm{d}W'(s)$ as n goes to infinity. Since we already know from Lemma 5.1 that $M_n'(t)$ converges weakly in $L^2(\Omega'; X^{-\beta})$ to $M'(t)$, this will complete the proof. ■

First, we make an observation that will be useful in the proof of convergence in the next paragraph. Since h belongs to $\mathbb{L}^\infty \cap \mathbb{W}^{1,3}$, the map $z \in \mathbb{H}^1 \mapsto z \times h$ is a bounded linear operator on $\mathbb{H}^1 = X^\beta$. This implies that the map defined on the dense subspace $\mathbb{L}^2$ of $X^{-\beta}$ by the formula $u \mapsto u \times h$ extends to a bounded linear operator on $X^{-\beta}$: for $u \in \mathbb{L}^2$ and $z \in X^\beta$, we have

$$|{}_{X^{-\beta}}\langle u \times h, z\rangle_{X^\beta}| = |{}_{X^{-\beta}}\langle u, z \times h\rangle_{X^\beta}| \le c_h |z|_{X^\beta} |u|_{X^{-\beta}},$$

where c_h is a positive real number depending only on h.

Let $\epsilon > 0$. Choose a natural number m such that for the partition $\{s_i^m := \frac{iT}{m}, i = 0, 1, \ldots, m\}$ of the interval $[0, T]$, we have

$$\left(\mathbb{E}' \int_0^t \left| u'(s) \times h - \sum_{j=0}^{m-1} (u'(s_j^m) \times h) \chi_{(s_j^m, s_{j+1}^m]}(s) \right|_{X^{-\beta}}^2 \mathrm{d}s\right)^{\frac{1}{2}} < \frac{\epsilon}{2}.$$

Then we have the following four facts which combine to tell us how close $M_n'(t)$ is to $\int_0^t (u'(s) \times h)\, \mathrm{d}W'(s)$ in $L^2(\Omega'; X^{-\beta})$:

(1)

$$\left(\mathbb{E}'\left[\left|\int_0^t (\pi_n(u_n'(s)\times h) - \sum_{j=0}^{m-1}\pi_n(u_n'(s_j^m)\times h)\chi_{(s_j^m,s_{j+1}^m]}(s))\,\mathrm{d}W_n'(s)\right|_{X^{-\beta}}^2\right]\right)^{\frac{1}{2}}$$
$$\leq \left(\mathbb{E}'\int_0^t |u_n'(s)\times h - u'(s)\times h|_{X^{-\beta}}^2\,\mathrm{d}s\right)^{\frac{1}{2}}$$
$$+\left(\mathbb{E}'\int_0^t \left|u'(s)\times h - \sum_{j=0}^{m-1}(u'(s_j^m)\times h)\chi_{(s_j^m,s_{j+1}^m]}(s)\right|_{X^{-\beta}}^2\,\mathrm{d}s\right)^{\frac{1}{2}}$$
$$+\left(\mathbb{E}'\int_0^t \left|\sum_{j=0}^{m-1}(u'(s_j^m) - u_n'(s_j^m))\times h\,\chi_{(s_j^m,s_{j+1}^m]}(s)\right|_{X^{-\beta}}^2\,\mathrm{d}s\right)^{\frac{1}{2}},$$

which is $<\frac{\epsilon}{2}$, for all sufficiently large n, since there is pointwise convergence of u_n' to u' in $C([0,T];X^{-\beta})\mathbb{P}'$ a.e.;

(2)

$$\mathbb{E}'\left[\left|\sum_{j=0}^{m-1}\pi_n(u_n'(s_j^m)\times h)(W_n'(t\wedge s_{j+1}^m) - W_n'(t\wedge s_j^m))\right.\right.$$
$$\left.\left. - \sum_{j=0}^{m-1}\pi_n(u'(s_j^m)\times h)(W'(t\wedge s_{j+1}^m) - W'(t\wedge s_j^m))\right|_{X^{-\beta}}^2\right]$$

converges to zero as n goes to infinity, thanks to pointwise convergence to zero of the integrand and uniform integrability;

(3)

$$\left(\mathbb{E}'\left[\left|\int_0^t (\pi_n(u'(s)\times h) - \sum_{j=0}^{m-1}\pi_n(u'(s_j^m)\times h)\chi_{(s_j^m,s_{j+1}^m]}(s))\,\mathrm{d}W'(s)\right|_{X^{-\beta}}^2\right]\right)^{\frac{1}{2}} < \frac{\epsilon}{2};$$

(4) and finally

$$\mathbb{E}'\left[\left|\int_0^t (\pi_n(u'(s)\times h) - (u'(s)\times h))\,\mathrm{d}W'(s)\right|_{X^{-\beta}}^2\right]$$

converges to zero as n goes to infinity.

Combining the four facts listed above, we have that $(\mathbb{E}'[|M_n'(t) - \int_0^t (u'(s)\times h)\,\mathrm{d}W'(s)|_{X^{-\beta}}^2])^{\frac{1}{2}} < \epsilon$ for all sufficiently large n. ∎

Once we prove that u' has property (2.11), we will have proved the existence of a weak martingale solution to Equation (1.7). We have that, for every $t \in [0, T]$, the process u' satisfies the equation

$$\begin{aligned} u'(t) = u_0 + \lambda_1 \int_0^t (u' \times \Delta u')(s)\,\mathrm{d}s \\ - \lambda_2 \int_0^t u'(s) \times (u' \times \Delta u')(s)\,\mathrm{d}s + \int_0^t (u'(s) \times h) \circ \mathrm{d}W'(s), \ \mathbb{P}'\text{a.s.}, \end{aligned} \tag{5.1}$$

where the first two integrals are the Bochner integrals of paths in $L^2(0, T; \mathbb{L}^2)$ and $L^2(0, T; X^{-\beta})$, respectively, and the stochastic integral is the Stratonovich integral in $\mathbb{L}^2$.

We now prove that u' has property (2.11) in our definition of a solution.

Proof of (2.11). We will apply Itô's formula in the form presented in Pardoux's fundamental work [31, Theorem 2]. To this end, let $\phi \in C_0^\infty(D, \mathbb{R})$. Then we consider a function

$$\psi : \mathbb{H} \ni u \mapsto \langle u, \phi u \rangle_{\mathbb{H}} \in \mathbb{R}.$$

Since $\psi'(u) = 2\phi u$ (where we identify $\mathbb{H}$ with its dual) and $\psi''(u)(v) = 2\langle \phi v, \cdot \rangle_{\mathbb{H}}$ for $u, v \in \mathbb{H}$, we can easily verify that ψ satisfies the assumptions (i)–(v) of Pardoux's theorem. Moreover, the process u' satisfies the assumptions of Pardoux's theorem, that is

$$\begin{aligned} &\mathbb{E}' \int_0^T |u'(t)|_V^2\,\mathrm{d}t < \infty \ \text{ by (4.12)}, \\ &\mathbb{E}' \int_0^T |(u' \times \Delta u')(t)|_{V'}^2\,\mathrm{d}t < \infty \ \text{ by (4.14)}, \\ &\mathbb{E}' \int_0^T |u'(t) \times (u' \times \Delta u')(t)|_{V'}^2\,\mathrm{d}t < \infty \ \text{ by (4.15)}, \\ &\mathbb{E}' \int_0^T |(u'(s) \times h) \times h|_{V'}^2\,\mathrm{d}t < \infty \ \text{ by (4.10)}, \\ &\mathbb{E}' \int_0^T |u'(s) \times h|_{\mathbb{H}}^2\,\mathrm{d}t < \infty \ \text{ by (4.10)}. \end{aligned}$$

Hence, we have that, for all $t \in [0, T]$, $\mathbb{P}'$-a.s.

$$\begin{aligned} &\langle u'(t), \phi u'(t) \rangle_{\mathbb{H}} - \langle u_0, \phi u_0 \rangle_{\mathbb{H}} \\ &= \int_0^t {}_{V'}\langle \lambda_1 (u' \times \Delta u')(s) - \lambda_2 u'(s) \times (u' \times \Delta u')(s) + \frac{1}{2}(u'(s) \times h) \times h, 2\phi u'(s) \rangle_V\,\mathrm{d}s \\ &\quad + \int_0^t \langle 2\phi u'(s), u'(s) \times h \rangle_{\mathbb{H}}\,\mathrm{d}W'(s) + \int_0^t \langle \phi u'(s) \times h, u'(s) \times h \rangle_{\mathbb{H}}\,\mathrm{d}s. \end{aligned} \tag{5.2}$$

The right-hand side of this equality vanishes for all $t \in [0, T]$, $\mathbb{P}'$ a.s. To see this, first note that, by Lemma 4.7,

$$\langle u' \times \Delta u', \phi u' \rangle_{\mathbb{H}} = 0 \quad \text{a.e. on } [0, T] \times \Omega'.$$

Thanks to the identity $\langle a \times b, a \rangle = 0 \ \forall a, b \in \mathbb{R}^3$ and the continuous imbedding of $\mathbb{H}^1$ into $\mathbb{L}^6$, we also have

$$_{V'}\langle u' \times (u' \times \Delta u'), \phi u' \rangle_V = {}_{\mathbb{L}^{\frac{6}{5}}}\langle u' \times (u' \times \Delta u'), \phi u' \rangle_{\mathbb{L}^6} = 0 \quad \text{a.e. on } [0, T] \times \Omega'.$$

The remaining Lebesgue integrals on the right-hand side of (5.2) cancel and the Itô integral vanishes. Thus, from equality (5.2), we have $\langle u'(t), \phi u'(t) \rangle_{\mathbb{H}} = \langle u_0, \phi u_0 \rangle_{\mathbb{H}}$ for all $t \in [0, T]$, $\mathbb{P}'$ a.s. Since ϕ is arbitrary and $|u_0(x)| = 1$ for a.e. $x \in D$ we infer that $|u'(t)(x)| = 1$ for a.e. $x \in D$ for all $t \in [0, T]$, $\mathbb{P}'$ a.s. ■

Now the just proved property (2.11) of the process u', in conjunction with (2.14), implies easily that

$$|u'(t, \omega) \times ((u' \times \Delta u')(t, \omega))|_{\mathbb{L}^2} \le |(u' \times \Delta u')(t, \omega)|_{\mathbb{L}^2} \tag{5.3}$$

for a.e. $(t, \omega) \in [0, T] \times \Omega'$ and $\mathbb{E}' \int_0^T |u'(t) \times (u' \times \Delta u')(t)|^2_{\mathbb{L}^2} \, dt < \infty$. Thus in fact all integrals on the right-hand side of Equation (5.1) are in $\mathbb{L}^2$ as asserted in Theorem 2.7.

Finally, we prove property (2.17) of Theorem 2.7. By (5.3) and (4.10) and a standard estimate for moments of Itô integrals (see [12, Lemma 7.2]), we have for any real $q > 1$ and $0 \le s < t \le T$:

$$\mathbb{E}'[|u'(t) - u'(s)|^{2q}_{\mathbb{H}}] \le |t - s|^q \left(C_1 \mathbb{E}' \left[\left(\int_0^T |(u' \times \Delta u')(\tau)|^2_{\mathbb{H}} \, d\tau \right)^q \right] + C_2 \right).$$

By (4.14), the expected value on the right-hand side of this inequality is finite, hence the Kolmogorov criterion applies.

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Appendix 1

For the reader's convenience we will recall some facts that are crucial for the proof of tightness of the approximating sequence (u_n). Let us recall first that the Sobolev space $W^{1,q}(0,T;E)$, where $q\in[1,\infty)$ and E is a separable Banach space is the space of all functions $u\in L^q(0,T;E)$ that are weakly differentiable and have a weak derivative u' also belonging to $L^q(0,T;E)$. If $\alpha\in(0,1)$ and $q\in[1,\infty)$, then the Besov–Slobodetski space $W^{\alpha,q}(0,T;E)$ is the space of all $u\in L^q(0,T;E)$ such that

$$\int_0^T\int_0^T \frac{|u(t)-u(s)|_E^q}{|t-s|^{1+\alpha q}}\,\mathrm{d}s\,\mathrm{d}t<\infty.$$

Then for $\alpha\in(0,1]$, the space $W^{\alpha,q}(0,T;E)$ endowed with the norm

$$|u|_{W^{\alpha,q}(0,T;E)}=\begin{cases}\left[\int_0^T|u(t)|_E^q\,\mathrm{d}t+\int_0^T\int_0^T\frac{|u(t)-u(s)|_E^q}{|t-s|^{1+\alpha q}}\,\mathrm{d}t\,\mathrm{d}s\right]^{1/q} & \text{if } \alpha\in(0,1),\\ \left[\int_0^T|u(t)|_E^q\,\mathrm{d}t+\int_0^T|u'(t)|_E^q\,\mathrm{d}t\right]^{\frac{1}{q}} & \text{if } \alpha=1\end{cases} \tag{A.1}$$

is a separable Banach space. It is known, see, for example, [33] for a direct treatment, that $W^{\alpha,q}(0,T;E)\hookrightarrow W^{\beta,q}(0,T;E)$ if $\beta\le\alpha\le 1$ and $W^{\alpha,q}(0,T;E)\hookrightarrow C^\delta([0,T];E)$ continuously provided that $\delta\ge 0$ and $\alpha>\delta+\frac{1}{q}$.

The following result is just Lemma 2.1 from [15].

Lemma A.1. Assume that E is a separable Hilbert space, $p\in[2,\infty)$ and $\alpha\in(0,\frac{1}{2})$. Then there exists a constant C depending on T and α, such that for all processes $\xi:[0,T]\times\Omega\to E$ for which the stochastic integral in (A.3) is defined and the integral on the right-hand side of (A.2) is finite, we have

$$\mathbb{E}|I(\xi)|^p_{W^{\alpha,p}(0,T;E)}\le C\,\mathbb{E}\int_0^T|\xi(r)|_E^p\,\mathrm{d}t, \tag{A.2}$$

where the process $I(\xi)$ is defined by

$$I(\xi):=\int_0^t\xi(s)\,\mathrm{d}W(s),\qquad t\ge 0. \tag{A.3}$$

In particular, $\mathbb{P}$-a.s. the trajectories of the process $I(\xi)$ belong to $W^{\alpha,2}(0,T;E)$. $\square$

Appendix 2

We will need the following two compactness results. For the first one, see [15, Theorem 2.1] which is a modification of results in [29, Section I.5; 35, Section 13.3]. The second one is related to [15, Theorem 2.2].

Lemma A.2. Assume that $B_0 \subset B \subset B_1$ are Banach spaces, B_0 and B_1 being reflexive. Assume that the embedding $B_0 \subset B$ is compact, $q \in (1, \infty)$ and $\alpha \in (0, 1)$. Then the embedding

$$L^p(0, T; B_0) \cap W^{\alpha,q}(0, T; B_1) \hookrightarrow L^p(0, T; B) \tag{A.4}$$

is compact. □

Lemma A.3. Assume that $X_0 \subset X$ are Banach spaces such that the embedding $X_0 \subset X$ is compact. Assume that $p \in (1, \infty)$ and $0 < \alpha < 1$ and $\alpha p > 1$. Then the embedding $W^{\alpha,p}(0, T; X_0) \subset C([0, T]; X)$ is compact. □

Appendix 3

A formulation of the BDG inequality for continuous local martingales can be found in, for example, [23, Theorem 17.7]. The upper bound of the BDG inequality for Itô integrals is stated here for the reader's convenience. Let $m \in (0, \infty)$. There exists a positive real constant K_m with the following property. Let $T \in (0, \infty)$ and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions and let $(W(t))_{t\in[0,T]}$ be a real-valued $(\mathcal{F}_t)$-Wiener process defined on this space. For any progressively measurable function $F : [0, T] \times \Omega \to \mathbb{R}$ such that $\mathbb{P}\{\int_0^T F^2(s)\,\mathrm{d}s < \infty\} = 1$, we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t F(s)\,\mathrm{d}W(s)\right|^{2m}\right] \leq K_m \mathbb{E}\left[\left(\int_0^T F^2(s)\,\mathrm{d}s\right)^m\right].$$

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