Weakest Precondition Reasoning for Expected Run–Times of Probabilistic Programs

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- Model probability distributions in machine learning

Syntax of Probabilistic Programs

$$C \quad \longrightarrow \quad \texttt{skip} \;\big|\; x \,\texttt{:=}\, E \;\big|\; C;\, C \;\big|\; \{C\} \,\Box\, \{C\}$$
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- E.g. $\frac{2}{3}\langle \text{true} \rangle + \frac{1}{3}\langle \text{false} \rangle$, $\frac{1}{2}\langle x > y \rangle + \frac{1}{2}\langle x \geq y \rangle$

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Better Question:

What is the expected run–time (ERT) of C on input σ?

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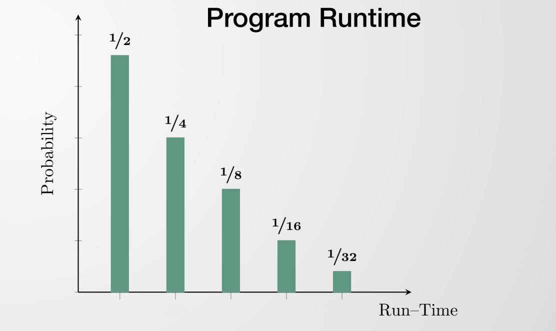

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- ERT of C can be infinite, even if C terminates almost–surely[1]

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[1] i.e. with probability 1

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- Complete partial order on $\mathbb{T}$:

$$t_1 \preceq t_2 \quad \text{iff} \quad \forall\, \sigma \in \Sigma\colon t_1(\sigma) \leq t_2(\sigma)$$

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$\uparrow$

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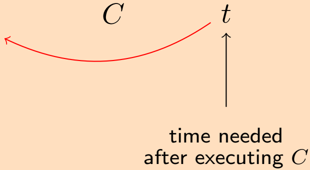

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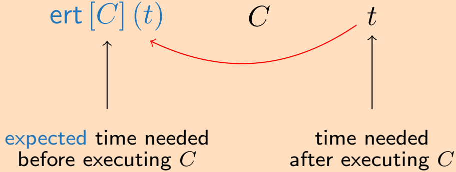

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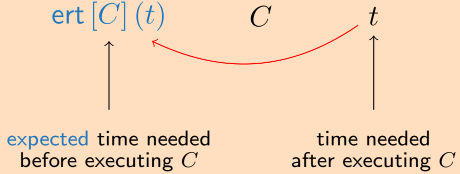

ERT in Terms of ert

$$\mathrm{ert}\,[C]\,(\mathbf{0})\,(\sigma) \;=\; \text{“ERT of } C \text{ on input } \sigma\text{”}$$

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Is the ert Calculus a Reasonable Run–Time Model?

- Correspondence to an operational semantics:
  - Operational model defined in terms of a reward MDP à la [QEST 2012] and [MFPS 2015]
  - ert coincides with expected reward in the operational MDP
  - Enables bounded model checking of expected run–times
- Nielson's Hoare–style logic for reasoning about run–time orders of magnitude of *deterministic programs*:
  - Nielson's logic relies on introducing additional *logical* variables
  - ert is sound and complete with respect to Nielson's logic
  - ert calculus is arguably easier to apply — no additional variables!

Case Study: The Coupon Collector's Problem

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Coupon collector's problem

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In probability theory, the **coupon collector's problem** describes the "collect all coupons and win" contests. It asks the following question: Suppose that there is an urn of n different coupons, from which coupons are being collected, equally likely, with replacement. What is the probability that more than t sample trials are needed to collect all n coupons? An alternative statement is: Given n coupons, how many coupons do you expect you need to draw with replacement before having drawn each coupon at least once? The mathematical analysis of the problem reveals that the expected number of trials needed grows as $\Theta(n\log(n))$.[1] For example, when $n = 50$ it takes about 225[2] trials to collect all 50 coupons.

Contents [show]

Understanding the problem [edit]

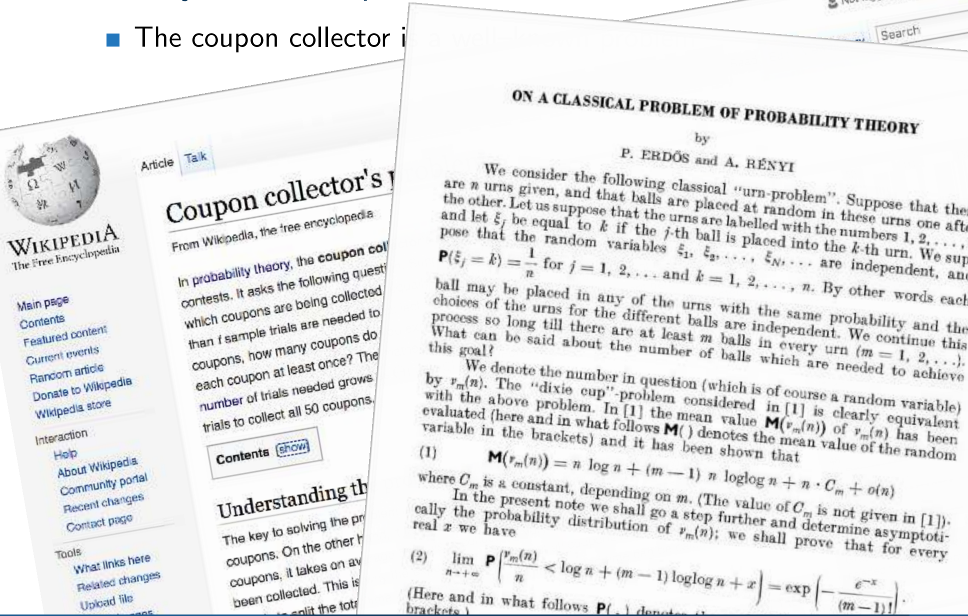
The key to solving the problem is understanding that it takes very little time to collect the first few coupons. On the other hand, it takes a long time to collect the last few coupons. In fact, for 50 coupons, it takes on average 50 trials to collect the very last coupon after the other 49 coupons have been collected. This is why the expected time to collect all coupons is much longer than 50. The idea ... split the total time into 50 intervals where the expected time can be calculated.

... sets of 1 to 100

Graph of number of coup... needed to collect them ...

Case Study: The Coupon Collector's Problem

- The coupon collector i



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- We model it by the following algorithm:

$$\begin{array}{l} cp \;\texttt{:=}\; [0, \ldots, 0];\; i \;\texttt{:=}\; 1;\; x \;\texttt{:=}\; N; \\ \texttt{while}\,(x > 0)\,\{ \\ \quad \texttt{while}\,(cp[i] \neq 0)\,\{\; i \;\texttt{:}\!\approx \texttt{Unif}[1 \ldots N]\;\}; \\ \quad cp[i] \;\texttt{:=}\; 1;\; x \;\texttt{:=}\; x - 1\;\} \end{array}$$

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- Coupon collector program runs in $\Theta(N \cdot \log N)$ for $N > 0$

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Thank you for your kind attention!

Backup Slides: The Actual Rule for Assignments

C	$\textbf{ert}\,[C]\,(t)$
$x :\approx \mu$	$\mathbf{1} + \lambda\sigma_\bullet\ \mathsf{E}_{[\![\mu]\!](\sigma)}\,(\lambda v.\, t\,[x/v]\,(\sigma))$

Backup Slides: ert Calculations and Proof Rule Application

Example 4 (Geometric distribution). Consider loop

$$C_{\mathtt{geo}}\colon\ \ \mathtt{while}\,(c=1)\,\{c :\approx {}^1\!/\!_2\cdot\langle 0\rangle + {}^1\!/\!_2\cdot\langle 1\rangle\}\ .$$

From the calculations below we conclude that $I = \mathbf{1} + [\![c=1]\!]\cdot\mathbf{4}$ is an upper invariant with respect to $\mathbf{0}$:

$$\begin{aligned}
&\mathbf{1} + [\![c\neq 1]\!]\cdot\mathbf{0} + [\![c=1]\!]\cdot \mathsf{ert}\,[c :\approx {}^1\!/\!_2\cdot\langle 0\rangle + {}^1\!/\!_2\cdot\langle 1\rangle]\,(I)\\
&\quad= \mathbf{1} + [\![c=1]\!]\cdot\big(\mathbf{1} + \tfrac{1}{2}\cdot I\,[c/0] + \tfrac{1}{2}\cdot I\,[c/1]\big)\\
&\quad= \mathbf{1} + [\![c=1]\!]\cdot\big(\mathbf{1} + \tfrac{1}{2}\cdot\underbrace{(\mathbf{1} + [\![0=1]\!]\cdot\mathbf{4})}_{=\mathbf{1}} + \tfrac{1}{2}\cdot\underbrace{(\mathbf{1} + [\![1=1]\!]\cdot\mathbf{4})}_{=\mathbf{5}}\big)\\
&\quad= \mathbf{1} + [\![c=1]\!]\cdot\mathbf{4} \;=\; I \;\preceq\; I
\end{aligned}$$

Then applying Theorem 3 we obtain

$$\mathsf{ert}\,[C_{\mathtt{geo}}]\,(\mathbf{0}) \;\preceq\; \mathbf{1} + [\![c=1]\!]\cdot\mathbf{4}\ .$$

In words, the expected run–time of $C_{\mathtt{geo}}$ is at most 5 from any initial state where $c=1$ and at most 1 from the remaining states. △

Backup Slides: Operational RMDP

C_{trunc}: if $(1/2 \cdot \langle \text{true} \rangle + 1/2 \cdot \langle \text{false} \rangle)$ $\{succ$:= true$\}$ else $\{$
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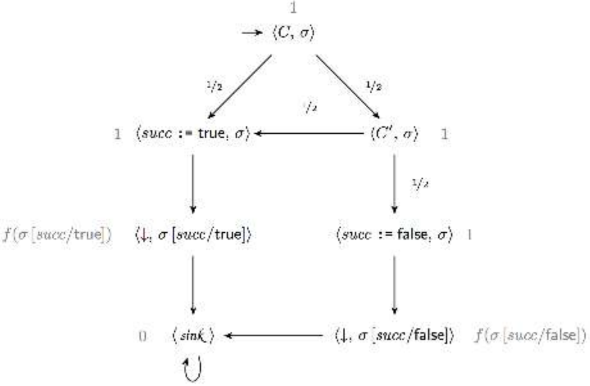
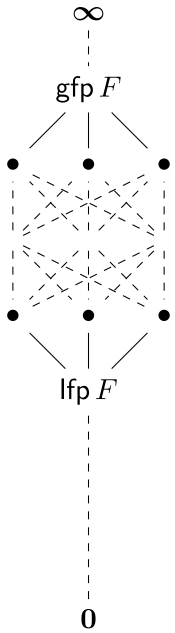


Backup Slides: Park's Lemma



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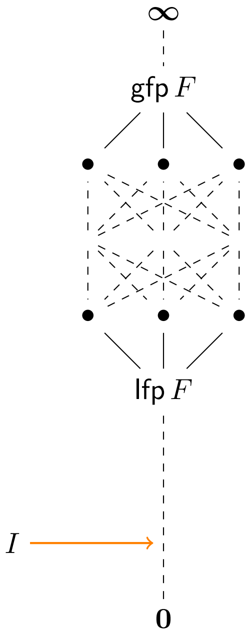

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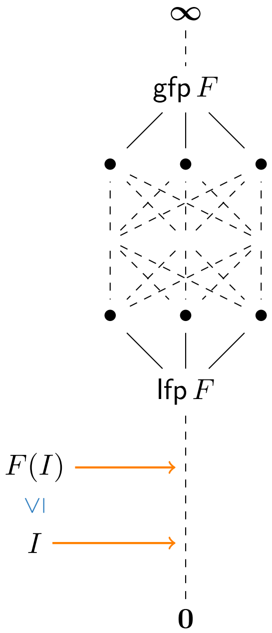

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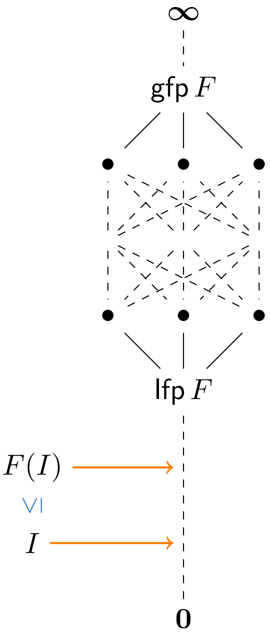

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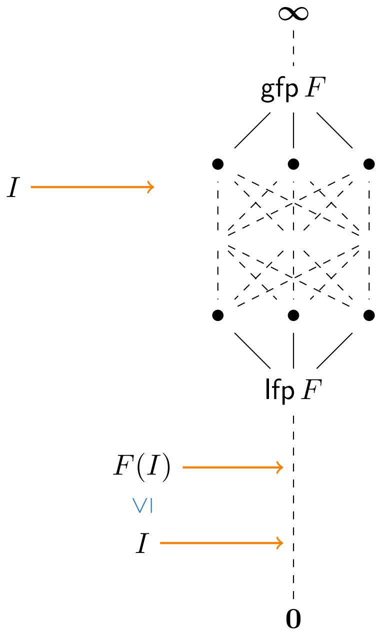

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