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WEAK SOLUTIONS TO A TIME-DEPENDENT HEAT EQUATION WITH NONLOCAL RADIATION BOUNDARY CONDITION AND ARBITRARY p-SUMMABLE RIGHT-HAND SIDE

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Abstract. We consider a model for transient conductive-radiative heat transfer in grey materials. Since the domain contains an enclosed cavity, nonlocal radiation boundary conditions for the conductive heat-flux are taken into account. We generalize known existence and uniqueness results to the practically relevant case of lower integrable heat-sources, and of nonsmooth interfaces. We obtain energy estimates that involve only the L^p norm of the heat sources for exponents p close to one. Such estimates are important for the investigation of models in which the heat equation is coupled to Maxwell's equations or to the Navier-Stokes equations (dissipative heating), with many applications such as crystal growth.

Keywords: radiative heat transfer, nonlinear parabolic equation, nonlocal boundary condition, right-hand side in L^1

MSC 2010: 35D05, 35K05, 35K15, 35K55

Introduction

Heat transfer processes that take place at high temperatures can be neither modeled nor simulated accurately without taking into account the phenomenon of heat radiation: the heating of high-temperatures furnaces in metallurgy or in crystal growth is one typical example of a relevant industrial problem. Models of radiative heat transfer have recently been studied from the point of view of applied mathematics in different publications: e.g. [16], [5], [4] for modeling and numerics, [11] for control theory, [7] for analysis.

In the present paper, we study from the analytical viewpoint the time-dependent heat transfer problem that consists in computing the temperature distribution resulting from the heating of several different opaque bodies separated from each other by an enclosed transparent medium. The essential purpose of the paper is to investigate for which regularity of the heat sources the problem admits weak solutions. In industrial applications, the heat sources are often obtained from coupled physical problems[1] so that regular right-hand sides hardly can be expected.

Problem description

We consider a finite number of bounded domains $\Omega_0, \dots, \Omega_m \subset \mathbb{R}^3$ $(m \geqslant 1)$ such that the set $\bigcup_{i=0}^{m} \overline{\Omega_i}$ is a simply connected Lipschitz domain and

$$\operatorname{dist}(\Omega_i, \Omega_j) > 0 \quad \text{for } i, j = 1, \dots, m \text{ and } i \neq j. \tag{0.1}$$

The computation domain $\Omega \subset \mathbb{R}^3$ is defined by $\overline{\Omega} := \bigcup_{i=1}^{m} \overline{\Omega_i}$, where $\Omega_1, \dots, \Omega_m$ represent opaque materials. Note the important feature that if $m \geqslant 2$, the set Ω is disconnected. The domain Ω_0 represents a transparent cavity.

We further assume that the transparent cavity Ω_0 is *enclosed* in Ω, that means, the geometry satisfies the enclosure property

$$\mathbb{R}^3 \setminus \Omega \text{ is disconnected.} \tag{0.2}$$

We define $\Sigma := \partial\Omega_0$ to be the boundary of the transparent cavity, where nonlocal radiation effects have to be modeled. We define $\Gamma := \partial\Omega \setminus \Sigma$. We assume that the surface Σ is at least pieciwise $\mathcal{C}^1$, and we in addition make the restriction that

$$\operatorname{dist}(\Gamma, \Sigma) > 0. \tag{0.3}$$

A typical geometrical situation is depicted in Fig. 1.

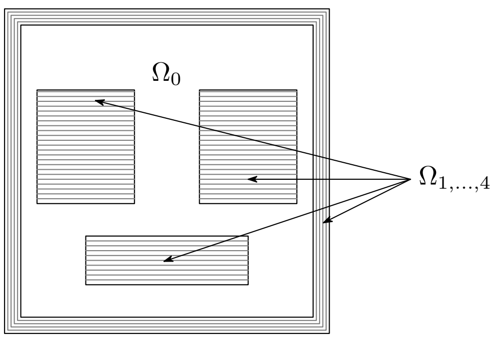

Figure 1. A typical geometry with the opaque bodies $\Omega_1, \dots, \Omega_4$ and the enclosed transparent cavity Ω_0.

[1] In the typical case of inductive heating, Maxwell's equations have to be solved.

We consider the problem

$$(0.4) \qquad \begin{cases} \dfrac{\partial\theta}{\partial t} - \operatorname{div}(\kappa(\theta)\nabla\theta) = f & \text{in } [0,T]\times\Omega_i \text{ for } i=1,\dots,m, \\ -\kappa(\theta)\dfrac{\partial\theta}{\partial\vec{n}} = R - J & \text{on } [0,T]\times\Sigma, \\ \theta = \theta_g & \text{on } [0,T]\times\Gamma \end{cases}$$

where θ is the absolute temperature, $\kappa = \kappa_i$ $(i = 1,\dots,m)$ denotes the temperature-dependent heat conductivity of the medium Ω_i, R is the outgoing radiation (radiosity), J is the incoming radiation, θ_g is the given temperature distribution, and f denotes the given heat source density. For $i = 1,\dots,m$, the unit normal pointing outwards to $\partial\Omega_i$ is denoted by $\vec{n}$.

The second relation in Problem (0.4) states that the conductive heat flux outgoing from each body has to balance the difference between the heat quantity brought to its surface by radiation, denoted by J, and the heat quantity leaving its surface due to radiation, denoted by R. Since R and J are in general unknown, additional relations are needed to close problem (0.4).

First, R and J are connected by the relation

$$(0.5) \qquad R = \varepsilon\sigma|\theta|^3\theta + (1-\varepsilon)J \quad \text{on } [0,T]\times\Sigma,$$

where the emissivity ε is a given function that takes values in $[0,1]$, and σ denotes the Stefan-Boltzmann constant. The relation (0.5) simply states that the outgoing radiation has to be the sum of the radiation *emitted* according to Stefan-Boltzmann's law, and of the *reflected* part of the incoming radiation.

Another constitutive relation between R and J is needed. If two points $z, y \in \Sigma$ are in each other's range of vision, then the radiation incoming at z from y, denoted by $j_y(z)$, is given by the inverse square law

$$j_y(z) = \frac{\vec{n}(z)\cdot(y-z)\,\vec{n}(y)\cdot(z-y)}{\pi|y-z|^4}\,R(y),$$

where $\vec{n}$ is a unit normal to Σ. The total radiation $J(z)$ is obtained by summing over the whole surface. For points pairs $(z,y) \in \Sigma\times\Sigma$ one introduces a *view factor* $w\colon\ \Sigma\times\Sigma \longrightarrow \mathbb{R}$ by setting

$$(0.6) \qquad w(z,y) := \begin{cases} \dfrac{\vec{n}(z)\cdot(y-z)\,\vec{n}(y)\cdot(z-y)}{\pi|y-z|^4}\,\Theta(z,y) & \text{if } z \neq y, \\ 0 & \text{if } z = y, \end{cases}$$

where Θ is the *visibility function* that penalizes the presence of opaque obstacles:

$$\Theta(z,y) = \begin{cases} 1 & \text{if }]z,y[\subset \Omega_0, \\ 0 & \text{else.} \end{cases} \tag{0.7}$$

Here we have used the notation $]z,y[:= \operatorname{conv}\{z,y\} \setminus \{z,y\}$. Observe that the view factor w is obviously well defined and nonnegative if the surface Σ has $\mathcal{C}^1$-regularity. This can be generalized to the case of a piecewise $\mathcal{C}^1$-boundary (see for example [3], [15], [2]).

The second constitutive relation between R and J is then given by

$$J = K(R) \quad \text{on } [0,T] \times \Sigma, \tag{0.8}$$

where

$$(K(R))(t,z) = \int_\Sigma w(z,y) R(t,y)\, \mathrm{d}S_y \quad \text{for } (t,z) \in [0,T] \times \Sigma. \tag{0.9}$$

The relations (0.5) and (0.8) are equivalent to an integral equation of the second kind, the *radiosity equation*

$$(I - (1-\varepsilon)K)(R) = \varepsilon\sigma|\theta|^3\theta \quad \text{on } [0,T] \times \Sigma, \tag{0.10}$$

where the symbol I denotes the identity mapping, and the functions ε, $1-\varepsilon$ in connection with integral operators simply mean multiplication.

The integro-differential problem

$$\begin{cases} \dfrac{\partial \theta}{\partial t} - \operatorname{div}(\kappa(\theta)\nabla\theta) = f & \text{in } [0,T] \times \Omega_i \text{ for } i = 1,\dots,m, \\ -\kappa(\theta)\dfrac{\partial \theta}{\partial \vec{n}} = (I-K)(R) & \text{on } [0,T] \times \Sigma, \\ (I - (1-\varepsilon)K)(R) = \varepsilon\sigma|\theta|^3\theta & \text{on } [0,T] \times \Sigma, \\ \theta = \theta_g & \text{on } [0,T] \times \Gamma \end{cases} \tag{0.11}$$

is well-defined and closed.

In a large class of applications, the formulation of (0.11) can be simplified. If the solution operator $(I - (1-\varepsilon)K)^{-1}$ of the integral equation (0.10) is well-defined (cf. Lemma A.2 (3)), it is possible to eliminate the unknown R. Introducing the linear operator

$$G := (I-K)(I-(1-\varepsilon)K)^{-1}\varepsilon, \tag{0.12}$$

we see that the problem (0.11) is equivalent to

$$
(0.13) \qquad (\mathrm{P}) \begin{cases} \dfrac{\partial\theta}{\partial t} - \operatorname{div}(\kappa(\theta)\nabla\theta) = f & \text{in } [0,T]\times\Omega_i \text{ for } i=1,\dots,m, \\ -\kappa(\theta)\dfrac{\partial\theta}{\partial\vec{n}} = G(\sigma|\theta|^3\theta) & \text{on } [0,T]\times\Sigma, \\ \theta = \theta_g & \text{on } [0,T]\times\Gamma, \end{cases}
$$

which only involves the one unknown θ. Throughout the paper, we focus on cases where the last formulation (P) is valid.

State of the research

The papers [15], [14] were devoted to the stationary equations corresponding to the problem (P). The existence of weak solutions was proved in the case that the transparent medium Ω_0 is *not enclosed*. In [10], a result was stated for the time-dependent problem under the same geometrical restriction.

The crucial point of the existence proof, the coercivity on a suitable Banach space of the nonlinear operator A defined by

$$
\langle A\theta, \psi\rangle := \int_\Omega \kappa\nabla\theta\cdot\nabla\psi + \int_\Sigma G(\sigma|\theta|^3\theta)\psi,
$$

turns out to have an elementary solution in geometries such that (0.2) fails.

In [7], new coercivity properties were established for the operator A, allowing to extend the previous results concerning the stationary problem to enclosures. Since the coercivity inequality proved in [7] relies on smoothing properties (compactness) of the integral operator K, the surface Σ has to be at least of class $\mathcal{C}^{1,\alpha}$ for some $\alpha > 0$. In the same paper [7], a paragraph was also devoted to the time-dependent problem, and an existence result was stated for $f \in L^2(]0,T[\times\Omega)$ in the case of a $\mathcal{C}^{1,\alpha}$ boundary.

In the present paper, we prove the existence of weak solutions to (P) in geometrical situations allowing for enclosures. Our main results nontrivially generalize the results of [7] in the following respects:

(1) We consider right-hand sides $f \in L^p(]0,T[\times\Omega)$, with arbitrary $1 \leqslant p \leqslant \infty$ (see Theorem 2.2 and Theorem 4.1).
(2) We include in our considerations the case of a piecewise smooth surface Σ (see Theorem 2.1).
(3) We propose new methods for proving existence in the case that $f \in L^2(]0,T[\times\Omega)$ (see Theorem 2.1).

The points (1) and (2) are especially relevant for the high-temperatures industrial applications mentioned at the beginning of the introduction. The main result of our paper, the existence of weak solutions for $f \in L^1(]0,T[\times\Omega)$, is also of interest for the theory of parabolic problems with L^1 right-hand side (see [1] and related publications), since the type of nonlocal nonlinearity introduced by the radiation boundary conditions have not yet been considered.

The paper is organized as follows. In the first section, we introduce the functional setting of the problem (P). The second section is devoted to existence results for the case that $f \in L^p(]0,T[\times\Omega)$ with $p > 1$ arbitrary. We then briefly address some regularity and uniqueness properties of weak solutions. The last section is concerned with the proof of existence in the case that $f \in L^1(]0,T[\times\Omega)$. In the appendix, we have gathered some auxiliary results needed throughout the paper.

1. Functional setting and definition of a weak solution

We use the notation

$$Q_t :=]0,t[\times\Omega, \quad \mathcal{S}_t :=]0,t[\times\Sigma, \quad \mathcal{C}_t :=]0,t[\times\Gamma.$$

We write Q instead of Q_T, $\mathcal{S}$ instead of $\mathcal{S}_T$, etc. For $1 \leqslant p,q < \infty$ we use the notation

$$L^{p,q}(Q) := \left\{u \in L^1(Q)\colon \int_0^T \left(\int_\Omega |u|^q \,\mathrm{d}x\right)^{p/q} \mathrm{d}t < \infty\right\},$$

and for $p = \infty$,

$$L^{\infty,q}(Q) := \left\{u \in L^1(Q)\colon \operatorname*{ess\,sup}_{t\in]0,T[} \left(\int_\Omega |u|^q \,\mathrm{d}x\right)^{1/q} < \infty\right\}.$$

Analogously, one can define the spaces $L^{p,q}(\mathcal{S})$. We write $L^p(Q)$, $L^p(\mathcal{S})$ instead of $L^{p,p}(Q)$, $L^{p,p}(\mathcal{S})$. For $1 \leqslant p < \infty$ we use the function spaces (see [6] for a general description)

$$W_p^{1,0}(Q) := \{u \in L^p(Q)\colon \exists\, u_{x_i} \in L^p(Q) \text{ for } i = 1,2,3\}$$

and

$$W_p^1(Q) := \{u \in W_p^{1,0}(Q)\colon \exists\, u_t \in L^p(Q)\},$$

where the partial derivatives u_{x_i}, u_t are intended in the weak sense. The space $V_2^{1,0}(Q)$ consists of all $u \in W_2^{1,0}(Q)$ such that $\operatorname*{ess\,sup}_{t\in]0,T[} \int_\Omega u^2(t,x)\,\mathrm{d}x < \infty$. We define

$$V^{p,q}(\Omega) := \{u \in W^{1,p}(\Omega)\colon \gamma(u) \in L^q(\Sigma)\},$$

where γ is the trace operator. For such p and q that the Sobolev space $W^{1,p}(\Omega)$ is not embedded in $L^q(\Sigma)$, the norm on $V^{p,q}$ is $\|\cdot\|_{W^{1,p}(\Omega)} + \|\gamma(\cdot)\|_{L^q(\Sigma)}$. The subscript Γ will indicate subspaces of functions that vanish on the surface Γ. We set

$$\mathcal{V}^{p,q}(Q) := \{u \in W_p^1(Q)\colon \gamma(u) \in L^q(\mathcal{S})\},$$
$$\mathcal{V}_0^{p,q}(Q) := \{u \in W_p^{1,0}(Q)\colon \gamma(u) \in L^q(\mathcal{S})\}.$$

Using the subscript $\mathcal{C}$, we denote the subspaces of functions that vanish on the surface $]0,T[\times\Gamma$. Throughout the paper, we assume that there exist positive constants κ_l, κ_u such that

$$0 < \kappa_l \leqslant \kappa_i(s) \leqslant \kappa_u < \infty \quad \text{for all } s \in \mathbb{R} \text{ for } i = 1, \ldots, m, \tag{1.1}$$

and a positive constant ε_l such that

$$0 < \varepsilon_l \leqslant \varepsilon(t,z) \leqslant 1 \quad \text{for } (t,z) \in\,]0,T[\times\Sigma. \tag{1.2}$$

The last hypothesis ensures that the operator G introduced in (0.12) is well-defined (see Lemma A.2 (3)). We note that for a real number $s \geqslant 1$, we denote throughout the paper by s' the conjugated exponent $s/(s-1)$. By convention, the numbers 1 and ∞ are conjugated.

With these preliminaries, and with help of Lemma A.2, we can show that the following definition is meaningful:

Definition 1.1. We call θ a *weak solution* to (P) if there exists $1 \leqslant s \leqslant \infty$ such that $\theta \in \mathcal{V}_0^{s,4}(Q)$, if $\theta = \theta_g$ almost everywhere on $\mathcal{C}$, and if the integral relation

$$-\int_Q \theta \frac{\partial \psi}{\partial t} + \int_Q \kappa(\theta)\nabla\theta\cdot\nabla\psi + \int_{\mathcal{S}} G(\sigma|\theta|^3\theta)\psi = \int_\Omega \theta_0\psi(0) + \int_Q f\psi$$

is valid for all $\psi \in \mathcal{V}_{\mathcal{C}}^{s',\infty}(Q)$ such that $\psi(T) = 0$ almost everywhere in Ω.

Remark 1.2. Since the data θ_g, θ_0 only play a subordinate role in applications, we restrict ourselves in the paper to the simplifying assumption

$$\theta_g = \text{const} = \theta_0, \tag{1.3}$$

which allows not to burden the proofs with technical details.

2. Existence of solutions

The main difficulty in proving the existence of weak solution to (P) is the strong growth of the term θ^4 on the boundary $\mathcal{S}$. In this section we show that the regularity of the right-hand side f and of the surface Σ are the two key-points to decide whether or not this term can be controlled.

(1) In the case of a piecewise smooth interface Σ, the existence of weak solutions to (P) has not yet been proved. That is the object of Theorem 2.1 below.

(2) In the case of a $\mathcal{C}^{1,\alpha}$-boundary Σ, the existence of weak solutions for right-hand sides f less regular than $f \in L^2(Q)$ has never been studied. That is the object of Theorem 2.2 below.

Theorem 2.1. *Let $\Sigma \in \mathcal{C}^1$ piecewise. Assume that $f \in L^{s_1,s_2}(Q)$, where $s_1, s_2 \in [1,\infty]$ are such that*

$$s_1 \in \begin{cases} \left]\dfrac{2s_2}{3(s_2-1)}, \infty\right] & \text{if } s_2 \leqslant \dfrac{3}{2}, \\ \left]\dfrac{2s_2}{3(s_2-1)}, \dfrac{2s_2}{2s_2-3}\right] & \text{if } s_2 > \dfrac{3}{2}. \end{cases}$$

Let θ_g, θ_0 satisfy (1.3), *let κ satisfy* (1.1), *and let ε satisfy* (1.2).

Then the number $\bar{q} := (5s_1s_2 - (3s_1 + 2s_2))/(3s_1 + 2s_2 - 2s_1s_2)$ satisfies $\bar{q} > 2$, and there exists a weak solution θ to (P) *such that $|\theta|^{(\bar{q}+1)/2} \in V_2^{1,0}(Q)$. In particular, we have*

$$\nabla\theta \in [L^s(Q)]^3, \quad \theta^4 \in L^{(\bar{q}+1)/3}(\mathcal{S}),$$

with $s := \min\{5(\bar{q}+1)/(\bar{q}+4), 2\}$.

We can deal with less regular right-hand sides if we assume that $\Sigma \in \mathcal{C}^{1,\alpha}$.

Theorem 2.2. *Let $\Sigma \in \mathcal{C}^{1,\alpha}$ for some $\alpha > 0$. Assume that $f \in L^{s_1,s_2}(Q)$, where $s_1 \in [1,\infty]$ and $s_2 \in [1, 3/2]$ are such that*

$$s_1 \in \left]\frac{2s_2}{5s_2-3}, \infty\right].$$

Let θ_g, θ_0 satisfy (1.3), *let κ satisfy* (1.1), *and let ε satisfy* (1.2).

Then the number $\bar{q} := (5s_1s_2 - (3s_1 + 2s_2))/(3s_1 + 2s_2 - 2s_1s_2)$ satisfies $\bar{q} > 0$, and there exists a weak solution θ to (P) *such that $|\theta|^{(\bar{q}+1)/2} \in V_2^{1,0}(Q)$. In particular, we have*

$$\nabla\theta \in [L^s(Q)]^3, \quad \theta \in L^{\bar{q}+4}(\mathcal{S}),$$

with $s := \min\{5(\bar{q}+1)/(\bar{q}+4), 2\}$.

The remainder of the section is devoted to the proof of Theorem 2.1 and of Theorem 2.2. We start the proof by constructing suitable approximate solutions in Proposition 2.3. In Propositions 2.5 and 2.6, we derive uniform estimates. Passage to the limit and existence proofs are given at the end of the section.

Proposition 2.3. *Let the assumptions of Theorem* 2.1 *or of Theorem* 2.2 *be satisfied. For* $\delta > 0$, *define* $f^{[\delta]} := \operatorname{sign}(f)\min\{|f|, 1/\delta\}$. *Let* $p \geqslant 5$ *be arbitrary.*

Then there exists $\theta \in L^p(0,T;W^{1,p}(\Omega))$ *such that* $\theta' \in L^{p'}(0,T;[W^{1,p}(\Omega)]^*)$, $\theta = \theta_g$ *in* $L^p(\mathcal{C})$, $\theta(0) = \theta_0$ *in* $L^p(\Omega)$ *and*

$$\langle \theta', \psi\rangle + \delta\int_0^T\int_\Omega (|\theta|^{p-2}\theta - |\theta_g|^{p-2}\theta_g)\psi + \int_0^T\int_\Omega (\delta|\nabla\theta|^{p-2} + \kappa(\theta))\nabla\theta\cdot\nabla\psi + \int_0^T\int_\Sigma G(\sigma|\theta|^3\theta)\psi = \int_0^T\int_\Omega f^{[\delta]}\psi \tag{2.1}$$

for all $\psi \in L^p(0,T;W^{1,p}_\Gamma(\Omega))$. *In addition,* $\theta \in L^\infty(Q)$.

P r o o f. We first introduce some notation. We define

$$\mathcal{V}_p := L^p(0,T;W^{1,p}_\Gamma(\Omega)), \quad L\xi := \xi',$$
$$D(L) := \{\xi \in L^p(0,T;W^{1,p}_\Gamma(\Omega))\colon\ \exists\,\xi' \in L^{p'}(0,T;[W^{1,p}_\Gamma(\Omega)]^*);\ \xi(0) = 0\}.$$

The symbol ξ' denotes the distributional time derivative of ξ. By classical results that can be found, for example, in [9] (see Ch. 3, Lem. 1.1), the operator L is a densely defined, maximal monotone linear operator from the linear subspace $D(L)$ of $\mathcal{V}_p$ into the dual $\mathcal{V}_p^*$.

For arbitrary $\xi \in \mathcal{V}_p$, we define $\hat{\xi} := \xi + \theta_g$ and we introduce an operator

$$\langle \mathcal{A}\xi, \psi\rangle := \delta\int_0^T\int_\Omega (|\hat{\xi}|^{p-2}\hat{\xi} - |\theta_g|^{p-2}\theta_g)\psi + \int_0^T\int_\Omega (\delta|\nabla\hat{\xi}|^{p-2} + \kappa(\hat{\xi}))\nabla\hat{\xi}\cdot\nabla\psi + \int_0^T\int_\Sigma G(\sigma|\hat{\xi}|^3\hat{\xi})\psi,$$

where the symbol $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathcal{V}_p$ and its dual $\mathcal{V}_p^*$.

We can show that $\mathcal{A}$ is a well-defined, bounded operator from $\mathcal{V}_p$ into $\mathcal{V}_p^*$. Observe that thanks to Lemma A.2

$$\begin{aligned}\left|\int_0^T\int_\Sigma G(\sigma|\hat{\xi}|^3\hat{\xi})\psi\right| &\leqslant \|G(\sigma|\hat{\xi}|^3\hat{\xi})\|_{L^{5/4}(0,T;L^{5/4}(\Sigma))}\|\psi\|_{L^5(0,T;L^5(\Sigma))}\\ &\leqslant \|G\|_{\mathcal{L}(5/4,5/4)}\sigma\|\hat{\xi}\|^4_{L^5(0,T;L^5(\Sigma))}\|\psi\|_{L^5(0,T;L^5(\Sigma))},\end{aligned}$$

where we use the notation (A.1). Since $p \geqslant 5$, we can use Hölder's and Young's inequalities to obtain that

$$\left|\int_0^T\int_\Sigma G(\sigma|\hat{\xi}|^3\hat{\xi})\psi\right| \leqslant c(1+\|\hat{\xi}\|^{p-1}_{L^p(0,T;W^{1,p}(\Omega))})\|\psi\|_{L^p(0,T;W^{1,p}_\Gamma(\Omega))}.$$

Estimating the other terms in $\mathcal{A}$ in a similar way, we verify that

$$\|\mathcal{A}\xi\|_{\mathcal{V}_p^*} \leqslant c_\delta(1+\|\xi\|^{p-1}_{\mathcal{V}_p}). \tag{2.2}$$

To prove the existence result, we show that there exists $\xi \in D(L)$ such that for all ψ in $\mathcal{V}_p$

$$\langle \xi', \psi\rangle + \langle \mathcal{A}\xi, \psi\rangle = \int_0^T\int_\Omega f^{[\delta]}\psi - \int_0^T\int_\Omega \frac{\partial\theta_g}{\partial t}\psi. \tag{2.3}$$

Then the function $\theta := \xi + \theta_g$ satisfies (2.1) and is the desired solution.

To prove that the mapping $\mathcal{F}$ given by

$$\langle \mathcal{F}, \psi\rangle := \int_0^T\int_\Omega f^{[\delta]}\psi - \int_0^T\int_\Omega \frac{\partial\theta_g}{\partial t}\psi$$

is a well-defined element of $\mathcal{V}_p^*$ is routine. Of course, under the assumption (1.3), the second term on the right-hand side is even zero.

Observe then that $\xi \in D(L)$ satisfies (2.3) if and only if the equation $(L+\mathcal{A})\xi = \mathcal{F}$ takes place in $\mathcal{V}_p^*$. Due to the theory of elliptic regularization (exposed for example in [9, Ch. 3, Th. 1.2]), it is sufficient to prove that $\mathcal{A}$ is coercive and pseudomonotone with respect to $D(L)$ to ensure the surjectivity of the operator $L+\mathcal{A}$ from $\mathcal{V}_p$ into $\mathcal{V}_p^*$.

For proving coercivity, we first verify that

$$\int_{\mathcal{S}} G(\sigma|\hat{\xi}|^3\hat{\xi})\hat{\xi} \geqslant (1-\|H\|_{\mathcal{L}(L^{5/4}(\mathcal{S}),L^{5/4}(\mathcal{S}))})\int_{\mathcal{S}}|\hat{\xi}|^5 \geqslant 0,$$

where we use Lemma A.2 (4). It follows that

$$\begin{aligned}\langle \mathcal{A}\xi,\xi\rangle &= \langle \mathcal{A}\xi, \hat{\xi}-\theta_g\rangle \\ &= \delta\int_Q(|\nabla\hat{\xi}|^p+|\hat{\xi}|^p) + \int_Q \kappa(\hat{\xi})|\nabla\hat{\xi}|^2 + \int_{\mathcal{S}} G(\sigma|\hat{\xi}|^3\hat{\xi})\hat{\xi} - \langle \mathcal{A}\xi,\theta_g\rangle \\ &\quad - \delta\int_Q(|\hat{\xi}|^{p-2}\hat{\xi}\theta_g + |\theta_g|^{p-2}\theta_g\hat{\xi} - |\theta_g|^p) \\ &\geqslant \delta\|\hat{\xi}\|^p_{L^p(0,T;W^{1,p}(\Omega))} - |\langle \mathcal{A}\xi,\theta_g\rangle| - \delta\int_Q(|\hat{\xi}|^{p-1}|\theta_g| + |\theta_g|^{p-1}|\hat{\xi}| + |\theta_g|^p).\end{aligned}$$

Using Hölder's and Young's inequalities, we obtain from the inequality (2.2) that

$$\begin{aligned}\langle \mathcal{A}\xi, \xi\rangle &\geqslant \frac{\delta}{2}\|\hat{\xi}\|^p_{L^p(0,T;W^{1,p}(\Omega))} - c_\delta(1+\|\xi\|^{p-1}_{\mathcal{V}_p})\|\theta_g\|_{L^p(0,T;W^{1,p}(\Omega))}\\ &\quad - \delta\tilde{c}\|\theta_g\|^p_{L^p(0,T;W^{1,p}(\Omega))}\\ &\geqslant \frac{\delta}{4}\,\|\hat{\xi}\|^p_{L^p(0,T;W^{1,p}(\Omega))} - C_\delta,\end{aligned}$$

with a constant C_δ that depends on δ but whose precise value is not needed. This proves the coercivity.

We now prove that $\mathcal{A}$ is pseudomonotone. Let $\xi_k \rightharpoonup \xi$ in $D(L)$. We assume that $\limsup_{k\to\infty}\langle \mathcal{A}\xi_k, \xi_k - \xi\rangle \leqslant 0$. The weak convergence in $D(L)$ means that

$$\xi_k \rightharpoonup \xi \ \text{ in } \mathcal{V}_p, \quad \xi'_k \rightharpoonup \xi' \ \text{ in } \mathcal{V}^*_p. \tag{2.4}$$

Applying the well-known compactness result of [9, Ch. 1, Th. 5.1], we can find a subsequence, still denoted by $\{\xi_k\}$, such that

$$\xi_k \longrightarrow \xi \quad \text{in } L^p(0,T;L^p(\Omega)). \tag{2.5}$$

The inequality

$$\|u\|_{L^p(\Sigma)} \leqslant \gamma\|u\|_{W^{1,p}(\Omega)} + c_\gamma\|u\|_{L^p(\Omega)} \tag{2.6}$$

holds for any u in $W^{1,p}(\Omega)$ and arbitrary small $\gamma > 0$. Therefore, from (2.4) and (2.5) we obtain the existence of a (not relabelled) subsequence such that

$$\xi_k \longrightarrow \xi \quad \text{in } L^p(0,T;L^p(\Sigma)).$$

Using the monotonicity of the p-Laplace terms, the property

$$\liminf_{k\to\infty}\langle \mathcal{A}\xi_k, \xi_k - \psi\rangle \geqslant \langle \mathcal{A}\xi, \xi - \psi\rangle$$

is readily verified for all ψ in $\mathcal{V}_p$, completing the proof of existence.

We finally prove the global boundedness of the solution $\theta := \xi + \theta_g$. Fix an arbitrary $t_1 < T$ and consider an arbitrary $0 < h < T - t_1$. Using the properties of the Steklov averaging operator, recalled in the appendix, we can prove the validity of the relation

$$\begin{aligned}&\int_0^{t_1}\!\!\int_\Omega \frac{\partial\theta_{(h)}}{\partial t}\psi + \delta\int_0^{t_1}\!\!\int_\Omega \{|\theta|^{p-2}\theta - |\theta_g|^{p-2}\theta_g\}_{(h)}\psi\\ &\quad + \int_0^{t_1}\!\!\int_\Omega \{(\delta|\nabla\theta|^{p-2} + \kappa(\theta))\nabla\theta\}_{(h)}\cdot\nabla\psi + \int_0^{t_1}\!\!\int_\Sigma \{G(\sigma|\theta|^3\theta)\}_{(h)}\psi\\ &= \int_0^{t_1}\!\!\int_\Omega f^{[\delta]}_{(h)}\psi\end{aligned} \tag{2.7}$$

for all $\psi \in L^p(0,T;W^{1,p}_\Gamma(\Omega))$. For every number $k > \max\{\operatorname{ess\,sup}_\Omega \theta_0, \operatorname{ess\,sup}_{\mathcal{C}} \theta_g\}$, the function $(\theta_{(h)} - k)^+$ belongs to $\mathcal{V}_p$ and can be used as a test function in (2.7).

Observe that

$$\begin{aligned}\int_0^{t_1}\int_\Omega \frac{\partial\theta_{(h)}}{\partial t}(\theta_{(h)} - k)^+ &= \int_0^{t_1}\int_\Omega \frac{\partial(\theta_{(h)} - k)}{\partial t}(\theta_{(h)} - k)^+ \\ &= \frac{1}{2}\int_\Omega [(\theta_{(h)} - k)^+(t_1)]^2 - \frac{1}{2}\int_\Omega [(\theta_{(h)} - k)^+(0)]^2.\end{aligned}$$

Our choice of k implies that $(\theta - k)^+(0) = (\theta_0 - k)^+ = 0$ almost everywhere in Ω. On the other hand, observe that $D(L) \subset C(0,T;L^2(\Omega))$. By the properties of the averaging operator $(\cdot)_{(h)}$, we thus have for all $t \in [0,T]$ that $\theta_{(h)}(t) \to \theta(t)$ in $L^2(\Omega)$ as $h \to 0$. Thus, as $h \to 0$,

$$\int_0^{t_1}\int_\Omega \frac{\partial\theta_{(h)}}{\partial t}(\theta_{(h)} - k)^+ \longrightarrow \frac{1}{2}\int_\Omega [(\theta - k)^+(t_1)]^2.$$

Passage to the limit with the remaining terms in (2.7) is an easy exercise. Observe that

$$\int_{Q_{t_1}} |\nabla\theta|^{p-2}\nabla\theta \cdot \nabla(\theta - k)^+ = \int_{Q_{t_1}} |\nabla\theta|^{p-2}|\nabla(\theta - k)^+|^2 \geqslant 0,$$

and that, due to the choice of the parameter k,

$$\int_{Q_{t_1}} (|\theta|^{p-2}\theta - |\theta_g|^{p-2}\theta_g)(\theta - k)^+ \geqslant 0.$$

So we obtain, for all $t_1 < T$, the relation

$$\begin{aligned}\frac{1}{2}\int_\Omega [(\theta - k)^+(t_1)]^2 + \int_{Q_{t_1}} \kappa(\theta)|\nabla(\theta - k)^+|^2 + \int_{\mathcal{S}_{t_1}} G(\sigma|\theta|^3\theta)(\theta - k)^+ \\ \leqslant \int_{Q_{t_1}} f^{[\delta]}(\theta - k)^+.\end{aligned}$$

In view of Lemma A.4 we have $\int_{\mathcal{S}_{t_1}} G(\sigma|\theta|^3\theta)(\theta - k)^+ \geqslant 0$, which implies that

$$\max_{t_1 \in [0,T]} \int_\Omega [(\theta - k)^+(t_1)]^2 + \int_Q \kappa(\theta)|\nabla(\theta - k)^+|^2 \leqslant 2\left|\int_Q f^{[\delta]}(\theta - k)^+\right|.$$

The results of [6] or of [13] prove the existence of an upper bound for θ in Q. A lower bound is obtained in the same way, by considering $(k - \theta)^-$ for $k < \min\{\operatorname{ess\,inf}_\Omega \theta_0, \operatorname{ess\,inf}_{\mathcal{C}} \theta_g\}$. It follows for all $r > 5/2$ that

$$\|\theta\|_{L^\infty(Q)} \leqslant \max\{\|\theta_g\|_{L^\infty(\mathcal{C})}, \|\theta_0\|_{L^\infty(\Omega)}\} + C_r\|f^{[\delta]}\|_{L^r(Q)}. \tag{2.8}$$

□

R e m a r k 2.4. The approximation method of Proposition 2.3 corresponds to the regularization of the problem (P) with a nonlinear Fourier-law for the heat flux. For δ small, the term $\delta(|\theta|^{p-2}\theta - |\theta_g|^{p-2}\theta_g)$ can be interpreted as a penalization of the heat sources at high-temperatures.

The next point consists in obtaining uniform estimates for sequences of approximate solutions.

Proposition 2.5. *Let the assumptions of Theorem* 2.1 *be satisfied, and define numbers* $\bar{q}$ *and* s *as in that theorem. Then for any sequence of solutions* $\{\theta_\delta\}$ *according to Proposition* 2.3 *we have*

$$\|\nabla\theta_\delta\|_{[L^s(Q)]^3} + \||\theta_\delta|^{(\bar{q}+1)/2}\|_{V_2^{1,0}(Q)} + \|\theta_\delta^4\|_{L^{(\bar{q}+1)/3}(\mathcal{S})} \leqslant C, \tag{2.9}$$

where the constant C *depends continuously on* $\|f\|_{L^{s_1,s_2}(Q)}$, *on* $\|\theta_0\|_{L^{\bar{q}+1}(\Omega)}$, *and on* $\|\theta_g\|_{L^\infty(Q)}$.

P r o o f. In the sequel we write for convenience θ instead of θ_δ. For the family of parameters $0 < q < \infty$, we first prove that

$$\begin{aligned} &\frac{1}{2(q+1)}\int_\Omega |\theta(t_1)|^{q+1} \\ &\quad + \int_{Q_{t_1}} \frac{4q}{(q+1)^2}\kappa(\theta)\big|\nabla|\theta|^{(q+1)/2}\big|^2 + \int_{\mathcal{S}_{t_1}} \sigma G(|\theta|^3\theta)|\theta|^{q-1}\theta \\ &\qquad \leqslant \int_{Q_{t_1}} f^{[\delta]}|\theta|^{q-1}\theta + \frac{c_q}{q+1}\int_\Omega |\theta_0|^{q+1}. \end{aligned} \tag{2.10}$$

The inequality (2.10) can be obtained by testing the approximate equation (2.3) with the signed powers $(|\theta|^{q-1}\theta - |\theta_g|^{q-1}\theta_g)$. This part of the proof is technical, and the reader will find it in Lemma C.1 below (see the appendix).

Define $w := |\theta|^{(q+1)/2}$. Since according to Lemma A.2, $G = I - H$ with a positive operator H, we have

$$\int_{\mathcal{S}_{t_1}} G(\sigma|\theta|^3\theta)|\theta|^{q-1}\theta \geqslant \int_{\mathcal{S}_{t_1}} G(\sigma|\theta|^4)|\theta|^q = \int_{\mathcal{S}_{t_1}} G(\sigma w^{8/(q+1)})w^{2q/(q+1)}.$$

Rewriting (2.10), we obtain that

$$\begin{aligned} &\int_\Omega w^2(t_1) + \int_{Q_{t_1}} |\nabla w|^2 + \int_{\mathcal{S}_{t_1}} G(\sigma w^{8/(q+1)})w^{2q/(q+1)} \\ &\qquad \leqslant c_q\Big(\int_\Omega |\theta_0|^{q+1} + \int_{Q_{t_1}} |f^{[\delta]}|w^{2q/(q+1)}\Big). \end{aligned} \tag{2.11}$$

We use Hölder's inequality to conclude that

$$\int_0^{t_1}\int_\Omega |f^{[\delta]}|w^{2q/(q+1)} \leqslant \|f\|_{L^{s_1,s_2}(Q)}\|w\|^{2q/(q+1)}_{L^{\beta_1,\beta_2}(Q)}, \tag{2.12}$$

where

$$\beta_1 := 2s_1'q/(q+1), \quad \beta_2 := 2s_2'q/(q+1). \tag{2.13}$$

Note that, in view of Proposition 2.3, we have $\theta \in L^\infty(Q)$, which ensures that $\|w\|_{L^{\beta_1,\beta_2}(Q)}$ is finite.

With help of (2.11), we first obtain that

$$\operatorname*{ess\,sup}_{t_1\in]0,T[}\int_\Omega w^2(t_1) \leqslant c_q\Big(\int_\Omega |\theta_0|^{q+1} + \|f\|_{L^{s_1,s_2}(Q)}\|w\|^{2q/(q+1)}_{L^{\beta_1,\beta_2}(Q)}\Big), \tag{2.14}$$

and reusing (2.11), it follows that

$$\|w\|^2_{V_2^{1,0}(Q)} \leqslant 2c_q\Big(\int_\Omega |\theta_0|^{q+1} + \|f\|_{L^{s_1,s_2}(Q)}\|w\|^{2q/(q+1)}_{L^{\beta_1,\beta_2}(Q)}\Big). \tag{2.15}$$

Assume now that the numbers β_1, β_2 can be chosen such as to satisfy the conditions of Lemma B.1 for the continuity of the embedding $V_2^{1,0}(Q) \hookrightarrow L^{\beta_1,\beta_2}(Q)$. It then follows, from Young's inequality, that

$$\|w\|^2_{L^{\beta_1,\beta_2}(Q)} \leqslant \tilde{c}_q\Big(\int_\Omega |\theta_0|^{q+1} + \|f\|^{q+1}_{L^{s_1,s_2}(Q)}\Big),$$

which can be inserted in (2.15) to obtain that

$$\|w\|^2_{V_2^{1,0}(Q)} \leqslant C(q, \|\theta_0\|_{L^{q+1}(\Omega)}, \|f\|_{L^{s_1,s_2}(Q)}). \tag{2.16}$$

Recalling the definition of w, we have obtained the estimate

$$\||\theta|^{(q+1)/2}\|^2_{V_2^{1,0}(Q)} \leqslant C(q, \|\theta_0\|_{L^{q+1}(\Omega)}, \|f\|_{L^{s_1,s_2}(Q)}). \tag{2.17}$$

The idea is now to control θ^4 on the boundary with help of Lemma B.1. Since $V_2^{1,0}(Q) \hookrightarrow L^{8/3}(\mathcal{S})$ with a continuous embedding, we now obtain a uniform bound

$$\|\theta^4\|_{L^{(q+1)/3}(\mathcal{S})} \leqslant \tilde{C}(q, \|\theta_0\|_{L^{q+1}(\Omega)}, \|f\|_{L^{s_1,s_2}(Q)}),$$

which makes sense provided that $(q+1)/3 \geqslant 1$.

Finally, we verify that if s_1, s_2 satisfy the hypothesis of Theorem 2.1, the choice

$$q := \frac{5s_1s_2 - (3s_1 + 2s_2)}{3s_1 + 2s_2 - 2s_1s_2}$$

ensures that $q \geqslant 2$, and that the continuity of the embedding $V_2^{1,0}(Q) \hookrightarrow L^{\beta_1,\beta_2}(Q)$ is valid.

We reuse the estimate (2.17) to obtain an estimate on $\nabla\theta$. If $r \leqslant 2$, we can write

$$\begin{aligned}\int_Q |\nabla\theta|^r &= \int_Q \frac{|\nabla\theta|^r}{|\theta|^{(1-q)r/2}}\,|\theta|^{(1-q)r/2}\\ &\leqslant \left(\int_Q \frac{|\nabla\theta|^2}{|\theta|^{1-q}}\right)^{r/2}\left(\int_Q |\theta|^{(1-q)r/(2-r)}\right)^{(2-r)/2}\\ &\leqslant c\|\nabla|\theta|^{(q+1)/2}\|^r_{L^2(Q)}\||\theta|^{(q+1)/2}\|^{(1-q)r/(q+1)}_{L^{2(1-q)r/((2-r)(q+1))}(Q)}.\end{aligned}$$

In view of estimate (2.17) and of Lemma B.1, we see that if the relation

$$2(1-q)r \leqslant \frac{10}{3}(2-r)(q+1)$$

is satisfied and $r \leqslant 2$, then $\nabla\theta$ will be uniformly bounded in $[L^r(Q)]^3$. This is true exactly for the range $1 \leqslant r \leqslant s$. $\square$

Proposition 2.6. *Let the assumptions of Theorem 2.2 be satisfied, and define the numbers $\bar q$ and s as in that theorem. Then for any sequence of solutions $\{\theta_\delta\}$ according to Proposition 2.3*

$$\||\theta_\delta|^{(\bar q+1)/2}\|_{V_2^{1,0}(Q)} + \|\nabla\theta_\delta\|_{[L^s(Q)]^3} + \|\theta_\delta\|_{L^{\bar q+4}(\mathcal{S})} \leqslant C, \tag{2.18}$$

where the constant C depends continuously on $\|f\|_{L^{s_1,s_2}(Q)}$, on $\|\theta_0\|_{L^{1+\bar q}(\Omega)}$, and on $\|\theta_g\|_{L^\infty(Q)}$.

P r o o f. We can of course use the same reasoning as in the proof of Proposition 2.5. However, we obtain an additional estimate thanks to the regularizing properties of the operator K on smooth surfaces.

We reconsider the relation (2.11). We apply Lemma A.3 with $\psi := w^{2\bar q/(\bar q+1)}$, $r := 4/\bar q$, and $s := (\bar q+1)/2\bar q$, and we obtain that

$$\begin{aligned}\int_0^{t_1}\int_\Sigma G(\sigma w^{8/(\bar q+1)})w^{2\bar q/(\bar q+1)}\\ \geqslant c_{1,\bar q}\int_0^{t_1}\int_\Sigma w^{2(\bar q+4)/(\bar q+1)} - c_{2,\bar q}\int_0^{t_1}\left(\int_\Sigma w\right)^{2(\bar q+4)/(\bar q+1)}.\end{aligned} \tag{2.19}$$

In view of Lemma B.1, the embedding $V_2^{1,0}(Q) \hookrightarrow L^{\infty,4/3}(\mathcal{S})$ is continuous. We obtain that

$$\int_0^T \Big(\int_\Sigma w\Big)^{2(\bar q+4)/(\bar q+1)} \leqslant T\|w\|^{2(\bar q+4)/(\bar q+1)}_{L^{\infty,1}(\mathcal{S})} \leqslant Tc\|w\|^{2(\bar q+4)/(\bar q+1)}_{V_2^{1,0}(Q)}$$
$$\leqslant T\tilde{C}(\bar q, \|\theta_0\|_{L^{\bar q+1}(\Omega)}, \|f\|_{L^{s_1,s_2}(Q)}),$$

in view of (2.16). We now obtain from (2.19) that

$$\|\theta\|^{\bar q+4}_{L^{\bar q+4}(\mathcal{S})} \leqslant C. \tag{2.20}$$

Unlike in the proof of Proposition 2.5, we are not bound to the condition $\bar q \geqslant 2$ to control θ^4 on the boundary. Therefore, a larger choice of the parameters s_1, s_2 is possible. □

In order to pass to the limit with the approximate solutions, we state in the following lemma an additional estimate.

Lemma 2.7. *Let the hypotheses of Proposition* 2.3 *be satisfied with* $p \geqslant s_1$, *and assume that the hypotheses either of Proposition* 2.5 *or of Proposition* 2.6 *are valid. Then the sequence* $\|\theta'_\delta\|_{L^1(0,T;[W^{1,p}_\Gamma(\Omega)]^*)}$ *is uniformly bounded.*

Proof. The proof is technical. The reader will find it at the end of the appendix. □

Proof of Theorem 2.1 and of Theorem 2.2. Thanks to the *a priori* estimates of Proposition 2.5 (or of Proposition 2.6) and of Lemma 2.7, the compactness theorems of [9] generalized in [12], imply the existence of a subsequence $\delta \to 0$ and of a function θ such that

$$\nabla\theta_\delta \rightharpoonup \nabla\theta \text{ in } [L^s(Q)]^3, \quad \theta_\delta \longrightarrow \theta \text{ in } L^2(Q), \quad \theta_\delta \longrightarrow \theta \text{ a.e. in } Q. \tag{2.21}$$

By means of the inequality (2.6), we also find subsequences such that

$$\theta_\delta \longrightarrow \theta \text{ in } L^s(\mathcal{S}), \quad \theta_\delta \longrightarrow \theta \text{ a.e. on } \mathcal{S}, \tag{2.22}$$

with $s := \min\{2, 5(\bar q+1)/\bar q + 4\}$.

In addition, we see that there exists a $u \in L^r(\mathcal{S})$ such that

$$\theta_\delta|\theta_\delta|^3 \rightharpoonup u \quad \text{in } L^r(\mathcal{S}) \tag{2.23}$$

with $r := (\bar q+1)/3 > 1$ in the case of Theorem 2.1, $r := (\bar q+4)/4 > 1$ in the case of Theorem 2.2. The fact that $r > 1$ implies that the weak limit u must be identical with the pointwise limit, if it exists. Then the convergence properties (2.22) imply that $u = \theta|\theta|^3$.

We test in (2.1) with an arbitrary ψ in $C^\infty(\overline{Q})$ that vanishes in $\{T\} \times \Omega$ and on $\mathcal{C}$. We can write

$$-\int_Q \theta_\delta \frac{\partial \psi}{\partial t} + \ldots + \int_Q \kappa(\theta_\delta)\nabla\theta_\delta \cdot \nabla\psi + \int_{\mathcal{S}} G(\sigma|\theta_\delta|^3\theta_\delta)\psi = \int_\Omega \theta_\delta(0)\psi(0) + \int_Q f^{[\delta]}\psi,$$

where $(\ldots)$ represents the terms involving the p-power. Passing to the limit in the last relation, we can easily show that

$$-\int_Q \theta \frac{\partial \psi}{\partial t} + \int_Q \kappa(\theta)\nabla\theta \cdot \nabla\psi + \int_{\mathcal{S}} G(\sigma|\theta|^3\theta)\psi = \int_\Omega \theta_0\psi(0) + \int_Q f\psi. \tag{2.24}$$

□

3. Some additional properties of weak solutions

In order to state the first result, we introduce the notation

$$\mathcal{W}(t_1, t_2, t_3; \Omega) := \{\psi \in W^{1,1}(\Omega)\colon\ \nabla\psi \in [L^{t_1}(\Omega)]^3,\ \psi \in L^{t_2}(\Omega),\ \gamma(\psi) \in L^{t_3}(\Sigma)\}.$$

As usual, the subscript Γ indicates vanishing on the surface Γ.

Lemma 3.1. *The weak solution θ constructed in Theorems 2.1 and 2.2 has a distributional time derivative $\theta' \in L^r(0,T;, [\mathcal{W}_\Gamma(t_1,t_2,t_3;\Omega)]^*)$, where in the case of Theorem 2.1,*

$$r := \min\Big\{\frac{5(\bar q+1)}{\bar q+4}, \frac{\bar q+1}{3}, s_1'\Big\},$$
$$t_1 := s', \quad t_2 := s_2', \quad t_3 := \frac{\bar q+1}{\bar q-2},$$

and in the case of Theorem 2.2,

$$r := \min\Big\{\frac{5(\bar q+1)}{\bar q+4}, \frac{\bar q+4}{4}, s_1'\Big\},$$
$$t_1 := s', \quad t_2 := s_2', \quad t_3 := \frac{\bar q+4}{\bar q}.$$

P r o o f. For $\psi \in C_c^\infty(0,T;C^\infty(\overline{\Omega}))$ such that $\psi = 0$ on Γ we can write

$$\left|\int_Q \theta\frac{\partial\psi}{\partial t}\right| \leqslant \int_Q \kappa(\theta)|\nabla\theta||\nabla\psi| + \int_{\mathcal{S}} |G(\sigma|\theta|^3|\theta|)||\psi| + \int_Q |f||\psi|.$$

We further estimate the right-hand side using the properties stated in Theorem 2.1 or Theorem 2.2. The proofs being quite similar to each other, we consider the case of Theorem 2.1. First, we have

$$\int_Q \kappa(\theta)|\nabla\theta||\nabla\psi| \leqslant \kappa_u \|\nabla\theta\|_{[L^s(Q)]^3}\|\nabla\psi\|_{[L^{s'}(Q)]^3},$$

$$\int_{\mathcal{S}} |G(\sigma|\theta|^3|\theta|)||\psi| \leqslant \|G\|_{\mathcal{L}((\bar{q}+1)/3,(\bar{q}+1)/3)}\|\theta^4\|_{L^{(\bar{q}+1)/3}(\mathcal{S})}\|\psi\|_{L^{(\bar{q}+1)/(\bar{q}-2)}(\mathcal{S})},$$

$$\int_Q |f||\psi| \leqslant \|f\|_{L^{s_1,s_2}(Q)}\|\psi\|_{L^{s_1',s_2'}(Q)}.$$

Thus, defining the numbers r, t_1, t_2 and t_3 as in the lemma, we obtain that

$$\left|\int_Q \theta\frac{\partial\psi}{\partial t}\right| \leqslant c(\|\nabla\psi\|_{[L^{r',t_1}(Q)]^3} + \|\psi\|_{L^{r',t_2}(Q)} + \|\psi\|_{L^{r',t_3}(\mathcal{S})}).$$

The claim follows. □

Lemma 3.2. *Let $f \in L^r(Q)$ for a $r > \frac{5}{2}$, and let $\theta_0 \in L^\infty(\Omega)$, as well as $\theta_g \in L^\infty(\mathcal{C})$. Then the weak solution θ of* (P) *constructed in the Theorems* 2.1 *and* 2.2 *is bounded in Q, and we have*

$$\|\theta\|_{L^\infty(Q)} \leqslant \max\{\|\theta_g\|_{L^\infty(\mathcal{C})}, \|\theta_0\|_{L^\infty(\Omega)}\} + C\|f\|_{L^r(Q)}.$$

Proof. The statement follows directly from the estimate (2.8) on the approximate solutions. □

In the case that $\kappa_i = \text{const}$ in Ω_i for $i = 1,\ldots,m$, the uniqueness of weak solutions in the class $\mathcal{V}_0^{2,5}(Q)$ has been proved in [7], together with an interesting *comparison principle*. The next lemma is not new but it extends the validity of this result to larger classes of weak solutions, and to the temperature-dependent heat conductivity, with an elementary method of proof.

Lemma 3.3. *Assume that κ_i is globally Lipschitz continuous for $i = 1,\ldots,m$. Let $f_1, f_2 \in L^{s_1,s_2}(Q)$ be such that $f_1 \geqslant f_2$ almost everywhere in Q.*

Then, if $\theta_j \in \mathcal{V}_0^{2,4}(Q) \cap C(0,T;L^1(\Omega))$ is a weak solution to (P) *corresponding to f_j ($j = 1,2$), we have $\theta_1 \geqslant \theta_2$ almost everywhere in Q. Consequently, the weak solution to* (P) *is unique in the class $\mathcal{V}_0^{2,4}(Q) \cap C(0,T;L^1(\Omega))$.*

P r o o f. Under the assumptions of the lemma, the difference $\theta_1 - \theta_2$ vanishes on $\mathcal{C}$, and the relation

$$(3.1)\qquad -\int_Q (\theta_1-\theta_2)\frac{\partial\psi}{\partial t} + \int_Q (\kappa(\theta_1)\nabla\theta_1 - \kappa(\theta_2)\nabla\theta_2)\cdot\nabla\psi + \int_{\mathcal{S}} G(\sigma[|\theta_1|^3\theta_1 - |\theta_2|^3\theta_2])\psi = \int_Q (f_1 - f_2)\psi$$

is valid for all $\psi \in \mathcal{V}^{2,\infty}_{\mathcal{C}}(Q)$. For parameters $\gamma > 0$ we consider the function $g_\gamma\colon \mathbb{R} \to \mathbb{R}$ defined by

$$(3.2)\qquad g_\gamma(t) := \frac{\operatorname{sign}(t)}{\gamma}\min\{|t|,\gamma\},$$

and we denote by F_γ the primitive of g_γ that vanishes at zero. Note that g_γ is monotonely increasing and globally bounded by 1. We apply Lemma B.2 with $u := (\theta_1-\theta_2)^-$, $\bar{u} = 0$ and we obtain for all $t_1 < T$ that

$$(3.3)\qquad \int_\Omega F_\gamma((\theta_1(t_1) - \theta_2(t_1))^-) + \int_{Q_{t_1}} (\kappa(\theta_1)\nabla\theta_1 - \kappa(\theta_2)\nabla\theta_2)\cdot\nabla g_\gamma((\theta_1-\theta_2)^-) + \int_{\mathcal{S}_{t_1}} G(\sigma[|\theta_1|^3\theta_1 - |\theta_2|^3\theta_2])g_\gamma((\theta_1-\theta_2)^-) \leqslant 0.$$

We write

$$\int_{Q_{t_1}} (\kappa(\theta_1)\nabla\theta_1 - \kappa(\theta_2)\nabla\theta_2)\cdot\nabla g_\gamma((\theta_1-\theta_2)^-) = \int_{Q_{t_1}} \kappa(\theta_1)\nabla(\theta_1-\theta_2)\cdot\nabla g_\gamma((\theta_1-\theta_2)^-) + \int_{Q_{t_1}} (\kappa(\theta_1)-\kappa(\theta_2))\nabla\theta_2\cdot\nabla g_\gamma((\theta_1-\theta_2)^-).$$

Observe that

$$\int_{Q_{t_1}} |\kappa(\theta_1)-\kappa(\theta_2)||\nabla\theta_2||\nabla g_\gamma((\theta_1-\theta_2)^-)| = \frac{1}{\gamma}\int_{\{(t,x)\in Q_{t_1}\,:\,0<|(\theta_1-\theta_2)^-(t,x)|<\gamma\}} |\kappa(\theta_1)-\kappa(\theta_2)||\nabla\theta_2||\nabla(\theta_1-\theta_2)^-|.$$

Pointwise in Q_{t_1} we have the estimate

$$\frac{1}{\gamma}\chi_{\{(t,x)\in Q_{t_1}\,:\,0<|(\theta_1-\theta_2)^-(t,x)|<\gamma\}}|\kappa(\theta_1)-\kappa(\theta_2)| \leqslant L_\kappa,$$

where L_κ is a Lipschitz constant of κ. Due to the dominated convergence theorem, we thus see that

$$\int_{Q_{t_1}} |\kappa(\theta_1) - \kappa(\theta_2)||\nabla\theta_2||\nabla g_\gamma((\theta_1 - \theta_2)^-)| \longrightarrow 0 \quad \text{as } \gamma \to 0.$$

Since g_γ monotonely increases, we have almost everywhere in Q that

$$\nabla(\theta_1 - \theta_2) \cdot \nabla g_\gamma((\theta_1 - \theta_2)^-) = |\nabla(\theta_1 - \theta_2)^-|^2 g'_\gamma((\theta_1 - \theta_2)^-) \geqslant 0.$$

From (3.3) we deduce the relation

$$\lim_{\gamma\to 0}\Big\{\int_\Omega F_\gamma((\theta_1(t_1) - \theta_2(t_1))^-) \\ + \int_{\mathcal{S}_{t_1}} G(\sigma[|\theta_1|^3\theta_1 - |\theta_2|^3\theta_2]) g_\gamma((\theta_1 - \theta_2)^-)\Big\} \leqslant 0.$$

We next observe that $g_\gamma((\theta_1 - \theta_2)^-) \to -\chi_{\{(t,x)\in\mathcal{S}\colon \theta_1<\theta_2\}}$ almost everywhere on $\mathcal{S}$ as $\gamma \to 0$. We obtain the inequality

$$\int_\Omega |(\theta_1(t_1) - \theta_2(t_1))^-| + \int_{\mathcal{S}_{t_1}} G(\sigma[|\theta_1|^3\theta_1 - |\theta_2|^3\theta_2])(-\chi_{\{(t,x)\in\mathcal{S}\colon \theta_1<\theta_2\}}) \leqslant 0.$$

Now expressing $G = I - H$, we conclude

$$\begin{aligned} -\int_{\mathcal{S}_{t_1}} & G(\sigma[|\theta_1|^3\theta_1 - |\theta_2|^3\theta_2])\chi_{\{(t,x)\in\mathcal{S}\colon \theta_1<\theta_2\}} \\ &= \int_{\mathcal{S}_{t_1}} \sigma|(|\theta_1|^3\theta_1 - |\theta_2|^3\theta_2)^-| \\ &\quad + \int_{\mathcal{S}_{t_1}} H(\sigma[|\theta_1|^3\theta_1 - |\theta_2|^3\theta_2])\chi_{\{(t,x)\in\mathcal{S}\colon \theta_1<\theta_2\}} \\ &\geqslant \int_{\mathcal{S}_{t_1}} \sigma|(|\theta_1|^3\theta_1 - |\theta_2|^3\theta_2)^-| \\ &\quad + \int_{\mathcal{S}_{t_1}} H(\sigma(|\theta_1|^3\theta_1 - |\theta_2|^3\theta_2)^-)\chi_{\{(t,x)\in\mathcal{S}\colon \theta_1<\theta_2\}} \\ &\geqslant (1 - \|H\|_{\mathcal{L}(1,1)}) \int_{\mathcal{S}_{t_1}} \sigma|(|\theta_1|^3\theta_1 - |\theta_2|^3\theta_2)^-| \geqslant 0. \end{aligned}$$

Thus, $\int_\Omega |(\theta_1(t_1) - \theta_2(t_1))^-| \leqslant 0$ for all $t_1 \in]0,T[$, and the claim follows. □

Remark 3.4. Under the assumptions of Theorem 2.1, we can introduce for fixed θ_0, θ_g with (1.3) the nonlinear solution operator $S\colon L^2(Q) \to \mathcal{V}_0^{2,5}(Q)$ to the problem (P) defined by the correspondence $f \mapsto \theta$. Since $\theta \in C(0,T;L^2(\Omega))$ is an easy consequence of Lemma 3.1, Lemma 3.3 shows that S is well-defined and monotone.

4. L^1-ESTIMATES

The result of Section 2 shows two cases such that the nonlocal radiation term $G(\theta^4)$ can be controlled to obtain a weak solution in the sense of Definition 1.1. In Theorem 2.1, we consider a nonsmooth surface Σ, and obtain the result by requiring a certain regularity of the right-hand f. In Theorem 2.2, right-hands $f \in L^{1+\varepsilon}(Q)$ for ε arbitrarily small are admissible, but we have to compensate this lack of regularity by using the smoothing properties of the operator K, valid only on regular surfaces.

In this section we will be interested in the limiting case that $f \in L^1(Q)$. As in the stationary case, we can prove the existence of weak solutions only if the surface Σ is sufficiently smooth (cf. [2]). In addition, we obtain uniform estimates only in the case of a nowhere vanishing reflexivity, that is

$$t \longmapsto \varepsilon(t,z) \in C([0,T]) \quad \text{for all } z \in \Sigma, \tag{4.1}$$

$$\forall\, t \in [0,T],\ \varepsilon(t,z) < 1 \quad \text{for all } z \in \Sigma. \tag{4.2}$$

For technical simplicity, we restrict ourselves in this section to the case that

$$f \geqslant 0 \quad \text{almost everywhere in } Q. \tag{4.3}$$

Theorem 4.1. *Let $\Sigma \in \mathcal{C}^{1,\alpha}$ ($\alpha > 0$), and let ε satisfy* (4.1) *and* (4.2). *Let $f \in L^1(Q)$ satisfy* (4.3), *and let θ_0, θ_g satisfy* (1.3). *There exists a weak solution $\theta \in \bigcap_{1\leqslant p<\frac{5}{4}} \mathcal{V}_0^{p,4}(Q) \cap L^{\infty,1}(Q)$ to* (P) *in the sense of Definition* 1.1. *In addition, $\operatorname*{ess\,inf}_Q \theta \geqslant \theta_g$.*

As to the structure of this section, we start the proof of Theorem 4.1 by constructing approximate solutions. Uniform estimates are derived in Proposition 4.3 and in Proposition 4.4. The proof of the main result 4.1 then follows.

Proposition 4.2. *Let Σ belong to the class $\mathcal{C}^1$ piecewise. Assume that $f \in L^1(Q)$, and that θ_0, θ_g satisfy* (1.3). *For $\delta > 0$, we define $f^{[\delta]} := \operatorname{sign}(f)\min\{|f|, 1/\delta\}$.*

Then there exists a unique $\theta_\delta \in \mathcal{V}_0^{2,5}(Q) \cap C(0,T;L^2(\Omega))$ such that $\theta_\delta = \theta_g$ on $\mathcal{C}$, and

$$-\int_Q \theta_\delta \frac{\partial \psi}{\partial t} + \int_Q \kappa(\theta_\delta)\nabla\theta_\delta \cdot \nabla\psi + \int_{\mathcal{S}} G(\sigma|\theta_\delta|^3\theta_\delta)\psi = \int_\Omega \theta_0\psi(0) + \int_Q f^{[\delta]}\psi \tag{4.4}$$

for all $\psi \in \mathcal{V}_{\mathcal{C}}^{2,5}(Q)$ such that $\psi(T) = 0$. In addition, we can assume that $\theta_\delta \in L^\infty(Q)$, and that there exists a distributional time-derivative $\theta'_\delta \in L^2(0,T;[V_\Gamma^{2,5}(\Omega)]^*)$. Under the additional assumption (4.3), *we can show that $\theta_\delta \geqslant \theta_g$ almost everywhere in Q.*

P r o o f. Existence in $\mathcal{V}_0^{2,5}(Q)$ follows from Theorem 2.1. The additional regularity follows from Lemma 3.1 and Lemma 3.2. Note that $\theta_\delta \in C(0,T;L^2(\Omega))$ is a well-known consequence of the existence of $\theta'_\delta \in L^2(0,T;[V_\Gamma^{2,5}(\Omega)]^*)$. The uniqueness is then derived from Lemma 3.3. Under the additional assumption (4.3), we can use Lemma 3.3 to verify that $\theta_\delta - \theta_g = S(f^{[\delta]}) - S(0) \geqslant 0$ almost everywhere in Q. □

Proposition 4.3. *Let the hypotheses of Proposition* 4.2 *be satisfied. For the sequence of approximate solutions* $\{\theta_\delta\}$ *that satisfy* (4.4), *the following uniform estimates are valid:*

(1) *There exists a positive constant* C_1 *such that* $\|\theta_\delta\|_{L^{\infty,1}(Q)} \leqslant C_1$.
(2) *For all* $1 \leqslant r < \frac{5}{4}$, *there exists a positive constant* $C_2 = C_2(r)$ *such that* $\|\theta_\delta\|_{W_r^{1,0}(Q)} \leqslant C_2$.
(3) *There exist a positive constant* C_3 *and a number* $1 < q < \infty$ *such that for all* $i = 1,\dots,m$, $\|\theta'_\delta\|_{L^1(0,T;[W_0^{1,q}(\Omega_i)]^*)} \leqslant C_3$.

The constants C_j $(j = 1,\dots,3)$ *depend continuously on* $\|f\|_{L^1(Q)}$, *on* $\|\theta_0\|_{L^1(\Omega)}$ *and on* $\|\theta_g\|_{L^\infty(Q)}$, *but are independent of* δ.

P r o o f. For the sake of notational simplicity, we write θ instead of θ_δ. For a parameter $\gamma > 0$, we consider functions $g = g_\gamma$, $F = F_\gamma \in C(\mathbb{R})$ given by

$$g_\gamma(s) := \frac{1}{\gamma}\operatorname{sign}(s)\min\{|s|,\gamma\}, \qquad F_\gamma(s) = \begin{cases} -s - \dfrac{\gamma}{2} & \text{if } s < -\gamma, \\ \dfrac{s^2}{2\gamma} & \text{if } -\gamma \leqslant s \leqslant \gamma, \\ s - \dfrac{\gamma}{2} & \text{if } s > \gamma. \end{cases}$$

Clearly, F is the primitive function of g that vanishes at zero. We are allowed to test the relation (4.4) with $\psi := g(\theta - \theta_g)$. Applying Lemma B.2 with $u := \theta - \theta_g$, we get the relation

$$\int_\Omega F_\gamma(\theta(t_1) - \theta_g) + \int_{Q_{t_1}} \kappa(\theta)\nabla\theta \cdot \nabla g_\gamma(\theta - \theta_g) + \int_{\mathcal{S}_{t_1}} G(\sigma|\theta|^3\theta) g_\gamma(\theta - \theta_g) \\ = \int_\Omega F_\gamma(\theta_0 - \theta_g) + \int_{Q_{t_1}} f^{[\delta]} g_\gamma(\theta - \theta_g).$$

Since g_γ is nondecreasing and θ_g is constant, we have $\nabla\theta \cdot \nabla g_\gamma(\theta - \theta_g) \geqslant 0$ almost everywhere in Q.

In view of Lemma A.4, we verify that

$$\int_{\mathcal{S}_{t_1}} G(\sigma|\theta|^3\theta) g_\gamma(\theta - \theta_g) \geqslant 0.$$

Letting $\gamma \to 0$ in the previous relation, we obtain the inequality

$$\int_\Omega |\theta(t_1) - \theta_g| \leqslant \|\theta_0\|_{L^1(\Omega)} + \|f\|_{L^1(Q)}. \tag{4.5}$$

This proves the estimate (1).

For the next estimate, we follow the techniques of [8]. For $n \in \mathbb{N}$ we consider the functions

$$g_n(t) := \begin{cases} -1 & \text{for } t < -(n+1), \\ t+n & \text{for } t \in [-(n+1), -n[, \\ 0 & \text{for } t \in [-n, n[, \\ t-n & \text{for } t \in [n, n+1[, \\ 1 & \text{for } t \geqslant n+1, \end{cases}$$

and define F_n as the primitive function of g_n that vanishes at zero. Observe that g_n is continuous, nondecreasing and bounded, and we are allowed to test the relation (4.4) with $\psi := g_n(\theta - \theta_g)$. Applying Lemma B.2 with $u := \theta - \theta_g$ and $\bar{u} := 0$, we obtain for all $t_1 < T$ that

$$\int_\Omega F_n(\theta(t_1) - \theta_g) + \int_{Q_{t_1}} \kappa(\theta)|\nabla\theta|^2 g_n'(\theta - \theta_g) + \int_{\mathbb{S}_{t_1}} G(\sigma|\theta|^3\theta) g_n(\theta - \theta_g)$$
$$= \int_\Omega F_n(\theta_0 - \theta_g) + \int_{Q_{t_1}} f^{[\delta]} g_n(\theta - \theta_g).$$

Recalling Lemma A.4, we verify that $\int_{\mathbb{S}_{t_1}} G(\sigma|\theta|^3\theta) g_n(\theta - \theta_g) \geqslant 0$. Letting $t_1 \to T$ yields the inequality

$$\int_Q g_n'(\theta - \theta_g)\kappa(\theta)|\nabla\theta|^2 \leqslant \int_\Omega F_n(\theta_0 - \theta_g) + \int_Q f^{[\delta]} g_n(\theta - \theta_g) \leqslant \|f\|_{L^1(Q)}, \tag{4.6}$$

where we have also used the fact that $\theta_0 = \theta_g$. As in Proposition B.3, we introduce

$$B_n := \{(t,x) \in Q \colon\ n \leqslant |\theta(t,x) - \theta_g| < n+1\}.$$

Relation (4.6) amounts to saying that

$$\int_{B_n} \kappa(\theta)|\nabla(\theta - \theta_g)|^2 \leqslant \|f\|_{L^1(Q)}.$$

Now, Proposition B.3 applies. Combined with (1), it gives (2).

Finally, we want to estimate the time derivatives. The relation (4.4) is equivalent to

$$\langle \theta'(t), \psi \rangle = -\int_\Omega \kappa(\theta(t))\nabla\theta(t) \cdot \nabla\psi - \int_\Sigma G(\sigma|\theta|^3\theta(t))\psi + \int_\Omega f^{[\delta]}(t)\psi \tag{4.7}$$

for almost all $t \in \,]0,T[$ and all $\psi \in V_\Gamma^{2,5}(\Omega)$. Here $\langle\cdot,\cdot\rangle$ is the duality pairing in $V^{2,5}(\Omega)$.

We recall that $\Omega = \bigcup_{i=1}^{m} \Omega_i$. In (4.7) we can choose any test function $\psi \in W_0^{1,q}(\Omega_i)$ ($q > 5$) that we extend by zero to the rest of Ω. For this test function it follows that

$$\langle\theta'(t),\psi\rangle = -\int_{\Omega_i} \kappa(\theta(t))\nabla\theta(t)\cdot\nabla\psi + \int_{\Omega_i} f^{[\delta]}(t)\psi.$$

We obtain

$$|\langle\theta'(t),\psi\rangle| \leqslant c(\|\nabla\theta(t)\|_{L^{q'}(\Omega_i)} + \|f(t)\|_{L^1(\Omega_i)})\|\psi\|_{W_0^{1,q}(\Omega_i)}.$$

Thus,

$$\|\theta'(t)\|_{[W_0^{1,q}(\Omega_i)]^*} \leqslant c(\|\nabla\theta(t)\|_{[L^{q'}(\Omega_i)]^3} + \|f(t)\|_{L^1(\Omega_i)}).$$

In view of the previous result (2), it follows for $q > 5$ that

$$\begin{aligned}\|\theta'\|_{L^1(0,T;[W_0^{1,q}(\Omega_i)]^*)} &\leqslant c(T^{1/q}\|\nabla\theta\|_{[L^{q'}(Q_i)]^3} + \|f\|_{L^1(Q)})\\ &\leqslant C(q,T,\|f\|_{L^1(Q)}).\end{aligned}$$

This is the last claim that we had to prove. □

In the next proposition we state two additional, technical estimates and derive the uniform estimate that will allow to control the surface integral.

Proposition 4.4. *Let the hypotheses of Proposition* 4.2 *be satisfied, and assume in addition that* $\Sigma \in \mathcal{C}^{1,\alpha}$, $\alpha > 0$, *and that* ε *satisfies* (4.1) *and* (4.2). *Let* $\{\theta_\delta\}$ *be the sequence of approximate solutions that satisfy* (4.4).

(1) *Let* $\tilde{H}$ *denote the positive operator introduced in Lemma* A.2 (6). *For every nonnegative* $h \in C_c^1(\mathbb{R})$ *we can find a positive constant* $C(h)$ *that depends on the data and on* h, *but is independent of* δ, *such that*

$$\int_{\mathcal{S}} \varepsilon\tilde{H}(\sigma\theta_\delta^4)h(\theta_\delta) \leqslant C(h).$$

(2) *There is a constant* c *independent of* δ *such that for all* $\lambda > 0$

$$\operatorname*{ess\,sup}_{t\in]0,T[}\{\mathrm{meas}(\{z\in\Sigma\colon\ \theta_\delta(t,z) > \lambda\})\} \leqslant c/\sqrt{\lambda}.$$

(3) *There exists a constant* C *independent of* δ *such that* $\|\theta_\delta^4\|_{L^1(\mathcal{S})} \leqslant C$.

P r o o f. (1) We first prove a preliminary estimate. For $\xi \in C(\mathbb{R})$ we denote by $\bar{\xi}$ the primitive function of ξ that vanishes at zero, and introduce the notation $M_\xi := \int_0^\infty |\xi(s)|^2 \, \mathrm{d}s$. We show that if $M_\xi < \infty$, then there exists a positive constant C_ξ such that

$$\|\nabla \bar{\xi}(\theta_\delta)\|_{[L^2(Q)]^3} \leqslant C_\xi. \tag{4.8}$$

We introduce the function $g = g_\xi \colon \mathbb{R}^+ \to \mathbb{R}^+$ defined by $g(t) := \int_0^t |\xi(s)|^2 \, \mathrm{d}s$, which is nonnegative and continuous, nondecreasing, and vanishes at zero. We denote by $F = F_\xi$ the primitive of g that vanishes at zero. Since g is globally bounded by the number M_ξ, its primitive F has at most linear growth at infinity. Starting from (4.4), we apply Lemma B.2 with $u := \theta_\delta$ and $\bar{u} := \theta_g$. For all $t_1 \in \,]0, T[$ we can derive the identity

$$\begin{aligned} \int_\Omega F(\theta_\delta(t_1)) + \int_{Q_{t_1}} \kappa(\theta_\delta) \nabla \theta_\delta \cdot \nabla g(\theta_\delta) + \int_{\mathcal{S}_{t_1}} G(\sigma |\theta_\delta|^3 \theta_\delta) g(\theta_\delta) \\ = \int_{Q_{t_1}} f^{[\delta]} g(\theta_\delta) + \int_\Omega F(\theta_0) + D, \end{aligned} \tag{4.9}$$

where

$$D := \int_{Q_{t_1}} \kappa(\theta_\delta) \nabla \theta_\delta \cdot \nabla g(\theta_g) + \int_{\mathcal{S}_{t_1}} G(\sigma |\theta_\delta|^3 \theta_\delta) g(\theta_g) + g(\theta_g) \int_\Omega (\theta_\delta(t_1) - \theta_\delta(0)).$$

Under the simplifying assumption (1.3) we see that

$$|D| \leqslant g(\theta_g) \|\theta_\delta\|_{L^{\infty,1}(Q)},$$

which remains bounded in view of Proposition 4.3. On the other hand, Lemma A.4 can be used to verify that $\int_{\mathcal{S}_{t_1}} G(\sigma \theta_\delta^4) g(\theta_\delta) \geqslant 0$. Thus, we can conclude that

$$\int_\Omega F(\theta_\delta(t_1)) + \int_{Q_{t_1}} \kappa(\theta_\delta) \nabla \theta_\delta \cdot \nabla g(\theta_\delta) \leqslant C(\|f\|_{L^1(Q)}, \theta_0, M_\xi).$$

We denote by $\bar{\xi}$ the primitive function of ξ that vanishes at zero. Obviously,

$$\int_{Q_{t_1}} \kappa(\theta_\delta) \nabla \theta_\delta \cdot \nabla g(\theta_\delta) = \int_{Q_{t_1}} \kappa(\theta_\delta) |\nabla \bar{\xi}(\theta_\delta)|^2,$$

proving (4.8). For all $h \in C_c^1(\mathbb{R})$, both $\xi := h$ and $\xi := \sqrt{|h'|}$ satisfy $M_\xi < \infty$. Setting $\tilde{h}(t) := \int_0^t \sqrt{|h'(s)|} \, \mathrm{d}s$, we obtain from (4.8) for every $h \in C_c^1(\mathbb{R})$ the existence of a constant $\tilde{C}(h)$ independent of δ such that

$$\|\nabla \bar{h}(\theta_\delta)\|_{[L^2(Q)]^3} + \|\nabla \tilde{h}(\theta_\delta)\|_{[L^2(Q)]^3} \leqslant \tilde{C}(h). \tag{4.10}$$

To prove (1), we apply Lemma B.2 to the relation (4.4) with $g := h$, $u := \theta_\delta$ and $\bar{u} := \theta_g$, and as in (4.9) we can deduce that

$$\int_\Omega \bar{h}(\theta_\delta(t_1)) + \int_{Q_{t_1}} h'(\theta_\delta)\kappa(\theta_\delta)|\nabla\theta_\delta|^2 + \int_{\mathcal{S}_{t_1}} G(\sigma|\theta_\delta|^3\theta_\delta)h(\theta_\delta) = \int_{Q_{t_1}} f^{[\delta]}h(\theta_\delta) + \int_\Omega F(\theta_0) + D, \tag{4.11}$$

where D, as in the case of (4.9), is uniformly bounded with respect to δ. On the other hand, due to (4.10) we have

$$\left| \int_{Q_{t_1}} h'(\theta_\delta)\kappa(\theta_\delta)|\nabla\theta_\delta|^2 \right| \leqslant \int_{Q_{t_1}} \kappa(\theta_\delta)|\nabla\tilde{h}(\theta_\delta)|^2 \leqslant \tilde{C}(h).$$

Note also that $\bar{h}(t) = \int_0^t h(s)\,\mathrm{d}s$ is globally bounded. Thus, decomposing $G = \varepsilon(I - \tilde{H})$ according to Lemma A.2 (6), we deduce from (4.11) that

$$\begin{aligned} \int_{\mathcal{S}} \varepsilon\tilde{H}(\sigma\theta_\delta^4)h(\theta_\delta) &\leqslant C(h) + \int_{\mathcal{S}} \varepsilon\sigma\theta_\delta^4 h(\theta_\delta) \\ &\leqslant C(h) + \sigma\,\mathrm{meas}(\mathcal{S})(\max_{s\in\mathrm{supp}(h)} |s|^4)(\max_{s\in\mathbb{R}} |h(s)|), \end{aligned} \tag{4.12}$$

proving the claim (1).

(2) For $\lambda > 0$ we consider the function $g = g_\lambda$ given by

$$g_\lambda(s) = \mathrm{sign}(s)\min\{|s|, \lambda\}.$$

We denote by F_λ the primitive of g_λ that vanishes at zero, that is

$$F_\lambda(s) = \begin{cases} \lambda|s| - \dfrac{\lambda^2}{2} & \text{if } |s| > \lambda, \\ \dfrac{s^2}{2} & \text{if } -\lambda \leqslant s \leqslant \lambda. \end{cases}$$

For all $t_1 \in \,]0, T[$, with help of Lemma B.2 we can derive the identity

$$\int_\Omega F_\lambda(\theta_\delta(t_1)) + \int_{Q_{t_1}} \kappa(\theta_\delta)\nabla\theta_\delta \cdot \nabla g_\lambda(\theta_\delta) + \int_{\mathcal{S}_{t_1}} G(\sigma|\theta_\delta|^3\theta_\delta)g_\lambda(\theta_\delta) = \int_{Q_{t_1}} f^{[\delta]}g_\lambda(\theta_\delta) + \int_\Omega F_\lambda(\theta_0) + D, \tag{4.13}$$

where

$$\begin{aligned} D := &\int_{Q_{t_1}} \kappa(\theta_\delta)\nabla\theta_\delta \cdot \nabla g_\lambda(\theta_g) \\ &+ \int_{\mathcal{S}_{t_1}} G(\sigma|\theta_\delta|^3\theta_\delta)g_\lambda(\theta_g) + g_\lambda(\theta_g)\int_\Omega (\theta_\delta(t_1) - \theta_\delta(0)). \end{aligned}$$

Under the simplifying assumption (1.3), we see that

$$|D| = \left| g_\lambda(\theta_g) \int_\Omega (\theta_\delta(t_1) - \theta_\delta(0)) \right| \leqslant \lambda \|\theta_\delta\|_{L^{\infty,1}(Q)}.$$

On the other hand, we observe that

$$\int_{\mathcal{S}_{t_1}} G(\sigma|\theta_\delta|^3\theta_\delta) g_\lambda(\theta_\delta) \geqslant 0,$$

due as usual to Lemma A.4. Observing that $F_\lambda(s) \geqslant g_\lambda^2(s)/2$ for all $s \in \mathbb{R}$, we deduce from (4.13) that

$$\begin{aligned} \frac{1}{2}\int_\Omega g_\lambda^2(\theta_\delta(t_1)) + \int_{Q_{t_1}} \kappa(\theta_\delta)|\nabla g_\lambda(\theta_\delta)|^2 \\ \leqslant \lambda(\|f\|_{L^1(Q)} + \theta_0 \operatorname{meas}(\Omega) + \|\theta_\delta\|_{L^{\infty,1}(Q)}). \end{aligned}$$

Using also Proposition 4.3, we see that $\|g_\lambda(\theta_\delta)\|^2_{V_2^{1,0}(Q)} \leqslant c\lambda$, where c does not depend on δ. Now, using the result of Lemma B.1, we obtain that

$$\|g_\lambda(\theta_\delta)\|^2_{L^{\infty,1}(\mathcal{S})} \leqslant c_0^2 c\lambda, \tag{4.14}$$

where c_0 is the embedding constant of $V_2^{1,0}(Q) \hookrightarrow L^{\infty,1}(\mathcal{S})$. For almost all $t \in]0,T[$, note that

$$\lambda \operatorname{meas}(\{z \in \Sigma\colon\ \theta_\delta(t,z) > \lambda\}) \leqslant \|g_\lambda(\theta_\delta)(t)\|_{L^1(\Sigma)},$$

and therefore

$$\lambda^2 (\operatorname*{ess\,sup}_{t\in]0,T[}\{\operatorname{meas}(\{z \in \Sigma\colon\ \theta_\delta(t,z) > \lambda\})\})^2 \leqslant (\operatorname*{ess\,sup}_{t\in]0,T[} \|g_\lambda(\theta_\delta)(t)\|_{L^1(\Sigma)})^2 \leqslant c_0^2 c\lambda,$$

in view of (4.14). This proves (2).

(3) For θ_δ, according to Proposition 4.2 we are allowed to introduce the solution R_δ of the radiosity equation (0.10),

$$R_\delta := (I - (1-\varepsilon)K)^{-1}(\varepsilon\sigma\theta_\delta^4).$$

In view of Lemma A.2 (3), we have $R_\delta \in L^{5/4}(\mathcal{S})$, and $R_\delta \geqslant 0$ almost everywhere on $\mathcal{S}$ due to the positivity of the operator $(I - (1-\varepsilon)K)^{-1}$.

On the other hand, by Lemma A.2 (6) we have the identity $K(R_\delta) = \tilde{H}(\sigma\theta_\delta^4)$. Using the result of Proposition 4.4 (1), it follows for all $h \in C_c^1(\mathbb{R})$ positive that

$$\int_{\mathcal{S}} \varepsilon K(R_\delta)h(\theta_\delta) = \int_{\mathcal{S}} \varepsilon \tilde{H}(\sigma\theta_\delta^4)h(\theta_\delta) \leqslant C(h). \tag{4.15}$$

Almost everywhere on $\mathcal{S}$ we have

$$R_\delta - (1-\varepsilon)K(R_\delta) = \varepsilon\sigma\theta_\delta^4. \tag{4.16}$$

Multiplying (4.16) by the function $h(\theta_\delta)$ and using the estimate (4.15) we can verify that

$$\begin{aligned} \|R_\delta h(\theta_\delta)\|_{L^1(\mathcal{S})} &\leqslant \sigma\|\theta_\delta^4 h(\theta_\delta)\|_{L^1(\mathcal{S})} + (1-\varepsilon_l)\|K(R_\delta)h(\theta_\delta)\|_{L^1(\mathcal{S})} \\ &\leqslant \sigma\,\mathrm{meas}(\mathcal{S})\Big(\max_{s\in\mathrm{supp}(h)}|s|^4\Big)\Big(\max_{s\in\mathbb{R}}|h(s)|\Big) + \frac{1-\varepsilon_l}{\varepsilon_l}C(h). \end{aligned} \tag{4.17}$$

We start again from the relation (4.16) valid for almost all $(t,z)\in\mathcal{S}$. For a fixed parameter $k>0$ we can choose a function $h_k\in C_c^1(\mathbb{R})$ with the following properties:

$$\begin{cases} h_k(s)=1, & s\in[0,\,k],\\ 1\geqslant h_k(s)\geqslant 0, & s\in\,]k,k+1],\\ h_k(s)=0, & s\in\,]k+1,\infty[, \end{cases}$$

and we can multiply (4.16) by the function $h_k(\theta_\delta)$. We obtain for almost all $(t,z)\in\mathcal{S}$ that

$$(1-\varepsilon(t,z))h_k(\theta_\delta(t,z))\int_\Sigma w(z,y)R_\delta(t,y)\,\mathrm{d}S_y \leqslant h_k(\theta_\delta(t,z))R_\delta(t,z). \tag{4.18}$$

We integrate the inequality (4.18) over Σ, and using Fubini's theorem together with the positivity of w, we obtain that

$$\begin{aligned} \int_\Sigma R_\delta(t,y)\Big(\int_\Sigma w(z,y)h_k(\theta_\delta(t,z))(1-\varepsilon(t,z))\,\mathrm{d}S_z\Big)\,\mathrm{d}S_y& \\ \leqslant \|h_k(\theta_\delta(t))R_\delta(t)\|_{L^1(\Sigma)}.& \end{aligned} \tag{4.19}$$

Note that for fixed k the right-hand side of (4.19) is uniformly bounded in $L^1(0,T)$ in view of the estimate (4.17). On the other hand, we can write for almost all $t\in\,]0,T[$ that

$$\begin{aligned} \mathrm{meas}(\{\xi\in\Sigma\colon\ \theta_\delta(t,\xi)\leqslant k\}) &= \mathrm{meas}(\Sigma) - \mathrm{meas}(\{\xi\in\Sigma\colon\ \theta_\delta(t,\xi)>k\}) \\ &\geqslant \mathrm{meas}(\Sigma) - c/\sqrt{k}, \end{aligned}$$

where we make use of the uniform estimate in Proposition 4.4 (2). We deduce that

$$\operatorname*{ess\,inf}_{t\in]0,T[}\{\mathrm{meas}(\{\xi\in\Sigma\colon\ \theta_\delta(t,\xi)\leqslant k\})\} \geqslant \mathrm{meas}(\Sigma) - c/\sqrt{k}. \tag{4.20}$$

We introduce

$$f_{k,t}^{\delta}(y) := \int_{\Sigma} w(z,y)\chi_{\{\xi\in\Sigma\colon\ \theta_\delta(t,\xi)\leqslant k\}}(y)(1-\varepsilon(t,z))\,\mathrm{d}S_z.$$

Defining δ_0 as in Lemma A.6, we can in view of (4.20) choose a $k_0 > 0$ independent of t, δ such that

$$\operatorname*{ess\,inf}_{t\in]0,T[}\{\operatorname{meas}(\{\xi\in\Sigma\colon\ \theta_\delta(t,\xi)\leqslant k_0\})\} \geqslant \operatorname{meas}\Sigma) - \delta_0.$$

We conclude from Lemma A.6 that

$$\operatorname*{ess\,inf}_{y\in\Sigma}\operatorname*{ess\,inf}_{t\in]0,T[} f_{k_0,t}^{\delta}(y) \geqslant \beta_0.$$

It now follows from (4.19) that

$$\beta_0\int_{\Sigma} R_\delta(t,y)\,\mathrm{d}S_y \leqslant \int_{\Sigma} R_\delta(t,y) f_{k_0,t}^{\delta}(y)\,\mathrm{d}S_y \leqslant \|h_{k_0}(\theta_\delta(t))R_\delta(t)\|_{L^1(\Sigma)}. \tag{4.21}$$

We integrate over $]0,T[$ to obtain

$$\beta_0\|R_\delta\|_{L^1(\mathcal{S})} \leqslant \|h_{k_0}(\theta_\delta)R_\delta\|_{L^1(\mathcal{S})} \leqslant C(k_0),$$

due to (4.17). Since $\theta_\delta^4 \leqslant R_\delta/(\sigma\varepsilon_l)$ uniformly on $\mathcal{S}$ due to (4.16), the claim follows. □

P r o o f of Theorem 4.1. Applying Proposition 4.3, we first find a sequence such that

$$\theta_\delta \rightharpoonup \theta \ \text{ in } \ W_r^{1,0}(Q) \quad \text{for } \ 1\leqslant r < \frac{5}{4}. \tag{4.22}$$

We now want to prove additional convergence properties. For the number q given in Proposition 4.3 (3), we have

$$W^{1,r}(\Omega_i) \hookrightarrow L^r(\Omega_i) \hookrightarrow [W_0^{1,q}(\Omega_i)]^*.$$

We introduce the notation $Q_i :=]0,T[\times\Omega_i$ and $S_i :=]0,T[\times\partial\Omega_i$. From Proposition 4.3 (3), and the generalized Lemma of Aubin-Lions, we get for all $i = 1,\dots,m$ that $\theta_\delta \longrightarrow \theta$ in $L^r(Q_i)$. In view of the inequality (2.6), we also find a subsequence such that $\theta_\delta \longrightarrow \theta$ in $L^r(S_i)$ and, after extracting subsequences, even

$$\begin{gathered}\theta_\delta \longrightarrow \theta \ \text{ in } \ L^r(Q), \quad \theta_\delta \longrightarrow \theta \ \text{ in } \ L^r(\mathcal{S}),\\ \theta_\delta \longrightarrow \theta \ \text{ pointwise a.e. in } Q \ \text{ and on } \mathcal{S}.\end{gathered} \tag{4.23}$$

In order to pass to the limit with the surface integrals in (4.4), the uniform bound derived in Proposition 4.4 (3) is not sufficient. Though we can prove the convergence in general settings, we can use the simplifying assumption (4.3) to significantly shorten matters. Under (4.3), observe that $f^{[\delta]} \nearrow f$ almost everywhere in Q as $\delta \searrow 0$. In view of the positivity property of Lemma 3.3, we see that $\theta_\delta \nearrow \theta$ almost everywhere in Q as $\delta \searrow 0$. Then the monotone convergence theorem and Proposition 4.4 (3) imply that

$$\int_{\mathcal{S}} \theta^4 = \lim_{\delta \searrow 0} \int_{\mathcal{S}} \theta_\delta^4 \leqslant C.$$

Therefore, $\theta^4 \in L^1(\mathcal{S})$, and we have even

$$\theta_\delta^4 \longrightarrow \theta^4 \quad \text{in } L^1(\mathcal{S}). \tag{4.24}$$

Thanks to (4.23) and (4.24), we can elementarily pass to the limit in (4.4) and get

$$-\int_Q \theta \frac{\partial \psi}{\partial t} + \int_Q \kappa(\theta) \nabla\theta \cdot \nabla\psi + \int_{\mathcal{S}} G(\sigma\theta^4)\psi = \int_\Omega \theta_0 \psi(0) + \int_Q f\psi \tag{4.25}$$

for all $\psi \in \mathcal{V}_{\mathcal{C}}^{q,\infty}(Q)$ $(q > 5)$ such that $\psi = 0$ in $\{T\} \times \Omega$. □

Appendix A. Essential properties of the radiation operators

Throughout this section we assume that $\Omega_0, \ldots, \Omega_m$ $(m \geqslant 1)$ are bounded domains such that $\bigcup_{i=0}^{m} \overline{\Omega_i}$ is simply connected, and such that the condition (0.1) is satisfied. As in the introduction of the paper, we set $\overline{\Omega} := \bigcup_{i=1}^{m} \overline{\Omega_i}$. We denote by Σ the boundary of the transparent cavity $\partial\Omega_0$, and by Γ the exterior boundary $\partial\Omega \setminus \Sigma$.

The following lemma has been proved in [3] for polyhedral surfaces, in [14] for piecewise $\mathcal{C}^1$−boundaries.

Lemma A.1. *Let* $\Sigma \in \mathcal{C}^1$ *piecewise. Let* $w\colon \Sigma \times \Sigma \to \mathbb{R}$ *denote the view-factor* (0.6). *Then, for almost all* $z \in \Sigma$,

$$\int_\Sigma w(z, y)\, \mathrm{d}S_y \leqslant 1.$$

In addition, the equality is valid if and only if the enclosure condition (0.2) *is satisfied.*

For $1 \leqslant p, q \leqslant \infty$, we introduce

$$\mathcal{L}(p, q) := \mathcal{L}(L^{p,q}(\mathcal{S}), L^{p,q}(\mathcal{S})), \tag{A.1}$$

the space of all linear continuous maps from $L^{p,q}(\mathcal{S})$ into itself. The following lemma states easily derived, but essential consequences of Lemma A.1.

Lemma A.2. *Let the hypotheses of Lemma* A.1 *be valid.*

(1) *For each $1 \leqslant p, q \leqslant \infty$ the operator K extends to a bounded linear operator from $L^{p,q}(\mathcal{S})$ into itself, and the norm estimate $\|K\|_{\mathcal{L}(p,q)} \leqslant 1$ is valid.*

(2) *The operator K is positive, in the sense that $K(f) \geqslant 0$ almost everywhere on $\mathcal{S}$ whenever $f \geqslant 0$ almost everywhere on $\mathcal{S}$. Moreover, K is positive semi-definite and selfadjoint in $L^2(\mathcal{S})$.*

(3) *If $\varepsilon\colon \mathcal{S} \to \mathbb{R}$ is such that*

$$0 < \varepsilon_l \leqslant \varepsilon(t,z) \leqslant 1 \quad \text{on }]0,T[\times\Sigma,$$

then the operator $[I-(1-\varepsilon)K]$ has a positive inverse in $\mathcal{L}(L^{p,q}(\mathcal{S}), L^{p,q}(\mathcal{S}))$ having the representation

$$[I-(1-\varepsilon)K]^{-1} = \sum_{i=0}^{\infty}(1-\varepsilon)^i K^i.$$

(4) *The operator G is positive semi-definite and selfadjoint in $L^2(\mathcal{S})$. The operator $H := I - G$ is positive, selfadjoint in $L^2(\mathcal{S})$, and satisfies for all $1 \leqslant p, q \leqslant \infty$ the norm estimate $\|H\|_{\mathcal{L}(p,q)} \leqslant 1$.*

(5) *Assume in addition that* (0.2) *is valid. Then the kernel of the operator G consists of the functions constant almost everywhere on Σ.*

(6) *The operator $\tilde{H} := I - G/\varepsilon$ has the representation $\tilde{H} = K[I-(1-\varepsilon)K]^{-1}\varepsilon$.*

Proof. Denote by S the surface measure on Σ. We can prove that the mapping $(z,y) \longmapsto w(z,y)$ is $S\times S$-measurable on $\Sigma\times\Sigma$ provided that Σ is a Lipschitz surface. This will ensure, for $f \in L^1(\mathcal{S})$, that the mapping

$$(t,z,y) \longmapsto w(z,y)f(t,y)$$

is $\lambda_1 \times S \times S$-measurable on $[0,T]\times\Sigma\times\Sigma$. Thus, by Fubini's theorem, we can easily derive the assertions of the lemma from the properties that were established in [15], [10], among others, for the stationary operators. □

We also need to recall two auxiliary lemmas.

Lemma A.3. *Let $\Sigma \in \mathcal{C}^{1,\alpha}$ $(\alpha > 0)$. Let $r, s > 0$ be two real numbers such that $s \leqslant r+1$. There exists a positive constant $c_{r,s}$ such that for all $\psi \in L^{r+1}(\Sigma)$,*

$$\int_\Sigma G(|\psi|^{r-1}\psi)\psi + \left(\int_\Sigma |\psi|^s\right)^{(r+1)/s} \geqslant c\|\psi\|^{r+1}_{L^{r+1}(\Sigma)}.$$

Proof. The proof in the case that (0.2) fails is trivial (see [15]), and valid even for nonsmooth boundaries. For the case that Ω is an enclosure, a proof is given in [2]. □

The next two statements are proved in [2].

Lemma A.4. *We assume that* (0.2) *is valid. Let* $F\colon \mathbb{R}\to\mathbb{R}$ *be a nondecreasing continuous function such that* $F(0)=0$ *and* $|F(t)|\leqslant C_0(1+|t|^s)$ *as* $|t|\to\infty$ *(*$0\leqslant s<\infty$*). Let* $0\leqslant r<\infty$ *be an arbitrary number. Then, for all* $\psi\in L^{r+s}(\Sigma)$*,*

$$\int_\Sigma G(|\psi|^{r-1}\psi)F(\psi)\geqslant 0.$$

Lemma A.5. *Let* $\Sigma\in\mathcal{C}^{1,\alpha}$ *(*$\alpha>0$*). For* $p>1/\alpha$*, the operators* K *and* $\tilde{H}$ *(Lemma* A.2 (6)*) are compact from* $L^p(\Sigma)$ *into* $C(\Sigma)$*.*

Lemma A.6. *Let* $\Sigma\in\mathcal{C}^{1,\alpha}$ *(*$\alpha>0$*) and assume that the emissivity* ε *according to* (1.2) *satisfies* (4.1) *and* (4.2)*. Then there exist positive constants* $\beta_0>0$ *and* $\delta_0>0$ *such that for all measurable* $A\subset\Sigma$ *with* $\operatorname{meas}(A)\geqslant\operatorname{meas}(\Sigma)-\delta_0$ *we have* $\int_A w(z,y)(1-\varepsilon(t,y))\,\mathrm{d}S_y\geqslant\beta_0$ *for all* $(t,z)\in[0,T]\times\Sigma$*.*

P r o o f. Assume that the claim is not true. Then there exists a sequence $\{A_n\}_{n\in\mathbb{N}}$ of measurable subsets of Σ such that $\chi_{A_n}\rightharpoonup 1$ in $L^p(\Sigma)$ ($p<\infty$ arbitrary), and a sequence $\{(t_n,z_n)\}\subset[0,T]\times\Sigma$ such that $(t_n,z_n)\to(t^*,z^*)$ for some $(t^*,z^*)\in[0,T]\times\Sigma$ and

$$K(\chi_{A_n}(1-\varepsilon(t_n)))(z_n)\leqslant 1/n. \tag{A.2}$$

Since the function ε is continuous in time and globally bounded, we can show that $\chi_{A_n}(1-\varepsilon(t_n))\rightharpoonup(1-\varepsilon(t^*))$ in $L^p(\Sigma)$ ($p<\infty$ arbitrary). In view of Lemma A.5, it therefore follows for a subsequence that $K(\chi_{A_n}(1-\varepsilon(t_n)))\to K(1-\varepsilon(t^*))$ uniformly on Σ. Passing to the limit in (A.2) and using the positivity of the function $K(1-\varepsilon(t^*))$, we obtain that

$$K(1-\varepsilon(t^*))(z^*)=0.$$

Since $K(1-\varepsilon(t^*))$ is a continuous function due to Lemma A.5, the statement of Lemma A.1 is valid everywhere on Σ, and we obtain that

$$\int_\Sigma w(z^*,y)\varepsilon(t^*,y)\,\mathrm{d}S_y=1.$$

Writing

$$\begin{aligned}1&=\int_\Sigma w(z^*,y)\varepsilon(t^*,y)\,\mathrm{d}S_y\\&=\int_{\{y\in\Sigma\colon\,\varepsilon(t^*,y)<1\}}w(z^*,y)\varepsilon(t^*,y)\,\mathrm{d}S_y+\int_{\{y\in\Sigma\colon\,\varepsilon(t^*,y)=1\}}w(z^*,y)\,\mathrm{d}S_y\end{aligned}$$

we clearly obtain a contradiction in view of (4.2) and Lemma A.1. □

The following embedding result is well known.

Lemma B.1. *Let* $\Omega \subset \mathbb{R}^3$ *be such that* $\partial\Omega \in \mathcal{C}^{0,1}$. *For* $T > 0$, *let* $Q :=]0,T[\times\Omega$. *If* r, q *satisfy*

$$r \in [2,\infty], \quad q \in [2,6], \quad \frac{1}{r} + \frac{3}{2q} = \frac{3}{4},$$

then there exists a positive constant $c_{r,q}$ *such that*

$$\|u\|_{L^{r,q}(Q)} \leqslant c\|u\|_{V_2^{1,0}(Q)}.$$

If $\tilde{r}, \tilde{q}$ *satisfy*

$$\tilde{r} \in [2,\infty], \quad \tilde{q} \in \Big[\frac{4}{3}, 4\Big], \quad \frac{1}{\tilde{r}} + \frac{1}{\tilde{q}} = \frac{3}{4},$$

then there exists a positive constant $\tilde{c}_{\tilde{r},\tilde{q}}$ *such that*

$$\|u\|_{L^{\tilde{r},\tilde{q}}(]0,T[\times\partial\Omega)} \leqslant \tilde{c}\|u\|_{V_2^{1,0}(Q)}.$$

P r o o f. See [6, Chapter II, § 3]. □

For functions defined in Q, we can introduce for all $h \in]0,T[$

$$u_{(h)}(x,t) := \frac{1}{h}\int_t^{t+h} u(x,\tau)\,\mathrm{d}\tau.$$

The function $u_{(h)}$ is called the *Steklov averaging* of u, and belongs to $W_2^1(Q_{T-h})$ whenever u belongs to $W_2^{1,0}(Q)$. Its fundamental properties are listed in [6, Chapter II, \$ 4]. The notation

$$u_{(\underline{h})}(x,t) := \frac{1}{h}\int_{t-h}^{t} u(x,\tau)\,\mathrm{d}\tau$$

makes sense if we extend u, for instance by zero, to the interval $[-h,0]$. For functions $u,\eta\colon Q \to \mathbb{R}$ such that η vanishes in the intervals $[-h,0]$ and $[T-h,T]$, and such that $\int_Q u\eta\,\mathrm{d}x\,\mathrm{d}t < \infty$, the relation

$$\text{(B.1)} \qquad \int_Q u\eta_{(\underline{h})}\,\mathrm{d}x\,\mathrm{d}t = \int_Q u_{(h)}\eta\,\mathrm{d}x\,\mathrm{d}t,$$

is valid. We now give a lemma that helps us to shorten some technical arguments.

Lemma B.2. *Let* $\xi_1 \in L^1(Q)$, $\xi_2 \in L^1(\mathcal{S})$, *and suppose that* $\xi_3 \in [L^p(Q)]^3$ *for some* $p > 1$. *Denoting as usual by* p' *the conjugated exponent to* p, *suppose that* $u \in W^{1,0}_{p'}(Q) \cap C(0,T;L^1(\Omega))$ *satisfies*

$$-\int_Q u\frac{\partial\psi}{\partial t} = \int_Q \xi_1\psi + \xi_3\cdot\nabla\psi + \int_{\mathcal{S}} \xi_2\psi \tag{B.2}$$

for all $\psi \in C^\infty_c(0,T;C^\infty(\Omega))$ *such that* $\psi = 0$ *on* $\mathcal{C}$.

Let $g\colon \mathbb{R} \to \mathbb{R}$ *be globally Lipschitz continuous and bounded, and let* F *denote the primitive function of* g *that vanishes at zero. Then, if* $\bar u \in W^1_s(Q) \cap C(0,T;L^1(\Omega))$, $s = \max\{p, p'\}$ *is such that* $u = \bar u$ *on* $\mathcal{C}$ *in the sense of traces, we have for all* $t_1 < T$ *the identity*

$$\begin{aligned}\int_\Omega F(u(t_1)) = \int_\Omega F(u(0)) &+ \int_{Q_{t_1}} \xi_1(g(u) - g(\bar u))\\ &+ \int_{Q_{t_1}} \xi_3\cdot\nabla(g(u) - g(\bar u)) + \int_{\mathcal{S}_{t_1}} \xi_2(g(u) - g(\bar u))\\ &- \int_{Q_{t_1}} u\frac{\partial g(\bar u)}{\partial t} + \int_\Omega u(t_1)g(\bar u(t_1)) - \int_\Omega u(0)g(\bar u(0)).\end{aligned}$$

If the function g *is not globally bounded but* $u, \bar u \in L^\infty(Q)$, *then the assertion remains valid.*

P r o o f. We denote by $C^\infty_\Gamma(\Omega)$ the set of all functions that are smooth in Ω and that vanish on Γ. We consider $t_1 < T$ arbitrary, and choose a positive number $h < T - t_1$. For an arbitrary $\tilde\psi \in C^\infty_c(0,t_1;C^\infty_\Gamma(\Omega))$ that we extend by zero to $[t_1, T]$ and $[-h, 0]$, the test function $\psi := \tilde\psi_{(\underline{h})}$ can be used in (B.2).

It is checked elementarily that

$$-\int_Q u\frac{\partial\tilde\psi_{(\underline{h})}}{\partial t} = \int_Q \frac{\partial u_{(h)}}{\partial t}\tilde\psi.$$

Using also the fact that the Steklov averaging operator commutes with derivative with respect to space, we transfer for each integral the Steklov averaging according to (B.1), and we obtain that

$$\int_Q \frac{\partial u_{(h)}}{\partial t}\tilde\psi = \int_Q (\xi_1)_{(h)}\tilde\psi + \int_Q (\xi_3)_{(h)}\cdot\nabla\tilde\psi + \int_{\mathcal{S}} (\xi_2)_{(h)}\tilde\psi \tag{B.3}$$

for all $\tilde\psi \in C^\infty_c(0,t_1;C^\infty_\Gamma(\Omega))$.

By assumption, the function $g(u_{(h)}) - g(\bar{u}_{(h)})$ belongs to the space $W^1_{p',\mathcal{C}}(Q_{t_1})$, and can therefore be approximated in the norm of $W^{1,0}_{p',\mathcal{C}}(Q_{t_1})$ by a sequence $\{\tilde{\psi}_k\} \subset C_c^\infty(0,t_1; C_\Gamma^\infty(\Omega))$. We insert $\tilde{\psi}_k$ in (B.3).

Passing to the limit $k \to \infty$ and observing that $(\partial u_{(h)}/\partial t) g(u_{(h)}) = (\partial/\partial t) F(u_{(h)})$, we obtain that

$$\begin{aligned}\int_\Omega F(u_{(h)}(t_1)) = \int_\Omega F(u_{(h)}(0)) &+ \int_{Q_{t_1}} (\xi_1)_{(h)} (g(u_{(h)}) - g(\bar{u}_{(h)})) \\ &+ \int_{Q_{t_1}} \frac{\partial u_{(h)}}{\partial t} g(\bar{u}_{(h)}) + \int_{Q_{t_1}} (\xi_3)_{(h)} \cdot \nabla (g(u_{(h)}) - g(\bar{u}_{(h)})) \\ &+ \int_{\mathcal{S}_{t_1}} (\xi_2)_{(h)} (g(u_{(h)}) - g(\bar{u}_{(h)})).\end{aligned}$$

Using integration by parts again, we have

$$\begin{aligned}&\int_{Q_{t_1}} \frac{\partial u_{(h)}}{\partial t} g(\bar{u}_{(h)}) \\ &\qquad = -\int_{Q_{t_1}} u_{(h)} \frac{\partial g(\bar{u}_{(h)})}{\partial t} + \int_\Omega u_{(h)}(t_1) g(\bar{u}_{(h)}(t_1)) - \int_\Omega u_{(h)}(0) g(\bar{u}_{(h)}(0)).\end{aligned}$$

Since $u \in C([0,T]; L^1(\Omega))$, we have for all $t \in [0,T]$ and $h \to 0$ that $u_{(h)}(t) \longrightarrow u(t)$ in $L^1(\Omega)$. Since the function g is globally bounded, its primitive F has at most linear growth at infinity, which implies that

$$F(u_{(h)}(t)) \longrightarrow F(u(t)) \quad \text{in } L^1(\Omega),$$

for all $t \in [0,T]$. The convergence of the right-hand side as $h \to 0$ is checked easily. This proves the claim. □

To obtain *a-priori* estimates in the L^1-case, we will need two further auxiliary results.

Proposition B.3. *For $n \in \mathbb{N}$ and $u \in W^{1,0}_p(Q) \cap L^{\infty,1}(Q)$, define*

$$B_n := \{(t,x) \in [0,T] \times \Omega \colon\ n \leqslant |u(t,x)| < n+1\}.$$

Suppose that there exists a positive constant C_* such that $\sup\limits_{n \in \mathbb{N}} \int_{B_n} |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t \leqslant C_*$. If $p < \frac{15}{4}$, then for all $1 \leqslant q < p - \frac{3}{4}$, we can find positive constants c_1, c_2 that depend only on Ω, q, p, such that for $s = (p-q)/3q$

$$\|\nabla u\|_{L^q(Q)} \leqslant c_1 + c_2 \|u\|^s_{L^{\infty,1}(Q)} C_*^{1/q}.$$

P r o o f. Similar results were proved in [1]. We can also follow the argumentation of [8]. □

Lemma C.1. *Assume that* θ *satisfies Proposition* 2.3. *Then, for all* $0 < q < \infty$, *the estimate* (2.10) *is valid.*

Proof. For a number $q \geqslant 1$ we consider the functions $g = g_q$, $F = F_q \in C(\mathbb{R})$ given by

$$g(s) := |s|^{q-1}s, \quad F(s) := \frac{1}{q+1}|s|^{q+1}.$$

The function F is the primitive function of g that vanishes at zero.

We want to test the relation (2.3) with the function $\psi := g(\theta) - g(\theta_g)$. Observe that $|\nabla g(\theta)| = q|\theta|^{q-1}|\nabla\theta|$. Since $\theta \in L^\infty(Q)$ according to Proposition 2.3, it follows that $g(\theta) \in L^p(0,T;W^{1,p}(\Omega))$. Since in addition $\psi = g(\theta) - g(\theta_g)$ vanishes on $\mathcal{C}$, the function ψ is admissible in (2.1).

Applying Lemma B.2, we can derive for all $t_1 < T$ the identity

$$\text{(C.1)} \quad \int_\Omega F(\theta(t_1)) + \int_{Q_{t_1}} (\delta|\nabla\theta|^{p-2} + \kappa(\theta))\nabla\theta \cdot \nabla(|\theta|^{q-1}\theta) + \int_{\mathcal{S}_{t_1}} G(\sigma|\theta|^3\theta)|\theta|^{q-1}\theta$$
$$= \int_{Q_{t_1}} f^{[\delta]}|\theta|^{q-1}\theta + \int_\Omega F(\theta(0)) + R,$$

where

$$\begin{aligned} R := &-\int_{Q_{t_1}} (|\theta|^{p-2}\theta - |\theta_g|^{p-2}\theta_g)(|\theta|^{q-1}\theta - |\theta_g|^{q-1}\theta_g) \\ &+ \int_{Q_{t_1}} (\delta|\nabla\theta|^{p-2} + \kappa(\theta))\nabla\theta \cdot \nabla(|\theta_g|^{q-1}\theta_g) \\ &+ \int_{\mathcal{S}_{t_1}} G(\sigma|\theta|^3\theta)|\theta_g|^{q-1}\theta_g + g(\theta_g)\int_\Omega (\theta(t_1) - \theta(0)). \end{aligned}$$

We now want to estimate the absolute value of R. The simplifying assumption (1.3) that θ_g is a constant, though not strictly necessary for the proof, will help us to shorten matters.

By Young's inequality we obtain that

$$\left|\int_\Omega (\theta(t_1) - \theta(0))\right| \leqslant \frac{1}{2(q+1)}\int_\Omega |\theta(t_1)|^{q+1} + c_q\int_\Omega |\theta_0|^{q+1}.$$

Due to the fact that G is selfadjoint we have

$$\int_{\mathcal{S}_{t_1}} G(\sigma|\theta|^3\theta)|\theta_g|^{q-1}\theta_g = \int_{\mathcal{S}_{t_1}} \sigma|\theta|^3\theta G(|\theta_g|^{q-1}\theta_g) = 0,$$

since G vanishes on constants. Thus, we obtain that

$$|R| \leqslant \frac{1}{2(q+1)} \int_\Omega |\theta(t_1)|^{q+1} + c_q \int_\Omega |\theta_0|^{q+1},$$

which together with (C.1) proves the claim for $q \geqslant 1$.

In the case $0 < q < 1$, we consider for a parameter $\alpha > 0$ the functions $g = g_{\alpha,q}$, $F = F_{\alpha,q} \in C(\mathbb{R})$ given by

$$g(s) := (|s| + \alpha)^{q-1} s,$$

$$F(t) := \begin{cases} -\dfrac{(\alpha - t)^q t}{q} - \dfrac{(\alpha - t)^{q+1}}{q(q+1)} + \dfrac{\alpha^{q+1}}{q(q+1)} & \text{if } t \leqslant 0, \\ \dfrac{(\alpha + t)^q t}{q} - \dfrac{(\alpha + t)^{q+1}}{q(q+1)} + \dfrac{\alpha^{q+1}}{q(q+1)} & \text{if } t > 0. \end{cases}$$

We again test the relation (2.3) with $\psi = g(\theta) - g(\theta_g)$. By similar arguments we obtain the relation (2.10), this time with $0 < q < 1$. □

Lemma C.2 (Lemma 2.7). *Let the hypotheses of Proposition* 2.3 *be satisfied, and assume that the hypotheses either of* 2.5 *or of Proposition* 2.6 *are valid. Then if* $p \geqslant s_1'$, *the sequence* $\|\theta_\delta'\|_{L^1(0,T;[W^{1,p}_\Gamma(\Omega)]^*)}$ *is uniformly bounded.*

Proof. For the sake of notational simplicity, we write θ instead of θ_δ. In (2.3) we test with $\theta(t) - \theta_g$, and by usual considerations we obtain the inequality

$$\langle \theta'(t), \theta(t)\rangle + \delta \|\theta(t)\|^p_{W^{1,p}(\Omega)} \leqslant \int_\Omega f^{[\delta]}(t)(\theta(t) - \theta_g) + \langle \theta'(t), \theta_g\rangle.$$

We integrate this inequality on $]0, t_1[$. We have

$$\int_0^{t_1} \langle \theta', \theta_g\rangle = \theta_g \int_\Omega (\theta(t_1) - \theta_0).$$

Therefore, by Young's inequality, we obtain that

$$\left| \int_0^{t_1} \langle \theta', \theta_g\rangle \right| \leqslant \frac{1}{4} \int_\Omega |\theta(t_1)|^2 + c \int_\Omega |\theta_0|^2.$$

Since $p > 3$, the space $W^{1,p}(\Omega)$ embeds continuously in the space of continuous functions, and we get

$$\frac{1}{4}\|\theta(t_1)\|^2_{L^2(\Omega)} + \delta \int_0^{t_1} \|\theta(t)\|^p_{W^{1,p}(\Omega)} \\ \leqslant c\|\theta_g\|^2_{L^2(\Omega)} + c\|f\|_{L^{p'}(0,T;L^1(\Omega))} \big(\|\theta\|_{L^p(0,T;W^{1,p}(\Omega))} + \|\theta_g\|_{L^p(0,T;W^{1,p}(\Omega))}\big).$$

Therefore,

$$\delta\|\theta\|^p_{L^p(0,T;W^{1,p}(\Omega))} \leqslant C + c\|f\|_{L^{s_1}(0,T;L^1(\Omega))}\|\theta\|_{L^p(0,T;W^{1,p}(\Omega))}.$$

If $\|\theta\|_{L^p(0,T;W^{1,p}(\Omega))} \geqslant 1$, then it follows that

$$\delta\|\theta\|^{p-1}_{L^p(0,T;W^{1,p}(\Omega))} \leqslant C + c\|f\|_{L^{s_1}(0,T;L^1(\Omega))}.$$

Thus, we get that

$$\|\delta^{1/(p-1)}\theta\|^{p-1}_{L^p(0,T;W^{1,p}(\Omega))} \leqslant \max\{\delta, C + c\|f\|_{L^{s_1}(0,T;L^1(\Omega))}\}. \tag{C.2}$$

Starting again from (2.3), for $\psi \in W^{1,p}_\Gamma(\Omega)$ we can write

$$\begin{aligned}
|\langle\theta'(t),\psi\rangle| \leqslant\ & \int_\Omega [\delta|\nabla\theta(t)|^{p-1} + \kappa(\theta(t))|\nabla\theta(t)|]|\nabla\psi| \\
& + \int_\Omega (\delta|\theta(t)|^{p-1} + \delta|\theta_g(t)|^{p-1} + |f(t)|)|\psi| \\
& + \int_\Sigma \big|G(\sigma|\theta(t)|^3\theta(t))\big||\psi| \\
\leqslant\ & \delta\|\theta(t)|^{p/p'}_{W^{1,p}(\Omega)}\|\psi\|_{W^{1,p}_\Gamma(\Omega)} + \kappa_u\|\nabla\theta(t)\|_{L^{p'}(\Omega)}\|\nabla\psi\|_{L^p(\Omega)} \\
& + \|f(t)\|_{L^1(\Omega)} \max_{\overline{\Omega}} |\psi| + c\|\theta(t)\|^4_{L^4(\Sigma)} \max_{\overline{\Omega}} |\psi|.
\end{aligned}$$

Using one more time the continuity of the embedding $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$, we get

$$\begin{aligned}
\|\theta'(t)\|_{[W^{1,p}_\Gamma(\Omega)]^*} \leqslant\ & c(\delta(\|\theta(t)\|^{p/p'}_{W^{1,p}(\Omega)} + \|\theta_g\|^{p/p'}_{W^{1,p}(\Omega)}) + \|\nabla\theta(t)\|_{L^{p'}(\Omega)} \\
& + \|f(t)\|_{L^1(\Omega)} + \|\theta(t)\|^4_{L^4(\Sigma)}).
\end{aligned} \tag{C.3}$$

We have $\delta\|\theta(t)\|^{p/p'}_{W^{1,p}(\Omega)} = \|\delta^{1/(p-1)}\theta(t)\|^{p-1}_{W^{1,p}(\Omega)}$, which, in view of (C.2), is uniformly bounded in the space $L^{p'}(0,T)$.

With Proposition 2.5 or 2.6 we find that the sequence $\{\|\theta_\delta\|^4_{L^4(\Sigma)}\}$ is bounded in the space $L^1(0,T)$. Thus, we get

$$\|\theta'_\delta\|_{L^1(0,T;[W^{1,p}_\Gamma(\Omega)]^*)} \leqslant C. \tag{C.4}$$

□

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