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WEAK SOLUTIONS TO A PHASE-FIELD MODEL WITH NON-CONSTANT THERMAL CONDUCTIVITY

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Abstract. We investigate the existence of weak solutions to a phase-field model when the thermal conductivity vanishes for some values of the order parameter. We obtain weak solutions for a general class of free energies, including non-differentiable ones. We also study the ω-limit set of these weak solutions, and investigate their convergence to a solution of a degenerate Cahn-Hilliard equation.

1. Introduction. This paper is concerned with a nonisothermal phase-field model for phase transitions with non-conserved order parameter. It describes the time evolution of an order parameter ϕ (which is the state variable characterizing the different phases) and the temperature u; it reads

$$\tau\phi_t - \xi^2\Delta\phi \in -F'(\phi) + w'(\phi)u \quad \text{in } \Omega \times (0,+\infty), \tag{1.1}$$

$$cu_t + w'(\phi)\phi_t = \operatorname{div}(B(\phi)\nabla u) \quad \text{in } \Omega \times (0,+\infty), \tag{1.2}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma \times (0,+\infty), \tag{1.3}$$

$$\phi(0) = \phi_0, \quad u(0) = u_0 \quad \text{in } \Omega, \tag{1.4}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ ($N \geq 1$) with smooth boundary Γ. Here, τ, ξ, and c are positive real numbers, and B denotes the thermal conductivity, and is assumed to depend only on the order parameter.

When both w' and B are constant ($B > 0$), the system (1.1)–(1.4) is the classical phase-field system, which has been studied in several papers: among them, we refer to [Ca1], [EZ], [BCH], [KN], and [BE], where existence and uniqueness of solutions to (1.1)–(1.4) are investigated for various functions F' (including cases where it may be singular or multivalued), while the long-time behaviour of these solutions is studied in [EZ], [BCH], and [KN].

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When B is a constant and w' a Lipschitz continuous function, the well-posedness of (1.1)–(1.4), together with the long-time behaviour of the solutions to (1.1)–(1.4), have been studied in [KN] and [La1].

But, as pointed out in [Ca1], the thermal conductivity B may differ from one phase to another, and could possibly be much larger in one phase than in the other one. The limit case is then to assume that the thermal conductivity B vanishes in one phase. Another motivation for considering vanishing thermal conductivity in (1.1)–(1.4) arises from the study of surface motion by surface diffusion. In [CT], J. Cahn and J. Taylor derive laws of motion for surface motion by surface diffusion, which involve the normal velocity v and the mean curvature κ of the surface. The simplest law they obtain reads

$$v = \left(\frac{1}{M}\Delta_s - \frac{1}{D} \right)^{-1} \Delta_s \kappa, \tag{1.5}$$

where $M > 0, D > 0$, and Δ_s denotes the surface Laplacian. They suggested that a phase-field approach to (1.5) (in the same spirit as that of [Ca2] for the relationship between the classical phase-field model and Stefan-like and Hele-Shaw problems) may involve a viscous Cahn-Hilliard equation with vanishing mobility, which is obtained from (1.1)–(1.4) by setting $c = 0$ and $w' = 1$.

Thus, our purpose in this work is to study (1.1)–(1.4) when the thermal conductivity B may vanish for some values of the order parameter ϕ. From a mathematical point of view, when B is allowed to vanish, the parabolic equation (1.2) becomes quasilinear and degenerates. Additional mathematical difficulties then arise in the study of (1.1)–(1.4). Hereafter, we investigate the existence of weak solutions to (1.1)–(1.4) (in a sense that will be made precise below), when B is only assumed to be a nonnegative function of ϕ. Because of the degeneracy of (1.2), the results we obtain in this paper are much weaker than the results obtained when B is a positive constant: more precisely, we have no uniqueness results and only poor regularity for the solution we construct (see Sec. 2).

We have already mentioned that, when we set $c = 0$ and $w' = 1$ in (1.1)–(1.4), we recover the degenerate viscous Cahn-Hilliard equation, while setting $c = \tau = 0$ and $w' = 1$ in (1.1)–(1.4) gives the degenerate Cahn-Hilliard equation. Existence of weak solutions to these two equations has been discussed in [EG]. The convergence of solutions to (1.1)–(1.4) to a solution of the degenerate Cahn-Hilliard equation when (τ, c) goes to zero is studied in a particular case in [La2] (see also Sec. 6).

We now describe the content of this paper: in Sec. 2, we state our assumptions and main results; in Sec. 3, we study a regularised problem, while Sec. 4 is devoted to the proofs of the results of Sec. 2. The main point here is to notice that there is enough regularity on ϕ, so that we may give a (weak) sense to the right-hand side of (1.2). We then study in Sec. 5 the ω-limit set of the solution to (1.1)–(1.4) that we construct in Sec. 2. Finally, we state in Sec. 6 a result on the convergence of solutions to (1.1)–(1.4) to a solution of the degenerate Cahn-Hilliard equation when (τ, c) goes to zero.

2. Main results. We now state our assumptions on the data in (1.1)–(1.4).

(A1) There exist a maximal monotone graph β on $\mathbb{R}$ with domain $D(\beta)$ satisfying $\mathrm{Int}(D(\beta)) \neq \varnothing$ and $0 \in \beta(0)$, and a function $F_0 \in C^2(I)$, where I denotes the closure of

$D(\beta)$ in $\mathbb{R}$, such that

$$F' = \beta + F_0'. \tag{2.1}$$

We further assume that there exist $c_1 \geq 0$ such that

$$|F_0''(r)| \leq c_1, \quad \forall r \in I. \tag{2.2}$$

We denote by $\hat{\beta}$ the convex function such that $\hat{\beta}(0) = 0$ and $\partial\hat{\beta} = \beta$ (here, $\partial\hat{\beta}$ denotes the subdifferential of $\hat{\beta}$).

(A2) $w \in C^2(I)$, and both w and w' are Lipschitz continuous functions, with Lipschitz constant L_w.

(A3) $B \in C(I)$ is a Lipschitz continuous function, and there exists $b > 0$ such that

$$0 \leq B(r) \leq b, \quad \forall r \in I. \tag{2.3}$$

Before stating our results, let us mention a few examples of function F' fulfilling assumption **(A1)** (see, e.g., [CT]):

(E1) $\beta(r) = r^3$ $(D(\beta) = \mathbb{R})$, $\quad F_0(r) = \frac{1}{4} - \frac{1}{2}r^2, \quad r \in \mathbb{R}$,

(E2) $\beta(r) = \ln(\frac{1+r}{1-r})$ $(D(\beta) = (-1,1))$, $\quad F_0(r) = 1 - r^2, \quad r \in [-1,1]$,

(E3) $\beta = \partial I_{[-1,1]}$ $(D(\beta) = [-1,1])$, $\quad F_0(r) = 1 - r^2, \quad r \in [-1,1]$,

where $I_{[-1,1]}$ denotes the indicator function of the interval $[-1,1]$ (i.e., $I_{[-1,1]}(r) = 0$ if $|r| \leq 1, I_{[-1,1]}(r) = +\infty$ otherwise).

Also, a possible example for B is

$$B(r) = \begin{cases} 0 & \text{if } r \leq -1, \\ \dfrac{1+r}{2} & \text{if } |r| \leq 1, \\ 1 & \text{if } r \geq 1. \end{cases}$$

We denote by V' the dual space $H^1(\Omega)'$ of $H^1(\Omega)$, and by $\langle \cdot, \cdot \rangle_{V',V}$ the duality pairing between $V = H^1(\Omega)$ and V'. We also put, for $T > 0$,

$$Q_T = \Omega \times (0,T), \qquad \Sigma_T = \Gamma \times (0,T).$$

We now state our main results.

THEOREM 2.1. Let $(\phi_0, u_0) \in H^1(\Omega) \times L^2(\Omega)$ be such that

$$\hat{\beta}(\phi_0) \in L^1(\Omega). \tag{2.4}$$

Under assumptions **(A1)**–**(A3)**, there exist functions (ϕ, ζ, u, J) satisfying, for any $T > 0$,

(i) $\phi \in W^{1,2}(0,T,L^2(\Omega)) \cap L^\infty(0,T,H^1(\Omega)) \cap L^2(0,T,H^2(\Omega)), \quad \phi(0) = \phi_0$,

(ii) $\zeta \in L^2(Q_T)$, $\zeta \in \beta(\phi)$ almost everywhere in Q_T,

(iii) $u \in W^{1,2}(0,T,V') \cap L^\infty(0,T,L^2(\Omega)), \quad u(0) = u_0$,

(iv) $J \in L^2(Q_T, \mathbb{R}^N), \quad J = \nabla(B(\phi)u) - u\nabla B(\phi)$,

and such that

$$\tau\phi_t + \zeta = \xi^2\Delta\phi - F_0'(\phi) + w'(\phi)u, \quad \text{a.e. in } Q_T, \tag{2.5}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \text{a.e. on } \Sigma_T, \tag{2.6}$$

$$c\int_0^T \langle u_t,\eta\rangle_{V',V}\,ds + \int_0^T\int_\Omega w'(\phi)\phi_t\eta\,dx\,ds + \int_0^T\int_\Omega J\cdot\nabla\eta\,dx\,ds = 0, \tag{2.7}$$

for any $\eta \in L^2(0,T,H^1(\Omega))$. Moreover,

$$\int_0^t\int_\Omega (\tau|\phi_t|^2 + |\widetilde{J}|^2)\,dx\,ds + \mathcal{L}(\phi(t),u(t)) \le \mathcal{L}(\phi_0,u_0) \tag{2.8}$$

holds, where $\widetilde{J} \in L^2(Q_T)$, $J = B(\phi)^{1/2}\widetilde{J}$, and

$$\mathcal{L}(\psi,v) = \int_\Omega \left(\frac{\xi^2}{2}|\nabla\psi|^2 + F(\psi) + \frac{c}{2}|v|^2\right)\,dx. \tag{2.9}$$

If, in addition, $B \ge b_0$ for some $b_0 > 0$, then $u \in L^2(0,T,H^1(\Omega))$ for each $T > 0$, and $J = B(\phi)\nabla u$, $\widetilde{J} = B(\phi)^{1/2}\nabla u$.

It follows from Theorem 2.1(ii) that the weak solution (ϕ,u) that we construct satisfies $\phi(x,t) \in D(\beta)$ for almost every (x,t) in $\Omega\times(0,+\infty)$. Notice also that it satisfies the Liapunov estimate (2.8).

If we only assume that $\phi_0 \in L^2(\Omega)$, but strengthen the assumption on $\beta(\phi_0)$, we still get an existence result of a solution to (1.1)–(1.4), but in a weaker sense.

PROPOSITION 2.2. Assume that **(A1)–(A3)** hold, and consider $(\phi_0,u_0) \in L^2(\Omega,\mathbb{R}^2)$ satisfying

$$\beta^0(\phi_0) \in L^2(\Omega), \tag{2.10}$$

where β^0 denotes the principal section of β. Then, there exist functions (ϕ,ζ,u,e,J) satisfying, for any $T > 0$,

(i) $\phi \in W^{1,2}(0,T,V') \cap L^\infty(0,T,L^2(\Omega)) \cap L^2(0,T,H^1(\Omega))$, $\quad \phi(0) = \phi_0$,
(ii) $\zeta \in L^2(Q_T)$, $\zeta \in \beta(\phi)$ almost everywhere in Q_T,
(iii) $e \in W^{1,2}(0,T,V')$, $u \in L^\infty(0,T,L^2(\Omega))$, $e = cu + w(\phi)$, $e(0) = cu_0 + w(\phi_0)$,
(iv) $J \in L^2(Q_T,\mathbb{R}^N)$, $\quad J = \nabla(B(\phi)u) - u\nabla B(\phi)$,

and such that

$$\begin{aligned}\tau\int_0^T \langle\phi_t,\eta\rangle_{V',V}\,ds &+ \xi^2\int_0^T\int_\Omega \nabla\phi\cdot\nabla\eta\,dx\,ds\\ &+ \int_0^T\int_\Omega (\zeta + F_0'(\phi) - w'(\phi)u)\eta\,dx\,ds = 0,\end{aligned} \tag{2.11}$$

$$\int_0^T \langle e_t,\eta\rangle_{V',V}\,ds + \int_0^T\int_\Omega J\cdot\nabla\eta\,dx\,ds = 0, \tag{2.12}$$

for any $\eta \in L^2(0,T,H^1(\Omega))$.

Note that since $\phi_0 \in L^2(\Omega)$, (2.10) and a convexity argument yield (2.4).

3. A regularised problem. In this section, we study the existence of solutions to (1.1)–(1.4) under the additional assumptions that the thermal conductivity B is bounded from below by a positive constant and that F' is a Lipschitz continuous function. More precisely, we assume the following:

(B1) $F \in C^1(\mathbb{R})$ and F' is a Lipschitz continuous function. We denote by Λ_F a Lipschitz constant for F'.

(B2) $w \in C^2(\mathbb{R})$, and both w and w' are Lipschitz continuous functions, with Lipschitz constant Λ_w.

(B3) $B \in C(\mathbb{R})$ is a Lipschitz continuous function and there exist positive constants m and M such that

$$0 < m \leq B(r) \leq M, \quad \forall r \in \mathbb{R}. \tag{3.1}$$

We then have the following existence result:

PROPOSITION 3.1. Let $(\phi_0, u_0) \in H^1(\Omega) \times L^2(\Omega)$. Under assumptions **(B1)**–**(B3)**, there exist functions (ϕ, u) satisfying, for any $T > 0$,

(i) $\phi \in W^{1,2}(0,T,L^2(\Omega)) \cap L^\infty(0,T,H^1(\Omega)) \cap L^2(0,T,H^2(\Omega)), \quad \phi(0) = \phi_0,$

(ii) $u \in W^{1,2}(0,T,V') \cap L^\infty(0,T,L^2(\Omega)) \cap L^2(0,T,H^1(\Omega)), \quad u(0) = u_0,$

and

$$\tau\phi_t - \xi^2\Delta\phi = -F'(\phi) + w'(\phi)u, \quad \text{a.e. in } Q_T, \tag{3.2}$$

$$\frac{\partial \phi}{\partial n} = 0, \quad \text{a.e. on } \Sigma_T, \tag{3.3}$$

$$c\int_0^T \langle u_t, \eta\rangle_{V',V}\, ds + \int_0^T \int_\Omega w'(\phi)\phi_t\eta\, dx\, ds + \int_0^T \int_\Omega B(\phi)\nabla u \cdot \nabla\eta\, dx\, ds = 0, \tag{3.4}$$

for any $\eta \in L^2(0,T,H^1(\Omega))$.

Proof of Proposition 3.1. The proof relies on the Galerkin method. We denote by

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots$$

the sequence of the eigenvalues of the operator $(-\Delta)$ with homogeneous Neumann boundary conditions, and by v_j, $j \geq 1$, the corresponding eigenfunctions such that $|v_j|_{L^2} = 1$. We also denote by V_j the vector space spanned by $\{v_k\}_{1\leq k\leq j}$, and by p_j the orthogonal projection of $L^2(\Omega)$ on V_j.

For each integer $j \geq 1$, we set

$$\phi_0^j = p_j\phi_0, \qquad e_0^j = p_j e_0,$$

where $e_0 = cu_0 + w(\phi_0) \in L^2(\Omega)$, since w is Lipschitz continuous. We then consider the approximate problem to find (ϕ^j, e^j) in $V_j \times V_j$ satisfying

$$\phi^j(0) = \phi_0^j, \qquad e^j(0) = e_0^j, \tag{3.5}$$

$$\begin{aligned}\tau\int_\Omega \phi_t^j v\, dx + \xi^2\int_\Omega \nabla\phi^j \cdot \nabla v\, dx + \int_\Omega F'(\phi^j)v\, dx \\ + \frac{1}{c}\int_\Omega w(\phi^j)w'(\phi^j)v\, dx = \frac{1}{c}\int_\Omega w'(\phi^j)e^j v\, dx,\end{aligned} \tag{3.6}$$

$$c\int_\Omega e_t^j\hat{v}\, dx + \int_\Omega B(\phi^j)\nabla e^j \cdot \nabla\hat{v}\, dx = \int_\Omega B(\phi^j)w'(\phi^j)\nabla\phi^j \cdot \nabla\hat{v}\, dx, \tag{3.7}$$

for any $(v, \hat{v}) \in V_j \times V_j$.

The problem (3.5)–(3.7) is in fact an initial value problem for a system of $2j$ ordinary differential equations for the components of (ϕ^j, e^j) on the basis of V_j. Since F', B, and w' are Lipschitz continuous functions, and since the (v_k) are smooth functions, (3.5)–(3.7) has a unique maximal solution (ϕ^j, e^j) defined on some time interval $[0, T_j)$, $T_j > 0$.

In order to prove that $T_j = +\infty$ and to pass to the limit as $j \to +\infty$, we need some estimates we derive now. In the following, we denote by C any positive constant depending only on $\Omega, N, \tau, \xi, c, F(0), F'(0), \Lambda_F, \Lambda_w, w(0), m, M, |\phi_0|_{H^1}$, and $|u_0|_{L^2}$, and by $C(T)$ any positive constant depending not only on the above mentioned data, but also on $T > 0$.

We put

$$\varepsilon_0 = \frac{m\xi^2}{M^2\Lambda_w^2}.$$

We take $v = \phi^j$ in (3.6), $\hat{v} = \varepsilon_0 e^j$ in (3.7), and add both; this gives, thanks to **(B1)**–**(B3)**,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (\tau|\phi^j|^2 + c\varepsilon_0|e^j|^2)\,dx + \int_\Omega (\xi^2|\nabla\phi^j|^2 + m\varepsilon_0|\nabla e^j|^2)\,dx \\
\leq C\int_\Omega (|\phi^j| + |\phi^j|^2 + |e^j|\,|\phi^j|)\,dx + M\Lambda_w\varepsilon_0\int_\Omega |\nabla\phi^j|\,|\nabla e^j|\,dx \\
\leq C\left(1 + \int_\Omega (|\phi^j|^2 + |e^j|^2)\,dx\right) + \frac{m\varepsilon_0}{2}\int_\Omega |\nabla e^j|^2\,dx + \frac{\xi^2}{2}\int_\Omega |\nabla\phi^j|^2\,dx.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (\tau|\phi^j|^2 + c\varepsilon_0|e^j|^2)\,dx + \int_\Omega (\xi^2|\nabla\phi^j|^2 + m\varepsilon_0|\nabla e^j|^2)\,dx \\
\leq C\left(1 + \int_\Omega (\tau|\phi^j|^2 + c\varepsilon_0|e^j|^2)\,dx\right).
\end{aligned} \tag{3.8}$$

It follows from (3.8) and Gronwall's lemma that

$$|\phi^j(t)|_{L^2(\Omega)} + |e^j(t)|_{L^2(\Omega)} \leq C(T), \quad 0 \leq t \leq T,\ t < T_j. \tag{3.9}$$

A first consequence of (3.9) is that $T_j = +\infty$ for each $j \geq 1$. Next, we infer from (3.8) and (3.9), after time integration, that

$$|\phi^j|_{L^2(0,T;H^1(\Omega))} + |e^j|_{L^2(0,T;H^1(\Omega))} \leq C(T). \tag{3.10}$$

Next, we take $v = \phi_t^j$ in (3.6) and find

$$\begin{aligned}
\tau\int_\Omega |\phi_t^j|^2\,dx + \frac{d}{dt}\int_\Omega \left(\frac{\xi^2}{2}|\nabla\phi^j|^2 + F(\phi^j)\right)\,dx \\
= \frac{1}{c}\int_\Omega w'(\phi^j)(e^j - w(\phi^j))\phi_t^j\,dx \\
\leq C\int_\Omega (1 + |\phi^j| + |e^j|)|\phi_t^j|\,dx \\
\leq \frac{\tau}{2}\int_\Omega |\phi_t^j|^2\,dx + C\left(1 + \int_\Omega (|\phi^j|^2 + |e^j|^2)\,dx\right).
\end{aligned}$$

After integration over $(0,t)$, $t \in (0,T)$, we get, thanks to **(B1)** and (3.10),

$$\tau \int_0^t \int_\Omega |\phi_t^j|^2 \, dx \, ds + \xi^2 \int_\Omega |\nabla \phi^j(t)|^2 \, dx \le C(T) + 2 \int_\Omega (F(\phi_0^j) - F(\phi^j)) \, dx \le C(T).$$

Thus,

$$|\phi_t^j|_{L^2(Q_T)} + |\phi^j|_{L^\infty(0,T,H^1(\Omega))} \le C(T). \tag{3.11}$$

We now take $v = -\Delta \phi^j$ in (3.6), and (3.10)–(3.11) and a straightforward computation yield

$$\int_0^T \int_\Omega |\Delta \phi^j|^2 \, dx \, ds \le C(T).$$

Hence, by standard elliptic theory,

$$|\phi^j|_{L^2(0,T,H^2(\Omega))} \le C(T). \tag{3.12}$$

Finally, we infer from **(B2)**, (3.10)–(3.11), and (3.7) that

$$|e_t^j|_{L^2(0,T,V')} \le C(T). \tag{3.13}$$

We are now able to pass to the limit as $j \to +\infty$. Let $T > 0$. We infer from (3.11) and (3.12) that (ϕ^j) is bounded in

$$\mathcal{W}_1 = \{v \in L^2(0,T,H^2(\Omega)),\ v_t \in L^2(0,T,L^2(\Omega))\},$$

and in

$$\mathcal{W}_2 = \{v \in L^\infty(0,T,H^1(\Omega)),\ v_t \in L^2(0,T,L^2(\Omega))\}.$$

Since the embedding of $H^2(\Omega)$ in $H^1(\Omega)$ and that of $H^1(\Omega)$ in $L^2(\Omega)$ are compact, it follows from [Si, Cor. 4] that $\mathcal{W}_1$ is compactly embedded in $L^2(0,T,H^1(\Omega))$, while $\mathcal{W}_2$ is compactly embedded in $\mathcal{C}([0,T],L^2(\Omega))$. Therefore,

$$\begin{aligned} &(\phi^j) \text{ is relatively compact in } L^2(0,T,H^1(\Omega)) \\ &\text{and in } \mathcal{C}([0,T],L^2(\Omega)). \end{aligned} \tag{3.14}$$

Similarly, it follows from (3.9) and (3.13) that (e^j) is bounded in

$$\mathcal{W}_3 = \{v \in L^\infty(0,T,L^2(\Omega)),\ v_t \in L^2(0,T,V')\},$$

which is compactly embedded in $\mathcal{C}([0,T],V')$ by [Si, Cor. 4]. Therefore,

$$(e^j) \text{ is relatively compact in } \mathcal{C}([0,T],V'). \tag{3.15}$$

It now follows from (3.9)–(3.15) that there exist

$$\phi \in W^{1,2}(0,T,L^2(\Omega)) \cap L^\infty(0,T,H^1(\Omega)) \cap L^2(0,T,H^2(\Omega)),$$
$$e \in W^{1,2}(0,T,V') \cap L^\infty(0,T,L^2(\Omega)) \cap L^2(0,T,H^1(\Omega)),$$

and a subsequence of (ϕ^j, e^j) (which we still denote by (ϕ^j, e^j)) satisfying

$$\begin{aligned} &\phi^j \to \phi \text{ in } L^2(0,T,H^1(\Omega)) \cap \mathcal{C}([0,T],L^2(\Omega)), \\ &\qquad \text{and a.e. in } Q_T, \\ &e^j \to e \text{ in } \mathcal{C}([0,T],V'), \\ &e^j \rightharpoonup e \text{ in } L^2(0,T,H^1(\Omega)). \end{aligned} \tag{3.16}$$

Since F' and w are Lipschitz continuous functions, and since w' is a bounded Lipschitz continuous function, we infer from (3.16) that $(F'(\phi^j))$ converges to $F'(\phi)$ in $L^2(Q_T)$, $(w(\phi^j))$ converges to $w(\phi)$ in $L^2(Q_T)$, while $(w'(\phi^j))$ converges to $w'(\phi)$ in $L^p(Q_T)$ for any $p \in [1, +\infty)$. We then pass to the limit in (3.6), and get

$$\tau\phi_t - \xi^2\Delta\phi + F'(\phi) = w'(\phi)\frac{e - w(\phi)}{c} \quad \text{a.e. in } Q_T, \tag{3.17}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{a.e. on } \Sigma_T. \tag{3.18}$$

We are left to pass to the limit in (3.7). Since B and (Bw') are bounded Lipschitz continuous functions, we infer from (3.16) that $(B(\phi^j))$ converges to $B(\phi)$ and $((Bw')(\phi^j))$ converges to $(Bw')(\phi)$ in $L^p(Q_T)$ for any $p \in [1, +\infty)$. These facts and (3.16) ensure that $(B(\phi^j)\nabla e^j)$ converges weakly to $(B(\phi)\nabla e)$ in $L^{3/2}(Q_T)$, while $((Bw')(\phi^j)\nabla\phi^j)$ converges weakly to $((Bw')(\phi)\nabla\phi)$ in $L^{3/2}(Q_T)$.

We may then pass to the limit in (3.7) and find that, for any $\eta \in L^2(0,T,H^1(\Omega))$,

$$c\int_0^T \langle e_t, \eta\rangle_{V',V}\, ds + \int_0^T \int_\Omega B(\phi)(\nabla e - w'(\phi)\nabla\phi)\cdot\nabla\eta\, dx\, ds = 0$$

holds. Setting

$$u = \frac{1}{c}(e - w(\phi)),$$

(3.17)–(3.18) and the above equality yield (3.2)–(3.4).

Finally, it follows from (3.5) and (3.16) that $\phi(0) = \phi_0$ and $e(0) = e_0$, which yields $u(0) = u_0$. Also, since w is a Lipschitz continuous function, the regularity of u follows at once from that of ϕ and e. □

4. Proofs. In this section, we prove Theorem 2.1 and Proposition 2.2. Here, β, F_0, w, and B are such that **(A1)**–**(A3)** hold. For any $\lambda > 0$, we consider the Yosida approximation β_λ of β: it follows from **(A1)** and classical properties of the Yosida approximation that β_λ is a maximal monotone graph of $\mathbb{R}$, which is Lipschitz continuous with Lipschitz constant λ^{-1} and $\beta_\lambda(0) = 0$ (see, e.g., [Br]). We also denote by $\hat\beta_\lambda$ the convex function such that $\hat\beta_\lambda(0) = 0$ and $\partial\hat\beta_\lambda = \beta_\lambda$. We finally put

$$F_\lambda = \hat\beta_\lambda + F_0.$$

Next, let $m \in \mathcal{C}([0, +\infty))$ be a nonnegative function such that

$$m(0) = 0, \quad m(\lambda) > 0, \quad \forall\lambda > 0. \tag{4.1}$$

We then define B_λ for each $\lambda > 0$ by

$$B_\lambda(r) = \begin{cases} B(\inf I) + m(\lambda) & \text{if } r \le \inf I \text{ (when } \inf I > -\infty), \\ B(r) + m(\lambda) & \text{if } r \in I, \\ B(\sup I) + m(\lambda) & \text{if } r \ge \sup I \text{ (when } \sup I < +\infty). \end{cases}$$

In the next lemma, we gather some properties of F_λ and B_λ that we will need in the sequel.

LEMMA 4.1. For each $\lambda \in (0,1), F_\lambda \in \mathcal{C}^1(\mathbb{R})$, and F'_λ is a Lipschitz continuous function with Lipschitz constant $(c_1 + \lambda^{-1})$, while B_λ is a Lipschitz continuous function. Moreover, $(\hat{\beta}_\lambda)$ converges to $\hat{\beta}$ in $\mathbb{R}$ in the sense of Mosco, and

$$F_\lambda(r) \ge F_0(0) + F'_0(0)r - \frac{c_1}{2}r^2, \quad r \in \mathbb{R}, \tag{4.2}$$

$$F'_\lambda(r)r \ge F'_0(0)r - c_1 r^2, \quad r \in \mathbb{R}, \tag{4.3}$$

$$F'_\lambda(r)r \ge F_\lambda(r) - F_0(0) - \frac{c_1}{2}r^2, \quad r \in \mathbb{R}, \tag{4.4}$$

$$m(\lambda) \le B_\lambda(r) \le b + |m|_{L^\infty(0,1)}, \quad r \in \mathbb{R}. \tag{4.5}$$

Recall that, if H is a Hilbert space and $(\Psi_\lambda)_{\lambda \ge 0}$ are convex functions of H, (Ψ_λ) converges to Ψ_0 in the sense of Mosco in H as $\lambda \to 0$ if the following hold:

(m1) for any $z \in D(\Psi_0)$, there exists a sequence (z_λ) in H such that (z_λ) converges strongly to z in H, while $(\Psi_\lambda(z_\lambda))$ converges to $\Psi_0(z)$ in $\mathbb{R}$;

(m2) if (Ψ_{λ_k}) is a subsequence of (Ψ_λ), and if (z_{λ_k}) is a sequence of H that converges weakly to z in H, then

$$\Psi_0(z) \le \liminf_{k \to +\infty} \Psi_{\lambda_k}(z_{\lambda_k})$$

(see, e.g., [Mo]).

Proof of Lemma 4.1. The estimates (4.2)–(4.5) are straightforward consequences of **(A1)**, convexity arguments and **(A3)**, while the Mosco convergence of the sequence $(\hat{\beta}_\lambda)$ to $\hat{\beta}$ in $\mathbb{R}$ follows from standard properties of the Yosida approximation (see, e.g., [Br]). □

We first prove Theorem 2.1.

Proof of Theorem 2.1. We consider $(\phi_0, u_0) \in H^1(\Omega) \times L^2(\Omega)$ such that (2.4) holds. It follows from Lemma 4.1 that, for each $\lambda \in (0,1)$, the functions $(F_\lambda, w, B_\lambda)$ satisfy assumptions **(B1)**–**(B3)** of Sec. 3. We then infer from Proposition 3.1 that, for each $\lambda \in (0,1]$, there exist functions $(\phi^\lambda, u^\lambda)$ satisfying, for any $T > 0$,

$$\phi^\lambda \in W^{1,2}(0,T,L^2(\Omega)) \cap L^\infty(0,T,H^1(\Omega)) \cap L^2(0,T,H^2(\Omega)), \quad \phi^\lambda(0) = \phi_0,$$
$$u^\lambda \in W^{1,2}(0,T,V') \cap L^\infty(0,T,L^2(\Omega)) \cap L^2(0,T,H^1(\Omega)), \quad u^\lambda(0) = u_0,$$

and

$$\tau\phi^\lambda_t - \xi^2\Delta\phi^\lambda = -F'_\lambda(\phi^\lambda) + w'(\phi^\lambda)u^\lambda, \quad \text{a.e. in } Q_T, \tag{4.6}$$

$$\frac{\partial \phi^\lambda}{\partial n} = 0, \quad \text{a.e. on } \Sigma_T, \tag{4.7}$$

$$c\int_0^T \langle u_t^\lambda,\eta\rangle_{V',V}\,ds+\int_0^T\int_\Omega w'(\phi^\lambda)\phi_t^\lambda\eta\,dx\,ds+\int_0^T\int_\Omega B_\lambda(\phi^\lambda)\nabla u^\lambda\cdot\nabla\eta\,dx\,ds=0, \tag{4.8}$$

for any $\eta\in L^2(0,T,H^1(\Omega))$.

Let $T>0$. In the following, we denote by C_T any positive constant depending only on $\Omega, N, \tau, \xi, c, b, c_1, F_0(0), F_0'(0), w(0)$, $|m|_{L^\infty(0,1)}$, $|\phi_0|_{H^1}$, $|\hat\beta(\phi_0)|_{L^1}$, $|u_0|_{L^2}$, and T.

LEMMA 4.2. There exists a constant C_T such that, for any $\lambda\in(0,1]$,

$$|\phi^\lambda|_{L^\infty(0,T,H^1(\Omega))}+|\phi_t^\lambda|_{L^2(Q_T)}+|u^\lambda|_{L^\infty(0,T,L^2(\Omega))}+\int_0^T\int_\Omega B_\lambda(\phi^\lambda)|\nabla u^\lambda|^2\,dx\,ds\le C_T. \tag{4.9}$$

Proof of Lemma 4.2. Since u^λ belongs to $L^2(0,T,H^1(\Omega))$, it is a valid test function in (4.8). We thus obtain for almost every $t\in(0,T)$,

$$\frac{c}{2}\int_\Omega |u^\lambda(t)|^2\,dx+\int_0^t\int_\Omega B_\lambda(\phi^\lambda)|\nabla u^\lambda|^2\,dx\,ds\le C_T-\int_0^t\int_\Omega w'(\phi^\lambda)\phi_t^\lambda u^\lambda\,dx\,ds. \tag{4.10}$$

Next, we take the scalar product in $L^2(\Omega)$ of (4.6) with $(\phi_t^\lambda+2c_1\tau^{-1}\phi^\lambda)$; after integration over $(0,t)$, $t\in(0,T)$, we find

$$\begin{aligned}&\int_\Omega\left(\frac{\xi^2}{2}|\nabla\phi^\lambda(t)|^2+F_\lambda(\phi^\lambda(t))+c_1|\phi^\lambda(t)|^2\right)dx\\&\qquad+\int_0^t\int_\Omega\left(\tau|\phi_t^\lambda|^2+\frac{2c_1\xi^2}{\tau}|\nabla\phi^\lambda|^2\right)dx\,ds\\&\quad\le\int_0^t\int_\Omega\left(w'(\phi^\lambda)\phi_t^\lambda u^\lambda+\frac{2c_1L_w}{\tau}|\phi^\lambda|\,|u^\lambda|-\frac{2c_1}{\tau}F_\lambda'(\phi^\lambda)\phi^\lambda\right)dx\,ds\\&\qquad+C_T+\int_\Omega(\hat\beta_\lambda(\phi_0)+F_0(\phi_0))\,dx.\end{aligned}$$

Since $\hat\beta_\lambda\le\hat\beta$, it follows from (4.3) and the above inequality that

$$\begin{aligned}&\int_\Omega\left(\frac{\xi^2}{2}|\nabla\phi^\lambda(t)|^2+F_\lambda(\phi^\lambda(t))+c_1|\phi^\lambda(t)|^2\right)dx\\&\qquad+\int_0^t\int_\Omega\left(\tau|\phi_t^\lambda|^2+\frac{2c_1\xi^2}{\tau}|\nabla\phi^\lambda|^2\right)dx\,ds\\&\quad\le\int_0^t\int_\Omega w'(\phi^\lambda)\phi_t^\lambda u^\lambda\,dx\,ds+C_T\left(1+\int_0^t\int_\Omega(|\phi^\lambda|^2+|u^\lambda|^2)\,dx\,ds\right).\end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11) yields

$$\begin{aligned}&\int_\Omega\left(\frac{\xi^2}{2}|\nabla\phi^\lambda(t)|^2+F_\lambda(\phi^\lambda(t))+c_1|\phi^\lambda(t)|^2+\frac{c}{2}|u^\lambda(t)|^2\right)dx\\&\qquad+\int_0^t\int_\Omega\left(\tau|\phi_t^\lambda|^2+\frac{2c_1\xi^2}{\tau}|\nabla\phi^\lambda|^2+B_\lambda(\phi^\lambda)|\nabla u^\lambda|^2\right)dx\,ds\\&\quad\le C_T\left(1+\int_0^t\int_\Omega(|\phi^\lambda|^2+|u^\lambda|^2)\,dx\,ds\right).\end{aligned}$$

Using (4.2) and the Young inequality, we obtain

$$\begin{aligned}\int_\Omega \left(\frac{\xi^2}{2}|\nabla\phi^\lambda(t)|^2 + c_1|\phi^\lambda(t)|^2 + \frac{c}{2}|u^\lambda(t)|^2\right) dx \\ + \int_0^t \int_\Omega (\tau|\phi_t^\lambda|^2 + B_\lambda(\phi^\lambda)|\nabla u^\lambda|^2)\, dx\, ds \\ \le \frac{3c_1}{4}\int_\Omega |\phi^\lambda(t)|^2\, dx + C_T\left(1 + \int_0^t \int_\Omega (|\phi^\lambda|^2 + |u^\lambda|^2)\, dx\, ds\right).\end{aligned}$$

Hence

$$\begin{aligned}|\phi^\lambda(t)|^2_{H^1(\Omega)} + |u^\lambda(t)|^2_{L^2(\Omega)} + \int_0^t \int_\Omega (|\phi_t^\lambda|^2 + B_\lambda(\phi^\lambda)|\nabla u^\lambda|^2)\, dx\, ds \\ \le C_T\left(1 + \int_0^t (|\phi^\lambda|^2_{H^1(\Omega)} + |u^\lambda|^2_{L^2(\Omega)})\, ds\right).\end{aligned} \tag{4.12}$$

We first infer from (4.12) and Gronwall's lemma that

$$|\phi^\lambda|_{L^\infty(0,T,H^1(\Omega))} + |u^\lambda|_{L^\infty(0,T,L^2(\Omega))} \le C_T. \tag{4.13}$$

Then, (4.9) follows from (4.12) and (4.13). □

LEMMA 4.3. There exists a constant $C_T > 0$ such that, for each $\lambda \in (0,1]$,

$$|\phi^\lambda|_{L^2(0,T,H^2(\Omega))} + |\beta_\lambda(\phi^\lambda)|_{L^2(Q_T)} \le C_T, \tag{4.14}$$

$$|u_t^\lambda|_{L^2(0,T,V')} \le C_T. \tag{4.15}$$

Proof of Lemma 4.3. It follows from (4.6) that ϕ^λ is a solution to

$$-\xi^2\Delta\phi^\lambda + F_\lambda'(\phi^\lambda) + c_1\phi^\lambda = f^\lambda \quad \text{in } Q_T, \qquad \frac{\partial\phi^\lambda}{\partial n} = 0 \quad \text{on } \Sigma_T,$$

where $f^\lambda = c_1\phi^\lambda + w'(\phi^\lambda)u^\lambda - \tau\phi_t^\lambda$. Since $F_\lambda' + c_1 Id$ is nondecreasing, a monotonicity argument yields

$$\xi|\Delta\phi^\lambda|_{L^2(Q_T)} + |F_\lambda'(\phi^\lambda) + c_1\phi^\lambda|_{L^2(Q_T)} \le |f^\lambda|_{L^2(Q_T)}.$$

But, we infer from **(A2)** and (4.9) that

$$|f^\lambda|_{L^2(Q_T)} \le C_T.$$

Then, (4.14) follows from the above two estimates, **(A1)**, (4.9), and standard elliptic arguments.

Next, (4.15) is a straightforward consequence of **(A2)**–**(A3)** and (4.8)–(4.9). □

We now infer from (4.9) and (4.14) that the sequence (ϕ^λ) is bounded in

$$\mathcal{W}_1 = \{v \in L^2(0,T,H^2(\Omega)),\ v_t \in L^2(0,T,L^2(\Omega))\},$$

and in

$$\mathcal{W}_2 = \{v \in L^\infty(0,T,H^1(\Omega)),\ v_t \in L^2(0,T,L^2(\Omega))\}.$$

Since $\mathcal{W}_1$ is compactly embedded in $L^2(0,T,H^1(\Omega))$, and $\mathcal{W}_2$ in $\mathcal{C}([0,T],L^2(\Omega))$ ([Si, Cor. 4]),

$$\begin{aligned}&(\phi^\lambda) \text{ is relatively compact in } L^2(0,T,H^1(\Omega))\\ &\text{and in } \mathcal{C}([0,T],L^2(\Omega)).\end{aligned} \tag{4.16}$$

We also infer from (4.9) and (4.15) that the sequence (u^λ) is bounded in

$$\mathcal{W}_3 = \{v \in L^\infty(0,T,L^2(\Omega)),\ v_t \in L^2(0,T,V')\},$$

which is compactly embedded in $\mathcal{C}([0,T],V')$ by [Si, Cor. 4]. Therefore,

$$(u^\lambda) \text{ is relatively compact in } \mathcal{C}([0,T],V'). \tag{4.17}$$

It follows from (4.9) and (4.14)–(4.17) that there exist

$$\begin{aligned}&\phi \in W^{1,2}(0,T,L^2(\Omega)) \cap L^\infty(0,T,H^1(\Omega)) \cap L^2(0,T,H^2(\Omega)), \quad \zeta \in L^2(Q_T),\\ &u \in W^{1,2}(0,T,V') \cap L^\infty(0,T,L^2(\Omega)), \quad J \in L^2(Q_T),\end{aligned}$$

and a subsequence of $(\phi^\lambda, u^\lambda)$ (which we still denote by $(\phi^\lambda, u^\lambda)$), such that

$$\begin{aligned}\phi^\lambda &\to \phi \quad \text{in } L^2(0,T,H^1(\Omega)),\ \text{in } \mathcal{C}([0,T],L^2(\Omega)),\\ &\qquad \text{and a.e. in } Q_T,\\ \phi^\lambda &\rightharpoonup \phi \quad \text{in } W^{1,2}(0,T,L^2(\Omega)),\\ \beta_\lambda(\phi^\lambda) &\rightharpoonup \zeta \quad \text{in } L^2(Q_T),\\ u^\lambda &\to u \quad \text{in } \mathcal{C}([0,T],V'),\\ u^\lambda &\rightharpoonup u \quad \text{in } L^2(Q_T),\\ B_\lambda(\phi^\lambda)\nabla u^\lambda &\rightharpoonup J \quad \text{in } L^2(Q_T).\end{aligned} \tag{4.18}$$

It now remains to identify ζ and J in terms of ϕ and u and to pass to the limit as λ decreases to zero in (4.6)–(4.8).

First, since $\beta \subset \liminf_{\lambda\to 0} \beta_\lambda$, it follows from (4.18) that

$$\phi \in D(\beta) \quad \text{and} \quad \zeta \in \beta(\phi) \quad \text{a.e. in } Q_T.$$

Next, since w' is bounded and Lipschitz continuous, it follows from (4.18) and the Lebesgue dominated convergence theorem that $(w'(\phi^\lambda))$ converges to $w'(\phi)$ in $L^p(Q_T)$ for any $p \in [1,+\infty)$. This fact and the weak convergence of (u^λ) in $L^2(Q_T)$ yield the weak convergence of $(w'(\phi^\lambda)u^\lambda)$ to $(w'(\phi)u)$ in $L^{3/2}(Q_T)$. We may then pass to the limit in (4.6)–(4.7) as $\lambda \to 0$ and obtain (2.5)–(2.6).

It remains to pass to the limit in (4.8) and to identify J in (4.18). For that purpose, we notice that, since B is a Lipschitz continuous function, and since (ϕ^λ) converges to ϕ in $L^2(0,T,H^1(\Omega))$, we have (see, e.g., [Ka, Thm. 16.7])

$$B(\phi^\lambda) \to B(\phi) \quad \text{in } L^2(0,T,H^1(\Omega)). \tag{4.19}$$

It first follows from (4.18)–(4.19) that

$$\begin{aligned}u^\lambda \nabla B(\phi^\lambda) &\rightharpoonup u\nabla B(\phi) \quad \text{in } L^1(Q_T),\\ B(\phi^\lambda)u^\lambda &\rightharpoonup B(\phi)u \quad \text{in } L^1(Q_T).\end{aligned} \tag{4.20}$$

Next, since $B(\phi^\lambda)$ belongs to $L^\infty(0,T,H^1(\Omega))$ and u^λ to $L^2(0,T,H^1(\Omega))$, we have

$$B(\phi^\lambda)u^\lambda \in L^2(0,T,W^{1,p}(\Omega)) \quad \text{where } p = \min\left\{2, \frac{N}{N-1}\right\},$$

and

$$\nabla(B(\phi^\lambda)u^\lambda) = B(\phi^\lambda)\nabla u^\lambda + u^\lambda \nabla B(\phi^\lambda).$$

It then follows from (4.20) that

$$B(\phi^\lambda)\nabla u^\lambda \rightharpoonup \nabla(B(\phi)u) - u\nabla B(\phi) \quad \text{in } \mathcal{D}'(Q_T). \tag{4.21}$$

Finally, since B is nonnegative, (4.1) and (4.9) yield

$$(m(\lambda)\nabla u^\lambda) \to 0 \quad \text{in } L^2(Q_T). \tag{4.22}$$

Now, since $B_\lambda = B + m(\lambda)$, we infer from (4.18), (4.21), and (4.22) that

$$J = \nabla(B(\phi)u) - u\nabla B(\phi).$$

Moreover, we may pass to the limit in (4.8) and obtain (2.7).

Finally, it follows from (4.18) that $\phi(0) = \phi_0$ and $u(0) = u_0$.

It remains to check the identity (2.8). For each $\lambda \in (0,1]$, the functions $(\phi^\lambda, u^\lambda)$ given by (4.6)–(4.8) satisfy

$$\int_0^t \int_\Omega (\tau|\phi_t^\lambda|^2 + B_\lambda(\phi^\lambda)|\nabla u^\lambda|^2)\, dx\, ds + \mathcal{L}_\lambda(\phi(t), u(t)) \le \mathcal{L}_\lambda(\phi_0, u_0), \tag{4.23}$$

where

$$\mathcal{L}_\lambda(\phi^\lambda, u^\lambda) = \int_\Omega \left(\frac{\xi^2}{2}|\nabla\phi^\lambda|^2 + F_\lambda(\phi^\lambda) + \frac{c}{2}|u^\lambda|^2\right) dx. \tag{4.24}$$

Indeed, we take the scalar product in $L^2(Q_t)$ of (4.6) with ϕ_t^λ, add it to (4.10), and thus get (4.23).

First, since $\hat{\beta}_\lambda \le \hat{\beta}$, we have

$$\mathcal{L}_\lambda(\phi_0, u_0) \le \mathcal{L}(\phi_0, u_0). \tag{4.25}$$

Next, we infer from (4.9) that, in addition to (4.18), we may assume that the sequence $(B_\lambda(\phi^\lambda)^{1/2}\nabla u^\lambda)$ converges weakly to some function $\widetilde{J}$ in $L^2(Q_T)$. Thus,

$$\int_0^t \int_\Omega |\widetilde{J}|^2\, dx\, ds \le \liminf_{\lambda\to 0} \int_0^t \int_\Omega B_\lambda(\phi^\lambda)|\nabla u^\lambda|^2\, dx\, ds. \tag{4.26}$$

Furthermore, (4.18) and the Lebesgue dominated convergence theorem ensure that the sequence $(B_\lambda(\phi^\lambda)^{1/2})$ converges strongly to $B(\phi)^{1/2}$ in $L^2(Q_T)$. Hence

$$J = B(\phi)^{1/2}\widetilde{J} \quad \text{a.e. in } Q_T.$$

It also follows from (4.18) that

$$\int_0^t \int_\Omega \tau|\phi_t|^2\, dx\, ds \le \liminf_{\lambda\to 0} \int_0^t \int_\Omega \tau|\phi_t^\lambda|^2\, dx\, ds. \tag{4.27}$$

Finally, since $(\hat{\beta}_\lambda)$ converges to $\hat{\beta}$ in the sense of Mosco in $\mathbb{R}$, the sequence of convex functions (Ψ_λ) given by

$$\Psi_\lambda(v) = \begin{cases} \dfrac{\xi^2}{2}\displaystyle\int_\Omega |\nabla v|^2\,dx + \int_\Omega \hat{\beta}_\lambda(v)\,dx, & \text{if } v \in H^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

converges in the sense of Mosco in $L^2(\Omega)$ to Ψ given by

$$\Psi(v) = \begin{cases} \dfrac{\xi^2}{2}\displaystyle\int_\Omega |\nabla v|^2\,dx + \int_\Omega \hat{\beta}(v)\,dx, & \text{if } v \in H^1(\Omega),\ \hat{\beta}(v) \in L^1(\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

It then follows from (4.18) and property **(m2)** of the Mosco convergence that, for $t \in (0,T)$,

$$\int_\Omega \left(\frac{\xi^2}{2}|\nabla\phi(t)|^2 + \hat{\beta}(\phi(t)) \right) dx \leq \liminf_{\lambda\to 0} \int_\Omega \left(\frac{\xi^2}{2}|\nabla\phi^\lambda(t)|^2 + \hat{\beta}_\lambda(\phi^\lambda(t)) \right) dx. \tag{4.28}$$

It also follows from **(A1)** and (4.18) that, for each $t \in [0,T]$,

$$\lim_{\lambda\to 0} \int_\Omega F_0(\phi^\lambda(t))\,dx = \int_\Omega F_0(\phi(t))\,dx. \tag{4.29}$$

We then infer from (4.23)–(4.29) and the weak-* convergence of the sequence (u^λ) in $L^\infty(0,T,L^2(\Omega))$ that (2.8) holds. The proof of Theorem 2.1 is then complete. $\square$

Proof of Proposition 2.2. Let $(\phi_0, u_0) \in L^2(\Omega,\mathbb{R}^2)$ be such that (2.10) holds. For $\lambda \in (0,1]$, we choose $\phi_0^\lambda \in H^1(\Omega)$ such that

$$|\phi_0^\lambda - \phi_0|_{L^2(\Omega)} \leq \lambda^2. \tag{4.30}$$

For each $\lambda \in (0,1]$, the functions $(F_\lambda, w, B_\lambda)$ defined at the beginning of Sec. 4 satisfy assumptions **(B1)**–**(B3)** of Sec. 3 by Lemma 4.1. We then infer from Proposition 3.1 that, for each $\lambda \in (0,1]$, there exist functions $(\phi^\lambda, u^\lambda)$ satisfying, for any $T > 0$,

$$\phi^\lambda \in W^{1,2}(0,T,L^2(\Omega)) \cap L^\infty(0,T,H^1(\Omega)) \cap L^2(0,T,H^2(\Omega)), \quad \phi^\lambda(0) = \phi_0^\lambda,$$
$$u^\lambda \in W^{1,2}(0,T,V') \cap L^\infty(0,T,L^2(\Omega)) \cap L^2(0,T,H^1(\Omega)), \quad u^\lambda(0) = u_0,$$

and (4.6)–(4.8).

Let $T > 0$. In the following, we denote by C_T any positive constant depending only on $\Omega, N, \tau, \xi, c, b, c_1, F_0(0), F_0'(0), w(0)$, $|m|_{L^\infty(0,1)}, |\phi_0|_{L^2}, |\beta^0(\phi_0)|_{L^2}, |u_0|_{L^2}$, and T.

LEMMA 4.4. There exists a constant C_T such that, for any $\lambda \in (0,1]$,

$$|\phi^\lambda|_{L^\infty(0,T,L^2(\Omega))} + |\phi^\lambda|_{L^2(0,T,H^1(\Omega))} + |e^\lambda|_{L^\infty(0,T,L^2(\Omega))} + \int_0^T \int_\Omega B_\lambda(\phi^\lambda)|\nabla u^\lambda|^2\,dx\,ds \leq C_T, \tag{4.31}$$

where $e^\lambda = cu^\lambda + w(\phi^\lambda)$.

Proof of Lemma 4.4. Since w is a Lipschitz continuous function, and ϕ^λ, u^λ both belong to $L^2(0,T,H^1(\Omega))$, $e^\lambda \in L^2(0,T,H^1(\Omega))$, and is thus a valid test function in (4.8).

Let $t \in (0,T)$. We take the scalar product in $L^2(Q_t)$ of (4.6) with ϕ^λ, take $\eta = \varepsilon e^\lambda$ in (4.8), where

$$\varepsilon = \frac{c\xi^2}{(b + |m|_{L^\infty(0,1)})L_w^2},$$

and add both; this gives

$$\begin{aligned}\int_\Omega \left(\frac{\tau}{2}|\phi^\lambda(t)|^2 + \frac{\varepsilon}{2}|e^\lambda(t)|^2\right) dx + \int_0^t \int_\Omega (\xi^2|\nabla\phi^\lambda|^2 + c\varepsilon B_\lambda(\phi^\lambda)|\nabla u^\lambda|^2)\,dx\,ds \\ \leq C_T + \int_0^t \int_\Omega (w'(\phi^\lambda)\phi^\lambda u^\lambda - F_\lambda'(\phi^\lambda)\phi^\lambda - \varepsilon B_\lambda(\phi^\lambda) w'(\phi^\lambda)\nabla\phi^\lambda \cdot \nabla u^\lambda)\,dx\,ds.\end{aligned}$$

Using (4.3), **(A2)**, and the Young inequality, we get

$$\begin{aligned}&\int_\Omega \left(\frac{\tau}{2}|\phi^\lambda(t)|^2 + \frac{\varepsilon}{2}|e^\lambda(t)|^2\right) dx + \int_0^t \int_\Omega (\xi^2|\nabla\phi^\lambda|^2 + c\varepsilon B_\lambda(\phi^\lambda)|\nabla u^\lambda|^2)\,dx\,ds \\ &\quad\leq C_T\left(1 + \int_0^t \int_\Omega (|\phi^\lambda|^2 + |u^\lambda|^2)\,dx\,ds\right) \\ &\qquad + \varepsilon L_w(b + |m|_{L^\infty(0,1)})^{1/2} \int_0^t \int_\Omega (B_\lambda(\phi^\lambda))^{1/2}|\nabla u^\lambda|\,|\nabla\phi^\lambda|\,dx\,ds \\ &\quad\leq C_T\left(1 + \int_0^t \int_\Omega (|\phi^\lambda|^2 + |u^\lambda|^2)\,dx\,ds\right) + \frac{\varepsilon c}{2}\int_0^t \int_\Omega B_\lambda(\phi^\lambda)|\nabla u^\lambda|^2\,dx\,ds \\ &\qquad + \frac{\varepsilon L_w^2(b + |m|_{L^\infty(0,1)})}{2c}\int_0^t \int_\Omega |\nabla\phi^\lambda|^2\,dx\,ds.\end{aligned}$$

Hence, thanks to the choice of ε,

$$\begin{aligned}\int_\Omega (\tau|\phi^\lambda(t)|^2 + \varepsilon|e^\lambda(t)|^2)\,dx + \int_0^t \int_\Omega (\xi^2|\nabla\phi^\lambda|^2 + c\varepsilon B_\lambda(\phi^\lambda)|\nabla u^\lambda|^2)\,dx\,ds \\ \leq C_T\left(1 + \int_0^t \int_\Omega (\tau|\phi^\lambda|^2 + \varepsilon|e^\lambda|^2)\,dx\,ds\right).\end{aligned} \tag{4.32}$$

Then, (4.31) follows from (4.32) and Gronwall's lemma. □

We next prove the following result:

LEMMA 4.5. There exists a constant C_T such that, for any $\lambda \in (0,1]$,

$$|\beta_\lambda(\phi^\lambda)|_{L^2(Q_T)} \leq C_T \tag{4.33}$$

holds.

Proof of Lemma 4.5. We put

$$f^\lambda = -F_0'(\phi^\lambda) + w'(\phi^\lambda)u^\lambda.$$

It follows from (4.31) and **(A1)**–**(A2)** that

$$|f^\lambda|_{L^2(Q_T)} \leq C_T. \tag{4.34}$$

By (4.6), ϕ^λ is a solution to

$$\tau\phi_t^\lambda - \xi^2\Delta\phi^\lambda + \beta_\lambda(\phi^\lambda) = f^\lambda \quad \text{in } Q_T, \tag{4.35}$$

$$\frac{\partial\phi^\lambda}{\partial n} = 0 \quad \text{on } \Sigma_T. \tag{4.36}$$

Since β_λ is Lipschitz continuous, $\beta_\lambda(\phi^\lambda)$ belongs to $L^2(0,T,H^1(\Omega))$. We take the scalar product in $L^2(Q_T)$ of (4.35) with $\beta_\lambda(\phi^\lambda)$ and find

$$\tau\int_\Omega (\hat\beta_\lambda(\phi^\lambda(T)) - \hat\beta_\lambda(\phi_0^\lambda))\,dx + \xi^2\int_0^T\int_\Omega \beta_\lambda'(\phi^\lambda)|\nabla\phi^\lambda|^2\,dx\,ds + \int_0^T\int_\Omega |\beta_\lambda(\phi^\lambda)|^2\,dx\,ds \le |f^\lambda|_{L^2(Q_T)}|\beta_\lambda(\phi^\lambda)|_{L^2(Q_T)}.$$

Hence, since β_λ is nondecreasing and $\hat\beta_\lambda \ge 0$,

$$\int_0^T\int_\Omega |\beta_\lambda(\phi^\lambda)|^2\,dx\,ds \le |f^\lambda|^2_{L^2(Q_T)} + 2\tau\int_\Omega \hat\beta_\lambda(\phi_0^\lambda)\,dx. \tag{4.37}$$

But $\hat\beta_\lambda$ is convex and vanishes at $r=0$. It then follows from (4.30) that

$$\begin{aligned}\int_\Omega \hat\beta_\lambda(\phi_0^\lambda)\,dx &\le \int_\Omega \beta_\lambda(\phi_0^\lambda)\phi_0^\lambda\,dx \\ &\le \int_\Omega (|\beta_\lambda(\phi_0^\lambda) - \beta_\lambda(\phi_0)| + |\beta^0(\phi_0)|)|\phi_0^\lambda|\,dx \\ &\le C_T.\end{aligned} \tag{4.38}$$

Combining (4.34) and (4.37)–(4.38) yields (4.33). □

Finally, it follows from (4.6), (4.8), (4.31), (4.33), and **(A1)**–**(A2)** that

$$|\phi_t^\lambda|_{L^2(0,T,V')} + |e_t^\lambda|_{L^2(0,T,V')} \le C_T. \tag{4.39}$$

We infer from (4.31) and (4.39) that (ϕ^λ) and (e^λ) are bounded in

$$\{v \in L^\infty(0,T,L^2(\Omega)),\ v_t \in L^2(0,T,V')\},$$

which is compactly embedded in $\mathcal{C}([0,T],V')$ by [Si, Cor. 4]. Therefore,

$$(\phi^\lambda) \text{ and } (e^\lambda) \text{ are relatively compact in } \mathcal{C}([0,T],V'). \tag{4.40}$$

We also infer from (4.31) and (4.39) that (ϕ^λ) is bounded in

$$\{v \in L^2(0,T,H^1(\Omega)),\ v_t \in L^2(0,T,V')\},$$

which is compactly embedded in $L^2(Q_T)$ by [Si, Cor. 4]. Therefore,

$$(\phi^\lambda) \text{ is relatively compact in } L^2(Q_T). \tag{4.41}$$

It follows from (4.31), (4.33), and (4.39)–(4.41) that there exist

$$\begin{aligned}&\phi \in W^{1,2}(0,T,V') \cap L^\infty(0,T,L^2(\Omega)) \cap L^2(0,T,H^1(\Omega)),\\ &e \in W^{1,2}(0,T,V') \cap L^\infty(0,T,L^2(\Omega)), \quad u \in L^\infty(0,T,L^2(\Omega)),\\ &\zeta \in L^2(Q_T), \quad J \in L^2(Q_T),\end{aligned}$$

and a subsequence of $(\phi^\lambda, u^\lambda)$ (which we still denote by $(\phi^\lambda, u^\lambda)$) such that

$$\begin{aligned} \phi^\lambda &\to \phi \quad \text{in } \mathcal{C}([0,T],V'),\ \text{in } L^2(Q_T),\ \text{and a.e. in } Q_T, \\ e^\lambda &\to e \quad \text{in } \mathcal{C}([0,T],V'), \\ \beta_\lambda(\phi^\lambda) &\rightharpoonup \zeta \quad \text{in } L^2(Q_T), \\ B_\lambda(\phi^\lambda)\nabla u^\lambda &\rightharpoonup J \quad \text{in } L^2(Q_T). \end{aligned} \tag{4.42}$$

Now we claim that in fact

LEMMA 4.6. The sequence (ϕ^λ) converges to ϕ in $L^2(0,T,H^1(\Omega))$.

Proof of Lemma 4.6. Let $(\lambda,\mu) \in (0,1)^2$. It follows from (4.6)–(4.7) that

$$\tau(\phi_t^\lambda - \phi_t^\mu) - \xi^2\Delta(\phi^\lambda - \phi^\mu) = f^{\lambda,\mu}, \qquad \frac{\partial(\phi^\lambda - \phi^\mu)}{\partial n} = 0, \tag{4.43}$$

where

$$f^{\lambda,\mu} = w'(\phi^\lambda)u^\lambda - w'(\phi^\mu)u^\mu - F_\lambda'(\phi^\lambda) + F_\mu'(\phi^\mu).$$

It follows from (4.31), (4.33), and **(A1)**–**(A2)** that

$$|f^{\lambda,\mu}|_{L^2(Q_T)} \le C_T. \tag{4.44}$$

We take the scalar product in $L^2(Q_T)$ of (4.43) with $(\phi^\lambda - \phi^\mu)$; this gives

$$\begin{aligned} &\frac{\tau}{2}|\phi^\lambda(T) - \phi^\mu(T)|_{L^2(\Omega)}^2 + \xi^2\int_0^T\int_\Omega |\nabla(\phi^\lambda - \phi^\mu)|^2\,dx\,ds \\ &\le \frac{\tau}{2}|\phi_0^\lambda - \phi_0^\mu|_{L^2(\Omega)}^2 + |f^{\lambda,\mu}|_{L^2(Q_T)}|\phi^\lambda - \phi^\mu|_{L^2(Q_T)} \\ &\le C_T(\lambda^4 + \mu^4 + |\phi^\lambda - \phi^\mu|_{L^2(Q_T)}), \end{aligned} \tag{4.45}$$

thanks to (4.30) and (4.44). It follows from (4.42) that the right-hand side of (4.45) converges to zero as $(\lambda,\mu) \to (0,0)$. Thus, (ϕ^λ) is a Cauchy sequence in $L^2(0,T,H^1(\Omega))$, hence the lemma. □

We now proceed in the same way as in the proof of Theorem 2.1 and complete the proof of Proposition 2.2. □

5. Long-time behaviour. In this section, we describe the ω-limit set in $L^2(\Omega,\mathbb{R}^2)$ of the weak solution to (1.1)–(1.4) that we obtain in Theorem 2.1. More precisely, we consider ϕ_0 in $H^1(\Omega)$ and u_0 in $L^2(\Omega)$ such that (2.4) holds, and denote by (ϕ,μ) the corresponding weak solution to (1.1)–(1.4) given by Theorem 2.1. The ω-limit set $\omega(\phi_0,u_0)$ of (ϕ_0,u_0) in $L^2(\Omega,\mathbb{R}^2)$ is then

$$\omega(\phi_0,u_0) = \left\{ (\phi_\infty,u_\infty) \in L^2(\Omega,\mathbb{R}^2),\ \begin{array}{l} \exists\ t_n \to +\infty \text{ such that} \\ (\phi(t_n),u(t_n)) \to (\phi_\infty,u_\infty) \text{ in } L^2(\Omega,\mathbb{R})^2 \end{array} \right\}.$$

We put

$$M_0 = \int_\Omega (cu_0 + w(\phi_0))\,dx.$$

PROPOSITION 5.1. Assume that **(A1)**–**(A3)** hold and that F_0 is nonnegative. If (ϕ_∞, u_∞) belongs to $\omega(\phi_0, u_0)$, it satisfies:

$$\phi_\infty \in H^2(\Omega), \qquad u_\infty \in L^2(\Omega), \qquad \int_\Omega (cu_\infty + w(\phi_\infty))\,dx = M_0,$$

and there exist $\zeta_\infty \in L^2(\Omega), J_\infty \in L^2(\Omega)$, such that

$$-\xi^2\Delta\phi_\infty + \zeta_\infty + F_0'(\phi_\infty) = w'(\phi_\infty)u_\infty \quad \text{in } \Omega, \tag{5.1}$$

$$\zeta_\infty \in \beta(\phi_\infty) \quad \text{in } \Omega, \tag{5.2}$$

$$\frac{\partial\phi_\infty}{\partial n} = 0 \quad \text{on } \Gamma, \tag{5.3}$$

$$\operatorname{div}(J_\infty) = 0 \quad \text{in } V', \tag{5.4}$$

$$J_\infty = \nabla(B(\phi_\infty)u_\infty) - u_\infty\nabla B(\phi_\infty) \quad \text{in } V'. \tag{5.5}$$

Proof of Proposition 5.1. We use the technique of [LP]. We consider (ϕ_∞, u_∞) in $\omega(\phi_0, u_0)$, and let (t_n) be a sequence of positive real numbers such that $t_n \to +\infty$, and

$$(\phi(t_n), u(t_n)) \to (\phi_\infty, u_\infty) \quad \text{in } L^2(\Omega, \mathbb{R}^2). \tag{5.6}$$

We denote by C_0 a constant such that

$$|\phi(t_n)|_{L^2(\Omega)} + |u(t_n)|_{L^2(\Omega)} \le C_0, \quad n \ge 1. \tag{5.7}$$

It follows from (2.7) and (5.6) that

$$\int_\Omega (cu_\infty + w(\phi_\infty))\,dx = M_0.$$

For each integer $n \ge 1$ and $t \in (0,1)$, we put

$$\phi_n(t) = \phi(t_n + t), \qquad \zeta_n(t) = \zeta(t_n + t),$$
$$u_n(t) = u(t_n + t), \qquad J_n(t) = J(t_n + t),$$

where ζ and J are given in Theorem 2.1.

In the following, we denote by C any positive constant depending only on $\Omega, N, \tau, \xi, c, b$, $L_w, |F_0(0)|, |F_0'(0)|, c_1, |\phi_0|_{H^1}, |\hat\beta(\phi_0)|_{L^1}, |u_0|_{L^2}$, and C_0 in (5.7).

We first gather some estimates in the next lemma.

LEMMA 5.2.

$$|\phi_n|_{L^\infty(0,1,H^1(\Omega))} + |\phi_{nt}|_{L^2(Q_1)} + |u_n|_{L^\infty(0,1,L^2(\Omega))} + |J_n|_{L^2(Q_1)} \le C, \tag{5.8}$$

$$|\zeta_n|_{L^2(Q_1)} + |\phi_n|_{L^2(0,1,H^2(\Omega))} \le C, \tag{5.9}$$

$$|\phi_t|_{L^2(0,+\infty,L^2(\Omega))} + |u_t|_{L^2(0,+\infty,V')} \le C. \tag{5.10}$$

Proof of Lemma 5.2. We infer from Theorem 2.1 and (2.3) that

$$\int_0^t \int_\Omega \left(\tau|\phi_t|^2 + \frac{1}{b}|J|^2\right)\,dx\,ds + \mathcal{L}(\phi(t), u(t)) \le \mathcal{L}(\phi_0, u_0)$$

holds, where $\mathcal{L}$ is given by (2.9). It follows from **(A1)** that

$$\mathcal{L}(\phi_0, u_0) \le C.$$

Since both $\hat{\beta}$ and F_0 are nonnegative functions, the above two estimates yield

$$|\phi_t|_{L^2(0,+\infty,L^2(\Omega))} + |J|_{L^2(0,+\infty,L^2(\Omega))} + |\nabla\phi|_{L^\infty(0,+\infty,L^2(\Omega))} + |u|_{L^\infty(0,+\infty,L^2(\Omega))} \le C. \tag{5.11}$$

A first consequence of (5.11), the boundedness of w', and (2.7) is

$$|u_t|_{L^2(0,+\infty,V')} \le C. \tag{5.12}$$

Combining (5.11) and (5.12) gives (5.10). It also follows from (5.11) that, for $t \in (t_n, t_n+1)$,

$$|\phi(t) - \phi(t_n)|_{L^2(\Omega)} \le (t-t_n)^{1/2} |\phi_t|_{L^2(t_n,t,L^2(\Omega))} \le C.$$

Hence, thanks to (5.7),

$$|\phi_n|_{L^\infty(0,1,L^2(\Omega))} \le C. \tag{5.13}$$

Then, (5.8) is a straightforward consequence of (5.11) and (5.13).

Finally, (5.9) follows from (5.11), (2.5), and a monotonicity argument. □

We next claim the following result:

LEMMA 5.3. The sequence (ϕ_n) converges to ϕ_∞ in $L^2(Q_1)$, while the sequence (u_n) converges to u_∞ in $L^2(0,1,V')$.

Proof of Lemma 5.3. We infer from (5.10) that, for $t \in (0,1)$, one has

$$\begin{aligned} |\phi_n(t) - \phi(t_n)|_{L^2(\Omega)} &\le t^{1/2} \left(\int_{t_n}^{t_n+t} |\phi_t|^2_{L^2(\Omega)}\, ds \right)^{1/2} \\ &\le \left(\int_{t_n}^{+\infty} |\phi_t|^2_{L^2(\Omega)}\, ds \right)^{1/2}, \end{aligned}$$

and the right-hand side of the above estimate decreases to zero as $t_n \to +\infty$. This fact, together with (5.6) yields that

$$|\phi_n(t) - \phi_\infty|_{L^2(\Omega)} \to 0 \quad \text{a.e. in } (0,1).$$

The convergence of (ϕ_n) to ϕ_∞ in $L^2(Q_1)$ then follows from (5.8) and the Lebesgue dominated convergence theorem.

Similarly, we prove that (u_n) converges to u_∞ in $L^2(0,1,V')$. □

It follows from Lemma 5.2 and Lemma 5.3 that we may assume that there exist ζ_∞ in $L^2(Q_1)$ and J_∞ in $L^2(Q_1)$ such that

$$\begin{aligned} &\phi_n \rightharpoonup \phi_\infty \quad \text{in } L^2(0,1,H^2(\Omega)), \quad \text{and in } W^{1,2}(0,1,L^2(\Omega)), \\ &\zeta_n \rightharpoonup \zeta_\infty \quad \text{in } L^2(Q_1), \\ &u_n \rightharpoonup u_\infty \quad \text{in } L^2(Q_1), \quad \text{and in } W^{1,2}(0,1,V'), \\ &J_n \rightharpoonup J_\infty \quad \text{in } L^2(Q_1). \end{aligned} \tag{5.14}$$

We first infer from (5.14), Lemma 5.3, and [Br, Prop. 2.5] that

$$\phi_\infty \in D(\beta), \quad \zeta_\infty \in \beta(\phi_\infty) \quad \text{a.e. in } Q_1.$$

We now identify J_∞ in (5.14). It follows from Lemma 5.3, (5.14), and an interpolation argument that

$$\phi_n \to \phi_\infty \quad \text{in } L^2(0,1,H^1(\Omega)).$$

Since B is a Lipschitz continuous function, we have also

$$B(\phi_n) \to B(\phi_\infty) \quad \text{in } L^2(0,1,H^1(\Omega)).$$

Since (u_n) converges weakly to u_∞ in $L^2(Q_1)$, we conclude as in Sec. 4 that J_∞ is given by (5.5).

Next, since w' is a bounded Lipschitz continuous function, it follows from Lemma 5.3 that $(w'(\phi_n))$ converges to $w'(\phi_\infty)$ in $L^p(Q_1)$ for any $p \in [1,+\infty)$. Thus, $(w'(\phi_n)u_n)$ converges weakly to $(w'(\phi_\infty)u_\infty)$ in $L^{3/2}(Q_1)$.

Next, consider $\rho \in \mathcal{D}(0,1), z \in \mathcal{D}(\Omega)$ and take $\eta(x,t) = \rho(t-t_n)z(x)$ in (2.7); this gives

$$c\int_0^1 \langle u_{nt}, z\rangle_{V',V}\rho(t)\,dt + \int_0^1\int_\Omega w'(\phi_n)\phi_{nt}\rho(t)z\,dx\,dt + \int_0^1\int_\Omega J_n\cdot\nabla z\rho(t)\,dx\,ds = 0.$$

Taking the limit as $n \to +\infty$ yields (5.4).

Similarly, it follows from (2.5)–(2.6) that

$$\tau\int_0^1\int_\Omega \phi_{nt}\rho(t)z\,dx\,dt + \int_0^1\int_\Omega (\zeta_n + F_0'(\phi_n))\rho(t)z\,dx\,dt + \xi^2\int_0^1\int_\Omega \nabla\phi_n\cdot\nabla z\rho(t)\,dx\,dt = \int_0^1\int_\Omega w'(\phi_n)u_n\rho(t)z\,dx\,dt. \tag{5.15}$$

We then pass to the limit in (5.15) and get (5.1)–(5.3). The proof of Proposition 5.1 is thus complete. □

6. Convergence to the degenerate Cahn-Hilliard equation. In this section, we investigate the limit of (1.1)–(1.4) when $\tau = c = \alpha$ and α decreases to zero in the following particular case: F is given by **(E2)** (see Sec. 2) and B by

$$B(r) = \begin{cases} 1-r^2 & \text{if } |r| \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hereafter, we only state the convergence result and refer the reader to [La2] for the complete proofs and statements.

We put

$$\beta(r) = \ln\left(\frac{1+r}{1-r}\right), \quad r \in (-1,1), \qquad F_0(r) = 1-r^2, \quad r \in [-1,1],$$

and $F = \hat\beta + F_0$, where

$$\hat\beta(r) = (1+r)\ln(1+r) + (1-r)\ln(1-r), \quad r \in [-1,1].$$

We next consider a family of initial data $(\phi_0^\alpha, u_0^\alpha)_{\alpha\in(0,1)}$ such that, for each $\alpha \in (0,1)$, $(\phi_0^\alpha, u_0^\alpha) \in H^1(\Omega) \times L^2(\Omega)$ and satisfy

$$\hat\beta(\phi_0^\alpha) \in L^1(\Omega),$$

$$\frac{\xi^2}{2}|\phi_0^\alpha|^2_{H^1(\Omega)} + \int_\Omega F(\phi_0^\alpha)\,dx + \frac{\alpha}{2}|u_0^\alpha|^2_{L^2(\Omega)} \le C_0,$$

for some constant $C_0 > 0$, and the sequence (ϕ_0^α) converges strongly in $L^2(\Omega)$ to some function $\phi_0 \in H^1(\Omega)$.

In order to state our convergence result, we need to specify how we construct the weak solution to (1.1)–(1.4) that we shall deal with in the sequel: for $\lambda \in (0,1)$, we put $B_\lambda = B + \lambda$, and fix $T > 0$.

We infer from Theorem 2.1 that, for each $(\alpha,\lambda) \in (0,1)^2$, there exist functions $(\phi^{\alpha,\lambda}, \zeta^{\alpha,\lambda}, u^{\alpha,\lambda})$ satisfying

(i) $\phi^{\alpha,\lambda} \in W^{1,2}(0,T,L^2(\Omega)) \cap L^\infty(0,T,H^1(\Omega)) \cap L^2(0,T,H^2(\Omega))$, $\quad \phi^{\alpha,\lambda}(0) = \phi_0^\alpha$,

(ii) $\zeta^{\alpha,\lambda} \in L^2(Q_T), \zeta^{\alpha,\lambda} = \beta(\phi^{\alpha,\lambda})$ a.e. in Q_T ($\phi^{\alpha,\lambda} \in (-1,1)$ a.e. in Q_T),

(iii) $u^{\alpha,\lambda} \in W^{1,2}(0,T,V') \cap L^\infty(0,T,L^2(\Omega)) \cap L^2(0,T,H^1(\Omega))$, $\quad u^{\alpha,\lambda}(0) = u_0^\alpha$,

and such that

$$\alpha\phi_t^{\alpha,\lambda} + \zeta^{\alpha,\lambda} = \xi^2\Delta\phi^{\alpha,\lambda} + 2\phi^{\alpha,\lambda} + u^{\alpha,\lambda} \quad \text{a.e. in } Q_T,$$

$$\frac{\partial\phi^{\alpha,\lambda}}{\partial n} = 0 \quad \text{a.e. on } \Sigma_T,$$

$$\alpha\int_0^T \langle u_t^{\alpha,\lambda}, \eta\rangle_{V',V}\,ds + \int_0^T\int_\Omega \phi_t^{\alpha,\lambda}\eta\,dx\,ds + \int_0^T\int_\Omega B_\lambda(\phi^{\alpha,\lambda})\nabla u^{\alpha,\lambda}\cdot\nabla\eta\,dx\,ds = 0,$$

for any $\eta \in L^2(0,T,H^1(\Omega))$.

Now, a proof similar to that of Theorem 2.1 yields:

Proposition 6.1. For any $\alpha \in (0,1)$, there is a subsequence of $(\phi^{\alpha,\lambda}, \zeta^{\alpha,\lambda}, u^{\alpha,\lambda})_{\lambda\in(0,1)}$ that converges as λ decreases to zero to functions $(\phi^\alpha, \zeta^\alpha, u^\alpha)$ which satisfies all the requirements of Theorem 2.1. In particular, $(B_\lambda(\phi^{\alpha,\lambda})\nabla u^{\alpha,\lambda})$ converges weakly to J^α in $L^2(Q_T)$, where J^α is given by Theorem 2.1(iv).

We then have the following result:

Theorem 6.2. There is a subsequence of (ϕ^α, J^α) that converges to (ϕ, J), where

(i) $\phi \in W^{1,2}(0,T,V') \cap \mathcal{C}([0,T],L^2(\Omega)) \cap L^\infty(0,T,H^1(\Omega)) \cap L^2(0,T,H^2(\Omega))$,

(ii) $\phi(0) = \phi_0$, $\frac{\partial\phi}{\partial n} = 0$ a.e. on Σ_T and $\phi \in [-1,1]$ a.e. in Q_T,

(iii) $J \in L^2(Q_T, \mathbb{R}^N)$,

and (ϕ, J) satisfies

$$\phi_t = \operatorname{div}(J) \quad \text{in } L^2(0,T,V'), \tag{6.1}$$

$$\begin{aligned}\int_0^T\int_\Omega J\cdot\eta\,dx\,ds = \xi^2\int_0^T\int_\Omega &\operatorname{div}(B(\phi)\eta)\Delta\phi\,dx\,ds \\ &+ \int_0^T\int_\Omega (2 - 2B(\phi))\eta\cdot\nabla\phi\,dx\,ds,\end{aligned} \tag{6.2}$$

for any $\eta \in L^2(0,T,H^1(\Omega,\mathbb{R}^N)) \cap L^\infty(Q_T,\mathbb{R}^N)$ such that $\eta \cdot n = 0$ on Σ_T.

Note that the limit (ϕ, J) in Theorem 6.2 is a weak solution to (6.1)–(6.2) which belongs to the same class as that of C. M. Elliott and H. Garcke ([EG]). Of course, the lack of uniqueness of weak solutions to (1.1)–(1.4) and (6.1)–(6.2) prevents us from getting more precise results.

A similar result has been proved by B. Stoth when B is a positive constant, and for a smooth double-well potential ([St]). But the method does not seem to apply here because of the degeneracy of B.

Let us finally mention that the proof of Theorem 6.2 relies strongly on the particular choice of B and the logarithmic free energy, and does not seem to extend to the general case.

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