Stellingen behorende bij het proefschrift:

Weakly Nonlinear Beam Equations: An Asymptotic Analysis

van Gerdineke Judith Boertjens

1. Laat f, w_0 en w_1 voldoen aan de voorwaarden, gedefiniëerd in (3.2.5)-(3.2.7) op bladzijde 49 en 50 van dit proefschrift. Voor elke ϵ en p die voldoen aan $0 < |\epsilon| < \epsilon_0 \ll 1$ en $p \geq 0$ heeft het begin-randwaarden probleem, gedefiniëerd in het stelsel vergelijkingen (3.2.1)-(3.2.4) op bladzijde 49 van dit proefschrift, een unieke en drie keer continu differentiëerbare oplossing met een continue vierde afgeleide naar x, voor alle $(x,t) \in \Omega_L = \{(x,t) \,|\, 0 \leq x \leq \pi, 0 \leq t \leq L|\epsilon|^{-1}\}$, met L een voldoende kleine positieve constante onafhankelijk van ϵ. Deze unieke oplossing hangt continu af van de gedefiniëerde beginvoorwaarden.
[Dit proefschrift, paragraaf 3.2]

2. Voor het begin-randwaarden probleem, gedefiniëerd in het stelsel vergelijkingen (2.1.4)-(2.1.7) op bladzijde 27 van dit proefschrift, geldt in de meeste gevallen dat essentiële energieuitwisseling plaatsvindt tussen eindig veel modes, wanneer er voor $t = 0$ in eindig veel modes energie aanwezig is. Voor het geval $p^2 = 0$ ontstaan er echter oneindig veel interne resonanties, d.w.z. energieuitwisselingen tussen oneindig veel modes, en is truncatie naar één of meerdere modes niet geldig.

3. $\frac{693}{152}$ is niet zo maar een breuk. [Dit proefschrift, hoofdstuk 2]

4. Wiskunde $\cap$ Politiek $\cap$ Telecommunicatie $= 435$[1].

5. *Vier uur creatief werk per dag is zo ongeveer het maximum voor een wiskundige.*
G.H. Hardy

6. *Vrouwen die gelijk willen zijn aan mannen missen ambitie.* Onbekend

7. *Het is beter een kaars aan te steken, dan te klagen over de duisternis.*
Een Chinese wijsheid

8. Politiek is eens omschreven[2] als *de kunst om het onvermijdelijke mogelijk te maken.* Een gedreven politicus echter beheerst de kunst om het mogelijke onvermijdelijk te maken.

9. Door de eeuwen heen heeft de schijnbare tegenstelling tussen schoonheid en intelligentie bij vrouwen aanleiding gegeven tot commentaar. Voorbeelden hiervan zijn: Spreuken 11:22, *Als een gouden ring in een varkenssnuit is een schone vrouw zonder verstand* [3], of meer recent, de onder (hoog opgeleide) mannen zeer populaire 'domme blondjes' moppen.

10. De invoering van de 'gratis' OV-jaarkaart voor studenten heeft een positief effect gehad op het ziekteverzuim van Delftse buschauffeurs.

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[1] In de Joodse traditie wordt aan de letters van het alfabet een getalwaarde toegekend (Gematria)
[2] Gaby van den Berghe (Vlaams journalist), 'Aan de haak', Albatros, Brussel, 1972
[3] Vertaling Nederlands Bijbelgenootschap

1. Suppose f, w_0, and w_1 satisfy the conditions, as defined in (3.2.5)-(3.2.7) on the pages 49 and 50 of this thesis. Then for every ϵ and p satisfying $0 < |\epsilon| < \epsilon_0 \ll 1$ and $p \geq 0$ the initial-boundary value problem, defined in the system of equations (3.2.1)-(3.2.4) on page 49 of this thesis, has a unique and three times continuously differentiable solution with a continuous fourth derivative with respect to x for $(x,t) \in \Omega_L = \{(x,t) \,|\, 0 \leq x \leq \pi, 0 \leq t \leq L|\epsilon|^{-1}\}$, with L a sufficiently small, positive constant independent of ϵ. This unique solution depends continuously on the initial values. [This thesis, section 3.2]

2. For the initial-boundary value problem, defined in the system of equations (2.1.4)-(2.1.7) on page 27 of this thesis, it holds for most cases that there is an essential energy exchange between a finite number of modes, if for $t = 0$ there is energy present in a finite number of modes. For the case $p^2 = 0$ however, infinitely many internal resonances occur, i.e. energy exchange occurs between infinitely many modes, and truncation to one or more modes is not valid.

3. $\frac{693}{152}$ is not just any fraction. [This thesis, chapter 2]

4. Mathematics $\cap$ Politics $\cap$ Telecommunication = 435^4.

5. *Four hours of creative work a day is about the limit for a mathematician.*
G.H. Hardy

6. *Women who seek to be equal to men lack ambition.* Unknown

7. *It is better to light a candle than to complain about the darkness.*
A Chinese saying

8. Politics has been described[5] as *the art of making the inevitable possible.* A driven politician however masters the art of making the possible inevitable.

9. Throughout the ages the seeming contradiction between beauty and intelligence of women has lead to comments. Examples are: Proverbs 11:22, *As a jewel of gold in a swine's snout, so is a fair woman which is without good sense*[6], or more recent, the among (highly educated) men very popular 'dumb blonds' jokes.

10. The introduction of the 'free' public transport card for students has had a positive effect on the rate of absenteeism due to sickness among bus drivers in Delft.

[4] In the Jewish tradition each letter of the alphabet has a numerical value (Gematria)
[5] Gaby van den Berghe (Flamish journalist), 'Aan de haak', Albatros, Brussels, 1972
[6] Authorised King James Version and The New English Bible

Weakly Nonlinear Beam Equations: An Asymptotic Analysis

Voor Opa de Weert

Weakly Nonlinear Beam Equations: An Asymptotic Analysis

PROEFSCHRIFT

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus Prof. ir. K.F. Wakker,
in het openbaar te verdedigen ten overstaan van een commissie,
door het College voor Promoties aangewezen,
op dinsdag 30 mei 2000 te 10.30 uur

door

Gerdineke Judith BOERTJENS,

wiskundig ingenieur,
geboren te Thesinge.

Dit proefschrift is goedgekeurd door de promotor:

Prof. dr. ir. J.W. Reyn

Samenstelling promotiecommissie:

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Prof. dr. ir. J.W. Reyn,	Technische Universiteit Delft, promotor
Dr. ir. W.T. van Horssen,	Technische Universiteit Delft, toegevoegd promotor
Prof. dr. Ph.P.J.E. Clément,	Technische Universiteit Delft
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Prof. dr. ir. J. Grasman,	Landbouwuniversiteit Wageningen
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Dr. ir. W.T. van Horssen heeft als toegevoegd promotor in belangrijke mate aan de totstandkoming van het proefschrift bijgedragen.

Boertjens, Gerdineke Judith

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Preface

This thesis is the reflection of almost 6 years of research within the Differential Equations group of the Department of Applied Mathematics at Delft University of Technology. This thesis would not have existed had it not been for the support of many people. Some of the support is reflected in the thesis, other support was more implicit, but as important.

First of all I would like to thank my supervisor and 'toegevoegd promotor', Wim van Horssen. His knowledge of the subject and interest in my research were indispensable in the completion of this thesis. Many thanks are also due to professor J.W. Reijn for the careful reading of the manuscript and the valuable suggestions for improvement. I am grateful he found the time to assist me even after his retirement. Thanks also to Adriaan van der Burgh for motivating me and encouraging me to present my research at various conferences abroad (Canada, USA, Germany, Indonesia and Italy!).
I thank Kees Lemmens for his support on any computer matter, and for putting up with a slightly paranoid person when it comes to learning new computer stuff ... Also thanks to Cees Korving for helping me with the dynamics in the mathematical formulation of the problem. Thanks are also due to all the colleagues I spent time with during the years, all the people with whom I drank coffee, had discussions on emancipation and other subjects, talked about soccer games and went on 'vakgroep-uitjes' (especially to Gerard Herman and Cees Korving, who carried me across the 'wad' on our trip to Ameland).

Special thanks to Timber for putting up with me as a roommate at the university. Thanks for all the movies at the 'filmhuis' we went to and the soccer games we watched together on TV and in the stadium (Het Kasteel). And of course thanks for all the dinners (homemade pasta) and Martini Rosso con ghiaccio we shared in the past, and will hopefully share more in the years to come! I am glad he and Agur kept their promise of becoming my 'paranimfs' during the defence of my thesis.

NWO, the J.M. Burgers Centre, Shell Nederland BV and the faculty are gratefully acknowledged for their financial support on all my foreign travels.

Last (but not least!) of all many thanks to the home front. First of all, a lot of love and kisses to Jeroen, who had to miss me a lot, especially the last year, when I worked nights and weekends to finish my thesis. I hope being able to show off with a girlfriend who is a doctor (in mathematics) makes up for it!
Furthermore, I would like to thank my parents and my family for their support and love. Dad, I guess all the games of LOCO and Memory finally payed off?! Mum, thanks for all the support and the listening ear whenever I needed it!

Delft, 11 april 2000 Judith Boertjens

Contents

Summary

Particular types of flexible structures, like tall buildings, suspension bridges or iced overhead transmission lines with bending stiffness, are subjected to oscillations due to various causes, such as (strong) winds or earthquakes. A classic example is the Tacoma Narrows suspension bridge. Who has not seen the video-film of the large-scale oscillations and the collapse of this bridge, shown in high school physics classes? A more recent example is the oscillation of the stays of the Erasmus Bridge in Rotterdam, during stormy and rainy weather.

In this thesis we consider a (simplified) model for nonlinear oscillations of a suspension bridge. For several (external) forces acting on the structure an initial-boundary value problem is defined, which describes the vertical displacement of the suspension bridge. For each case the initial-boundary value problem is studied, using a multiple timescale perturbation method. Formal approximations, i.e. functions that satisfy the differential equation and the initial-boundary values up to some order in ϵ, are constructed in the form of a power series for each case. Furthermore it is shown whether or not internal resonances occur. In all cases a justification is given whether a (so-called Galerkin) truncation of the infinite series for the formal approximations of the solution is valid or not.
For one class of initial-boundary value problems the existence and uniqueness of solutions and the asymptotic validity of approximations on a large time-scale are proven, using Banach's fixed point theorem.

The results of this thesis can, among other purposes, be used to consider whether Galerkin truncation is applicable or not, when analysing this type of problems (numerically). Also, using the results of this thesis, a justification of the asymptotic validity of approximations can be given, and some remarks can be made on the difference between approximations and exact solutions.

Chapter 1

Introduction

1.1 Relevant literature

Particular types of flexible structures, like tall buildings, suspension bridges or iced overhead transmission lines with bending stiffness, are subjected to oscillations due to various causes, such as (strong) winds or earthquakes. A classic example is the Tacoma Narrows suspension bridge. Who has not seen the video-film of the large-scale oscillations and the collapse of this bridge, shown in high school physics classes? A more recent example is the oscillation of the stays of the Erasmus Bridge in Rotterdam, during stormy and rainy weather. Simple models which describe these oscillations involve nonlinear second order partial differential equations (wave equations), as can be seen for example in [1]-[4], second order ordinary coupled differential equations, as can be seen in [5]-[8], or nonlinear fourth order partial differential equations (beam equations), as can be seen for example in [9]-[13].

Nonlinear beam equations are considered in various papers. In [14]-[19] several experimental models are discussed, such as canti-levered beams ([14]-[17]), canti-levered pipes conveying fluid ([18]) or heat-exchanger tubes ([19]). In [20]-[25] several theoretical analyses are discussed.

In many cases perturbation methods can be used to construct approximations for solutions of this type of second or fourth order equations. Initial-boundary value problems for second order PDE's have been considered for a long time, for instance in [1]-[4], [26]-[30], using a multiple scales perturbation method, in [31]-[32], using a (Galerkin) averaging method, or in [33]-[34], using both methods. More recently initial-boundary value problems for fourth order PDE's have been considered, for example in [9]-[25]. We will discuss these papers in more detail in the next paragraph. For fourth order strongly nonlinear PDE's numerical finite element methods

can for instance be used, as is done for example in [35].
In [9] the following equation is given to describe the deflection of a suspension bridge modeled by a nonlinear beam:

$$u_{tt} + u_{xxxx} + \delta u_t = -ku^+ + W(x) + \epsilon f(x,t),$$

with simply supported boundary conditions for u, the vertical deflection of the bridge, $W(x)$ the weight of the bridge per unit length, f an external forcing term, $u^+ = u$ when $u > 0$ and $= 0$ when $u < 0$, δ a damping coefficient and k a spring constant. In [9] a single mode representation is used, together with a numerical analysis. Some remarks are made on existence and multiplicity of periodic solutions to linear PDE's.
In [10] and [11] the following system of equations is used to describe the galloping amplitude of canti-levered beams in a wind-field:

$$\begin{aligned} EIw_{1,xxxx} + Cw_{1,t} + mw_{1,tt} &= D(x,t), \\ EIw_{2,xxxx} + Cw_{2,t} + mw_{2,tt} &= L(x,t), \end{aligned}$$

where EI is the flexural rigidity of the beam, C a damping coefficient, m the mass of the beam per unit length, w_1 (w_2) the deflection along (across) the wind direction and D (L) a drag (lift) force. Single mode representations are used for w_1 and w_2 and a multiple scales method is used to approximate solutions to the system of equations.
In [12] an equation is given to describe nonlinear oscillations of a suspension bridge:

$$K_2 u_{xxxx} + u_{tt} - K_1 u_{xxtt} + K_3 u^+ = 1 + k\cos(x) + \epsilon h(x,t),$$

with simply supported boundary conditions for u, the vertical deflection of the bridge. K_1, K_2, K_3 and k are constants and h is an external excitation. An eigenvalue problem is defined and some results on existence of solutions in the nonlinear case are given.
In [13] a mathematical analysis of dynamical models of suspension bridges is given. The following equations are given for the vibration z of the road-bed of a suspension bridge in vertical direction and the vibration y of the main cable of the suspension bridge from which the road-bed is suspended by:

$$\begin{aligned} m_b z_{tt} + \alpha D^4 z + F_0(y-z) &= m_b g + f_1(t), \\ m_c y_{tt} - \beta D^2 y + F_0(y-z) &= m_c g + f_2(t), \end{aligned}$$

where D^k denotes the spatial derivative of order k, m_b and m_c are the masses per unit length of the road bed and the cable respectively, α and β the flexural rigidity of the structure and the coefficient of tensile strength

of the cable respectively, F_0 a restraining force and f_1, f_2 external, non-conservative forces. The boundary conditions considered are clamped on both edges or clamped-hinged. A variational method and an energy functional are used to prove existence of solutions, a Cauchy problem is formulated for a more general form of the nonlinearity considered and used to prove existence of solutions and uniqueness of solutions when the nonlinearity satisfies certain conditions. Also some numerical results are given.
In [14]-[17] an elastic beam equation is considered with nonlinear boundary conditions (clamped-free):

$$Dv_{xxxx} = -mv_{tt} - mV_{0,tt},$$

with v the lateral displacement of the beam relative to the support, V_0 the displacement of the clamped-end support, m the mass of the beam per unit length and D the bending stiffness of the beam. A single mode Galerkin expansion is used to approximate solutions of this beam equation and some experimental results are given, stating that chaotic vibrations occur.
In [18] the nonlinear dynamics of planar motions of canti-levered pipes conveying fluid are considered. The equations of motion of the pipe system in horizontal and vertical direction are

$$\begin{aligned} EI\frac{\partial^4 x}{\partial s^4} + \frac{\partial}{\partial s}\left[\frac{\partial x}{\partial s}(P + EI\kappa^2)\right] + m\frac{\partial^2 x}{\partial t^2} + M\frac{D^2 x}{Dt^2} - (m+M)g &= 0, \\ EI\frac{\partial^4 y}{\partial s^4} + \frac{\partial}{\partial s}\left[\frac{\partial y}{\partial s}(P + EI\kappa^2)\right] + m\frac{\partial^2 y}{\partial t^2} + M\frac{D^2 y}{Dt^2} - (m+M)g &= 0, \end{aligned}$$

where EI is the flexural rigidity of the pipe, x and y the horizontal and vertical displacements of the pipe respectively, m and M are the pipe and fluid mass per unit length respectively, g is the acceleration due to gravity, P is the axial force exerted on the pipe and κ is the curvature, s the curvilinear coordinate along the centreline of the pipe, $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial s}$, with u the flow velocity of the fluid. The equations are coupled by the non-constant term $P + EI\kappa^2$. A two-mode Galerkin representation and an eigenvalue approach are used to analyze the coupled system of equations.
In [19] the dynamics of heat exchanger tubes is considered. The equation of motion discussed in [19] is

$$EIw_{xxxx} + cw_t + mw_{tt} + \delta(x - x_b)f(w) = F(w, w_t, w_{tt}).$$

The system under consideration has clamped boundary conditions and a loose support at the middle of the tube, at $x = x_b$. w is the cross stream lateral displacement of the tube, EI the flexural rigidity of the tube, m the mass of the tube per unit length, c a damping coefficient, f the nonlinear force due to the constraint at $x = x_b$, with $\delta(x-x_b)$ the Dirac delta function,

x_b the location of the constraint and F the flow induced force. A Galerkin mode expansion is used and some numerical results are given.
In [20] non-planar oscillations of a simply supported beam are described by

$$\begin{aligned} u_{tt} + u_{xxxx} + u_t - \left[a + \epsilon \int_0^\pi (u_x^2 + v_x^2)dx\right] u_{xx} &= p(x,t), \\ v_{tt} + v_{xxxx} + v_t - \left[a + \epsilon \int_0^\pi (u_x^2 + v_x^2)dx\right] v_{xx} &= q(x,t), \end{aligned}$$

with simply supported boundary conditions for u and v, where u and v are the displacements in x- and y-direction respectively, a a constant and p and q the x- and y-components of the external force considered. The coupled system of equations is analyzed using a finite projection method, and some results on existence of solutions are given. Also some results are given for a rotating beam, using a finite truncation method.
In [21] time-periodic free vibrations of an extensible beam with rotational inertia are discussed, using the following equation:

$$\begin{aligned} Cu_{tt} - u_{ttxx} + u_{xxxx} + f_1(x,u,u_x) - \frac{\partial}{\partial x} f_2(x,u,u_x) & \\ - \left(\int_0^\pi u_s^2(s,t)ds\right) u_{xx} &= 0, \end{aligned}$$

with simply supported boundary conditions for u, the transversal displacement of the beam, C a constant and f_1, f_2 nonlinear functions. A variational approach and a finite dimensional approximation are used to prove the existence of infinitely many time periodic solutions.
In [22] nonlinear transversal vibrations of a uniform beam, which is axially restrained and forced by a two-mode harmonic function, are considered. The equation of motion for w, the deflection of the beam in transversal direction, is

$$\rho h w_{tt} + EI w_{yyyy} - \frac{Eh}{2I}\left[\int_0^l w_y^2 dy\right] w_{yy} = F(y,t),$$

where w satisfies simply supported boundary conditions, l is the length of the beam, h the thickness of the beam, E Young's modulus, I the moment of inertia, ρ the mass density of the beam and F the lateral applied load. A Galerkin expansion method and a finite difference method are used to approximate solutions of the PDE.
In [23] the transversal deflection u of an extensible beam of length L is given by

$$u_{tt} + \alpha u_{xxxx} + \left(\beta + \int_0^L u_\xi^2(\xi,t)d\xi\right) u_{xx} = 0,$$

with α and β constants, $\alpha > 0$. This equation is semi-linear since the integral term can be seen as a function of t only. Some results on existence and uniqueness of solutions to this semi-linear equation are given using a finite Galerkin expansion method.

In [24] the forced oscillations of beams with nonlinear damping are considered. The following equation for the transversal displacement u of the beam is given:

$$u_{tt} + u_{xxxx} + \beta(u_t) = f(x,t),$$

with u a 2π-periodic function in t, which satisfies simply supported boundary conditions, $\beta(u_t)$ a super-linear increasing function of u_t and $f(x,t)$ a 2π-periodic linear function of t. A one-mode Galerkin expansion, an eigenvalue approach and a difference method are used to approximate solutions. In addition some numerical results are given.

In [25] vibrations of a beam resting on elastic foundations are considered. The following equation is given for the lateral displacement w of the beam:

$$EIw_{xxxx} - \frac{\partial}{\partial x}\left[K_1(x)ww_x\right] + K(x)w = -\rho A w_{tt},$$

where EI is the flexural rigidity of the beam, $K(x)$ the Winkler foundation stiffness along the beam, $K_1(x)$ the second foundation parameter along the beam, ρ the material density of the beam, A the cross-sectional area of the beam. Various boundary conditions are considered. Truncation to one mode is applied, a power series method with a stiffness matrix and a finite element method are used to approximate solutions to the PDE.

In most of the papers considered above a truncation to one or several modes is applied, without giving a validation, and no remarks are made on the relation between the approximation and the exact solution.

1.2 Mathematical formulation of the problem

To derive the equations of motion for an elastic beam we will follow part of the analysis given in [37]. We consider an elastic beam of length l, simply supported at the ends, in vertical direction. An external force is be applied at the ends of the beam such that no vertical displacement is possible. Oscillations are possible due to the strain of the beam. The x-axis is taken along the beam axis, such that the left end of the beam corresponds with $x = 0$. The z-axis is taken vertically. The y-axis is perpendicular to the (x,z)-plane. We assume that the beam can move in the x- and z-direction only. We introduce the following symbols: μ is the mass of the beam per unit length, ρ the mass density of the beam, A the area of the cross-section

Q of the beam perpendicular to the x-axis (so $\mu = \rho A$), E the elasticity modulus (Young's modulus), I the axial moment of inertia of the cross-section. The inertial axes of the cross-section Q are the y- and z-axes, so $I = \int\int_Q z^2 dydz$. The vertical displacement of the beam from rest is $w = w(x,t)$, the horizontal displacement of the beam is $u = u(x,t)$. The curvature of the beam in the (x,z)-plane can be approximated by w_{xx} as follows. We consider an element of the beam of length Δx in static state. In deformed state the arc length of this element is Δs, where $\Delta s \approx \mathcal{R}\Delta\varphi$, with $\Delta\varphi$ the arc angle, as can be seen in Figure 1.1 and $\mathcal{R}$ the radius of

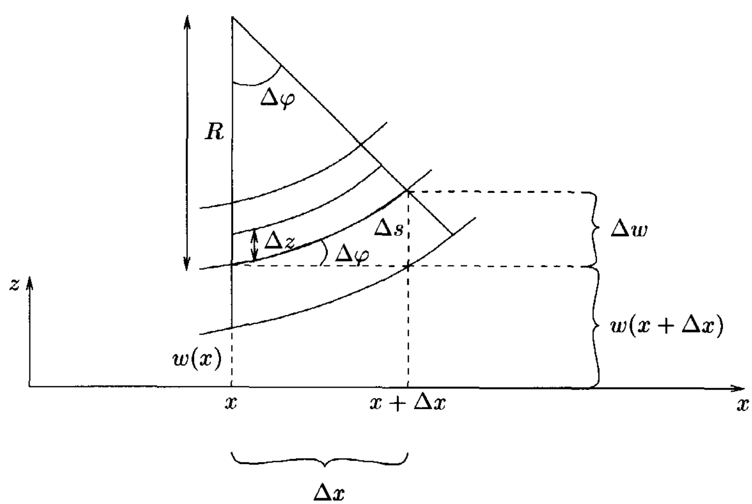

Figure 1.1: The bending of a line-element Δx.

curvature of the beam axis in deformed state at position x. Furthermore, $\Delta\varphi \approx \tan\Delta\varphi \approx \Delta w/\Delta x$ and $\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta w)^2}$. For $\Delta x \to 0$ this gives us $\mathcal{R} = (1 + w_x^2)^{\frac{3}{2}}/w_{xx}$. Assuming that w_x is small with respect to 1, we can approximate the curvature, which is equal to $1/\mathcal{R}$, by w_{xx}. Using this, the strain ε_{xx} due to 'pure' bending of a line-element of the beam at a distance z from the line of centroids (the x-axis) is given by

$$\varepsilon_{xx} = \frac{(\mathcal{R} - z)\Delta\varphi - \mathcal{R}\Delta\varphi}{\mathcal{R}\Delta\varphi} = -\frac{z}{\mathcal{R}} \approx -zw_{xx}.$$

Furthermore, the strain ε_{x0} due to stretching of the line of centroids of a line-element of the beam can be approximated by $u_x + \frac{1}{2}w_x^2$ as follows. From Figure 1.2 and the definition of strain due to stretching, which can be found in any standard textbook on mechanics (see for example [38]) we have the following expression for ε_{x0}:

$$\varepsilon_{x0} = \frac{\sqrt{(\Delta x + \Delta u)^2 + (\Delta w)^2} - \Delta x}{\Delta x}.$$

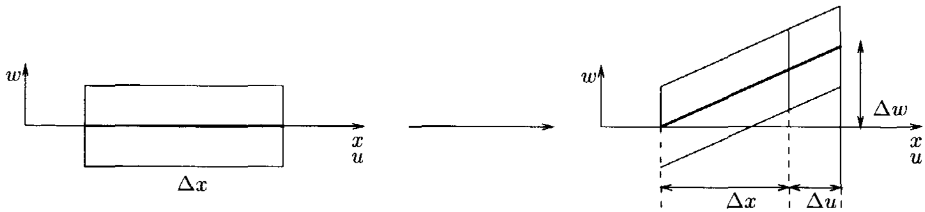

Figure 1.2: The stretching of a line-element Δx.

For $\Delta x \to 0$ this gives us $\varepsilon_{x0} = \sqrt{1 + 2u_x + u_x^2 + w_x^2} - 1$. By assuming that u_x^2 is small with respect to u_x, and by expanding the square-root as a Taylor series, we have $\varepsilon_{x0} \approx u_x + 1/2w_x^2$. The total strain of a line-element of the beam at a distance z from the x-axis is given by $\varepsilon_x = \varepsilon_{x0} + \varepsilon_{xx} = u_x + \frac{1}{2}w_x^2 - zw_{xx}$. It is shown in [37] that, using Hooke's Law, the work performed to deflect the beam from its initial position, is

$$\mathcal{A}(t) = \frac{1}{2}EA\int_0^l \left[u_x + \frac{1}{2}w_x^2\right]^2 dx + \frac{1}{2}EI\int_0^l (w_{xx})^2\, dx. \tag{1.2.1}$$

The kinetic energy of the beam is given by

$$\mathcal{E}_k(t) = \frac{1}{2}\mu\int_0^l \left[u_t^2 + w_t^2\right] dx. \tag{1.2.2}$$

Using (1.2.1) and (1.2.2) the Hamiltonian integral is

$$\begin{aligned} \mathcal{F} &= \mathcal{F}(t_2) - \mathcal{F}(t_1) = \int_{t_1}^{t_2} (\mathcal{A}(t) - \mathcal{E}_k(t))dt \qquad (1.2.3) \\ &= \frac{1}{2}\int_{t_1}^{t_2}\int_0^l \{EA[u_x + \frac{1}{2}w_x^2]^2 + EI(w_{xx})^2 - \mu[u_t^2 + w_t^2]\}dxdt. \end{aligned}$$

Using Hamilton's Principle, which states that the variation of $\mathcal{F}$ is equal to 0, the Euler equations for this problem are

$$\mu u_{tt} - EA\frac{\partial}{\partial x}\left[u_x + \frac{1}{2}w_x^2\right] = 0, \tag{1.2.4}$$

$$\mu w_{tt} + EIw_{xxxx} - EA\frac{\partial}{\partial x}\left\{w_x\left[u_x + \frac{1}{2}w_x^2\right]\right\} = 0. \tag{1.2.5}$$

The system given by (1.2.4)-(1.2.5) can be simplified by the following assumption, introduced by Kirchhoff (see [39]): the velocity of the beam in x-direction, u_t, is small compared to w_t and can be neglected in (1.2.3), so $\mathcal{F} = \frac{1}{2}\int_{t_1}^{t_2}\int_0^l\{EA[u_x + \frac{1}{2}w_x^2]^2 + EI(w_{xx})^2 - \mu[w_t^2]\}dxdt$. The system given by (1.2.4)-(1.2.5) can now be simplified to

$$EA\frac{\partial}{\partial x}\left[u_x + \frac{1}{2}w_x^2\right] = 0, \tag{1.2.6}$$

$$\mu w_{tt} + EIw_{xxxx} - EAw_{xx}\left[u_x + \frac{1}{2}w_x^2\right] = 0. \tag{1.2.7}$$

From (1.2.6) we get $u_x + \frac{1}{2}w_x^2 = \varepsilon_{x0}$ is a function of t only. Integrating ε_{x0} with respect to x from 0 to l gives us $\int_0^l (u_x + \frac{1}{2}w_x^2)dx = \varepsilon_{x0}l$, which means $u(l,t) - u(0,t) + \frac{1}{2}\int_0^l w_x^2 dx = \varepsilon_{x0}l = (u_x + \frac{1}{2}w_x^2)l$. Substituting this into (1.2.7) gives us the following equation for the vertical displacement w:

$$\mu w_{tt} + EIw_{xxxx} - \frac{EA}{l}\left[u(l,t) - u(0,t) + \frac{1}{2}\int_0^l w_x^2 dx\right] w_{xx} = 0. \quad (1.2.8)$$

If other external forces are considered, the right-hand side of (1.2.8) becomes nonzero, as we will show in the next part.

In [9] a survey of literature on oscillations of suspension bridges is given. Using a similar analysis, we will derive a simplified model for nonlinear oscillations in suspension bridges, where the vertical displacement of an elastic beam is given by (1.2.8). We model the suspension bridge as a beam of length l. The stays of the bridge are treated as *nonlinear* springs, as sketched in Figure 1.3. The torsional vibrations of the beam are not

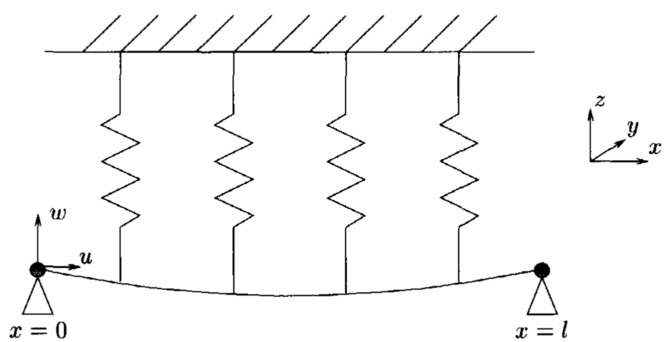

Figure 1.3: A simple model of a suspension bridge.

taken into account (i.e. are considered to be small compared to the vertical vibrations). We neglect internal damping. Furthermore, we consider a uniform wind flow, which causes nonlinear drag and lift forces (F_D, F_L) acting on the structure per unit length. We introduce F_s, the force per unit length acting on the beam due to the springs, and W, the weight of the bridge per unit length, which we consider to be constant, i.e. $W = \mu g$, where g is the gravitational acceleration. The equation describing the vertical displacement of the beam in this case is

$$\begin{aligned} \mu w_{tt} + EIw_{xxxx} + F_s(w) = & \quad (1.2.9) \\ -\mu g + F_D + F_L + \frac{EA}{l}\left[u(l,t) - u(0,t) + \frac{1}{2}\int_0^l w_x^2 dx\right] w_{xx}. & \end{aligned}$$

We assume that the spring force F_s can be expanded in a Taylor series, with $F_s(0) = 0$ and can be approximated by the first two terms, $F_s(w) = \kappa_1 w +$

$\kappa_2 w^2$, which means we assume that higher order terms can be neglected. κ_1 and κ_2 are spring constants. If $\kappa_2 \neq 0$ we in fact assume that the springs have a different behaviour for compression and expansion, i.e. for $w < 0$ and $w > 0$. In section 4 of [1] it has been shown that $F_D + F_L$ can be approximated by

$$\frac{\rho_{\mathrm{a}} d v_\infty^2}{2}\left(a_0 + \frac{a_1}{v_\infty} w_t + \frac{a_2}{v_\infty^2} w_t^2 + \frac{a_3}{v_\infty^3} w_t^3\right),$$

where ρ_{a} is the density of air, d is the diameter of the cross section of the beam, v_∞ is the uniform wind flow velocity and a_0, a_1, a_2, a_3 depend on certain drag and lift coefficients and are given explicitly in [1].
Equation (1.2.9) will be simplified by eliminating the term $-\mu g$ using $w = \tilde{w} + \frac{\mu g}{\kappa_1} s(x)$, where $s(x)$ satisfies the following time-independent linear equation with boundary conditions:

$$\begin{aligned} &s^{(4)}(x) + \frac{\kappa_1}{EI} s(x) = -\frac{\kappa_1}{EI}, \quad 0 < x < l, \\ &s(0) = s(l) = 0, \ s^{(2)}(0) = s^{(2)}(l) = 0. \end{aligned}$$

It can be shown that $s(x) = \cos(\beta x)\cosh(\beta x) + (\sin(\beta l)\sin(\beta x)\cosh(\beta x) - \sinh(\beta l)\cos(\beta x)\sinh(\beta x))/(\cos(\beta l) + \cosh(\beta l)) - 1$, with $\beta = (\frac{\kappa_1}{4EI})^{\frac{1}{4}}$. The term $\frac{\mu g}{\kappa_1} s(x)$ represents the deflection of the beam in static state due to gravity. Using the dimensionless variables

$$\bar{w} = \frac{\pi}{l}\frac{c}{v_\infty}\tilde{w}, \ \bar{x} = \frac{\pi}{l} x, \ \bar{t} = \frac{\pi}{l} ct, \ \bar{u} = \frac{4}{l}\frac{c^2}{v_\infty^2} u,$$

with $c = \frac{\pi}{l}\sqrt{\frac{EI}{\mu}}$, (1.2.9) becomes

$$\begin{aligned} \bar{w}_{\bar{t}\bar{t}} + \bar{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} + \left(\frac{l}{\pi}\right)^4 \frac{\kappa_1}{EI}\bar{w} &= \frac{\rho_{\mathrm{a}} d l}{2\pi\mu}\frac{v_\infty}{c}\left(a_0 + a_1 \bar{w}_{\bar{t}} + a_2 \bar{w}_{\bar{t}}^2 + a_3 \bar{w}_{\bar{t}}^3\right) \\ &+ \frac{1}{4}\frac{A}{I}\frac{v_\infty}{c}\frac{l}{\pi}\left(\frac{v_\infty}{c}\frac{l}{\pi}\left(\bar{u}(\pi,\bar{t}) - \bar{u}(0,\bar{t}) + \frac{2}{\pi}\int_0^\pi \bar{w}_{\bar{x}}^2 d\bar{x}\right)\bar{w}_{\bar{x}\bar{x}} + \frac{\mu g}{\kappa_1}\mathcal{H}\right), \\ &- \frac{\kappa_2}{EI}\left(\frac{l}{\pi}\right)^4\left(\frac{v_\infty}{c}\frac{l}{\pi}\bar{w}^2 + \frac{\mu g}{\kappa_1}\mathcal{J}\right), \end{aligned} \tag{1.2.10}$$

with

$$\begin{aligned} \mathcal{H} &= \left(\bar{u}(\pi,\bar{t}) - \bar{u}(0,\bar{t}) + \frac{2}{\pi}\int_0^\pi \bar{w}_{\bar{x}}^2 d\bar{x}\right) s^{(2)}(\frac{l}{\pi}\bar{x}) \\ &+ \frac{4}{\pi}\left[\int_0^\pi \bar{w}_{\bar{x}} s^{(1)}(\frac{l}{\pi}\bar{x}) d\bar{x}\left(\bar{w}_{\bar{x}\bar{x}} + \frac{\mu g}{\kappa_1}\frac{\pi}{l}\frac{c}{v_\infty} s^{(2)}(\frac{l}{\pi}\bar{x})\right)\right] \\ &+ \frac{2}{\pi}\frac{\mu g}{\kappa_1}\frac{\pi}{l}\frac{c}{v_\infty}\left[\int_0^\pi \left(s^{(1)}(\frac{l}{\pi}\bar{x})\right)^2 d\bar{x}\left(\bar{w}_{\bar{x}\bar{x}} + \frac{\mu g}{\kappa_1}\frac{\pi}{l}\frac{c}{v_\infty} s^{(2)}(\frac{l}{\pi}\bar{x})\right)\right], \end{aligned}$$

and

$$\mathcal{J} = 2s(\frac{l}{\pi}\bar{x}) + \frac{\pi}{l}\frac{c}{v_\infty}\frac{\mu g}{\kappa_1}s^2(\frac{l}{\pi}\bar{x}).$$

Assuming that v_∞, the uniform wind velocity, is small with respect to $\frac{\pi}{l}c$, we put $\tilde{\epsilon} = \frac{v_\infty}{c}\frac{l}{\pi}$, where $\tilde{\epsilon}$ a small parameter. Furthermore, we assume that the deflection of the beam in static state due to gravity, $\frac{\mu g}{\kappa_1}s$, is small with respect to the vertical displacement $\tilde{w}$, which is of order $\tilde{\epsilon}$. This means we assume that $\frac{\mu g}{\kappa_1}$ is $\mathcal{O}(\tilde{\epsilon}^n)$, with $n > 1$, since $s(x)$ is of order 1 (as well as $s^{(1)}(x)$ and $s^{(2)}(x)$), as can be seen from the expression for s which is given above. It can be shown easily that $\mathcal{H} = \mathcal{O}(1)$ and $\mathcal{J} = \mathcal{O}(1)$. Equation (1.2.10) now becomes

$$\begin{aligned}
\bar{w}_{\bar{t}\bar{t}} + \bar{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} + \left(\frac{l}{\pi}\right)^4 \frac{\kappa_1}{EI}\bar{w} &= \frac{\rho_a d}{2\mu}\tilde{\epsilon}\left(a_0 + a_1\bar{w}_{\bar{t}} + a_2\bar{w}_{\bar{t}}^2 + a_3\bar{w}_{\bar{t}}^3\right) \\
&+ \frac{1}{4}\frac{A}{I}\tilde{\epsilon}^2\left(\bar{u}(\pi,\bar{t}) - \bar{u}(0,\bar{t}) + \frac{2}{\pi}\int_0^\pi \bar{w}_{\bar{x}}^2 d\bar{x}\right)\bar{w}_{\bar{x}\bar{x}} - \frac{\kappa_2}{EI}\left(\frac{l}{\pi}\right)^4\tilde{\epsilon}\bar{w}^2 \\
&+ \frac{1}{4}\frac{A}{I}\mathcal{O}(\tilde{\epsilon}^{m_1}) - \frac{\kappa_2}{EI}\left(\frac{l}{\pi}\right)^4\mathcal{O}(\tilde{\epsilon}^{m_2}),\ m_1 > 2, m_2 > 1.
\end{aligned} \tag{1.2.11}$$

In section 4 of [1] it has been shown that the first term in the right hand side of (1.2.11) can be approximated as follows:

$$\frac{\rho_a d}{2\mu}\tilde{\epsilon}\left(a_0 + a_1\bar{w}_{\bar{t}} + a_2\bar{w}_{\bar{t}}^2 + a_3\bar{w}_{\bar{t}}^3\right) = \frac{\rho_a d}{2\mu}\tilde{\epsilon}\left(a\bar{w}_{\bar{t}} - b\bar{w}_{\bar{t}}^3\right) + \mathcal{O}(\tilde{\epsilon}^{m_3}),$$

with $m_3 > 1$ and where a and b are specific combinations of drag and lift coefficients which are given explicitely in [1], and are of order 1. This means that (1.2.11) now becomes

$$\begin{aligned}
\bar{w}_{\bar{t}\bar{t}} + \bar{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} + \left(\frac{l}{\pi}\right)^4 \frac{\kappa_1}{EI}\bar{w} &= \frac{\rho_a d}{2\mu}\tilde{\epsilon}\left(a\bar{w}_{\bar{t}} - b\bar{w}_{\bar{t}}^3\right) \\
&+ \frac{1}{4}\frac{A}{I}\tilde{\epsilon}^2\left(\bar{u}(\pi,\bar{t}) - \bar{u}(0,\bar{t}) + \frac{2}{\pi}\int_0^\pi \bar{w}_{\bar{x}}^2 d\bar{x}\right)\bar{w}_{\bar{x}\bar{x}} - \frac{\kappa_2}{EI}\left(\frac{l}{\pi}\right)^4\tilde{\epsilon}\bar{w}^2 \\
&+ \frac{1}{4}\frac{A}{I}\mathcal{O}(\tilde{\epsilon}^{m_1}) - \frac{\kappa_2}{EI}\left(\frac{l}{\pi}\right)^4\mathcal{O}(\tilde{\epsilon}^{m_2}) + \frac{\rho_a d}{2\mu}\mathcal{O}(\tilde{\epsilon}^{m_3}),
\end{aligned} \tag{1.2.12}$$

with $m_1 > 2$, $m_2 > 1$ and $m_3 > 1$. Using the transformation $\hat{w} = \sqrt{\frac{3b}{a}}\bar{w}$ and $\hat{u} = \frac{3b}{a}\bar{u}$ (1.2.12) becomes

$$\begin{aligned}
\hat{w}_{\bar{t}\bar{t}} + \hat{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} + \left(\frac{l}{\pi}\right)^4 \frac{\kappa_1}{EI}\hat{w} &= \frac{\rho_a d}{2\mu}a\tilde{\epsilon}\left(\hat{w}_{\bar{t}} - \frac{1}{3}\hat{w}_{\bar{t}}^3\right) \\
&+ \frac{1}{4}\frac{A}{I}\frac{a}{3b}\tilde{\epsilon}^2\left(\hat{u}(\pi,\bar{t}) - \hat{u}(0,\bar{t}) + \frac{2}{\pi}\int_0^\pi \hat{w}_{\bar{x}}^2 d\bar{x}\right)\hat{w}_{\bar{x}\bar{x}} - \frac{\kappa_2}{EI}\left(\frac{l}{\pi}\right)^4\sqrt{\frac{a}{3b}}\tilde{\epsilon}\hat{w}^2
\end{aligned}$$

$$+\sqrt{\frac{3b}{a}}\left(\frac{1}{4}\frac{A}{I}\mathcal{O}(\tilde{\epsilon}^{m_1})-\frac{\kappa_2}{EI}\left(\frac{l}{\pi}\right)^4\mathcal{O}(\tilde{\epsilon}^{m_2})+\frac{\rho_a d}{2\mu}\mathcal{O}(\tilde{\epsilon}^{m_3})\right), \quad (1.2.13)$$

with $m_1 > 2$, $m_2 > 1$ and $m_3 > 1$. Taking

$$\epsilon_1 = \frac{\rho_a d}{2\mu}a\tilde{\epsilon} = \frac{\rho_a d}{2\mu}\frac{v_\infty}{c}\frac{l}{\pi}a, \quad (1.2.14)$$

$$\epsilon_2 = \frac{1}{4}\frac{A}{I}\frac{a}{3b}\tilde{\epsilon}^2 = \frac{1}{4}\frac{A}{I}\left(\frac{v_\infty}{c}\frac{l}{\pi}\right)^2\frac{3b}{a}, \quad (1.2.15)$$

$$\epsilon_3 = -\frac{\kappa_2}{EI}\left(\frac{l}{\pi}\right)^4\sqrt{\frac{a}{3b}}\tilde{\epsilon} = -\frac{\kappa_2}{EI}\left(\frac{l}{\pi}\right)^4\sqrt{\frac{a}{3b}}\frac{v_\infty}{c}\frac{l}{\pi}, \quad (1.2.16)$$

(1.2.13) becomes

$$\begin{aligned}\hat{w}_{\bar{t}\bar{t}} + \hat{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} + p^2\hat{w} &= \epsilon_1\left(\hat{w}_{\bar{t}} - \frac{1}{3}\hat{w}_{\bar{t}}^3\right)\\ &+\epsilon_2\left(\hat{u}(\pi,\bar{t}) - \hat{u}(0,\bar{t}) + \frac{2}{\pi}\int_0^\pi \hat{w}_{\bar{x}}^2 d\bar{x}\right)\hat{w}_{\bar{x}\bar{x}} + \epsilon_3\hat{w}^2\\ &+\epsilon_1 o(1) + \epsilon_2 o(1) + \epsilon_3 o(1),\end{aligned}$$

with $p^2 = \left(\frac{l}{\pi}\right)^4\frac{k}{EI}$ of order 1. Four different cases for $\epsilon_1, \epsilon_2, \epsilon_3$ given by (1.2.14)-(1.2.16) are considered in this thesis. The case $\epsilon_2 \ll \epsilon_1$ and $\epsilon_3 \ll \epsilon_1$ is considered in Chapter 2 (up to $\mathcal{O}(\epsilon_1^n)$, $n > 1$). The case $\epsilon_1 \ll \epsilon_3$ and $\epsilon_2 \ll \epsilon_3$ is considered in Chapter 3 (up to $\mathcal{O}(\epsilon_3^n)$, $n > 1$), using a slightly different scaling of w. The case $\epsilon_1 \ll \epsilon_2$ and $\epsilon_3 \ll \epsilon_2$ is considered in Chapter 4 (up to $\mathcal{O}(\epsilon_2^n)$, $n > 1$), using a slightly different scaling of w. The case $\epsilon_2 \approx \epsilon_1$ and $\epsilon_3 \ll \epsilon_1$ is considered in Chapter 5 (up to $\mathcal{O}(\epsilon_1^n)$, $n > 1$).

1.3 An analytical approximation method

In this thesis we consider the following initial-boundary value problem, which describes, up to $\mathcal{O}(\epsilon^n)$, $n > 1$, the vertical displacement of an elastic beam with certain external forces acting on the beam:

$$w_{tt} + w_{xxxx} + p^2 w = \epsilon\mathcal{F}(x,t,w,w_t,w_{xx}), \quad (1.3.1)$$

$$w(0,t) = w(\pi,t) = 0, \qquad t \geq 0, \quad (1.3.2)$$

$$w_{xx}(0,t) = w_{xx}(\pi,t) = 0, \qquad t \geq 0, \quad (1.3.3)$$

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \qquad 0 < x < \pi, \quad (1.3.4)$$

where different cases are considered for $\mathcal{F}$, as we saw in the previous paragraph. For all cases formal approximations, i.e. functions that satisfy the differential equation and the initial-boundary values up to some order in

ϵ, of solutions of (1.3.1)-(1.3.4) are constructed. When straightforward ϵ-expansions are used to approximate solutions, secular terms can occur in the approximations. To avoid these secular terms a two time-scales perturbation method is used. The initial-boundary value problem (1.3.1)-(1.3.4) is extended to an initial value problem by extending all functions in x. The boundary conditions imply that w should be extended as an odd, 2π-periodic function in x, i.e. we write w as a Fourier sine-series in x

$$w(x,t) = \sum_{m=1}^{\infty} q_m(t)\sin(mx). \tag{1.3.5}$$

This extension implies that all terms in (1.3.1) should be extended as odd, 2π-periodic functions in x. In this thesis we consider four different cases for $\mathcal{F}$

$$\begin{aligned}
&(i) && \mathcal{F} = w_t - 1/3w_t^3,\\
&(ii) && \mathcal{F} = w^2,\\
&(iii) && \mathcal{F} = (u(\pi,t) - u(0,t) + 2/\pi\int_0^\pi w_x^2 dx)w_{xx},\\
&(iv) && \mathcal{F} = w_t - 1/3w_t^3 + \delta(u(\pi,t) - u(0,t) + 2/\pi\int_0^\pi w_x^2 dx)w_{xx},
\end{aligned}$$

where δ is a constant of order 1. For the cases (i), (iii) and (iv) the extension in x is straightforward, but for case (ii) the nonlinearity on the right hand side of (1.3.1) has to be rewritten as follows:

$$w_{tt} + w_{xxxx} + p^2 w = \epsilon h(x) w^2, \tag{1.3.6}$$

where the function h, defined on $\mathbb{R}$, is given by $h(x) = 1$ for $0 < x < \pi$, $h(0) = h(\pi) = 0$, and h is odd and 2π-periodic in x. The function $h(x)$ can then be written as a Fourier sine-series

$$h(x) = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\sin((2j+1)x)}{2j+1}. \tag{1.3.7}$$

We emphasize that these quadratic nonlinearities are usually not treated correctly in the literature.

As stated above, terms that give rise to secular terms can occur in the approximations. To eliminate these terms we introduce two timescales, $t_0 = t$ and $t_1 = \epsilon t$, and assume that q_n can be expanded in a formal power series in ϵ, that is, $q_n(t) = q_{n,0}(t_0,t_1) + \epsilon q_{n,1}(t_0,t_1) + \epsilon^2 q_{n,2}(t_0,t_1) + \cdots$. Equations for $q_{n,0}, q_{n,1}, \ldots$ are derived and can be solved one by one.

In the literature, for example in [27], similar systems are analysed using muliple scale methods or averaging methods. In this thesis, however, we

choose to use the method of multiple scales, because in our opinion this method is more efficient and more pleasant to work with.
As stated above, we construct formal approximations of solutions of the initial-boundary value problem (1.3.1)-(1.3.4). In Chapter 3 an asymptotic theory is presented for a restricted case of (1.3.1)-(1.3.4), where $\mathcal{F} = f(x,t,w;\epsilon)$ and where the initial values (1.3.4) can also depend on ϵ. We will show that this initial-boundary value problem is well-posed in the classical sense, i.e. we will show that there exists a unique classical solution for this initial-boundary value problem. Also the asymptotic validity of the constructed approximations will be given for this case, which means that in Chapter 3 an order ϵ approximation for the solution of (1.3.1)-(1.3.4) is constructed on a timescale of order $1/\epsilon$.

Chapter 2

A Weakly Nonlinear Beam Equation with a Rayleigh Perturbation†

Abstract In this chapter an initial-boundary value problem for a weakly nonlinear beam equation with a Rayleigh perturbation will be studied. It will be shown that the calculations to find internal resonances in this case are much more complicated than and differ substantially from the calculations for the weakly nonlinear wave equation with a Rayleigh perturbation as for instance presented in [1] or [26]. The initial-boundary value problem can be regarded as a simple model describing wind-induced oscillations of flexible structures like suspension bridges or iced overhead transmission lines. Using a two time-scales perturbation method approximations for solutions of this initial-boundary problem will be constructed.

2.1 Introduction

Flexible structures, like tall buildings, suspension bridges or iced overhead transmission lines with bending stiffness, are subjected to oscillations due to wind forces or other causes (e.g. earthquakes). Simple models which describe these oscillations can involve nonlinear second and fourth order PDE's, as can be seen for example in [1] or [9]. In most cases asymptotic methods can be used to construct approximations for solutions of this type of equations. Initial-boundary value problems for second order PDE's have been considered for a long time, for instance in [26]-[29] and [33]. These problems have been studied in [1]-[3],[31] and [32], using a two time-scales

†This chapter is a revised version of [40] *On Mode Interactions for a Weakly Nonlinear Beam Equation*, Nonlinear Dynamics, 17 (1998), pp. 23 40

perturbation method or a Galerkin-averaging method to construct approximations. For example, in [1] an asymptotic theory for second order PDE's is presented for models describing the growth of wind-induced oscillations of iced overhead transmission lines. For fourth order PDE's the analysis is more complex. In a number of papers ([9]-[11] and [43]) approximations for solutions of initial-boundary value problems for fourth order PDE's are constructed using perturbation methods. In most cases single mode representations are used, without justification. In this chapter approximations are constructed using a two time-scales perturbation method and a justification is given in which cases mode truncation is valid.

In [9] a survey of literature on oscillations in suspension bridges is given. We will derive a simple model for nonlinear oscillations in suspension bridges using Equation (18) from [9], with some modifications. We consider the suspension bridge to be a beam of length l. The vertical displacement is $w = w(x,t)$. The stays of the bridge are treated as two-sided springs, as sketched in Figure 2.1. We introduce the following symbols: $\mu = \rho A$ the

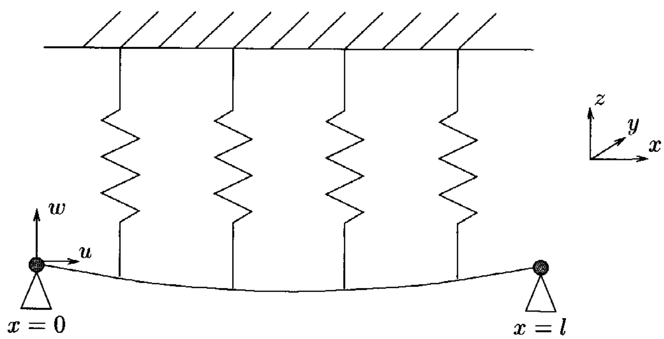

Figure 2.1: A simple model of a suspension bridge.

density of the beam per unit length, ρ the density of the beam, A the area of the cross-section of the beam perpendicular to the x-axis, E the elasticity modulus, I the axial momentum of inertia of the cross-section and k the spring constant of the stays of the bridge. We neglect internal damping and consider the weight W of the bridge per unit length to be constant ($W = \mu g$, g is the gravitational acceleration). We consider a uniform wind flow in y-direction, which causes nonlinear drag and lift forces (F_D, F_L) to act on the structure per unit length. The equation describing the vertical displacement of the beam is

$$\mu w_{tt} + EIw_{xxxx} + kw = -\mu g + F_D + F_L. \tag{2.1.1}$$

In Section 4 of [1] it is shown that F_D and F_L can be approximated by

$$\frac{\rho_{\mathrm{a}} d v_\infty^2}{2}\left(a_0 + \frac{a_1}{v_\infty} w_t + \frac{a_2}{v_\infty^2} w_t^2 + \frac{a_3}{v_\infty^3} w_t^3\right), \tag{2.1.2}$$

where ρ_a is the density of air, d is the diameter of the cross section of the beam, v_∞ is the uniform wind flow velocity in y-direction and a_0, a_1, a_2, a_3 depend on certain drag and lift coefficients and are given explicitly in [1]. Equation (2.1.1) will be simplified by eliminating the term $-g$ using $w = \tilde{w} + \frac{\mu g}{k} s(x)$, where $s(x)$ satisfies the following time-independent linear equation with boundary conditions:

$$
\begin{aligned}
&s^{(4)}(x) + \frac{k}{EI} s(x) = -\frac{k}{EI}, \qquad 0 < x < l, \\
&s(0) = s(l) = 0, \qquad s^{(2)}(0) = s^{(2)}(l) = 0.
\end{aligned}
$$

It can be shown that $s(x) = \cos(\alpha x)\cosh(\alpha x) + (\sin(\alpha l)\sin(\alpha x)\cosh(\alpha x) - \sinh(\alpha l)\cos(\alpha x)\sinh(\alpha x))/(\cos(\alpha l) + \cosh(\alpha l)) - 1$, with $\alpha^4 = \frac{k}{EI}$. The term $\frac{\mu g}{k} s(x)$ represents the deflection of the beam in static state due to gravity. We substitute $w = \tilde{w} + \frac{\mu g}{k} s(x)$ into (2.1.1), divide by μ, use (2.1.2) and introduce the dimension-less variables, $\bar{x} = \frac{\pi}{l} x$, $\bar{t} = \frac{\pi}{l} ct$, $\bar{w} = \frac{\pi}{l} \frac{c}{v_\infty} \tilde{w}$, with $c = \frac{\pi}{l}\sqrt{\frac{EI}{\mu}}$. Equation (2.1.1) now becomes

$$
\bar{w}_{\bar{t}\bar{t}} + \bar{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} + \frac{k}{EI}\left(\frac{l}{\pi}\right)^4 \bar{w} = \frac{\rho_a d l}{2\pi\mu}\frac{v_\infty}{c}\left(a_0 + a_1 \bar{w}_{\bar{t}} + a_2 \bar{w}_{\bar{t}}^2 + a_3 \bar{w}_{\bar{t}}^3\right). \tag{2.1.3}
$$

Assuming v_∞, the uniform wind velocity, is small with respect to $\frac{\pi}{l} c$, we put $\tilde{\epsilon} = \frac{v_\infty}{c}\frac{l}{\pi}$, with $\tilde{\epsilon}$ a small parameter. In Section 4 of [1] it is shown that the right hand side of (2.1.3) up to order $\tilde{\epsilon}$ is equal to $\frac{\rho_a d}{2\mu}\tilde{\epsilon}\left(a\bar{w}_{\bar{t}} - b\bar{w}_{\bar{t}}^3\right)$, where a, b are specific combinations of drag and lift coefficients which are given explicitly in [1]. Using the transformation $\hat{w} = \sqrt{\frac{3b}{a}}\bar{w}$ (2.1.3) becomes $\hat{w}_{\bar{t}\bar{t}} + \hat{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} + p^2 \hat{w} = \epsilon\left(\hat{w}_{\bar{t}} - \frac{1}{3}\hat{w}_{\bar{t}}^3\right) + \mathcal{O}(\epsilon^n)$, $n > 1$, where $p^2 = \left(\frac{l}{\pi}\right)^4 \frac{k}{EI}$ and $\epsilon = \frac{\rho_a d}{2\mu} a\tilde{\epsilon}$ a small dimension-less parameter.

We can now introduce the following initial-boundary value problem, which describes up to $\mathcal{O}(\epsilon^n)$, $n > 1$, the vertical displacement of a beam with a uniform wind flow (which causes nonlinear forces) acting on it:

$$
\begin{aligned}
w_{tt} + w_{xxxx} + p^2 w &= \epsilon\left(w_t - \frac{1}{3}w_t^3\right), && 0 < x < \pi,\ t > 0, && (2.1.4)\\
w(0,t) = w(\pi,t) &= 0, && t \geq 0, && (2.1.5)\\
w_{xx}(0,t) = w_{xx}(\pi,t) &= 0, && t \geq 0, && (2.1.6)\\
w(x,0) = w_0(x), \quad w_t(x,0) &= w_1(x), && 0 < x < \pi, && (2.1.7)
\end{aligned}
$$

where ϵ and p are constants with $0 < |\epsilon| \ll 1$ and $0 < p^2 < 10$, $w = w(x,t)$ is the vertical displacement of the beam, x is the coordinate along the beam, $w_0(x)$ is the initial displacement of the beam in vertical direction and $w_1(x)$ is the initial velocity of the beam in vertical direction. All functions are assumed to be sufficiently smooth. The first two terms on the left hand side of the PDE (2.1.4) are the linear part of the beam equation, p^2w represents a linear restoring force and the right hand side is a so-called Rayleigh perturbation, which describes an external (wind)force (see [1]). The boundary conditions describe a simply supported beam. As we showed above, the initial-boundary problem (2.1.4)-(2.1.7) can be considered as a simple model for nonlinear oscillations in suspension bridges. In this chapter formal approximations, i.e. functions that satisfy the differential equation and the initial and boundary values up to some order in ϵ, will be constructed for the initial-boundary value problem (2.1.4)-(2.1.7), using a Fourier mode expansion and a two time-scales perturbation method. The interaction between the different modes will be considered. It will be shown that for some specific values of p^2 'extra' interactions occur that cause complicated internal resonances. Resonances of this type are for example analyzed in [2] and [36]. For these p^2-values truncation to one or two modes, as given in [10] or [11], is not valid in all cases. We will show that for instance for $p^2 = 9$ energy transfer occurs between modes 1 and 3, and for $p^2 = \frac{693}{152}$ energy transfer occurs between the first four modes. To our knowledge these mode interactions for weakly nonlinear beam equations have not yet been studied thoroughly.

The outline of this chapter is as follows. In Section 2.2 a two time-scales perturbation method is applied to the initial-boundary value problem (2.1.4)-(2.1.7). We show that for most p^2-values mode interactions occur only between modes with non-zero initial energy (up to $\mathcal{O}(\epsilon)$). For some specific values, $p^2 = \frac{693}{152}$ and $p^2 = 9$, also modes with zero initial energy are excited. In Section 2.3 we construct formal approximations for solutions of the initial-boundary value problem for the cases $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$ and $p^2 = 9$. To understand the transition from $p^2 \neq 9$ to $p^2 = 9$ we also study the detuning case $p^2 = 9 + \epsilon\alpha$ with $\alpha \in \mathbb{R}$. In Section 2.4 we make some general remarks on these results and related literature: the surprising new mode interactions, the difference between the results for the beam equation with a Rayleigh perturbation and the wave equation with a Rayleigh perturbation, and the validity of the Galerkin-truncation of modes for approximations of solutions.

2.2 The Perturbation Method

In the next two sections we construct formal approximations for solutions of the initial-boundary value problem (2.1.4)-(2.1.7). When straightforward ϵ-expansions are used to approximate solutions, secular terms may occur in the approximations. To avoid these secular terms we use a two time-scales

perturbation method.
The boundary conditions imply that w can be written as a Fourier sine-series in x, i.e., $w(x,t) = \sum_{m=1}^{\infty} q_m(t)\sin(mx)$. Substituting this series into (2.1.4), we obtain the following system of equations:

$$\sum_{m=1}^{\infty}\left(\ddot{q}_m + (m^4+p^2)q_m\right)\sin(mx) = \epsilon\left(\sum_{m=1}^{\infty}\dot{q}_m\sin(mx) - \frac{1}{3}\sum_{m,k,j=1}^{\infty}\dot{q}_m\dot{q}_k\dot{q}_j\sin(mx)\sin(kx)\sin(jx)\right). \tag{2.2.1}$$

In the appendix we show that the equation for each q_n then is

$$\ddot{q}_n + (n^4+p^2)q_n = \epsilon\left[\dot{q}_n - \frac{1}{4}\left(\sum_{n=m+k-j} - \sum_{n=-m-k+j} - \frac{1}{3}\sum_{n=m+k+j}\right)\dot{q}_m\dot{q}_k\dot{q}_j\right], \tag{2.2.2}$$

for n=1,2,3,... . q_n must satisfy the initial conditions given by (2.1.7) $q_n(0) = \frac{2}{\pi}\int_0^{\pi} w_0(x)\sin(nx)dx$, $\dot{q}_n(0) = \frac{2}{\pi}\int_0^{\pi} w_1(x)\sin(nx)dx$. We introduce the two time-scales, $t_0 = t$ and $t_1 = \epsilon t$, and assume that q_n can be expanded in a formal power-series in ϵ, i.e., $q_n(t) = q_{n,0}(t_0,t_1) + \epsilon q_{n,1}(t_0,t_1) + \epsilon^2 q_{n,2}(t_0,t_1) + \cdots$. We substitute this into (2.2.2) and collect equal powers in ϵ. The $\mathcal{O}(\epsilon^0)$-problem becomes

$$\frac{\partial^2}{\partial t_0^2}q_{n,0} + \omega_{n_p}^2 q_{n,0} = 0,\ t > 0, \tag{2.2.3}$$

$$q_{n,0}(0,0) = \frac{2}{\pi}\int_0^{\pi} w_0(x)\sin(nx)dx, \tag{2.2.4}$$

$$\frac{\partial}{\partial t_0}q_{n,0}(0,0) = \frac{2}{\pi}\int_0^{\pi} w_1(x)\sin(nx)dx, \tag{2.2.5}$$

for $n = 1,2,3,\ldots$. The general solution for (2.2.3)-(2.2.5) is

$$q_{n,0}(t_0,t_1) = A_{n,0}(t_1)\cos(\omega_{n_p}t_0) + B_{n,0}(t_1)\sin(\omega_{n_p}t_0), \tag{2.2.6}$$

where $\omega_{n_p} = (n^4+p^2)^{\frac{1}{2}}$, and $A_{n,0}, B_{n,0}$ satisfy the initial conditions

$$A_{n,0}(0) = q_{n,0}(0,0),\ B_{n,0}(0) = \frac{1}{\omega_{n_p}}\frac{\partial}{\partial t_0}q_{n,0}(0,0).$$

Next we consider the $\mathcal{O}(\epsilon^1)$-problem

$$\frac{\partial^2}{\partial t_0^2}q_{n,1} + \omega_{n_p}^2 q_{n,1} = -2\frac{\partial^2}{\partial t_0\partial t_1}q_{n,0} + \frac{\partial}{\partial t_0}q_{n,0} \tag{2.2.7}$$

$$-\frac{1}{4}\left(\sum_{n=m+k-j} - \sum_{n=-m-k+j} -\frac{1}{3}\sum_{n=m+k+j}\right)\frac{\partial}{\partial t_0}q_{m,0}\frac{\partial}{\partial t_0}q_{k,0}\frac{\partial}{\partial t_0}q_{j,0},$$

$$q_{n,1}(0,0)=0, \qquad \frac{\partial}{\partial t_0}q_{n,1}(0,0)=-\frac{\partial}{\partial t_1}q_{n,0}(0,0), \tag{2.2.8}$$

for $n=1,2,3,\ldots$. We substitute (2.2.6) into (2.2.7) and get

$$\frac{\partial^2}{\partial t_0^2}q_{n,1}+\omega_{n_p}^2 q_{n,1}=2\omega_{n_p}\left(\frac{dA_{n,0}}{dt_1}\sin(\omega_{n_p}t_0)-\frac{dB_{n,0}}{dt_1}\cos(\omega_{n_p}t_0)\right) \tag{2.2.9}$$

$$+H_n-\frac{1}{4}\left(\sum_{n=m+k-j} - \sum_{n=-m-k+j} -\frac{1}{3}\sum_{n=m+k+j}\right)H_mH_kH_j,$$

with $H_l{=}\omega_{l_p}(B_{l,0}\cos(\omega_{l_p}t_0){-}A_{l,0}\sin(\omega_{l_p}t_0))$. Since $\cos(\omega_{n_p}t_0)$ and $\sin(\omega_{n_p}t_0)$ are part of the homogeneous solution of (2.2.9), we want the coefficients of $\cos(\omega_{n_p}t_0)$ and $\sin(\omega_{n_p}t_0)$ on the right hand side of (2.2.9) to be equal to zero (elimination of secular terms). This gives us equations for $A_{n,0}$ and $B_{n,0}$. Finding solutions for $A_{n,0}, B_{n,0}$ and thus for $q_{n,0}$ gives us a zero order approximation of the exact solution of the initial-boundary value problem (2.1.4)-(2.1.7). In the appendix we show that in order to find the equations for $A_{n,0}, B_{n,0}$ we have to determine the secular terms in (2.2.9), by solving the Diophantine-like equations

$$\begin{cases} n=m+k-j \ \vee \ n=-m-k+j \ \vee \ n=m+k+j, \\ \pm(n^4+p^2)^{\frac{1}{2}}=\pm(m^4+p^2)^{\frac{1}{2}}\pm(k^4+p^2)^{\frac{1}{2}}\pm(j^4+p^2)^{\frac{1}{2}}. \end{cases} \tag{2.2.10}$$

Only specific combinations of m,k,j will give solutions to (2.2.10). In the appendix we show that for some values of $p^2\in]0,10[$ there are solutions to (2.2.10), which give additional contributions to the equations for $A_{n,0}, B_{n,0}$. These values are $p^2=\frac{693}{152}$ and $p^2=9$. The case $p^2=0$ also has solutions which give additional contributions but will not be considered in this chapter.

For $p^2\in]0,10[\setminus\{\frac{693}{152},9\}$, it is shown in the appendix ((2.5.11)-(2.5.12)) that the contributions are similar to the wave-equation case, described in [26]. Introducing $\overline{A}_{n,0}=\omega_{n_p}A_{n,0}$ and $\overline{B}_{n,0}=\omega_{n_p}B_{n,0}$ we get the following equations for $\overline{A}_{n,0},\overline{B}_{n,0}$:

$$\frac{d\overline{A}_{n,0}}{dt_1}=\frac{1}{2}\overline{A}_{n,0}\left(1+\frac{1}{16}\left(\overline{A}_{n,0}^2+\overline{B}_{n,0}^2\right)-\frac{1}{4}\sum_{k=1}^{\infty}\left(\overline{A}_{k,0}^2+\overline{B}_{k,0}^2\right)\right), \tag{2.2.11}$$

$$\frac{d\overline{B}_{n,0}}{dt_1}=\frac{1}{2}\overline{B}_{n,0}\left(1+\frac{1}{16}\left(\overline{A}_{n,0}^2+\overline{B}_{n,0}^2\right)-\frac{1}{4}\sum_{k=1}^{\infty}\left(\overline{A}_{k,0}^2+\overline{B}_{k,0}^2\right)\right), \tag{2.2.12}$$

for $n=1,2,3,\ldots$. From (2.2.11)-(2.2.12) we can see that if $\overline{A}_{n,0}(0)=\overline{B}_{n,0}(0)=0$ then $\overline{A}_{n,0}(t_1)=\overline{B}_{n,0}(t_1)=0\ \forall t_1>0$. So if we start with zero initial energy in the nth mode there will be no energy present up to $\mathcal{O}(\epsilon)$. We say the coupling between the modes is of $\mathcal{O}(\epsilon)$. This allows truncation to those modes that have non-zero initial energy. We will discuss (2.2.11)-(2.2.12) in more detail in Section 2.3.1.

For $p^2 = \frac{693}{152}$ extra contributions in the equations for $A_{n,0}$ and $B_{n,0}$ occur for $n = 1, 2, 3, 4$. Introducing $\overline{A}_{n,0} = \omega_{n_p} A_{n,0}$ and $\overline{B}_{n,0} = \omega_{n_p} B_{n,0}$ we get the following equations for $\overline{A}_{n,0}, \overline{B}_{n,0}$:

$$\begin{aligned} \frac{d\overline{A}_{n,0}}{dt_1} &= \frac{1}{2}\overline{A}_{n,0}\left(1 + \frac{1}{16}(\overline{A}^2_{n,0} + \overline{B}^2_{n,0}) - \frac{1}{4}\sum_{k=1}^{\infty}(\overline{A}^2_{k,0} + \overline{B}^2_{k,0})\right) \\ &\quad - \frac{1}{32}\mathcal{F}_n(\overline{A}_{1,0}, \overline{B}_{1,0}, \overline{A}_{2,0}, \overline{B}_{2,0}, \overline{A}_{3,0}, \overline{B}_{3,0}, \overline{A}_{4,0}, \overline{B}_{4,0}), \qquad (2.2.13) \\ \frac{d\overline{B}_{n,0}}{dt_1} &= \frac{1}{2}\overline{B}_{n,0}\left(1 + \frac{1}{16}(\overline{A}^2_{n,0} + \overline{B}^2_{n,0}) - \frac{1}{4}\sum_{k=1}^{\infty}(\overline{A}^2_{k,0} + \overline{B}^2_{k,0})\right) \\ &\quad + \frac{1}{32}\mathcal{G}_n(\overline{A}_{1,0}, \overline{B}_{1,0}, \overline{A}_{2,0}, \overline{B}_{2,0}, \overline{A}_{3,0}, \overline{B}_{3,0}, \overline{A}_{4,0}, \overline{B}_{4,0}), \qquad (2.2.14) \end{aligned}$$

for $n \leq 4$. $\mathcal{F}_n, \mathcal{G}_n$ are given explicitly for $n = 1, 2, 3, 4$ in the appendix ((2.5.13)). For $n \geq 5$ $\overline{A}_{n,0}$ and $\overline{B}_{n,0}$ have to satisfy (2.2.11)-(2.2.12). In this case there is an $\mathcal{O}(1)$ coupling between the modes $1, 2, 3, 4$. For instance, if there is initial energy present in the first three modes, then in general energy will be transfered to the fourth mode. In this case truncation to three modes (or less) is not valid. All four modes have to be taken into account. We will not discuss these equations in more detail.

For $p^2 = 9$ extra contributions occur in the equations for $A_{n,0}$ and $B_{n,0}$ for $n = 1, 3$. Introducing $\overline{A}_{n,0} = \omega_{n_p} A_{n,0}$ and $\overline{B}_{n,0} = \omega_{n_p} B_{n,0}$ we get the following equations for $n = 1, 3$:

$$\begin{aligned} \frac{d\overline{A}_{1,0}}{dt_1} &= \frac{1}{2}\overline{A}_{1,0}\left(1 - \frac{3}{16}(\overline{A}^2_{1,0} + \overline{B}^2_{1,0}) - \frac{1}{4}(\overline{A}^2_{3,0} + \overline{B}^2_{3,0})\right) \qquad (2.2.15) \\ &\quad - \frac{1}{8}\overline{A}_{1,0}\sum_{k\neq 1,3}\left(\overline{A}^2_{k,0} + \overline{B}^2_{k,0}\right) - \frac{1}{32}\left((\overline{A}^2_{1,0} - \overline{B}^2_{1,0})\overline{A}_{3,0} + 2\overline{A}_{1,0}\overline{B}_{1,0}\overline{B}_{3,0}\right), \\ \frac{d\overline{B}_{1,0}}{dt_1} &= \frac{1}{2}\overline{B}_{1,0}\left(1 - \frac{3}{16}(\overline{A}^2_{1,0} + \overline{B}^2_{1,0}) - \frac{1}{4}(\overline{A}^2_{3,0} + \overline{B}^2_{3,0})\right) \qquad (2.2.16) \\ &\quad - \frac{1}{8}\overline{B}_{1,0}\sum_{k\neq 1,3}\left(\overline{A}^2_{k,0} + \overline{B}^2_{k,0}\right) + \frac{1}{32}\left(2\overline{A}_{1,0}\overline{B}_{1,0}\overline{A}_{3,0} + (\overline{B}^2_{1,0} - \overline{A}^2_{1,0})\overline{B}_{3,0}\right), \\ \frac{d\overline{A}_{3,0}}{dt_1} &= \frac{1}{2}\overline{A}_{3,0}\left(1 - \frac{1}{4}(\overline{A}^2_{1,0} + \overline{B}^2_{1,0}) - \frac{3}{16}(\overline{A}^2_{3,0} + \overline{B}^2_{3,0})\right) \qquad (2.2.17) \\ &\quad - \frac{1}{8}\overline{A}_{3,0}\sum_{k\neq 1,3}\left(\overline{A}^2_{k,0} + \overline{B}^2_{k,0}\right) - \frac{1}{96}\overline{A}_{1,0}\left(\overline{A}^2_{1,0} - 3\overline{B}^2_{1,0}\right), \end{aligned}$$

$$\frac{d\overline{B}_{3,0}}{dt_1} = \frac{1}{2}\overline{B}_{3,0}\left(1 - \frac{1}{4}(\overline{A}_{1,0}^2 + \overline{B}_{1,0}^2) - \frac{3}{16}(\overline{A}_{3,0}^2 + \overline{B}_{3,0}^2)\right) \tag{2.2.18}$$
$$-\frac{1}{8}\overline{B}_{3,0}\sum_{k\neq 1,3}\left(\overline{A}_{k,0}^2 + \overline{B}_{k,0}^2\right) + \frac{1}{96}\overline{B}_{1,0}\left(\overline{B}_{1,0}^2 - 3\overline{A}_{1,0}^2\right).$$

For $n = 2, 4, 5, \ldots$ $\overline{A}_{n,0}$ and $\overline{B}_{n,0}$ have to satisfy (2.2.11)-(2.2.12). In this case there is an $\mathcal{O}(1)$ coupling between the modes 1 and 3 which indicates an internal resonance between these modes. This means that if there is initial energy present in mode 1 an energy transfer occurs between modes 1 and 3. Truncation to one mode is not valid, both mode 1 and 3 have to be taken into account. We will discuss (2.2.15)-(2.2.18) in more detail in Section 2.3.2. As we saw above for $p^2 \in]0, 10[$ internal resonances between certain modes occur for $p^2 = \frac{693}{152}$ or $p^2 = 9$. Truncation to a few modes is not valid, unless all modes concerned have zero initial energy. For $p^2 \neq \frac{693}{152}$ and $p^2 \neq 9$ the results found are similar to the results given in [26] for the nonlinear wave equation. In the next section we will review these results briefly in order to understand the detuning from the case $p^2 = 9$ better. Since for the case $p^2 = 9$ the equations for modes 1 and 3 are examined in more detail, we will also discuss the equations for modes 1 and 3 explicitly in the case $p^2 \neq 9$. In this chapter we will only consider the case $p^2 = 9$ with detuning since this case seems to be less complicated than the case $p^2 = \frac{693}{152}$. For $p^2 \geq 10$ other internal resonances can be found in a similar way for specific values of p^2.

2.3 Modal Interaction

2.3.1 The case $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$

As stated above we only consider the equations for $n = 1$ and $n = 3$. We introduce polar coordinates to transform (2.2.11)-(2.2.12)

$$\overline{A}_{n,0} = r_n\cos(\phi_n)\ ,\quad \overline{B}_{n,0} = r_n\sin(\phi_n)\ , \tag{2.3.1}$$

with $r_n = r_n(t_1)$ the amplitude and $\phi_n = \phi_n(t_1)$ the phase of the oscillation and get the following equations for r_1, r_3, ϕ_1, ϕ_3:

$$\dot{r}_1 = \frac{1}{2}r_1(1 - \frac{3}{16}r_1^2 - \frac{1}{4}r_3^2), \tag{2.3.2}$$
$$\dot{r}_3 = \frac{1}{2}r_3(1 - \frac{1}{4}r_1^2 - \frac{3}{16}r_3^2), \tag{2.3.3}$$
$$r_1^2\dot{\phi}_1 = 0, \qquad r_3^2\dot{\phi}_3 = 0, \tag{2.3.4}$$

where a dot represents differentiation with respect to t_1. We can analyse (2.3.2)-(2.3.4) in the (r_1, r_3)-plane. Elementary analysis gives us the behaviour of the solutions of (2.3.2)-(2.3.4) in the (r_1, r_3)-phase plane as shown in Figure 2.2. We see that depending on the initial values the system

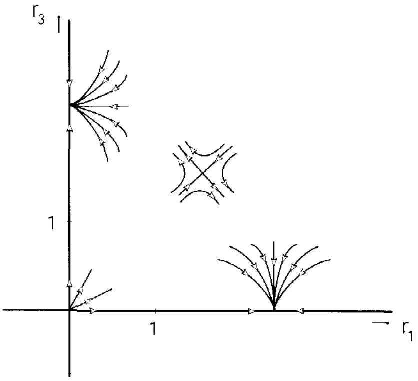

Figure 2.2: Phase plane for $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$.

tends (as $t \to \infty$) to oscillate in only one mode. In the next subsection we will consider the case $p^2 = 9$. We introduce polar coordinates for each mode. The analysis of the equations for $\overline{A}_{1,0}$, $\overline{A}_{3,0}$, $\overline{B}_{1,0}$, $\overline{B}_{3,0}$ can be brought back to an analysis in a 3-dimensional space with r_1, r_3 and $\psi = \phi_3 - 3\phi_1$ as variables. Looking at the above case in this 3-dimensional space will give us the same behaviour as drawn in Figure 2.2 for every ψ-value. We will then have lines of critical points in the 3-dimensional space.

2.3.2 The case $p^2 = 9$

In Section 2.2 (2.2.15)-(2.2.18) were given. Introducing polar coordinates for each mode we transform these equations using (2.3.1)

$$\dot{r}_1 = \frac{1}{2}r_1(1 - \frac{3}{16}r_1^2 - \frac{1}{4}r_3^2) - \frac{1}{32}r_1^2 r_3 \cos(\phi_3 - 3\phi_1), \tag{2.3.5}$$

$$\dot{r}_3 = \frac{1}{2}r_3(1 - \frac{1}{4}r_1^2 - \frac{3}{16}r_3^2) - \frac{1}{96}r_1^3 \cos(\phi_3 - 3\phi_1), \tag{2.3.6}$$

$$r_1^2\dot{\phi}_1 = -\frac{1}{32}r_1^3 r_3 \sin(\phi_3 - 3\phi_1), \tag{2.3.7}$$

$$r_3^2\dot{\phi}_3 = \frac{1}{96}r_1^3 r_3 \sin(\phi_3 - 3\phi_1). \tag{2.3.8}$$

$\dot{r}_1, \dot{r}_3$ represent the changes of energy in the different modes and $r_1^2\dot{\phi}_1$, $r_3^2\dot{\phi}_3$ represent the changes of angular momentum, where a dot represents differentiation with respect to t_1. We can analyse (2.3.5)-(2.3.8) in the (r_1, r_3, ψ)-space, with $\psi = \phi_3 - 3\phi_1$

$$\dot{r}_1 = \frac{1}{2}r_1(1 - \frac{3}{16}r_1^2 - \frac{1}{4}r_3^2) - \frac{1}{32}r_1^2 r_3 \cos(\psi), \tag{2.3.9}$$

$$\dot{r}_3 = \frac{1}{2}r_3(1-\frac{1}{4}r_1^2-\frac{3}{16}r_3^2)-\frac{1}{96}r_1^3\cos(\psi), \tag{2.3.10}$$

$$\dot{\psi} = \frac{1}{96}\frac{r_1}{r_3}\left(r_1^2+9r_3^2\right)\sin(\psi). \tag{2.3.11}$$

For $r_3 = 0$ (no energy present in mode 3) (2.3.9)-(2.3.11) do not hold. In that case we have to analyse the original differential equations (2.2.15)-(2.2.18). We analyse (2.3.9)-(2.3.11) in the (r_1, r_3, ψ) phase space. We find two lines of critical points and three isolated critical points for $\psi \in [0, 2\pi[$ (compared to 4 lines of critical points in the case $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$). The critical points are listed in Table 2.1. The behavior of the solutions of

Critical point	Behavior
$(0, 0, \psi)$	unstable $2d$-node for each ψ
$(0, \sqrt{\frac{16}{3}}, \psi)$	stable $2d$-node for each ψ
$(1.5930, 1.2625, 0)$	$3d$-saddle-node
$(1.2249, 1.8555, \pi)$	$3d$-saddle-node
$(2.2977, 0.63794, \pi)$	stable $3d$-node

Table 2.1: Critical points for $p^2 = 9$.

(2.3.9)-(2.3.11) in the (r_1, r_3, ψ)-phase space is sketched in Figure 2.3. The phase space is a $3d$-projection of the behavior of the $4d$-solutions of (2.2.15)-(2.2.18). The system is 2π-periodic in ψ. In the plane $\psi = 0$ there is a stable

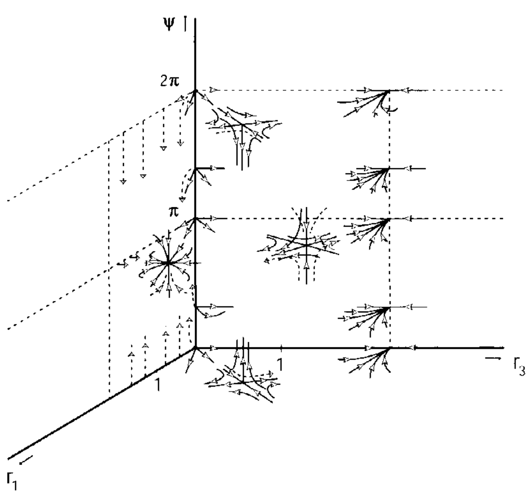

Figure 2.3: Phase space for $p^2 = 9$.

node $(0,\sqrt{\frac{16}{3}},0)$ and a saddle (1.593,1.262,0). The r_1-axis seems to attract, but as soon as a solution curve intersects this axis, a phase jump occurs to the plane $\psi = \pi$, imposed by the differential equations (2.2.15)-(2.2.18). The line $(0,0,\psi)$ is a line of unstable singular points. The line $(0,\sqrt{\frac{16}{3}},\psi)$ is a line of stable singular points, which is structurally unstable, as can be seen in Section 2.3.3 (for $\alpha \neq 0$ the line of singular points disappears). The system tends ($t \to \infty$) to oscillate in the third mode with a phase depending on the initial values, or in a combined oscillation of the first and third mode with a phase difference π.

2.3.3 The case $p^2 = 9 + \epsilon\alpha$, $\alpha \neq 0$

To understand the detuning from $p^2 = 9$ we consider $p^2 = 9 + \epsilon\alpha$ with $\epsilon \ll 1$ and $\alpha \neq 0$ of $\mathcal{O}(1)$. The equations for $\overline{A}_{1,0}, \overline{B}_{1,0}, \overline{A}_{3,0}, \overline{B}_{3,0}$ are

$$\begin{aligned}
\frac{d\overline{A}_{1,0}}{dt_1} &= \frac{1}{2}\overline{A}_{1,0}\left(1 - \frac{3}{16}(\overline{A}^2_{1,0} + \overline{B}^2_{1,0}) - \frac{1}{4}(\overline{A}^2_{3,0} + \overline{B}^2_{3,0})\right) \\
&\quad + \frac{\alpha}{2\sqrt{10}}\,\overline{B}_{1,0} + \frac{1}{32}\left((\overline{B}^2_{1,0} - \overline{A}^2_{1,0})\overline{A}_{3,0} - 2\overline{A}_{1,0}\overline{B}_{1,0}\overline{B}_{3,0}\right), \quad (2.3.12) \\
\frac{d\overline{B}_{1,0}}{dt_1} &= \frac{1}{2}\overline{B}_{1,0}\left(1 - \frac{3}{16}(\overline{A}^2_{1,0} + \overline{B}^2_{1,0}) - \frac{1}{4}(\overline{A}^2_{3,0} + \overline{B}^2_{3,0})\right) \\
&\quad - \frac{\alpha}{2\sqrt{10}}\,\overline{A}_{1,0} + \frac{1}{32}\left(2\overline{A}_{1,0}\overline{B}_{1,0}\overline{A}_{3,0} + (\overline{B}^2_{1,0} - \overline{A}^2_{1,0})\overline{B}_{3,0}\right), \quad (2.3.13) \\
\frac{d\overline{A}_{3,0}}{dt_1} &= \frac{1}{2}\overline{A}_{3,0}\left(1 - \frac{1}{4}(\overline{A}^2_{1,0} + \overline{B}^2_{1,0}) - \frac{3}{16}(\overline{A}^2_{3,0} + \overline{B}^2_{3,0})\right) \\
&\quad + \frac{\alpha}{6\sqrt{10}}\,\overline{B}_{3,0} + \frac{1}{96}\overline{A}_{1,0}\left(3\overline{B}^2_{1,0} - \overline{A}^2_{1,0}\right), \quad (2.3.14) \\
\frac{d\overline{B}_{3,0}}{dt_1} &= \frac{1}{2}\overline{B}_{3,0}\left(1 - \frac{1}{4}(\overline{A}^2_{1,0} + \overline{B}^2_{1,0}) - \frac{3}{16}(\overline{A}^2_{3,0} + \overline{B}^2_{3,0})\right) \\
&\quad - \frac{\alpha}{6\sqrt{10}}\,\overline{A}_{3,0} + \frac{1}{96}\overline{B}_{1,0}\left(\overline{B}^2_{1,0} - 3\overline{A}^2_{1,0}\right). \quad (2.3.15)
\end{aligned}$$

For $n = 2, 4, 5, \ldots$ (2.2.11) and (2.2.12) (with an extra α-term as above) hold. We transform (2.3.12)-(2.3.15) by introducing polar coordinates, using (2.3.1)

$$\dot{r}_1 = \frac{1}{2}r_1(1 - \frac{3}{16}r_1^2 - \frac{1}{4}r_3^2) - \frac{1}{32}r_1^2 r_3\cos(\phi_3 - 3\phi_1), \quad (2.3.16)$$

$$\dot{r}_3 = \frac{1}{2}r_3(1 - \frac{1}{4}r_1^2 - \frac{3}{16}r_3^2) - \frac{1}{96}r_1^3\cos(\phi_3 - 3\phi_1), \quad (2.3.17)$$

$$r_1^2\dot{\phi}_1 = -\frac{1}{2}\frac{\alpha}{\sqrt{10}}r_1^2 - \frac{1}{32}r_1^3 r_3\sin(\phi_3 - 3\phi_1), \quad (2.3.18)$$

$$r_3^2\dot{\phi}_3 = -\frac{1}{6}\frac{\alpha}{\sqrt{10}}r_3^2 + \frac{1}{96}r_1^3 r_3\sin(\phi_3 - 3\phi_1), \quad (2.3.19)$$

where the dot represents differentiation with respect to t_1. We can analyze (2.3.16)-(2.3.19) in the (r_1, r_3, ψ)-space, with $\psi = \phi_3 - 3\phi_1$

$$\dot{r}_1 = \frac{1}{2}r_1(1 - \frac{3}{16}r_1^2 - \frac{1}{4}r_3^2) - \frac{1}{32}r_1^2 r_3 \cos(\psi), \tag{2.3.20}$$

$$\dot{r}_3 = \frac{1}{2}r_3(1 - \frac{1}{4}r_1^2 - \frac{3}{16}r_3^2) - \frac{1}{96}r_1^3 \cos(\psi), \tag{2.3.21}$$

$$\dot{\psi} = \frac{4}{3}\frac{\alpha}{\sqrt{10}} + \frac{1}{96}\frac{r_1}{r_3}\left(r_1^2 + 9r_3^2\right)\sin(\psi). \tag{2.3.22}$$

For $r_3 = 0$ (2.3.20)-(2.3.22) do not hold. In that case we have to analyze the original differential equations (2.3.12)-(2.3.15). We examine the (r_1, r_3, ψ)-phase space for several values of $\alpha > 0$. For $\alpha < 0$ a similar analysis holds. Numerical analysis indicates that for $\alpha \geq \frac{1}{10}$ the effects of the α-terms in (2.3.20)-(2.3.22) are significant. We start with $\alpha = \frac{1}{10}$. The critical points are listed in Table 2.2. The behavior of the solutions of (2.3.20)-(2.3.22)

Critical point	Behavior
$(0, 0, \psi)$	unstable $2d$-node for each ψ
$(1.5928, 1.2658, 6.0923)$	$3d$-saddle-node
$(1.2312, 1.8499, 3.3309)$	$3d$-saddle-node
$(2.2976, 0.63438, 3.2675)$	stable $3d$-node

Table 2.2: Critical points for $p^2 = 9 + \epsilon\alpha$ with $\alpha = \frac{1}{10}$.

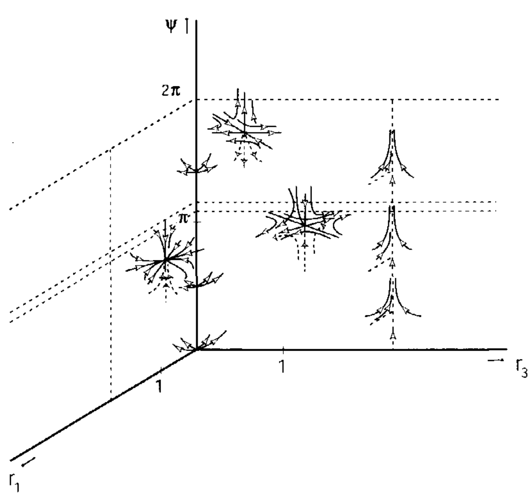

Figure 2.4: Phase space for $p^2 = 9 + \epsilon\alpha$ with $\alpha = \frac{1}{10}$.

for $\alpha = \frac{1}{10}$ in the (r_1, r_3, ψ)-phase space is sketched in Figure 2.4. For a point on the line $(0,\sqrt{\frac{16}{3}},\psi)$ we have $\dot{r}_1 = \dot{r}_3 = 0$ and $\dot{\psi} > 0$. This line is no longer a line of singular points but is a line which attracts in the r_1, r_3-direction and with movement in ψ-direction. We see that compared to the phase space for $p^2 = 9$ the $3d$-saddle-node in the plane $\psi = 2\pi$ moves downwards and the $3d$-saddle-node in the plane $\psi = \pi$ upwards. It can be shown that the stable $3d$-node changes from to a $3d$-spiral-node for some value of $\alpha > 0.1$. With increasing α we see that the two $3d$-saddle-nodes move closer towards each other. The $3d$-spiral-node moves upwards and towards the plane $r_3 = 0$. The line $(0,\sqrt{\frac{16}{3}},\psi)$ is still an attracting line with movement along it. The line $(0,0,\psi)$ remains a line of singular points. The two $3d$-saddle-nodes coincide for $\alpha \approx 0.5649$ and vanish for $\alpha > 0.5649$. The critical points for $\alpha \approx 0.5649$ are listed in Table 2.3. We find the behavior of the solutions of (2.3.20)-(2.3.22) for $\alpha \approx 0.5649$

Critical point	Behavior
$(0, 0, \psi)$	unstable $2d$-node for each ψ
$(1.5073, 1.5200, 4.6874)$	higher order singularity
$(2.2971, 0.51108, 3.8717)$	$3d$-spiral-node

Table 2.3: Critical points for $p^2 = 9 + \epsilon\alpha$ with $\alpha \approx 0.5649$.

in the (r_1, r_3, ψ)-phase space as sketched in Figure 2.5. For $\alpha > 0.5649$ a separating surface remains. The $3d$-spiral-node moves further upwards and

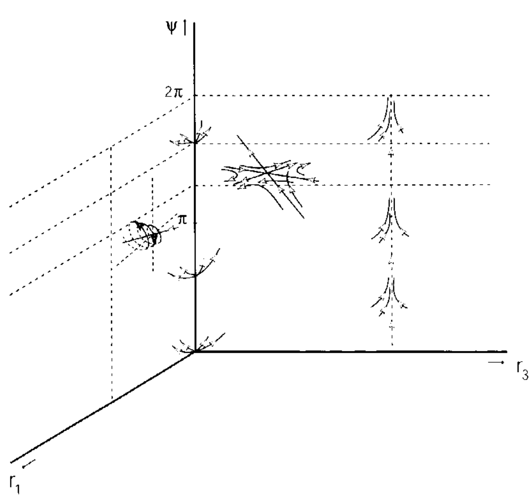

Figure 2.5: Phase space for $p^2 = 9 + \epsilon\alpha$ with $\alpha \approx 0.5649$.

towards the plane $r_3 = 0$, and the line $(0,\sqrt{\frac{16}{3}},\psi)$ remains an attracting line with movement along it. For $\alpha = 1$ the critical points are listed in Table 2.4. The behavior of the solutions of (2.3.20)-(2.3.22) for $\alpha = 1$ in the (r_1, r_3, ψ)-phase space is sketched in Figure 2.6.

All the phases paces in this section are constructed by numerical integration of the differential equations.

Critical point	Behavior
$(0, 0, \psi)$	unstable $2d$-node for each ψ
$(2.3029, 0.31905, 4.2651)$	$3d$-spiral-node

Table 2.4: Critical points for $p^2 = 9 + \epsilon\alpha$ with $\alpha = 1$.

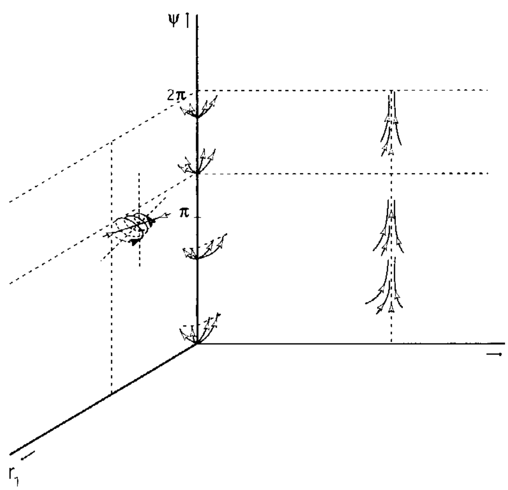

Figure 2.6: Phase space for $p^2 = 9 + \epsilon\alpha$ with $\alpha = 1$.

2.4 Conclusions

In this chapter we considered an initial-boundary value problem for a weakly nonlinear beam equation. We have constructed formal approximations of order ϵ and considered the interaction between different oscillation modes. We showed that for most p^2-values the behavior of solutions of the Rayleigh beam equation is similar to that of solutions of the Rayleigh wave equation as presented in [26]. In that case mode interactions occur only between modes with non-zero initial energy (up to $\mathcal{O}(\epsilon)$). We then say the coupling between the modes is of $\mathcal{O}(\epsilon)$, and truncation is allowed, restricted to those modes that have non-zero initial energy. We showed that for large times the system will oscillate in only one mode up to $\mathcal{O}(\epsilon)$. However, for some p^2-values extra mode interactions occur, which cause complicated internal resonances. Physically this means that in some cases (depending on the value of p^2, which depends on the spring constant k of the stays of the bridge and on certain properties of the beam), if the beam initially oscillates in a high vibration mode lower vibration modes can be excited, that can cause large amplitude oscillations of the suspension bridge. For $0 < p^2 < 10$ these values are $p^2 = 9$ and $p^2 = \frac{693}{152}$. For $p^2 \geq 10$ other internal resonances can be found in a similar way for special values of p^2. For $p^2 = 9$ the equations have been studied in detail including the detuning case $p^2 = 9 + \epsilon\alpha$. Energy transfer occurs between modes 1 and 3 even if mode 3 initially has zero energy. We call this coupling of $\mathcal{O}(1)$. Truncation to one mode will give loss of information, and approximations will not be valid. Examining the behavior of the oscillations, we showed that a new equilibrium state arises, namely a combined oscillation of modes 1 and 3. The analysis performed in this chapter can be extended to other p^2-values. In this chapter we constructed formal approximations that satisfy the PDE and the initial conditions up to $\mathcal{O}(\epsilon^2)$. In [1] an asymptotic theory is presented for a wave equation with similar nonlinearities. The formal approximations constructed for that problem were shown to be asymptotically valid, i.e. the difference between the approximation and the exact solution is of order ϵ on a time-scale of $1/\epsilon$ as $\epsilon \downarrow 0$. It is beyond the scope of this chapter to give the asymptotic analysis for the beam equation we discussed. We expect that the asymptotic validity of the constructed approximations can be shown in a way similar to the analysis presented in [1].

2.5 Appendix - Determination and Elimination of Secular Terms

In this appendix we construct equations for the functions q_n such that no secular terms occur in the approximations for the displacement function $w(x,t)$. In Section 2.2 the system of equations (2.2.1) is given

$$\sum_{m=1}^{\infty}\left(\ddot{q}_m + (m^4 + p^2)q_m\right)\sin(mx) = \tag{2.5.1}$$
$$\epsilon\left(\sum_{m=1}^{\infty}\dot{q}_m\sin(mx) - \frac{1}{3}\sum_{m,k,j=1}^{\infty}\dot{q}_m\dot{q}_k\dot{q}_j\sin(mx)\sin(kx)\sin(jx)\right).$$

The last summation on the right hand side of (2.5.1) can be rewritten using the goniometric formula $\sin(mx)\sin(kx)\sin(jx) = \frac{1}{4}(\sin((m+k-j)x) -$

$\sin((m-k-j)x) - \sin((m+j+k)x) + \sin((m-k+j)x))$.
We obtain the equations for q_n by multiplying (2.5.1) with $\frac{2}{\pi}\sin(nx)$ and then by integrating the so-obtained equation with respect to x from 0 to π. Using orthogonality relations and the symmetry in m, k, j we obtain the following equation for each q_n:

$$\ddot{q}_n + (n^4+p^2)q_n = \tag{2.5.2}$$
$$\epsilon\left[\dot{q}_n - \frac{1}{4}\left(\sum_{n=m+k-j} - \sum_{n=-m-k+j} - \frac{1}{3}\sum_{n=m+k+j}\right)\dot{q}_m\dot{q}_k\dot{q}_j\right],$$

for $n = 1, 2, 3, \ldots$. To avoid secular terms in $q_n(t)$ a two time-scales perturbation method is introduced in Section 2.2 and $q_n(t)$ is expanded in $q_n(t) = q_{n,0}(t_0,t_1) + \epsilon q_{n,1}(t_0,t_1) + \epsilon^2 q_{n,2}(t_0,t_1) + \cdots$, where $t_0 = t$ and $t_1 = \epsilon t$. It has been shown in Section 2.2 that $q_{n,1}$ has to satisfy (2.2.9)

$$\frac{\partial^2}{\partial t_0^2}q_{n,1} + \omega_{n_p}^2 q_{n,1} = 2\omega_{n_p}\left(\frac{dA_{n,0}}{dt_1}\sin(\omega_{n_p}t_0) - \frac{dB_{n,0}}{dt_1}\cos(\omega_{n_p}t_0)\right) \tag{2.5.3}$$
$$+H_n - \frac{1}{4}\left(\sum_{n=m+k-j} - \sum_{n=-m-k+j} - \frac{1}{3}\sum_{n=m+k+j}\right)H_mH_kH_j,$$

with $H_l = \omega_{l_p}\left(B_{l,0}\cos(\omega_{l_p}t_0) - A_{l,0}\sin(\omega_{l_p}t_0)\right)$ and $\omega_{n_p} = (n^4+p^2)^{\frac{1}{2}}$. $A_{n,0}$ and $B_{n,0}$ are still arbitrary functions in t_1. The functions $A_{n,0}$ and $B_{n,0}$ will now be determined such that no secular terms occur in $q_{n,1}$.
The term $H_mH_kH_j$ in (2.5.3) can be expanded using goniometric formula's

$$\begin{aligned} H_mH_kH_j = \frac{1}{4}\omega_{m_p}\omega_{k_p}\omega_{j_p}\,[& \\ & (-\mathcal{C}_1A_{j,0} + \mathcal{C}_2B_{j,0})\cos\left(\left(\omega_{m_p}+\omega_{k_p}+\omega_{j_p}\right)t_0\right) \\ & + (-\mathcal{C}_2A_{j,0} - \mathcal{C}_1B_{j,0})\sin\left(\left(\omega_{m_p}+\omega_{k_p}+\omega_{j_p}\right)t_0\right) \\ & + (\mathcal{C}_1A_{j,0} + \mathcal{C}_2B_{j,0})\cos\left(\left(\omega_{m_p}+\omega_{k_p}-\omega_{j_p}\right)t_0\right) \\ & + (\mathcal{C}_2A_{j,0} - \mathcal{C}_1B_{j,0})\sin\left(\left(\omega_{m_p}+\omega_{k_p}-\omega_{j_p}\right)t_0\right) \\ & + (\mathcal{C}_3A_{j,0} + \mathcal{C}_4B_{j,0})\cos\left(\left(\omega_{m_p}-\omega_{k_p}+\omega_{j_p}\right)t_0\right) \\ & + (-\mathcal{C}_4A_{j,0} + \mathcal{C}_3B_{j,0})\sin\left(\left(\omega_{m_p}-\omega_{k_p}+\omega_{j_p}\right)t_0\right) \\ & + (-\mathcal{C}_3A_{j,0} + \mathcal{C}_4B_{j,0})\cos\left(\left(\omega_{m_p}-\omega_{k_p}-\omega_{j_p}\right)t_0\right) \\ & + (\mathcal{C}_4A_{j,0} + \mathcal{C}_3B_{j,0})\sin\left(\left(\omega_{m_p}-\omega_{k_p}-\omega_{j_p}\right)t_0\right)\,], \end{aligned} \tag{2.5.4}$$

with

$$\mathcal{C}_1 = A_{m,0}B_{k,0} + B_{m,0}A_{k,0}, \qquad \mathcal{C}_2 = -A_{m,0}A_{k,0} + B_{m,0}B_{k,0},$$
$$\mathcal{C}_3 = -A_{m,0}B_{k,0} + B_{m,0}A_{k,0}, \qquad \mathcal{C}_4 = A_{m,0}A_{k,0} + B_{m,0}B_{k,0}.$$

As stated in Section 2.2, $\cos(\omega_{n_p}t_0)$ and $\sin(\omega_{n_p}t_0)$ are part of the homogeneous solution of $q_{n,1}$. We want the coefficients of $\cos(\omega_{n_p}t_0)$ and $\sin(\omega_{n_p}t_0)$ on the right hand side of (2.5.3) to be equal to zero in order to eliminate secular terms. The terms in $H_mH_kH_j$ given in (2.5.4) can cause secular terms if $\pm(n^4+p^2)^{\frac{1}{2}} = \pm(m^4+p^2)^{\frac{1}{2}} \pm (k^4+p^2)^{\frac{1}{2}} \pm (j^4+p^2)^{\frac{1}{2}}$. To determine the contribution of the summations in (2.5.3) to the coefficients of $\cos(\omega_{n_p}t_0)$ and $\sin(\omega_{n_p}t_0)$ on the right hand side of (2.5.3) we have to examine the following Diophantine-like equations:

$$I : \begin{cases} n = m+k+j, \\ \pm(n^4+p^2)^{\frac{1}{2}} = \pm(m^4+p^2)^{\frac{1}{2}} \pm (k^4+p^2)^{\frac{1}{2}} \pm (j^4+p^2)^{\frac{1}{2}}, \end{cases} \tag{2.5.5}$$

$$II : \begin{cases} n = -m-k+j, \\ \pm(n^4+p^2)^{\frac{1}{2}} = \pm(m^4+p^2)^{\frac{1}{2}} \pm (k^4+p^2)^{\frac{1}{2}} \pm (j^4+p^2)^{\frac{1}{2}}, \end{cases} \tag{2.5.6}$$

$$III : \begin{cases} n = m+k-j, \\ \pm(n^4+p^2)^{\frac{1}{2}} = \pm(m^4+p^2)^{\frac{1}{2}} \pm (k^4+p^2)^{\frac{1}{2}} \pm (j^4+p^2)^{\frac{1}{2}}, \end{cases} \tag{2.5.7}$$

where $p^2 \in]0,10[$ and $m,k,j \geq 1$. We want to find out which combinations have solutions, and thus give rise to secular terms. In the following we will use the inequality

$$x^2 < (x^4+p^2)^{\frac{1}{2}} \leq x^2 - a^2 + (a^4+p^2)^{\frac{1}{2}}, \tag{2.5.8}$$

for $x \geq a$. We will now determine the solutions of (2.5.5)-(2.5.7). We will show that there are some solutions that hold for all p^2-values and some solutions that hold only for certain values for p^2. We can see directly that (2.5.5)-(2.5.7) do not hold if $(n^4+p^2)^{\frac{1}{2}} = -(m^4+p^2)^{\frac{1}{2}} - (k^4+p^2)^{\frac{1}{2}} - (j^4+p^2)^{\frac{1}{2}}$. This leaves us with the following possibilities:

Case $I(i)$: $n = m+k+j$, $(n^4+p^2)^{\frac{1}{2}} = (m^4+p^2)^{\frac{1}{2}} + (k^4+p^2)^{\frac{1}{2}} + (j^4+p^2)^{\frac{1}{2}}$. From (2.5.8) we know that $n^2 < (n^4+p^2)^{\frac{1}{2}} \leq m^2+k^2+j^2-3+3(1+p^2)^{\frac{1}{2}}$, since $n,m,k,j \geq 1$. We also know $n^2 = m^2+k^2+j^2+2(mk+mj+kj)$. This gives us: $2(mk+mj+kj) < 3((1+p^2)^{\frac{1}{2}}-1)$. On the other hand $n^2 - 1 + (1+p^2)^{\frac{1}{2}} \geq (n^4+p^2)^{\frac{1}{2}} > m^2+k^2+j^2$, which gives us $2(mk+mj+kj) > -((1+p^2)^{\frac{1}{2}}-1)$. Let $\lambda = 2(mk+mj+kj)$. Then we have $n^2 = m^2+k^2+j^2+\lambda$, with

$$-((1+p^2)^{\frac{1}{2}}-1) < \lambda < 3((1+p^2)^{\frac{1}{2}}-1). \tag{2.5.9}$$

Since λ is even and $m,k,j \geq 1$ we have $\lambda \geq 6$. If $p^2 \leq 8$ then (2.5.9) does not hold, so there is no solution. For larger values of p^2, solutions can occur. If we take $m=k=j=1$, then $n=3$ and $(3^4+p^2)^{\frac{1}{2}} = 3(1+p^2)^{\frac{1}{2}}$, has a solution for $p^2 = 9$. For $p^2 \in]0,10[$ it follows from (2.5.9) that $\lambda < 3(\sqrt{11}-1) < 7$. On the other hand for $m \neq 1$, $k \neq 1$ or $j \neq 1$ we have

$\lambda \geq 10$. So for $p^2 \in]0, 10[$ there is only one solution: $m = k = j = 1$ with $p^2 = 9$ (it can be shown that more contributions occur for $p^2 > 10$).
The case $I(ii)$: $n = m+k+j$, $(n^4+p^2)^{\frac{1}{2}} = (m^4+p^2)^{\frac{1}{2}}+(k^4+p^2)^{\frac{1}{2}}-(j^4+p^2)^{\frac{1}{2}}$, has no solutions. This can be shown using a similar analysis as in case $I(i)$. Due to symmetry there are no contributions for all cases with one minus sign.
The case $I(iii)$: $n = m + k + j$, $(n^4 + p^2)^{\frac{1}{2}} = (m^4 + p^2)^{\frac{1}{2}} - (k^4 + p^2)^{\frac{1}{2}} - (j^4 + p^2)^{\frac{1}{2}}$, has no solutions. This can be shown using a similar analysis as in case $I(i)$. Due to symmetry there are no contributions for all cases with two minus signs.
The case II, $n = -m - k + j$, is the same as I, with n and j switched.
Case $III(i)$: $n = m+k-j$, $(n^4+p^2)^{\frac{1}{2}} = (m^4+p^2)^{\frac{1}{2}}+(k^4+p^2)^{\frac{1}{2}}+(j^4+p^2)^{\frac{1}{2}}$.
We know $n^2 = m^2 + k^2 + j^2 + 2(mk - mj - kj)$. We introduce $\lambda = 2(mk - mj - kj)$ and $N = n^2$, $M = m^2$, $K = k^2$, $J = j^2$. So $N = M + K + J + \lambda$, and

$$(N^2 + p^2)^{\frac{1}{2}} = (M^2 + p^2)^{\frac{1}{2}} + (K^2 + p^2)^{\frac{1}{2}} + (J^2 + p^2)^{\frac{1}{2}}. \qquad (2.5.10)$$

From (2.5.10) we see that $K < N$, so (proof similar to a proof in [26])

$$\begin{aligned} &\frac{1}{N + (N^2 + p^2)^{\frac{1}{2}}} < \frac{1}{K + (K^2 + p^2)^{\frac{1}{2}}} \\ &< \frac{1}{K + (K^2 + p^2)^{\frac{1}{2}}} + \frac{1}{M + (M^2 + p^2)^{\frac{1}{2}}} + \frac{1}{J + (J^2 + p^2)^{\frac{1}{2}}}, \\ \iff& \\ &(N^2 + p^2)^{\frac{1}{2}} < (M^2 + p^2)^{\frac{1}{2}} + (K^2 + p^2)^{\frac{1}{2}} + (J^2 + p^2)^{\frac{1}{2}} + \lambda. \end{aligned}$$

We see that when $\lambda \leq 0$ (2.5.10) cannot hold. We will show that for $\lambda > 0$ solutions can exist, depending on the value of p^2. In the same way as in case $I(i)$ we see that λ must satisfy $\lambda < 3((1 + p^2)^{\frac{1}{2}} - 1)$. We know λ is even and > 0, so $\lambda \geq 2$. With $p^2 < 10$ possible values of λ are $2, 4, 6$. We can show that it suffices to examine $\lambda = 2$. Due to symmetry in m, k we can assume $m \geq k$. For $m = k$, $\lambda = 2m(m - 2j)$, which means that (2.5.10) has no solutions for $\lambda < 8$ and $0 < p^2 < 10$. So we can consider $m > k$. From the definition of λ we have $m(k - j) = \frac{\lambda}{2} + kj > 0$, so $(k - j) > 0$, $k > j$. This means $m > k > j \geq 1$, and therefore $j \geq 1$, $k \geq 2$, $m \geq 3$ and $n \geq 4$. Using (2.5.8) we can improve the upper-bound of λ: $n^2 < (n^4+p^2)^{\frac{1}{2}} \leq m^2-9+(3^4+p^2)^{\frac{1}{2}}+k^2-4+(2^4+p^2)^{\frac{1}{2}}+j^2-1+(1+p^2)^{\frac{1}{2}}$, so $\lambda < (3^4+p^2)^{\frac{1}{2}} + (2^4+p^2)^{\frac{1}{2}} + (1+p^2)^{\frac{1}{2}} - 14$. For $p^2 < 10$ this means $\lambda < 3.95\ldots$. We now examine the case $\lambda = 2$. This means $mk - mj - kj = 1$. If $j = 1$ then $m = 1 + \frac{2}{k-1}$. The only possible solution is $m = 3$, $k = 2$. This can be continued for $j \geq 2$. It can be shown that for these solutions $p^2 > 10$. This means that for $p^2 < 10$ we have only one solution $j = 1$,

$k=2$, $m=3$, $n=4$, with $p^2=\frac{693}{152}$. Due to symmetry in m,k m and k can be switched. It can be shown that more solutions exist for $p^2>10$.
The case $III(ii)(a)$: $n=m+k-j$, $(n^4+p^2)^{\frac{1}{2}}=(m^4+p^2)^{\frac{1}{2}}+(k^4+p^2)^{\frac{1}{2}}-(j^4+p^2)^{\frac{1}{2}}$ has the obvious solutions $k=n\wedge j=m$. Due to symmetry in m,k m and k can be switched.
The case $III(ii)(b)$: $n=m+k-j$, $(n^4+p^2)^{\frac{1}{2}}=(m^4+p^2)^{\frac{1}{2}}-(k^4+p^2)^{\frac{1}{2}}+(j^4+p^2)^{\frac{1}{2}}$, has the obvious solutions $m=n\wedge j=k$.
The case $III(ii)(c)$: $n=m+k-j$, $(n^4+p^2)^{\frac{1}{2}}=-(m^4+p^2)^{\frac{1}{2}}+(k^4+p^2)^{\frac{1}{2}}+(j^4+p^2)^{\frac{1}{2}}$, is the same as $III(ii)(b)$ with m and k switched.
Using a similar analysis as in case $III(i)$ it can be shown that the solutions given in $III(ii)(a)-(c)$ are the only solutions.
The case $III(iii)(a)$: $n=m+k-j$, $(n^4+p^2)^{\frac{1}{2}}=-(m^4+p^2)^{\frac{1}{2}}-(k^4+p^2)^{\frac{1}{2}}+(j^4+p^2)^{\frac{1}{2}}$ is the same as $I(i)$ with n and j switched.
The case $III(iii)(b)$: $n=m+k-j$, $(n^4+p^2)^{\frac{1}{2}}=(m^4+p^2)^{\frac{1}{2}}-(k^4+p^2)^{\frac{1}{2}}-(j^4+p^2)^{\frac{1}{2}}$, is the same as $III(i)$ with n and m, k and j switched.
The case $III(iii)(c)$: $n=m+k-j$, $(n^4+p^2)^{\frac{1}{2}}=-(m^4+p^2)^{\frac{1}{2}}+(k^4+p^2)^{\frac{1}{2}}-(j^4+p^2)^{\frac{1}{2}}$, is the same as $III(iii)(b)$, with m and k switched.

We will now determine the equations for $A_{n,0},B_{n,0}$ in each case.
For $p^2\in]0,10[\backslash\{\frac{693}{152},9\}$ the only solutions of (2.5.5)-(2.5.7) are the solutions found in case $III(ii)$. By eliminating secular terms we get the following equations for $\overline{A}_{n,0},\overline{B}_{n,0}$ ($\overline{A}_{n,0}=\omega_{n_p}A_{n,0}$ and $\overline{B}_{n,0}=\omega_{n_p}B_{n,0}$):

$$\frac{d\overline{A}_{n,0}}{dt_1}=\frac{1}{2}\overline{A}_{n,0}\left(1+\frac{1}{16}\left(\overline{A}^2_{n,0}+\overline{B}^2_{n,0}\right)-\frac{1}{4}\sum_{k=1}^{\infty}\left(\overline{A}^2_{k,0}+\overline{B}^2_{k,0}\right)\right), \quad (2.5.11)$$
$$\frac{d\overline{B}_{n,0}}{dt_1}=\frac{1}{2}\overline{B}_{n,0}\left(1+\frac{1}{16}\left(\overline{A}^2_{n,0}+\overline{B}^2_{n,0}\right)-\frac{1}{4}\sum_{k=1}^{\infty}\left(\overline{A}^2_{k,0}+\overline{B}^2_{k,0}\right)\right), \quad (2.5.12)$$

for $n=1,2,3,\ldots$.
For $p^2=\frac{693}{152}$ the equations for $\overline{A}_{n,0},\overline{B}_{n,0}$, $n\le 4$, are

$$\frac{d\overline{A}_{n,0}}{dt_1}=\frac{1}{2}\overline{A}_{n,0}\left(1+\frac{1}{16}(\overline{A}^2_{n,0}+\overline{B}^2_{n,0})-\frac{1}{4}\sum_{k=1}^{\infty}(\overline{A}^2_{k,0}+\overline{B}^2_{k,0})\right)-\frac{1}{32}\mathcal{F}_n$$
$$\frac{d\overline{B}_{n,0}}{dt_1}=\frac{1}{2}\overline{B}_{n,0}\left(1+\frac{1}{16}(\overline{A}^2_{n,0}+\overline{B}^2_{n,0})-\frac{1}{4}\sum_{k=1}^{\infty}(\overline{A}^2_{k,0}+\overline{B}^2_{k,0})\right)+\frac{1}{32}\mathcal{G}_n$$

For $n\ge 5$ $\overline{A}_{n,0}$ and $\overline{B}_{n,0}$ have to satisfy (2.5.11)-(2.5.12). $\mathcal{F}_n$ and $\mathcal{G}_n$ satisfy

$$\mathcal{F}_1=2(-\overline{A}_{2,0}\overline{A}_{3,0}+\overline{B}_{2,0}\overline{B}_{3,0})\overline{A}_{4,0}-2(\overline{A}_{2,0}\overline{B}_{3,0}+\overline{B}_{2,0}\overline{A}_{3,0})\overline{B}_{4,0},$$
$$\mathcal{F}_2=-2(\overline{A}_{1,0}\overline{A}_{4,0}+\overline{B}_{1,0}\overline{B}_{4,0})\overline{A}_{3,0}+2(-\overline{A}_{1,0}\overline{B}_{4,0}+\overline{B}_{1,0}\overline{A}_{4,0})\overline{B}_{3,0},$$

$$
\begin{aligned}
\mathcal{F}_3 &= -2(\overline{A}_{1,0}\overline{A}_{4,0}+\overline{B}_{1,0}\overline{B}_{4,0})\overline{A}_{2,0}+2(-\overline{A}_{1,0}\overline{B}_{4,0}+\overline{B}_{1,0}\overline{A}_{4,0})\overline{B}_{2,0},\\
\mathcal{F}_4 &= 2(-\overline{A}_{2,0}\overline{A}_{3,0}+\overline{B}_{2,0}\overline{B}_{3,0})\overline{A}_{1,0}+2(\overline{A}_{2,0}\overline{B}_{3,0}+\overline{B}_{2,0}\overline{A}_{3,0})\overline{B}_{1,0},\\
\mathcal{G}_1 &= 2(\overline{A}_{2,0}\overline{B}_{3,0}+\overline{B}_{2,0}\overline{A}_{3,0})\overline{A}_{4,0}+2(-\overline{A}_{2,0}\overline{A}_{3,0}+\overline{B}_{2,0}\overline{B}_{3,0})\overline{B}_{4,0},\\
\mathcal{G}_2 &= 2(\overline{A}_{1,0}\overline{B}_{4,0}-\overline{B}_{1,0}\overline{A}_{4,0})\overline{A}_{3,0}-2(\overline{A}_{1,0}\overline{A}_{4,0}+\overline{B}_{1,0}\overline{B}_{4,0})\overline{B}_{3,0},\\
\mathcal{G}_3 &= 2(\overline{A}_{1,0}\overline{B}_{4,0}-\overline{B}_{1,0}\overline{A}_{4,0})\overline{A}_{2,0}-2(\overline{A}_{1,0}\overline{A}_{4,0}+\overline{B}_{1,0}\overline{B}_{4,0})\overline{B}_{2,0},\\
\mathcal{G}_4 &= 2(\overline{A}_{2,0}\overline{B}_{3,0}+\overline{B}_{2,0}\overline{A}_{3,0})\overline{A}_{1,0}-2(-\overline{A}_{2,0}\overline{A}_{3,0}+\overline{B}_{2,0}\overline{B}_{3,0})\overline{B}_{1,0}. \quad (2.5.13)
\end{aligned}
$$

For $p^2 = 9$ the equations for $\overline{A}_{n,0}$ and $\overline{B}_{n,0}$, $n = 1, 3$, are

$$
\begin{aligned}
\frac{d\overline{A}_{n,0}}{dt_1} &= \frac{1}{2}\overline{A}_{n,0}\left(1+\frac{1}{16}(\overline{A}_{n,0}^2+\overline{B}_{n,0}^2)-\frac{1}{4}\sum_{k=1}^{\infty}(\overline{A}_{k,0}^2+\overline{B}_{k,0}^2)\right)-\frac{1}{32}\mathcal{H}_n\\
\frac{d\overline{B}_{n,0}}{dt_1} &= \frac{1}{2}\overline{B}_{n,0}\left(1+\frac{1}{16}(\overline{A}_{n,0}^2+\overline{B}_{n,0}^2)-\frac{1}{4}\sum_{k=1}^{\infty}(\overline{A}_{k,0}^2+\overline{B}_{k,0}^2)\right)+\frac{1}{32}\mathcal{J}_n
\end{aligned}
$$

For $n = 2, 4, 5, \ldots$ $\overline{A}_{n,0}$ and $\overline{B}_{n,0}$ satisfy (2.5.11)-(2.5.12). $\mathcal{H}_n, \mathcal{J}_n$ satisfy

$$
\begin{aligned}
\mathcal{H}_1 &= (\overline{A}_{1,0}^2-\overline{B}_{1,0}^2)\overline{A}_{3,0}+2\overline{A}_{1,0}\overline{B}_{1,0}\overline{B}_{3,0}, \quad \mathcal{H}_3 = \frac{1}{3}\overline{A}_{1,0}(\overline{A}_{1,0}^2-3\overline{B}_{1,0}^2),\\
\mathcal{J}_1 &= 2\overline{A}_{1,0}\overline{B}_{1,0}\overline{A}_{3,0}+(\overline{B}_{1,0}^2-\overline{A}_{1,0}^2)\overline{B}_{3,0}, \quad \mathcal{J}_3 = \frac{1}{3}\overline{B}_{1,0}(\overline{B}_{1,0}^2-3\overline{A}_{1,0}^2).
\end{aligned}
$$

Chapter 3

An Asymptotic Theory for a Weakly Nonlinear Beam Equation with a Quadratic Perturbation†

Abstract In this chapter an initial-boundary value problem for a weakly nonlinear beam equation with a quadratic nonlinearity will be studied. The initial-boundary value problem can be regarded as a simple model describing free oscillations of flexible structures like suspension bridges. Using a two time-scales perturbation method an approximation for the solution of this initial-boundary problem will be constructed. For a class of initial-boundary value problems the existence and uniqueness of solutions and the asymptotic validity of approximations on a large time-scale are shown. It will be shown that for specific values of the beam parameters complicated internal resonances occur.

3.1 Introduction

Particular types of flexible structures, like tall buildings, suspension bridges, or iced overhead transmission lines with bending stiffness, can be subjected to oscillations due to various causes. Simple models which describe these oscillations are given in the form of nonlinear second- and fourth-order partial differential equations, as can be seen for example in [1] or [9]. Usually asymptotic methods can be used to construct approximations for

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†This chapter is a revised version of [41] *An Asymptotic Theory for a Weakly Nonlinear Beam Equation with a Quadratic Perturbation*, SIAM J. Appl. Math., 60 (2000), pp. 602-632

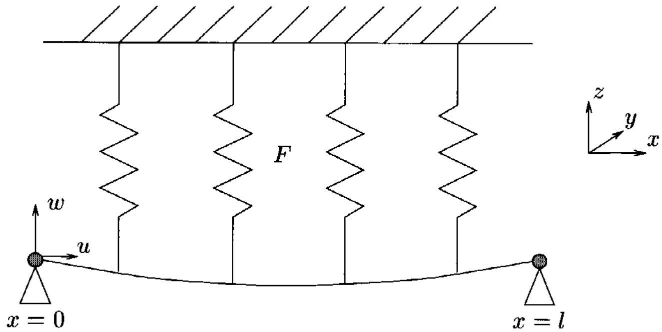


Figure 3.1: A simple model of a suspension bridge.

solutions of this type of second- or fourth-order partial differential equation. Initial-boundary value problems for nonlinear second-order partial differential equations have been studied already for a long time, for example, in [26]-[30], [33] and [34]. In [1]-[3], [31] and [32] these problems were studied using a two time-scales perturbation method or a Galerkin-averaging method to construct approximations. Asymptotic theories which support these approximations are presented in [1]-[3], [28], [29], [31] and [45]. For fourth-order partial differential equations the analysis becomes more complex. In [12], [23] and [44] the existence of periodic solutions for fourth-order partial differential equations is discussed, using an eigenvalue/eigenvector notation. In [9]-[11] and [43] approximations for solutions of initial-boundary value problems for fourth-order partial differential equations are constructed using perturbation methods. Little or nothing is said about the difference between the approximations and the exact solutions. In most cases single-mode representations are used, without giving a justification. To our knowledge an asymptotic theory, which shows the existence and uniqueness of solutions as well as the asymptotic validity of constructed approximations, has not yet been developed for fourth-order nonlinear partial differential equations. In this chapter an asymptotic theory is presented for an initial-boundary value problem for a weakly nonlinear beam equation, which can be seen as a simple model for free oscillations in suspension bridges. In [9] a survey of the literature on oscillations in suspension bridges is given. We will derive a simple model for nonlinear oscillations in suspension bridges, using model equation (18) from [9]. The equations of motion for a beam can also be found in standard textbooks such as [37] or [38]. We consider the suspension bridge to be modeled as a beam of length l. The vertical displacement is $w = w(x,t)$. The stays of the bridge are treated as *nonlinear* springs, as sketched in Figure 3.1. We introduce the following notation: μ is the mass of the beam per unit length, ρ the density of the beam, A the area of the cross-section of the beam perpendicular to the x-axis (so $\mu = \rho A$), E the elasticity modulus, I the moment of inertia of the cross-section, and F the force per unit length acting on the beam due to the springs. We neglect internal damping and consider the weight W of the bridge per unit length to be constant ($W = \mu g$, g is the gravitational acceleration). Since we are interested in free oscillations, no external forces are present. The equation describing the vertical displacement of the beam is

$$\mu w_{tt} + EIw_{xxxx} + F(w) = -\mu g. \tag{3.1.1}$$

We assume that the spring force F can be expanded in a Taylor series, with $F(0) = 0$: $F(w) = kw + bw^2 + \cdots$, where k and b are spring constants. We assume that the springs have a different behavior for compression and expansion, i.e., for $w < 0$ and $w > 0$. Since we consider "small" (compared to the length l) vertical displacements, we neglect all terms with order three and higher. Substituting this into (3.1.1) and dividing by μ gives us the following equation:

$$w_{tt} + \frac{EI}{\mu}w_{xxxx} + \frac{k}{\mu}w + \frac{b}{\mu}w^2 = -g. \tag{3.1.2}$$

We simplify (3.1.2) by eliminating the term $-g$ using $w = \tilde{w} + \frac{\mu g}{k}s(x)$, where $s(x)$ satisfies the following time-independent linear equation with boundary conditions:

$$\begin{aligned} &s^{(4)}(x) + \frac{k}{EI}s(x) = -\frac{k}{EI}, \qquad 0 < x < l, \\ &s(0) = s(l) = 0, \ s^{(2)}(0) = s^{(2)}(l) = 0. \end{aligned}$$

It can be shown that $s(x) = \cos(\beta x)\cosh(\beta x) + (\sin(\beta l)\sin(\beta x)\cosh(\beta x) - \sinh(\beta l)\cos(\beta x)\sinh(\beta x))/(\cos(\beta l) + \cosh(\beta l)) - 1$, with $\beta = \left(\frac{k}{4EI}\right)^{\frac{1}{4}}$. The term $\frac{\mu g}{k}s(x)$ represents the deflection of the beam in static state due to gravity. Using the dimension-less variables $\bar{w} = \frac{l}{A}\tilde{w}$, $\bar{x} = \frac{\pi}{l}x$, $\bar{t} = \left(\frac{\pi}{l}\right)^2\left(\frac{EI}{\mu}\right)^{\frac{1}{2}}t$, (3.1.2) becomes

$$\begin{aligned} &\bar{w}_{\bar{t}\bar{t}} + \bar{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} \\ &+\frac{l^4}{\pi^4 EI}\left(k\bar{w} + \frac{bA}{l}\bar{w}^2 + 2\frac{b\mu g}{k}s\left(\frac{l}{\pi}\bar{x}\right)\bar{w} + \frac{bl}{A}\left(\frac{\mu g}{k}s\left(\frac{l}{\pi}\bar{x}\right)\right)^2\right) = 0. \end{aligned} \tag{3.1.3}$$

Assuming that the area A of the cross-section is small compared to the length l, we put $\tilde{\epsilon} = \frac{A}{l}$ with $\tilde{\epsilon}$ a small parameter. Furthermore, we assume that the deflection of the beam in static state due to gravity, $\frac{\mu g}{k}s$, is small with respect to the vertical displacement $\tilde{w}$, which is of order $\tilde{\epsilon}$. This means we assume $\frac{\mu g}{k}s$ is $\mathcal{O}(\tilde{\epsilon}^n)$ with $n > 1$. Setting $\epsilon = -b\tilde{\epsilon}(\frac{l}{\pi})^4\frac{1}{EI}$, (3.1.3) becomes $\bar{w}_{\bar{t}\bar{t}} + \bar{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} + p^2\bar{w} = \epsilon\bar{w}^2 + \mathcal{O}(\epsilon^n)$, with $n > 1$, $p^2 = (\frac{l}{\pi})^4\frac{k}{EI}$, and ϵ a small dimension-less parameter. We can now formulate the following initial-boundary value problem, which describes, up to $\mathcal{O}(\epsilon^n)$, $n > 1$, the vertical displacement of a beam with a nonlinear spring force acting on it:

$$w_{tt} + w_{xxxx} + p^2 w = \epsilon w^2, \qquad 0 < x < \pi,\ t > 0, \tag{3.1.4}$$
$$w(0,t) = w(\pi,t) = 0, \qquad t \geq 0, \tag{3.1.5}$$
$$w_{xx}(0,t) = w_{xx}(\pi,t) = 0, \qquad t \geq 0, \tag{3.1.6}$$
$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \qquad 0 < x < \pi, \tag{3.1.7}$$

where ϵ and p are constants with $0 < |\epsilon| \ll 1$ and $p > 0$, $w = w(x,t)$ is the vertical displacement of the beam, x is the coordinate along the beam, $w_0(x)$ is the initial displacement of the beam in vertical direction, and $w_1(x)$ is the initial velocity of the beam in vertical direction. In this chapter we will assume that $p^2 < 100$. All functions are assumed to be sufficiently smooth. The first two terms on the left-hand side of (3.1.4) are the linear part of the beam equation and $p^2w - \epsilon w^2$ represents the quadratic restoring force of the spring. The boundary conditions describe a simply supported beam. Since there are no external forces considered, the initial-boundary value problem (3.1.4)-(3.1.7) can be considered as a simple model to describe free oscillations of suspension bridges, where the stays of the bridge are modeled as two-sided springs with a quadratic nonlinearity, which indicates a different spring behavior for $w < 0$ and $w > 0$.

In this chapter an asymptotic theory will be presented for a more general case of (3.1.4)-(3.1.7), where the nonlinearity (of the springs or otherwise) on the right-hand side of (3.1.4) is of the form $f(x,t,w;\epsilon)$ and where the initial values (3.1.7) can also depend on ϵ. We will show that this initial-boundary value problem is well-posed in the classical sense, i.e., we will show that there exists a unique classical solution for this initial-boundary value problem. We will also show the asymptotic validity of approximations, which are constructed using formal perturbation methods. We will construct an order ϵ approximation for the solution of (3.1.4)-(3.1.7), using a Fourier mode expansion and a two time-scales perturbation method. The interaction (energy exchange) between different oscillation modes will be considered for different values of the parameter p^2. For almost all values of p^2 only an interaction of order ϵ between different oscillation modes will occur on a time-scale of order ϵ^{-1}. It will be shown that only for certain values of p^2 mode interactions of order 1 will occur on an ϵ^{-1} time-scale, i.e., energy transfer of order 1 occurs between two or more modes on an ϵ^{-1} time-scale. For these p^2-values, truncation to one or two modes, as for instance performed in [10] or [11], is not valid in all cases. We will show

that, for instance, for $p^2 \approx \frac{17}{3}$ an energy transfer occurs between modes 2 and 3. For $p^2 \approx -91 + \frac{10}{3}\sqrt{2457}$ an energy transfer occurs between modes 1, 2, and 4, and for $p^2 \approx \frac{77}{3}$ an energy transfer occurs between modes 1, 2, 3, and 4. For these three critical values of p^2 we will examine the interactions for $p^2 = p_{cr}^2 + \epsilon\alpha$, with $\alpha = 0$ and $\alpha \neq 0$ (detuning). Resonances of this type have been studied in [36], but to our knowledge these mode interactions for weakly nonlinear beam equations have not yet been studied thoroughly.
The outline of this chapter is as follows. In Section 3.2 the well-posedness of a more general case of the initial-boundary value problem (3.1.4)-(3.1.7) is considered and established on a time-scale of order ϵ^{-1}. In Section 3.3 the asymptotic validity of approximations of solutions of this general initial-boundary value problem is studied. In Sections 3.4 and 3.5 the asymptotic theory is applied to the specific initial-boundary value problem (3.1.4)-(3.1.7). On a time-scale of order ϵ^{-1} an order ϵ asymptotic approximation, as $\epsilon \to 0$, for the solution of (3.1.4)-(3.1.7) is constructed using a two time-scales perturbation method. We will show that for most p^2-values no mode interactions occur (up to $\mathcal{O}(\epsilon)$). For some specific values of p^2, modes with zero initial energy are also excited. For three different values of p^2 these mode interactions are considered explicitly in Section 3.5. In Section 3.6 some conclusions and general remarks are given.

3.2 The well-posedness of the problem

In this section we consider the well-posedness in a classical sense of the following class of weakly nonlinear initial-boundary value problems for a real valued function $w(x,t)$:

$$w_{tt} + w_{xxxx} + p^2 w = \epsilon f(x,t,w;\epsilon), \qquad 0 < x < \pi,\ \ t > 0, \quad (3.2.1)$$

$$w(0,t) = w(\pi,t) = 0, \qquad t \geq 0, \quad (3.2.2)$$

$$w_{xx}(0,t) = w_{xx}(\pi,t) = 0, \qquad t \geq 0, \quad (3.2.3)$$

$$w(x,0) = w_0(x;\epsilon), \quad w_t(x,0) = w_1(x;\epsilon), \qquad 0 < x < \pi, \quad (3.2.4)$$

where ϵ, p are constants, $\epsilon \in [-\epsilon_0, \epsilon_0]$, and $p \geq 0$, and where f, w_0, w_1 satisfy

f and all first-, second-, and third-order partial derivatives of f with respect to x, t, w are $\in C([0,\pi] \times [0,\infty] \times \mathbb{R} \times [-\epsilon_0,\epsilon_0], \mathbb{R})$, and $f(0,t,0;\epsilon) = f(\pi,t,0;\epsilon) \equiv 0$ for $t \geq 0$, (3.2.5)

$$w_0, \frac{\partial w_0}{\partial x}, \frac{\partial^2 w_0}{\partial x^2}, \frac{\partial^3 w_0}{\partial x^3}, \frac{\partial^4 w_0}{\partial x^4}, w_1, \frac{\partial w_1}{\partial x}, \frac{\partial^2 w_1}{\partial x^2} \in C([0,\pi] \times [-\epsilon_0,\epsilon_0], \mathbb{R}),$$

with $w_0(0;\epsilon) = w_0(\pi;\epsilon) = \dfrac{\partial^2 w_0}{\partial x^2}(0;\epsilon) = \dfrac{\partial^2 w_0}{\partial x^2}(\pi;\epsilon) \equiv 0$ and

$$w_1(0;\epsilon) = w_1(\pi;\epsilon) = \frac{\partial^2 w_1}{\partial x^2}(0;\epsilon) = \frac{\partial^2 w_1}{\partial x^2}(\pi;\epsilon) \equiv 0, \tag{3.2.6}$$

f and all first-, second-, and third-order partial derivatives of f with respect to x, t, w are uniformly bounded for all x, t, ϵ considered. (3.2.7)

We define a classical solution as a function that is three times continuously differentiable on $[0,\pi] \times [0,\infty]$, for which the fourth partial derivative with respect to x is continuous on $[0,\pi] \times [0,\infty]$ and that satisfies (3.2.1)-(3.2.4), where f, w_0, w_1 satisfy (3.2.5)-(3.2.7). In order to prove existence and uniqueness of a classical solution of the initial-boundary value problem (3.2.1)-(3.2.4) an equivalent integral equation will be used. We obtain this equation using the Green's function G for the linear operator $\frac{\partial^4}{\partial x^4} + \frac{\partial^2}{\partial t^2} + p^2$ with simply supported boundary conditions (see Appendix A, (3.7.7)-(3.7.8)):

$$w(x,t) = \tag{3.2.8}$$
$$\epsilon \int_0^t \int_0^\pi G(\xi,\tau;x,t) f(\xi,\tau,w;\epsilon) d\xi d\tau + w_l(x,t;\epsilon) \equiv (Tw)(x,t),$$

where

$$G(\xi,\tau;x,t) = \tag{3.2.9}$$
$$\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4+p^2}} \sin\left[\sqrt{n^4+p^2}(t-\tau)\right] H(t-\tau) \sin n\xi \sin nx$$

for $\xi, x \in [0,\pi]$, $\tau, t \geq 0$, where the Heaviside function $H(a)$ is equal to 1 for $a > 0$ and equal to 0 for $a < 0$. It can be easily seen that G is uniformly bounded and continuous in x, t. The solution of the initial-boundary value problem (3.2.1)-(3.2.4) with $f \equiv 0$ is

$$w_l(x,t;\epsilon) = \int_0^\pi \{G(\xi,0;x,t)w_1(\xi;\epsilon) - G_\tau(\xi,0;x,t)w_0(\xi;\epsilon)\}\, d\xi. \tag{3.2.10}$$

It can be shown elementarily that the integral equation (3.2.8) and the initial-boundary value problem (3.2.1)-(3.2.4) are equivalent when three times continuously differentiable functions with a continuous fourth derivative with respect to x (on $[0,\pi] \times [0,\infty]$) are considered. This means that if $w(x,t)$ is a three times continuously differentiable solution of the initial-boundary value problem (3.2.1)-(3.2.4) and w_{xxxx} is continuous, then $w(x,t)$ is also a solution of the integral equation (3.2.8) and that, if $v(x,t)$ is a three times continuously differentiable solution of the integral equation (3.2.8) and v_{xxxx} is continuous, then $v(x,t)$ is also a solution of the initial-boundary value problem (3.2.1)-(3.2.4).

We will start with some definitions. Let

$$\Omega_L = \left\{(x,t) \,\middle|\, 0 \le x \le \pi, 0 \le t \le L|\epsilon|^{-1}\right\}, \tag{3.2.11}$$

with L a sufficiently small positive constant independent of ϵ. Let the Banach space $\mathcal{B}$ of all real-valued continuous functions w on Ω_L be given and let $C_M(\Omega_L)$ be the closed subset

$$C_M(\Omega_L) = \left\{ w \in \mathcal{B} \,\middle|\, \|w\| = \max_{(x,t)\in\Omega_L} |w(x,t)| \le M \right\}.$$

We now state the following theorem.

Theorem 3.1 *Suppose f, w_0, and w_1 satisfy* (3.2.5)-(3.2.7). *Then for every ϵ and p satisfying $0 < |\epsilon| < \epsilon_0 \ll 1$ and $p \ge 0$ the initial-boundary value problem* (3.2.1)-(3.2.4) *has a unique and three times continuously differentiable solution with a continuous fourth derivative with respect to x for $(x,t) \in \Omega_L$, with L a sufficiently small, positive constant independent of ϵ. This unique solution depends continuously on the initial values.*

Proof. As stated above the initial-boundary value problem (3.2.1)-(3.2.4) is equivalent to the integral equation (3.2.8). To prove existence and uniqueness of the solution of (3.2.8) a fixed point theorem will be used. Using the fact that w_0, $\frac{\partial^2 w_0}{\partial x^2}$, and w_1 are continuous on the closed and bounded interval $[0,\pi] \times [-\epsilon_0, \epsilon_0]$ and therefore uniformly bounded on that interval, and using (3.8.10) obtained in Appendix B, it follows that there is a constant M_1 independent of ϵ such that, for fixed w_0 and w_1,

$$\|w_l\| \le \frac{1}{2} M_1, \tag{3.2.12}$$

i.e., w_l, given by (3.2.10), is bounded. Since f and $\frac{\partial f}{\partial w}$ are assumed to be continuous and uniformly bounded for those values of x, t, ϵ under consideration, there are constants M_2 and M_3 independent of ϵ such that

$$|f(x,t,w;\epsilon)| \le M_2, \tag{3.2.13}$$
$$|f(x,t,v_1;\epsilon) - f(x,t,v_2;\epsilon)| \le M_3\|v_1 - v_2\| \tag{3.2.14}$$

for all $(x,t) \in \Omega_L$, $\epsilon \in [-\epsilon_0, \epsilon_0]$, and $w, v_1, v_2 \in C_{M_1}(\Omega_L)$. Since G is uniformly bounded for those values of ξ, τ, x, t under consideration, there is a constant M_4 such that

$$|G(\xi,\tau;x,t)| \le M_4 \tag{3.2.15}$$

for all (ξ,τ) and $(x,t) \in \Omega_L$. Since G, given by (3.2.9), is a continuous function in x and t, and since f is bounded, it follows that the integral operator maps $C_{M_1}(\Omega_L)$ into the space of continuous functions on Ω_L. Using

(3.2.12), (3.2.13), (3.2.14), (3.2.15), and the fact that $(x,t) \in \Omega_L$, we can show that $Tw \in C_{M_1}(\Omega_L)$, i.e., the integral operator T maps $C_{M_1}(\Omega_L)$ into itself:

$$\begin{aligned}
|(Tw)(x,t)| &\leq \left|\epsilon \int_0^t \int_0^\pi G(\xi,\tau;x,t) f(\xi,\tau,w;\epsilon)d\xi d\tau\right| + |w_l(x,t;\epsilon)| \\
&\leq |\epsilon| \left|\int_0^t \int_0^\pi |G(\xi,\tau;x,t)|\,|f(\xi,\tau,w;\epsilon)|d\xi d\tau\right| + \frac{1}{2}M_1 \\
&\leq |\epsilon| \left|\int_0^t \int_0^\pi M_4 M_2 d\xi d\tau\right| + \frac{1}{2}M_1 \\
&\leq |\epsilon| t\pi M_4 M_2 + \frac{1}{2}M_1 \leq \pi L M_2 M_4 + \frac{1}{2}M_1.
\end{aligned}$$

If the maximum of $|Tw|$ on the left-hand side is taken for $(x,t) \in \Omega_L$, we obtain

$$\|Tw\| \leq \pi L M_2 M_4 + \frac{1}{2}M_1.$$

If we take the constant L such that $\pi L M_2 M_4 \leq \frac{1}{2}M_1$, it follows that

$$\|Tw\| \leq M_1 \text{ for all } w \in C_{M_1}(\Omega_L).$$

Hence it follows that $T : C_{M_1}(\Omega_L) \longrightarrow C_{M_1}(\Omega_L)$. In a similar way we can show that the integral operator T is a contraction on $C_{M_1}(\Omega_L)$. Let $v_1, v_2 \in C_{M_1}(\Omega_L)$; then

$$\begin{aligned}
&|(Tv_1)(x,t) - (Tv_2)(x,t)| \\
&\leq |\epsilon| \left|\int_0^t \int_0^\pi G(\xi,\tau;x,t)\left\{f(\xi,\tau,v_1;\epsilon) - f(\xi,\tau,v_2;\epsilon)\right\} d\xi d\tau\right| \\
&\leq |\epsilon| \left|\int_0^t \int_0^\pi |G(\xi,\tau;x,t)|\,|f(\xi,\tau,v_1;\epsilon) - f(\xi,\tau,v_2;\epsilon)|d\xi d\tau\right| \\
&\leq |\epsilon| t\pi M_4 M_3 \|v_1 - v_2\| \leq \pi L M_3 M_4 \|v_1 - v_2\|.
\end{aligned}$$

If we take the constant L such that $\pi L M_2 M_4 \leq \frac{1}{2}M_1$ and $\pi L M_3 M_4 \leq k$ with $0 < k < 1$, it follows that

$$\|Tv_1 - Tv_2\| \leq k\|v_1 - v_2\| \text{ with } 0 < k < 1 \text{ for all } v_1, v_2 \in C_{M_1}(\Omega_L),$$

i.e., T is a contraction on $C_{M_1}(\Omega_L)$. Then, Banach's fixed point theorem implies that the integral operator T has a unique fixed point $w \in C_{M_1}(\Omega_L)$, i.e., a continuous function w on Ω_L satisfying the integral equation (3.2.8). It can be shown elementarily that T maps C^i functions into C^i functions for $i = 1, 2, 3$. Furthermore, it can be shown that T maps C^3 functions into C^3 functions that have a continuous fourth derivative with respect to x. So, the unique solution of the integral equation (3.2.8) is a three times continuously differentiable function, with a continuous fourth derivative with respect to x. Since the integral equation (3.2.8) and the initial-boundary value problem (3.2.1)-(3.2.4) are equivalent, it follows that the initial-boundary value problem has a unique and three times continuously differentiable solution w, with w_{xxxx} continuous. This proves the first part of Theorem 3.1.

Next we will show that the solution of the initial-boundary value problem depends continuously on the initial values. Let $w(x,t)$ satisfy (3.2.1)-(3.2.4) and let $\tilde{w}(x,t)$ satisfy (3.2.1)-(3.2.3) and the initial conditions $\tilde{w}(x,0) = \tilde{w}_0(x;\epsilon)$, $\tilde{w}_t(x,0) = \tilde{w}_1(x;\epsilon)$, where $\tilde{w}_0$ and $\tilde{w}_1$ satisfy the same properties (3.2.6) as w_0 and w_1. Using the equivalent integral equation (3.2.8), (3.2.14), (3.2.15), (3.8.1), the fact that $(x,t) \in \Omega_L$, and taking $w, \tilde{w} \in C_{M_1}(\Omega_L)$ we obtain

$$
\begin{aligned}
&|w(x,t) - \tilde{w}(x,t)| \\
\leq\ & |\epsilon| \left| \int_0^t \int_0^\pi G(\xi,\tau;x,t) \left\{ f(\xi,\tau,w;\epsilon) - f(\xi,\tau,\tilde{w};\epsilon) \right\} d\xi d\tau \right| \\
&+ \left| \int_0^\pi G(\xi,0;x,t) \left\{ w_1(\xi;\epsilon) - \tilde{w}_1(\xi;\epsilon) \right\} d\xi \right| \\
&+ \left| \int_0^\pi G_\tau(\xi,0;x,t) \left\{ w_0(\xi;\epsilon) - \tilde{w}_0(\xi;\epsilon) \right\} d\xi \right| \\
\leq\ & |\epsilon| \left| \int_0^t \int_0^\pi |G(\xi,\tau;x,t)| \, |f(\xi,\tau,w;\epsilon) - f(\xi,\tau,\tilde{w};\epsilon)| d\xi d\tau \right| \\
&+ \left| \int_0^\pi |G(\xi,0;x,t)| \, |w_1(\xi;\epsilon) - \tilde{w}_1(\xi;\epsilon)| d\xi \right| \\
&+ \left| \int_0^\pi |G_\tau(\xi,0;x,t)| \, |w_0(\xi;\epsilon) - \tilde{w}_0(\xi;\epsilon)| d\xi \right| \\
\leq\ & |\epsilon| t\pi M_4 M_3 \|w - \tilde{w}\| + \pi M_4 \|w_1 - \tilde{w}_1\| \\
&+ \pi^2 \max_{0 \leq x \leq \pi} \left\{ p|w_0(x;\epsilon) - \tilde{w}_0(x;\epsilon)| + \left| \frac{\partial^2}{\partial x^2} w_0(x;\epsilon) - \frac{\partial^2}{\partial x^2} \tilde{w}_0(x;\epsilon) \right| \right\} \\
\leq\ & \pi L M_3 M_4 \|w - \tilde{w}\| + \pi M_4 \|w_1 - \tilde{w}_1\| + \pi^2 p \|w_0 - \tilde{w}_0\| \\
&+ \pi^2 \left\| \frac{\partial^2 w_0}{\partial x^2} - \frac{\partial^2 \tilde{w}_0}{\partial x^2} \right\| \\
\leq\ & k\|w - \tilde{w}\| + \pi^2 \left\{ \frac{M_4}{\pi} \|w_1 - \tilde{w}_1\| + p\|w_0 - \tilde{w}_0\| + \left\| \frac{\partial^2 w_0}{\partial x^2} - \frac{\partial^2 \tilde{w}_0}{\partial x^2} \right\| \right\}
\end{aligned}
$$

for all $(x,t) \in \Omega_L$ and with $0 < k < 1$. If the maximum of $|w - \tilde{w}|$ on the left-hand side is taken for $(x,t) \in \Omega_L$, we obtain

$$
\|w - \tilde{w}\| \leq \frac{\pi}{1-k} \left\{ \pi p \|w_0 - \tilde{w}_0\| + \pi \left\| \frac{\partial^2 w_0}{\partial x^2} - \frac{\partial^2 \tilde{w}_0}{\partial x^2} \right\| + M_4 \|w_1 - \tilde{w}_1\| \right\}.
$$

This means that small differences between the initial values cause small differences between the solutions w and $\tilde{w}$ on Ω_L. This completes the proof of Theorem 3.1. □

3.3 On the asymptotic validity of formal approximations

In Section 3.4 we will construct an approximation of the solution of the initial-boundary value problem (3.2.1)-(3.2.4) for $f(x,t,w;\epsilon) = w^2$. This approximation is a formal approximation, i.e., a function which satisfies the partial differential equation (3.2.1) and the initial conditions (3.2.4) up to some order depending on the small parameter ϵ. The formal approximation in Section 3.4 satisfies (3.2.1) and (3.2.4) up to $\mathcal{O}(\epsilon^2)$. In this section we will show that a formal approximation of the solution of the initial-boundary value problem (3.2.1)-(3.2.4) is also an asymptotic approximation, i.e., the difference between the formal approximation and the exact solution $\to 0$ as $\epsilon \to 0$, on a time-scale of $1/\epsilon$.

Suppose we construct a three times continuously differentiable function $v(x,t;\epsilon)$ on Ω_L with v_{xxxx} continuous and which satisfies

$$v_{tt} + v_{xxxx} + p^2 v = \epsilon f(x,t,v;\epsilon) + |\epsilon|^m \mathcal{R}_1(x,t;\epsilon),\ 0 < x < \pi, t > 0, \quad (3.3.1)$$

$$v(0,t;\epsilon) = v(\pi,t;\epsilon) = 0,\ t \geq 0, \quad (3.3.2)$$

$$v_{xx}(0,t;\epsilon) = v_{xx}(\pi,t;\epsilon) = 0,\ t \geq 0, \quad (3.3.3)$$

$$v(x,0;\epsilon) = w_0(x;\epsilon) + |\epsilon|^{m-1}\mathcal{R}_2(x;\epsilon) = v_0(x;\epsilon),\ 0 < x < \pi, \quad (3.3.4)$$

$$v_t(x,0;\epsilon) = w_1(x;\epsilon) + |\epsilon|^{m-1}\mathcal{R}_3(x;\epsilon) = v_1(x;\epsilon),\ 0 < x < \pi, \quad (3.3.5)$$

with $m > 1$, where ϵ, p, f, w_0, w_1 satisfy the same conditions as in Section 3.2, equations (3.2.5)-(3.2.7) and where $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ satisfy

$\mathcal{R}_1$ and all first-, second-, and third-order partial derivatives of $\mathcal{R}_1$ with respect to x, t are $\in C([0,\pi] \times [0,\infty] \times [-\epsilon_0,\epsilon_0], \mathbb{R})$ and $\mathcal{R}_1(0,t;\epsilon) = \mathcal{R}_1(\pi,t;\epsilon) \equiv 0$ for $t \geq 0$, (3.3.6)

$$\mathcal{R}_2, \frac{\partial \mathcal{R}_2}{\partial x}, \frac{\partial^2 \mathcal{R}_2}{\partial x^2}, \frac{\partial^3 \mathcal{R}_2}{\partial x^3}, \frac{\partial^4 \mathcal{R}_2}{\partial x^4}, \mathcal{R}_3, \frac{\partial \mathcal{R}_3}{\partial x}, \frac{\partial^2 \mathcal{R}_3}{\partial x^2} \in C([0,\pi] \times [-\epsilon_0,\epsilon_0], \mathbb{R})$$

with $\mathcal{R}_2(0;\epsilon) = \mathcal{R}_2(\pi;\epsilon) = \dfrac{\partial^2 \mathcal{R}_2}{\partial x^2}(0;\epsilon) = \dfrac{\partial^2 \mathcal{R}_2}{\partial x^2}(\pi;\epsilon) \equiv 0$, and

$$\mathcal{R}_3(0;\epsilon) = \mathcal{R}_3(\pi;\epsilon) = \frac{\partial^2 \mathcal{R}_3}{\partial x^2}(0;\epsilon) = \frac{\partial^2 \mathcal{R}_3}{\partial x^2}(\pi;\epsilon) \equiv 0, \quad (3.3.7)$$

$\mathcal{R}_1$ and all first-, second-, and third-order partial derivatives of $\mathcal{R}_1$ with respect to x, t are uniformly bounded for all x, t, ϵ considered. (3.3.8)

We now state the following theorem.

Theorem 3.2 *Let v satisfy* (3.3.1)-(3.3.5), *where f, w_0, and w_1 satisfy* (3.2.5)-(3.2.7) *and $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ satisfy* (3.3.6)-(3.3.8). *Then for $m > 1$ the formal approximation v is an asymptotic approximation (as $\epsilon \to 0$) of the solution w of the nonlinear initial-boundary value problem* (3.2.1)-(3.2.4), *for $(x,t) \in \Omega_L$. This means that, as $\epsilon \to 0$,*

$$|w(x,t) - v(x,t;\epsilon)| = \mathcal{O}(|\epsilon|^{m-1}) \text{ for } 0 \leq x \leq \pi \text{ and } 0 \leq t \leq L|\epsilon|^{-1},$$

in which L is a sufficiently small, positive constant independent of ϵ.

Proof. Let $\hat{f}(x,t,v;\epsilon) = f(x,t,v;\epsilon) + |\epsilon|^{m-1}\mathcal{R}_1(x,t;\epsilon)$, and let v_l be

$$v_l(x,t;\epsilon) = \int_0^\pi \{G(\xi,0;x,t)v_1(\xi;\epsilon) - G_\tau(\xi,0;x,t)v_0(\xi;\epsilon)\}\, d\xi.$$

Suppose v_l satisfies $\|v_l\| \leq \frac{1}{2}M_1$ and $\hat{f}$ satisfies (3.2.13)-(3.2.14). It then follows from Theorem 3.1 that (3.3.1)-(3.3.5) has a unique, three times continuously differentiable solution $v(x,t;\epsilon)$ on Ω_L, with v_{xxxx} continuous; v is also a solution of the equivalent integral equation

$$v(x,t;\epsilon) = \epsilon \int_0^t \int_0^\pi G(\xi,\tau;x,t)\hat{f}(\xi,\tau,v;\epsilon)d\xi d\tau + v_l(x,t;\epsilon) \equiv (Tv)(x,t;\epsilon). \tag{3.3.9}$$

Since the functions $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ satisfy (3.3.6)-(3.3.8), it follows that there are constants M_5, M_6, M_7, M_8 such that

$$|\mathcal{R}_1(x,t;\epsilon)| \leq M_5, \quad |\mathcal{R}_2(x;\epsilon)| \leq M_6, \tag{3.3.10}$$

$$\left|\frac{\partial^2 \mathcal{R}_2(x;\epsilon)}{\partial x^2}\right| \leq M_7, \quad |\mathcal{R}_3(x;\epsilon)| \leq M_8 \tag{3.3.11}$$

for all $(x,t) \in \Omega_L$, $\epsilon \in [-\epsilon_0, \epsilon_0]$, and $w, v_1, v_2 \in C_{M_1}(\Omega_L)$. Subtracting the integral equation (3.3.9) from the integral equation (3.2.8), using (3.2.14), (3.2.15), (3.3.10), (3.3.11), (3.8.1), the fact that $w, v \in C_{M_1}(\Omega_L)$, and the fact that $(x,t) \in \Omega_L$, it follows that

$$\begin{aligned}
&|w(x,t) - v(x,t;\epsilon)| \\
&\leq |\epsilon| \left| \int_0^t \int_0^\pi G(\xi,\tau;x,t) \left\{ f(\xi,\tau,w;\epsilon) - \hat{f}(\xi,\tau,v;\epsilon) \right\} d\xi d\tau \right| \\
&\quad + |w_l(x,t;\epsilon) - v_l(x,t;\epsilon)| \\
&\leq |\epsilon| \left| \int_0^t \int_0^\pi G(\xi,\tau;x,t) \{ f(\xi,\tau,w;\epsilon) - f(\xi,\tau,v;\epsilon) \} d\xi d\tau \right|
\end{aligned}$$

$$
\begin{aligned}
& +|\epsilon|^m \left| \int_0^t \int_0^\pi G(\xi,\tau;x,t)\mathcal{R}_1(\xi,\tau;\epsilon)d\xi d\tau \right| \\
& + \left| \int_0^\pi G(\xi,0;x,t)\left\{w_1(\xi;\epsilon) - v_1(\xi;\epsilon)\right\} d\xi \right| \\
& + \left| \int_0^\pi G_\tau(\xi,0;x,t)\left\{w_0(\xi;\epsilon) - v_0(\xi;\epsilon)\right\} d\xi \right| \\
\leq\ & |\epsilon| \left| \int_0^t \int_0^\pi |G(\xi,\tau;x,t)|\,|f(\xi,\tau,w;\epsilon) - f(\xi,\tau,v;\epsilon)| d\xi d\tau \right| \\
& +|\epsilon|^m \left| \int_0^t \int_0^\pi |G(\xi,\tau;x,t)|\,|\mathcal{R}_1(\xi,\tau;\epsilon)| d\xi d\tau \right| \\
& +|\epsilon|^{m-1} \left| \int_0^\pi |G(\xi,0;x,t)|\,|\mathcal{R}_3(\xi;\epsilon)| d\xi \right| \\
& +|\epsilon|^{m-1} \left| \int_0^\pi G_\tau(\xi,0;x,t)\,\mathcal{R}_2(\xi;\epsilon) d\xi \right| \\
\leq\ & |\epsilon| \left| \int_0^t \int_0^\pi M_4 M_3 \|w-v\| d\xi d\tau \right| + |\epsilon|^m \left| \int_0^t \int_0^\pi M_4 M_5 d\xi d\tau \right| \\
& +|\epsilon|^{m-1} \left| \int_0^\pi M_4 M_8 d\xi \right| + |\epsilon|^{m-1}\pi^2 \max_{0\leq x\leq\pi} \left\{ p|\mathcal{R}_2(x;\epsilon)| + \left| \frac{\partial^2 \mathcal{R}_2(x;\epsilon)}{\partial x^2} \right| \right\} \\
\leq\ & |\epsilon| t\pi M_3 M_4 \|w-v\| + |\epsilon|^m t\pi M_4 M_5 + |\epsilon|^{m-1}\pi\left(M_4 M_8 + \pi p M_6 + \pi M_7\right) \\
\leq\ & \pi L M_3 M_4 \|w-v\| + |\epsilon|^{m-1}\pi\left(L M_4 M_5 + M_4 M_8 + \pi p M_6 + \pi M_7\right) \\
\leq\ & k\|w-v\| + |\epsilon|^{m-1}\pi\left(L M_4 M_5 + M_4 M_8 + \pi p M_6 + \pi M_7\right)
\end{aligned}
$$

for all $(x,t) \in \Omega_L$ and with $0 < k < 1$. If the maximum of $|w - v|$ on the left-hand side is taken for $(x,t) \in \Omega_L$, we obtain

$$
\|w - v\| \leq |\epsilon|^{m-1} \frac{\pi}{1-k} \left(L M_4 M_5 + M_4 M_8 + \pi p M_6 + \pi M_7\right).
$$

So, for $(x,t) \in \Omega_L$, $|w(x,t) - v(x,t;\epsilon)| = \mathcal{O}(|\epsilon|^{m-1})$ as $\epsilon \to 0$. Hence, for $m > 1$ the function v is an asymptotic approximation (as $\epsilon \to 0$) of the solution w of the initial-boundary value problem (3.2.1)-(3.2.4). This completes the proof of Theorem 3.2. □

3.4 Construction of asymptotic approximations - general case

In this and the following section we will construct an asymptotic approximation of the solution of the initial-boundary value problem (3.1.4)-(3.1.7). When straightforward ϵ-expansions are used to approximate solutions, secular terms can occur for specific values of p^2. To avoid these secular terms we use a two time-scales perturbation method. We extend the initial-boundary value problem (3.1.4)-(3.1.7) to an initial value problem by extending all functions in x. The boundary conditions impose that w should be extended as an odd, 2π-periodic function in x, i.e. we write w as a Fourier sine-series in x:

$$w(x,t) = \sum_{m=1}^{\infty} q_m(t)\sin(mx). \tag{3.4.1}$$

This extension implies that all terms in (3.1.4) should be extended as odd, 2π-periodic functions in x. For the nonlinearity on the right-hand side of (3.1.4) this means that we have to rewrite (3.1.4) as

$$w_{tt} + w_{xxxx} + p^2 w = \epsilon h(x) w^2, \tag{3.4.2}$$

where the function h, defined on $\mathbb{R}$, is given by $h(x) = 1$ for $0 < x < \pi$, $h(0) = h(\pi) = 0$, and h is odd and 2π-periodic in x. The function $h(x)$ can then be written as a Fourier sine-series:

$$h(x) = \frac{4}{\pi}\sum_{j=0}^{\infty}\frac{\sin((2j+1)x)}{2j+1}. \tag{3.4.3}$$

By substituting (3.4.1) and (3.4.3) into (3.4.2), we obtain

$$\begin{aligned}&\sum_{m=1}^{\infty}\left(\ddot{q}_m + (m^4+p^2)q_m\right)\sin(mx) \\ &\quad = \epsilon\frac{4}{\pi}\sum_{m,k=1}^{\infty}\sum_{j=0}^{\infty}\frac{1}{2j+1}q_m q_k \sin(mx)\sin(kx)\sin((2j+1)x).\end{aligned} \tag{3.4.4}$$

Using orthogonality properties of the sine-functions on $[0,\pi]$ it is shown in Appendix C, equation (3.9.2), that the equation for each q_n is

$$\begin{aligned}&\ddot{q}_n + (n^4+p^2)q_n \\ &\quad = \frac{\epsilon}{\pi}\left(2\sum_{n=m-k+2j+1} - 2\sum_{n=m-k-2j-1} + \sum_{n=m+k-2j-1}\right. \\ &\qquad\qquad \left. - \sum_{n=m+k+2j+1} - \sum_{n=-m-k+2j+1}\right)\frac{1}{2j+1}q_m q_k,\end{aligned} \tag{3.4.5}$$

for $n = 1, 2, 3, \ldots$, where q_n must satisfy the following initial conditions:

$$q_n(0) = \frac{2}{\pi}\int_0^{\pi} w_0(x)\sin(nx)dx, \quad \dot{q}_n(0) = \frac{2}{\pi}\int_0^{\pi} w_1(x)\sin(nx)dx\ .$$

In the literature, for example in [27], systems similar to (3.4.5) are analyzed using averaging methods. In this chapter, however, we prefer to use the method of multiple scales for its efficiency and wider applicability.

As stated above, terms that give rise to secular terms may occur in the right-hand side of (3.4.5). To eliminate these terms we introduce two time-scales, $t_0 = t$ and $t_1 = \epsilon t$, and assume that q_n can be expanded in a formal power series in ϵ, that is, $q_n(t) = q_{n,0}(t_0, t_1) + \epsilon q_{n,1}(t_0, t_1) + \epsilon^2 q_{n,2}(t_0, t_1) + \cdots$. We substitute this into (3.4.5) and collect equal powers in ϵ. The $\mathcal{O}(\epsilon^0)$-problem becomes

$$\frac{\partial^2}{\partial t_0^2} q_{n,0} + \omega_{n_p}^2 q_{n,0} = 0, \ t > 0, \tag{3.4.6}$$

$$q_{n,0}(0,0) = \frac{2}{\pi}\int_0^\pi w_0(x)\sin(nx)dx, \tag{3.4.7}$$

$$\frac{\partial}{\partial t_0} q_{n,0}(0,0) = \frac{2}{\pi}\int_0^\pi w_1(x)\sin(nx)dx \tag{3.4.8}$$

for $n = 1, 2, 3, \ldots$ with $\omega_{n_p} = \sqrt{n^4 + p^2}$. The general solution for (3.4.6)-(3.4.8) is

$$q_{n,0}(t_0, t_1) = A_{n,0}(t_1)\cos(\omega_{n_p} t_0) + B_{n,0}(t_1)\sin(\omega_{n_p} t_0), \tag{3.4.9}$$

where $A_{n,0}, B_{n,0}$ satisfy the following initial conditions:

$$A_{n,0}(0) = q_{n,0}(0,0), \ B_{n,0}(0) = \frac{1}{\omega_{n_p}}\frac{\partial}{\partial t_0} q_{n,0}(0,0).$$

Next we consider the $\mathcal{O}(\epsilon^1)$-problem

$$\begin{aligned}\frac{\partial^2}{\partial t_0^2} q_{n,1} + \omega_{n_p}^2 q_{n,1} &= -2\frac{\partial^2}{\partial t_0 \partial t_1} q_{n,0} \\ &+ \frac{1}{\pi}\Bigg(2\sum_{n=m-k+2j+1} -2\sum_{n=m-k-2j-1} + \sum_{n=m+k-2j-1} \\ &- \sum_{n=m+k+2j+1} - \sum_{n=-m-k+2j+1}\Bigg)\frac{1}{2j+1} q_{m,0} q_{k,0},\end{aligned} \tag{3.4.10}$$

$$q_{n,1}(0,0) = 0, \quad \frac{\partial}{\partial t_0} q_{n,1}(0,0) = -\frac{\partial}{\partial t_1} q_{n,0}(0,0), \tag{3.4.11}$$

for $n = 1, 2, 3, \ldots$. We substitute (3.4.9) into (3.4.10) and get

$$\begin{aligned}\frac{\partial^2}{\partial t_0^2} q_{n,1} + \omega_{n_p}^2 q_{n,1} &= 2\omega_{n_p}\left(\frac{dA_{n,0}}{dt_1}\sin(\omega_{n_p} t_0) - \frac{dB_{n,0}}{dt_1}\cos(\omega_{n_p} t_0)\right) \\ &+ \frac{1}{\pi}\Bigg(2\sum_{n=m-k+2j+1} -2\sum_{n=m-k-2j-1} + \sum_{n=m+k-2j-1} \\ &- \sum_{n=m+k+2j+1} - \sum_{n=-m-k+2j+1}\Bigg)\frac{1}{2j+1}\mathcal{H},\end{aligned} \tag{3.4.12}$$

where

$$\mathcal{H} = \left(A_{m,0}\cos(\omega_{m_p}t_0) + B_{m,0}\sin(\omega_{m_p}t_0)\right)\left(A_{k,0}\cos(\omega_{k_p}t_0) + B_{k,0}\sin(\omega_{k_p}t_0)\right).$$

Since $\cos(\omega_{n_p}t_0)$ and $\sin(\omega_{n_p}t_0)$ are part of the homogeneous solution of $q_{n,1}$, we want the coefficients of $\cos(\omega_{n_p}t_0)$ and $\sin(\omega_{n_p}t_0)$ on the right-hand side of (3.4.12) to be equal to zero (elimination of secular terms). This gives us differential equations for $A_{n,0}$ and $B_{n,0}$. When $A_{n,0}$ and $B_{n,0}$ have been determined, and thus $q_{n,0}$, we have constructed an approximation v of the exact solution w of (3.1.4)-(3.1.7):

$$v(x,t;\epsilon) = \sum_{n=1}^{\infty}\left(q_{n,0}(t_0,t_1) + \epsilon q_{n,1}(t_0,t_1)\right)\sin(nx)$$

with $q_{n,0}(t_0,t_1) = A_{n,0}(t_1)\cos(\omega_{n_p}t_0) + B_{n,0}(t_1)\sin(\omega_{n_p}t_0)$ and $q_{n,1}(t_0,t_1) = q_{n,1}^{inh} + A_{n,1}\cos(\omega_{n_p}t_0) + B_{n,1}\sin(\omega_{n_p}t_0)$, with $q_{n,1}^{inh}$ the in-homogeneous solution of (3.4.12). $A_{n,1}$ and $B_{n,1}$ can be used to avoid singular terms in the $\mathcal{O}(\epsilon^2)$ approximation. Since we are interested in the $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ approximations, we consider the functions $A_{n,1}$ and $B_{n,1}$ to be constant. It can be easily seen that v satisfies (3.1.4) up to order ϵ^2:

$$v_{tt} + v_{xxxx} + p^2 v - \epsilon v^2 = |\epsilon|^2\mathcal{R}_1, \tag{3.4.13}$$

with

$$\begin{aligned}
\mathcal{R}_1(x,t;\epsilon) = \\
&\left\{\sum_{n=1}^{\infty}\left(2\frac{\partial^2}{\partial t_0\partial t_1}q_{n,1} + \frac{\partial^2}{\partial t_1^2}q_{n,0}\right)\sin(nx)\right. \\
&\qquad\qquad \left. - \sum_{m,k=1}^{\infty} 2q_{m,0}q_{k,1}\sin(mx)\sin(kx)\right\} \\
&+\epsilon\left\{\sum_{n=1}^{\infty}\frac{\partial^2}{\partial t_1^2}q_{n,1}\sin(nx) - \sum_{m,k=1}^{\infty} q_{m,1}q_{k,1}\sin(mx)\sin(kx)\right\}.
\end{aligned}$$

v satisfies the boundary conditions (3.1.5)-(3.1.6) and the following initial conditions:

$$v(x,0;\epsilon) = w_0(x) + |\epsilon|\mathcal{R}_2(x;\epsilon) \text{ and } v_t(x,0;\epsilon) = w_1(x) + |\epsilon|\mathcal{R}_3(x;\epsilon) \tag{3.4.14}$$

with $\mathcal{R}_2(x;\epsilon) \equiv 0$ and $\mathcal{R}_3(x;\epsilon) = |\epsilon|\sum_{n=1}^{\infty}\frac{\partial}{\partial t_1}q_{n,1}(0,0)\sin(nx)$. It can be easily seen that the functions $\mathcal{R}_1, \mathcal{R}_2$, and $\mathcal{R}_3$ satisfy (3.3.6)-(3.3.8). Therefore, using Theorem 3.2, we know that the constructed approximation v, which satisfies (3.4.13)-(3.4.14) and (3.1.5)-(3.1.6), is an asymptotic approximation (as $\epsilon \to 0$) of the exact solution w of the initial boundary value problem (3.1.4)-(3.1.7), i.e.,

$$|w(x,t) - v(x,t;\epsilon)| = \mathcal{O}(|\epsilon|^{m-1}) \text{ for } 0 \leq x \leq \pi \text{ and } 0 \leq t \leq L|\epsilon|^{-1},$$

with L a sufficiently small, positive constant independent of ϵ and $m = 2$ in this case.

In Appendix C we show that to find the equations for $A_{n,0}, B_{n,0}$, we have to determine the terms in (3.4.12) that give rise to secular terms in the approximations by solving the Diophantine-like equations

$$\begin{cases} n = m + k \pm (2j+1) \ \vee \ n = m - k \pm (2j+1) \ \vee \\ n = -m - k + 2j + 1, \\ \pm\sqrt{n^4 + p^2} = \pm\sqrt{m^4 + p^2} \pm \sqrt{k^4 + p^2}. \end{cases} \tag{3.4.15}$$

Only specific combinations of m, k, j, and p^2 will give solutions to (3.4.15). In Appendix C we determine those values of $p^2 \in (0, 100)$ (which we will refer to as critical values) for which (3.4.15) admits solutions. Only for these critical values of p^2, i.e., p^2_{cr}, will the differential equations for $A_{n,0}$ and $B_{n,0}$ be nontrivial and give rise to internal resonances (mode interactions). In Table 3.1 the first seven values of p^2_{cr} are given, together with the corresponding interacting modes. For three different values of p^2_{cr} we

$p^2_{cr} = \frac{17}{3}$	≈ 5.67	$\{2,3\}$
$p^2_{cr} = \frac{77}{3}$	≈ 25.67	$\{1,2,3,4\}$
$p^2_{cr} = -\frac{10913}{3} + \frac{10}{3}\sqrt{1211497}$	≈ 31.27	$\{4,8,9\}$
$p^2_{cr} = -299 + 2\sqrt{31369}$	≈ 55.23	$\{2,4,5\}$
$p^2_{cr} = -\frac{236593}{3} + \frac{10}{3}\sqrt{560787241}$	≈ 72.16	$\{6,18,19\}$
$p^2_{cr} = -91 + \frac{10}{3}\sqrt{2457}$	≈ 74.23	$\{1,2,4\}$
$p^2_{cr} = -\frac{1937}{3} + \frac{2}{3}\sqrt{1229761}$	≈ 93.63	$\{2,5,6\}$

Table 3.1: Critical values of $p^2 \leq 100$ and corresponding interacting modes.

will discuss the differential equations for $A_{n,0}$ and $B_{n,0}$ explicitly. For these values ($p^2_{cr} = \frac{17}{3}$, $p^2_{cr} = -91 + \frac{10}{3}\sqrt{2457}$, and $p^2_{cr} = \frac{77}{3}$) it will be shown that mode interactions occur between, respectively, two, three, and four

modes. To understand the detuning from each of these values of p_{cr}^2, we will consider $p^2 = p_{cr}^2 + \epsilon\alpha$ with $\alpha \in \mathbb{R}$ and $\alpha = \mathcal{O}(1)$. The case where (3.4.15) has no solutions, i.e., $p^2 \neq p_{cr}^2$, is discussed briefly below.
In the general case ($p^2 \neq p_{cr}^2$), there are no solutions to (3.4.15). As shown in the last part of Appendix C this means the equations for $A_{n,0}, B_{n,0}$ are

$$\frac{dA_{n,0}}{dt_1} = \frac{dB_{n,0}}{dt_1} \equiv 0 \text{ for } n = 1, 2, 3, \dots,$$

which means $A_{n,0}(t_1) \equiv A_{n,0}(0)$ and $B_{n,0}(t_1) \equiv B_{n,0}(0)$ for all n. So, if we start with zero initial energy in the nth mode, there will be no energy present up to $\mathcal{O}(\epsilon)$ on a time-scale of order ϵ^{-1}. We say the coupling between the modes is of $\mathcal{O}(\epsilon)$. This allows truncation to those modes that have nonzero initial energy.
For $p^2 = p_{cr}^2 + \epsilon\alpha = \frac{17}{3} + \epsilon\alpha$ extra contributions in the equations for $A_{n,0}$ and $B_{n,0}$ occur for $n = 2, 3$. The equations for $A_{2,0}, B_{2,0}, A_{3,0}, B_{3,0}$ can be determined easily, as shown in the last part of Appendix C.

$$\frac{dA_{2,0}}{dt_1} = \frac{\alpha}{2\sqrt{2^4 + p_{cr}^2}} B_{2,0} - \frac{16}{21\pi} \frac{1}{\sqrt{2^4 + p_{cr}^2}} \left(A_{2,0}B_{3,0} - B_{2,0}A_{3,0}\right), \quad (3.4.16)$$

$$\frac{dB_{2,0}}{dt_1} = -\frac{\alpha}{2\sqrt{2^4 + p_{cr}^2}} A_{2,0} + \frac{16}{21\pi} \frac{1}{\sqrt{2^4 + p_{cr}^2}} \left(A_{2,0}A_{3,0} + B_{2,0}B_{3,0}\right), \quad (3.4.17)$$

$$\frac{dA_{3,0}}{dt_1} = \frac{\alpha}{2\sqrt{3^4 + p_{cr}^2}} B_{3,0} - \frac{16}{21\pi} \frac{1}{\sqrt{3^4 + p_{cr}^2}} \left(2A_{2,0}B_{2,0}\right), \quad (3.4.18)$$

$$\frac{dB_{3,0}}{dt_1} = -\frac{\alpha}{2\sqrt{3^4 + p_{cr}^2}} A_{3,0} + \frac{16}{21\pi} \frac{1}{\sqrt{3^4 + p_{cr}^2}} \left(A_{2,0}^2 - B_{2,0}^2\right). \quad (3.4.19)$$

For $n = 1, 4, 5, \dots,$ $A_{n,0}$ and $B_{n,0}$ satisfy

$$\frac{dA_{n,0}}{dt_1} = \frac{\alpha}{\sqrt{n^4 + p_{cr}^2}} B_{n,0}, \qquad \frac{dB_{n,0}}{dt_1} = -\frac{\alpha}{\sqrt{n^4 + p_{cr}^2}} A_{n,0}. \quad (3.4.20)$$

From (3.4.20) we see that if $A_{n,0}(0) = B_{n,0}(0) = 0$, then for all $t_1 > 0$ $A_{n,0}(t_1) = B_{n,0}(t_1) \equiv 0$ for $n = 1, 4, 5, \dots$. So if we start with zero initial energy in the nth mode ($n = 1, 4, 5, \dots$), there will be no energy present up to $\mathcal{O}(\epsilon)$ on a time-scale of order ϵ^{-1}. We say the coupling between the modes $n = 1, 4, 5, \dots$ is of $\mathcal{O}(\epsilon)$. This means that modes with zero initial energy do not have to be taken into account (for $n = 1, 4, 5, \dots$). On the other hand, there is an $\mathcal{O}(1)$ coupling in this case between modes 2 and 3. This means that if there is initial energy present in mode 2 an energy transfer occurs between modes 2 and 3. Truncation to one mode is not valid: both modes 2 and 3 have to be taken into account, even if mode 3 has zero initial energy. We will discuss (3.4.16)-(3.4.19) in more detail in Section 3.5.1.

For $p^2 = p_{cr}^2 + \epsilon\alpha = -91 + \frac{10}{3}\sqrt{2457} + \epsilon\alpha (\approx 74.23 + \epsilon\alpha)$ extra contributions occur in the equations for $A_{n,0}, B_{n,0}$ for $n = 1, 2, 4$. The equations for $A_{1,0}, B_{1,0}$, $A_{2,0}, B_{2,0}, A_{4,0}, B_{4,0}$ can be determined easily, as shown in the last part of Appendix C.

$$\frac{dA_{1,0}}{dt_1} = \frac{\alpha}{2\sqrt{1+p_{cr}^2}} B_{1,0} + \frac{32}{105\pi\sqrt{1+p_{cr}^2}} \left(A_{2,0}B_{4,0} - B_{2,0}A_{4,0}\right), \quad (3.4.21)$$

$$\frac{dB_{1,0}}{dt_1} = -\frac{\alpha}{2\sqrt{1+p_{cr}^2}} A_{1,0} - \frac{32}{105\pi\sqrt{1+p_{cr}^2}} \left(A_{2,0}A_{4,0} + B_{2,0}B_{4,0}\right), \quad (3.4.22)$$

$$\frac{dA_{2,0}}{dt_1} = \frac{\alpha}{2\sqrt{2^4+p_{cr}^2}} B_{2,0} + \frac{32}{105\pi\sqrt{2^4+p_{cr}^2}} \left(A_{1,0}B_{4,0} - B_{1,0}A_{4,0}\right), \quad (3.4.23)$$

$$\frac{dB_{2,0}}{dt_1} = -\frac{\alpha}{2\sqrt{2^4+p_{cr}^2}} A_{2,0} - \frac{32}{105\pi\sqrt{2^4+p_{cr}^2}} \left(A_{1,0}A_{4,0} + B_{1,0}B_{4,0}\right) \quad (3.4.24)$$

$$\frac{dA_{4,0}}{dt_1} = \frac{\alpha}{2\sqrt{4^4+p_{cr}^2}} B_{4,0} + \frac{32}{105\pi\sqrt{4^4+p_{cr}^2}} \left(A_{1,0}B_{2,0} + B_{1,0}A_{2,0}\right), \quad (3.4.25)$$

$$\frac{dB_{4,0}}{dt_1} = -\frac{\alpha}{2\sqrt{4^4+p_{cr}^2}} A_{4,0} - \frac{32}{105\pi\sqrt{4^4+p_{cr}^2}} \left(A_{1,0}A_{2,0} - B_{1,0}B_{2,0}\right) \quad (3.4.26)$$

For $n = 3, 5, 6, \ldots,$ $A_{n,0}$ and $B_{n,0}$ satisfy (3.4.20). In this case there is an $\mathcal{O}(1)$ coupling between modes 1, 2, and 4. If there is initial energy present in only one of modes 1, 2, and 4 no energy transfer occurs, but if there is initial energy present in two of the three modes an energy transfer occurs between these three modes. In that case truncation to one or two modes is not valid: all three modes have to be taken into account, even if one of them has zero initial energy. We will discuss (3.4.21)-(3.4.26) in more detail in Section 3.5.2.

For $p^2 = p_{cr}^2 + \epsilon\alpha, p_{cr}^2 \in \{-\frac{10913}{3} + \frac{10}{3}\sqrt{1211497}, -299 + 2\sqrt{31369}, -\frac{1937}{3} + \frac{2}{3}\sqrt{1229761}, -\frac{236593}{3} + \frac{10}{3}\sqrt{560787241}\}$ the analysis is similar to the previous case with modal interaction between three modes. We will not discuss these cases in more detail.

For $p^2 = p_{cr}^2 + \epsilon\alpha = \frac{77}{3} + \epsilon\alpha$ extra contributions in the equations for $A_{n,0}$ and $B_{n,0}$ occur for $n = 1, 2, 3, 4$. The equations for $A_{1,0}, B_{1,0}, A_{2,0}, B_{2,0}, A_{3,0}, B_{3,0}$, $A_{4,0}, B_{4,0}$ can be determined easily, as shown in the last part of Appendix C.

$$\frac{dA_{1,0}}{dt_1} = \frac{\alpha}{2\sqrt{1+p_{cr}^2}} B_{1,0} + \frac{12}{45\pi\sqrt{1+p_{cr}^2}} \left(A_{1,0}B_{3,0} - B_{1,0}A_{3,0}\right), \quad (3.4.27)$$

$$\frac{dB_{1,0}}{dt_1} = -\frac{\alpha}{2\sqrt{1+p_{cr}^2}} A_{1,0} - \frac{12}{45\pi\sqrt{1+p_{cr}^2}} \left(A_{1,0}A_{3,0} + B_{1,0}B_{3,0}\right), \quad (3.4.28)$$

$$\frac{dA_{2,0}}{dt_1} = \frac{\alpha}{2\sqrt{2^4+p_{cr}^2}} B_{2,0} - \frac{96}{135\pi\sqrt{2^4+p_{cr}^2}} \left(A_{3,0}B_{4,0} - B_{3,0}A_{4,0}\right), \quad (3.4.29)$$

$$\frac{dB_{2,0}}{dt_1} = -\frac{\alpha}{2\sqrt{2^4+p_{cr}^2}}A_{2,0} + \frac{96}{135\pi\sqrt{2^4+p_{cr}^2}}\left(A_{3,0}A_{4,0}+B_{3,0}B_{4,0}\right) \quad (3.4.30)$$

$$\frac{dA_{3,0}}{dt_1} = \frac{\alpha}{2\sqrt{3^4+p_{cr}^2}}B_{3,0} + \frac{12}{45\pi\sqrt{3^4+p_{cr}^2}}\left(2A_{1,0}B_{1,0}\right) - \frac{96}{135\pi}\frac{1}{\sqrt{3^4+p_{cr}^2}}\left(A_{2,0}B_{4,0}-B_{2,0}A_{4,0}\right), \quad (3.4.31)$$

$$\frac{dB_{3,0}}{dt_1} = -\frac{\alpha}{2\sqrt{3^4+p_{cr}^2}}A_{3,0} - \frac{12}{45\pi\sqrt{3^4+p_{cr}^2}}\left(A_{1,0}^2-B_{1,0}^2\right) + \frac{96}{135\pi}\frac{1}{\sqrt{3^4+p_{cr}^2}}\left(A_{2,0}A_{4,0}+B_{2,0}B_{4,0}\right), \quad (3.4.32)$$

$$\frac{dA_{4,0}}{dt_1} = \frac{\alpha}{2\sqrt{4^4+p_{cr}^2}}B_{4,0} - \frac{96}{135\pi\sqrt{4^4+p_{cr}^2}}\left(A_{2,0}B_{3,0}+B_{2,0}A_{3,0}\right), \quad (3.4.33)$$

$$\frac{dB_{4,0}}{dt_1} = -\frac{\alpha}{2\sqrt{4^4+p_{cr}^2}}A_{4,0} + \frac{96}{135\pi\sqrt{4^4+p_{cr}^2}}\left(A_{2,0}A_{3,0}-B_{2,0}B_{3,0}\right) \quad (3.4.34)$$

For $n \geq 5$, $A_{n,0}$ and $B_{n,0}$ satisfy (3.4.20). In this case there is an $\mathcal{O}(1)$ coupling between the modes $1, 2, 3, 4$. This means, for example, that if there is initial energy present in modes 1 and 2 an energy transfer will occur between all four modes. Truncation to one or two modes is not valid: all four modes have to be taken into account. We will discuss (3.4.27)-(3.4.34) briefly in Section 3.5.3.

As we saw above, internal resonances between certain modes occur for special values of $p^2 \in (0, 100)$. Truncation to one or several mode(s) is not valid in general: all modes have to be taken into account. In the next section we will discuss the behavior of the solutions for the differential equations for $A_{n,0}$ and $B_{n,0}$ when $p^2 = \frac{17}{3} + \epsilon\alpha$ and $p^2 = -91 + \frac{10}{3}\sqrt{2457} + \epsilon\alpha$ with $\alpha = 0$ and $\alpha \neq 0$ (detuning). We will also briefly discuss the behavior of the solutions when $p^2 = \frac{77}{3} + \epsilon\alpha$. It can be shown that other internal resonances occur in a similar way for specific values of $p^2 \geq 100$.

3.5 Construction of asymptotic approximations - specific p^2-values

3.5.1 The case $p^2 = p_{cr}^2 + \epsilon\alpha = \frac{17}{3} + \epsilon\alpha$

In this case there is an interaction between modes 2 and 3. We will therefore consider only the equations for $n = 2$ and $n = 3$. We assume all other modes have zero initial energy. This truncation is allowed, as stated in Section 3.4. We transform (3.4.16)-(3.4.19) using $A_i(t_1) = C_i a_i(\tau)$ and $B_i(t_1) = C_i b_i(\tau)$, where

$$t_1 = \frac{21\pi}{16}\tau, \quad C_i = \left(\frac{(2^4+p_{cr}^2)^2(3^4+p_{cr}^2)}{(i^2+p_{cr}^2)}\right)^{\frac{1}{4}} \text{ for } i = 2,3,$$

and get the following equations:

$$\dot{a}_2 = \frac{\beta}{\sqrt{2^4+p_{cr}^2}}b_2 - a_2b_3 + b_2a_3, \tag{3.5.1}$$

$$\dot{b}_2 = -\frac{\beta}{\sqrt{2^4+p_{cr}^2}}a_2 + a_2a_3 + b_2b_3, \tag{3.5.2}$$

$$\dot{a}_3 = \frac{\beta}{\sqrt{3^4+p_{cr}^2}}b_3 - 2a_2b_2, \tag{3.5.3}$$

$$\dot{b}_3 = -\frac{\beta}{\sqrt{3^4+p_{cr}^2}}a_3 + a_2^2 - b_2^2, \tag{3.5.4}$$

where $\beta = \frac{21\pi}{32}\alpha$ (the dot represents differentiation with respect to τ). We introduce polar coordinates

$$a_n = r_n\cos(\phi_n), \quad b_n = r_n\sin(\phi_n), \tag{3.5.5}$$

with the amplitude function $r_n = r_n(\tau)$ and the phase function $\phi_n = \phi_n(\tau)$. In the polar coordinates (3.5.5) for $n = 2,3$, (3.5.1)-(3.5.4) become

$$\dot{r}_2 = -r_2r_3\sin(\phi_3 - 2\phi_2), \tag{3.5.6}$$

$$\dot{\phi}_2 = -\frac{\beta}{\sqrt{2^4+p_{cr}^2}} + r_3\cos(\phi_3 - 2\phi_2), \tag{3.5.7}$$

$$\dot{r}_3 = r_2^2\sin(\phi_3 - 2\phi_2), \tag{3.5.8}$$

$$\dot{\phi}_3 = -\frac{\beta}{\sqrt{3^4+p_{cr}^2}} + \frac{r_2^2}{r_3}\cos(\phi_3 - 2\phi_2). \tag{3.5.9}$$

Multiplying (3.5.6) with r_2 and (3.5.8) with r_3 and adding both equations we obtain $r_2\dot{r}_2 + r_3\dot{r}_3 = 0$, which means

$$r_2^2 + r_3^2 = c_1^2. \tag{3.5.10}$$

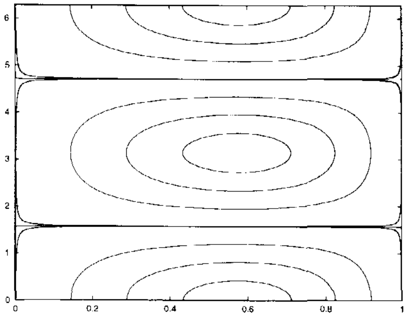
Using (3.5.10) we can analyze (3.5.6)-(3.5.9) in the (r_3, ψ) phase space, with $\psi = \phi_3 - 2\phi_2$:

$$\dot{r}_3 = (c_1^2 - r_3^2)\sin(\psi), \tag{3.5.11}$$

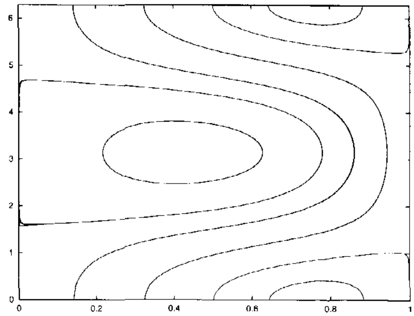
$$\dot{\psi} = \gamma + \frac{1}{r_3}\left(c_1^2 - 3r_3^2\right)\cos(\psi), \tag{3.5.12}$$

where $\gamma = (-\frac{1}{\sqrt{3^4+p_{cr}^2}} + \frac{2}{\sqrt{2^4+p_{cr}^2}})\beta$. For $r_3 = 0$, (3.5.11)-(3.5.12) do not hold. In that case we have to analyze the original differential equations

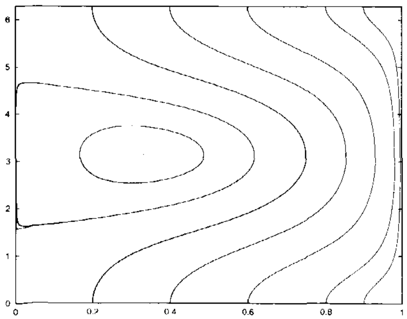
(3.5.1)-(3.5.4). We will determine the critical points of (3.5.11)-(3.5.12) analytically for all values of γ. For several values of γ, phase spaces are calculated using a numerical integration method.
We start with $\gamma = 0$. We analyze (3.5.11)-(3.5.12) in the (r_3, ψ) phase space. The system is 2π-periodic in ψ. We find four critical points: $(\frac{1}{3}\sqrt{3}c_1, 0)$ and $(\frac{1}{3}\sqrt{3}c_1, \pi)$, both centers, and $(c_1, \frac{\pi}{2})$ and $(c_1, \frac{3\pi}{2})$, both saddles. The behavior of the solutions of (3.5.11)-(3.5.12) in the (r_3, ψ) phase space for $\gamma = 0$ is given in Figure 3.2(a). For the sake of convenience, we have taken $c_1 = 1$ in all phase spaces given below. The figures are essentially the same for $c_1 \neq 1$. In the exceptional case, when r_3 becomes 0, the original differential equations (3.5.1)-(3.5.4) impose a phase jump for ψ, to $\psi = \frac{\pi}{2}$, as can be seen in Figure 3.2(a). We see that the system oscillates around an equilibrium state, which is a combination of two modes. Next we consider the detuning from the previous case, i.e., $\gamma \neq 0$. We start with $\gamma > 0$. As γ increases, one center starts moving toward the $r_3 = 0$ axis and the other center toward the $r_3 = c_1$ axis. One saddle moves toward $(c_1, 0)$ and the other toward $(c_1, 2\pi)$. For a certain value of γ, $\gamma = 2c_1$, three critical points coincide. We therefore start with $0 < \gamma < 2c_1$.



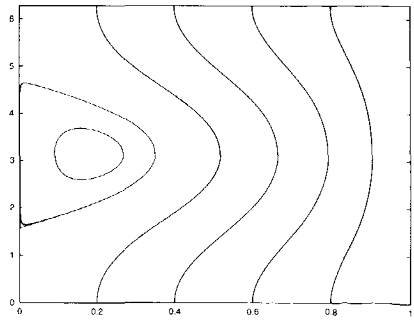
(a) $\gamma = 0$



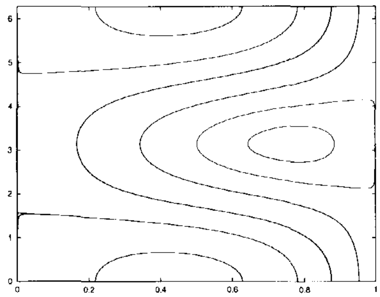
(b) $\gamma = 1$



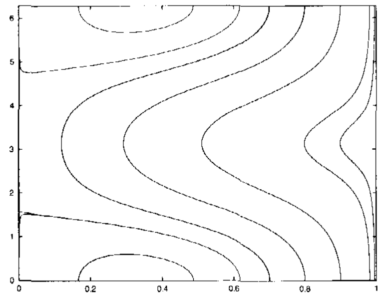
(c) $\gamma = 2$



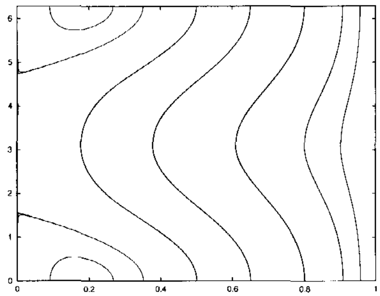
(d) $\gamma = 5$



(e) $\gamma = -1$



(f) $\gamma = -2$



(g) $\gamma = -5$

Figure 3.2: Phase space for $p^2 = \frac{17}{3} + \epsilon\alpha$, $\gamma = (-\frac{1}{\sqrt{3^4+p^2}} + \frac{2}{\sqrt{2^4+p^2}})\frac{21\pi}{32}\alpha$, with r_3 (horizontal) from 0 to 1, ψ (vertical) from 0 to 2π.

We again find four critical points: $(\overline{r}_3, 0)$ and $(\widetilde{r}_3, \pi)$, both centers, and $(c_1, \overline{\psi})$ and $(c_1, \widetilde{\psi})$, both saddles, where $\overline{r}_3 = \frac{\gamma}{6} + \frac{1}{6}\sqrt{\gamma^2 + 12c_1^2} > \frac{1}{3}\sqrt{3}c_1$, $\widetilde{r}_3 = -\frac{\gamma}{6} + \frac{1}{6}\sqrt{\gamma^2 + 12c_1^2} < \frac{1}{3}\sqrt{3}c_1$, and where $\overline{\psi}$ and $\widetilde{\psi}$ are solutions of $\cos(\psi) = \frac{\gamma}{2c_1}$, with $\overline{\psi} < \frac{\pi}{2}$ and $\widetilde{\psi} > \frac{3\pi}{2}$. The behavior of the solutions of (3.5.11)-(3.5.12) in the (r_3, ψ) phase space for $0 < \gamma < 2c_1$ is given in Figure 3.2(b) for $c_1 = 1$. For $\gamma = 2c_1$ three critical points coincide, which means two critical points remain: $(c_1, 0)$, a higher order singularity, and $(\frac{1}{3}c_1, \pi)$, a center. The behavior of the solutions of (3.5.11)-(3.5.12) in the (r_3, ψ) phase space for $\gamma = 2c_1$ is given in Figure 3.2(c) for $c_1 = 1$. For $\gamma > 2c_1$ only one critical point remains: $(\widehat{r}_3, \pi)$, a center, where $\widehat{r}_3 < \frac{1}{3}c_1$ and $\widehat{r}_3$ moves toward the $r_3 = 0$ axis as γ increases. The behavior of the solutions of (3.5.11)-(3.5.12) in the (r_3, ψ) phase space for $\gamma > 2c_1$ is given in Figure 3.2(d) for $c_1 = 1$. For $\gamma < 0$ a similar analysis can be given. The behavior of the solutions of (3.5.11)-(3.5.12) in the (r_3, ψ) phase space for the cases $-2c_1 < \gamma < 0$, $\gamma = -2c_1$, and $\gamma < -2c_1$ is given in Figure 3.2(e)-(g).

3.5.2 The case $p^2 = p_{cr}^2 + \epsilon\alpha = -91 + \frac{10}{3}\sqrt{2457} + \epsilon\alpha \approx 74.23 + \epsilon\alpha$

In this case there is an interaction between modes 1, 2, and 4. We will therefore consider only the equations for $n = 1, 2, 4$. We assume all other modes have zero initial energy. This truncation is allowed, as stated in Section 3.4. We transform (3.4.21)-(3.4.26) using $A_i(t_1) = C_i a_i(\tau)$ and $B_i(t_1) = C_i b_i(\tau)$, where

$$t_1 = \frac{105\pi}{32}\tau, \qquad C_i = \left(\frac{(1^4 + p_{cr}^2)(2^4 + p_{cr}^2)(4^4 + p_{cr}^2)}{(i^2 + p_{cr}^2)}\right)^{\frac{1}{4}} \quad \text{for } i = 1, 2, 4,$$

and we get the following equations:

$$\begin{aligned}
\dot{a}_1 &= \frac{\beta}{\sqrt{1^4 + p_{cr}^2}} b_1 + a_2 b_4 - b_2 a_4, && (3.5.13)\\
\dot{b}_1 &= -\frac{\beta}{\sqrt{1^4 + p_{cr}^2}} a_1 - (a_2 a_4 + b_2 b_4), && (3.5.14)\\
\dot{a}_2 &= \frac{\beta}{\sqrt{2^4 + p_{cr}^2}} b_2 + a_1 b_4 - b_1 a_4, && (3.5.15)\\
\dot{b}_2 &= -\frac{\beta}{\sqrt{2^4 + p_{cr}^2}} a_2 - (a_1 a_4 + b_1 b_4), && (3.5.16)\\
\dot{a}_4 &= \frac{\beta}{\sqrt{4^4 + p_{cr}^2}} b_4 + a_2 b_1 + b_2 a_1, && (3.5.17)\\
\dot{b}_4 &= -\frac{\beta}{\sqrt{4^4 + p_{cr}^2}} a_4 - a_1 a_2 + b_1 b_2, && (3.5.18)
\end{aligned}$$

where $\beta = \frac{105\pi}{64}\alpha$. (A dot represents differentiation with respect to τ.) In the polar coordinates (3.5.5) for $n = 1, 2, 4$, (3.5.13)-(3.5.18) become

$$\begin{aligned}
\dot{r}_1 &= r_2 r_4 \sin(\phi_4 - (\phi_1 + \phi_2)), && (3.5.19)\\
\dot{\phi}_1 &= -\frac{\beta}{\sqrt{1^4 + p_{cr}^2}} - \frac{r_2 r_4}{r_1}\cos(\phi_4 - (\phi_1 + \phi_2)), && (3.5.20)\\
\dot{r}_2 &= r_1 r_4 \sin(\phi_4 - (\phi_1 + \phi_2)), && (3.5.21)\\
\dot{\phi}_2 &= -\frac{\beta}{\sqrt{2^4 + p_{cr}^2}} - \frac{r_1 r_4}{r_2}\cos(\phi_4 - (\phi_1 + \phi_2)), && (3.5.22)\\
\dot{r}_4 &= -r_1 r_2 \sin(\phi_4 - (\phi_1 + \phi_2)), && (3.5.23)\\
\dot{\phi}_4 &= -\frac{\beta}{\sqrt{4^4 + p_{cr}^2}} - \frac{r_1 r_2}{r_4}\cos(\phi_4 - (\phi_1 + \phi_2)). && (3.5.24)
\end{aligned}$$

Multiplying (3.5.19) with r_1 and (3.5.23) with r_4 and adding both equations we obtain $r_1\dot{r}_1 + r_4\dot{r}_4 = 0$. Applying the same procedure for (3.5.21) and (3.5.23) gives us $r_2\dot{r}_2 + r_4\dot{r}_4 = 0$. This means

$$r_1^2 + r_4^2 = c_1^2, \qquad r_2^2 + r_4^2 = c_2^2. \qquad (3.5.25)$$

Using (3.5.25) we can analyze (3.5.19)-(3.5.24) in the (r_4, ψ) phase space with $\psi = \phi_4 - (\phi_1 + \phi_2)$:

$$\begin{aligned} \dot{r}_4 &= -\sqrt{c_1^2 - r_4^2}\sqrt{c_2^2 - r_4^2}\ \sin(\psi), && (3.5.26)\\ \dot{\psi} &= \gamma + \frac{1}{r_4\sqrt{c_1^2 - r_4^2}\sqrt{c_2^2 - r_4^2}}\Big(r_4^2(c_1^2 + c_2^2 - 2r_4^2) && (3.5.27)\\ &\qquad -(c_1^2 - r_4^2)(c_2^2 - r_4^2)\Big)\cos(\psi), \end{aligned}$$

where

$$\gamma = \left(\frac{1}{\sqrt{1^4 + p_{cr}^2}} + \frac{1}{\sqrt{2^4 + p_{cr}^2}} - \frac{1}{\sqrt{4^4 + p_{cr}^2}}\right)\beta.$$

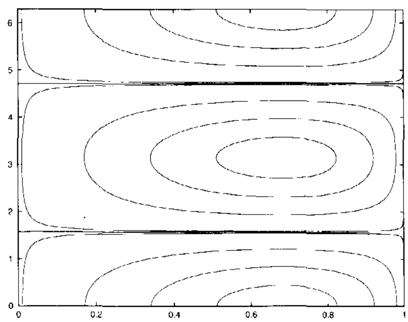
From (3.5.25) it follows that $r_4 \in [0, \min(c_1, c_2)]$. For $r_4 = 0$ and $r_4 = \min(c_1, c_2)$, (3.5.26)-(3.5.27) do not hold. In these cases we have to analyze the original differential equations (3.5.13)-(3.5.18). We will determine the critical points of (3.5.26)-(3.5.27) analytically for all values of γ. For several values of γ, phase spaces are calculated using a numerical integration method.

We consider two cases: $c_1 = c_2$ and $c_1 \neq c_2$. For $c_1 = c_2$ we have $r_1 = r_2$, $\phi_1 - \phi_2 =$ constant. The analysis for this case is similar to the case with mode interaction between two modes, as discussed in the previous subsection and it will not be discussed further. For $c_1 \neq c_2$ the analysis is different. We start with $\gamma = 0$. We analyze (3.5.26)-(3.5.27) in the (r_4, ψ) phase space. We find two critical points: $(N_{c_1,c_2}, 0)$ and (N_{c_1,c_2}, π), both centers, where $N_{c_1,c_2} = \sqrt{\frac{c_1^2+c_2^2}{3} - \frac{1}{3}\sqrt{c_1^4 - c_1^2c_2^2 + c_2^4}}$. The behavior of the solutions of (3.5.26)-(3.5.27) in the (r_4, ψ) phase space for $\gamma = 0$ is given in Figure 3.3(a). For the sake of convenience, we have taken $c_1 = 1$ and $c_2 = 2$ in all phase spaces given below. The figures are essentially the same for other values of c_1 and c_2. Phase jumps occur for $r_4 = 0$, to $\psi = \frac{3\pi}{2}$ and for $r_4 = \min(c_1, c_2)$, to $\psi = \frac{\pi}{2}$, imposed by the original differential equations (3.5.13)-(3.5.18). We see that the system oscillates around an equilibrium state, which is a combination of three modes.

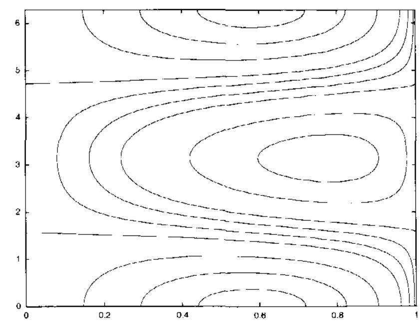
Next we consider the detuning from the previous case, i.e., $\gamma \neq 0$. We start with $\gamma > 0$. Again we find two critical points: $(\overline{r}_4, 0)$ and $(\widetilde{r}_4, \pi)$, both centers, where $\overline{r}_4$ and $\widetilde{r}_4$ are the two real roots of

$$\begin{aligned} 9r_4^8 - \left(12(c_1^2 + c_2^2) + \gamma^2\right) r_4^6 + \left(4c_1^4 + 14c_1^2c_2^2 + 4c_2^4 + \gamma^2(c_1^2 + c_2^2)\right) r_4^4 \\ -c_1^2c_2^2\left(4(c_1^2 + c_2^2) + \gamma^2\right) r_4^2 + c_1^4c_2^4 = 0, \end{aligned}$$

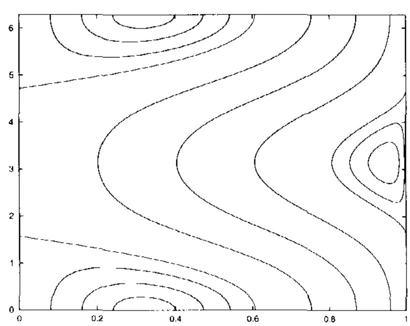
with $\overline{r}_4 < N_{c_1,c_2}$ and $\widetilde{r}_4 > N_{c_1,c_2}$, and with $\overline{r}_4 \downarrow 0$ and $\widetilde{r}_4 \uparrow \min(c_1, c_2)$ as γ increases. The behavior of the solutions of (3.5.26)-(3.5.27) in the (r_4, ψ) phase space for $\gamma > 0$ is given in Figure 3.3(b)-(c).



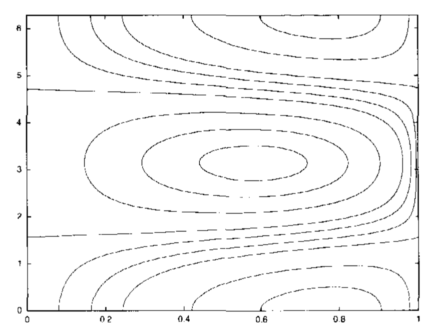
(a) $\gamma = 0$, $c_1 = 1$, $c_2 = 2$



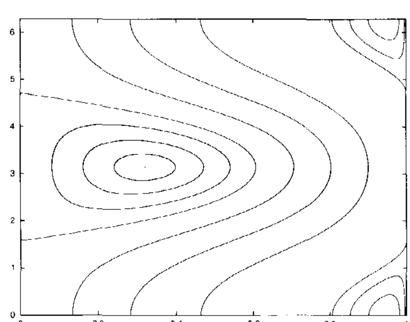
(b) $\gamma = 1$, $c_1 = 1$, $c_2 = 2$



(c) $\gamma = 5$, $c_1 = 1$, $c_2 = 2$



(d) $\gamma = -1$, $c_1 = 1$, $c_2 = 2$



(e) $\gamma = -5$, $c_1 = 1$, $c_2 = 2$

Figure 3.3: Phase space for $p^2 \approx 74.23 + \epsilon\alpha$, $\gamma = (\frac{1}{\sqrt{1^4+p_{cr}^2}} + \frac{1}{\sqrt{2^4+p_{cr}^2}} - \frac{1}{\sqrt{4^4+p_{cr}^2}})\frac{105\pi}{64}\alpha$, with r_3 (horizontal) from 0 to 1, ψ (vertical) from 0 to 2π.

For $\gamma < 0$ a similar analysis can be given. The behavior of the solutions of (3.5.26)-(3.5.27) in the (r_4, ψ) phase space for $\gamma < 0$ is given in Figure 3.3(d)-(e).

3.5.3 The case $p^2 = p_{cr}^2 + \epsilon\alpha = \frac{77}{3} + \epsilon\alpha$

In this case there is an interaction between modes 1, 2, 3, and 4. We will therefore consider only the equations for $n = 1, 2, 3, 4$. We assume all other modes have zero initial energy. This truncation is allowed, as stated in Section 3.4. We transform (3.4.27)-(3.4.34) using $A_i(t_1) = C_i a_i(\tau)$ and $B_i(t_1) = C_i b_i(\tau)$, where

$$t_1 = \frac{135\pi}{96}\tau, \qquad C_i = \left(\frac{(2^4 + p_{cr}^2)(3^4 + p_{cr}^2)(4^4 + p_{cr}^2)}{(i^2 + p_{cr}^2)}\right)^{\frac{1}{4}} \quad \text{for } i = 1, 2, 3, 4,$$

and get the following equations:

$$\dot{a}_1 = \frac{\beta}{\sqrt{1^4 + p_{cr}^2}} b_1 + \frac{3}{32}\sqrt{65}\,(a_1 b_3 - b_1 a_3), \tag{3.5.28}$$

$$\dot{b}_1 = -\frac{\beta}{\sqrt{1^4 + p_{cr}^2}} a_1 - \frac{3}{32}\sqrt{65}\,(a_1 a_3 + b_1 b_3), \tag{3.5.29}$$

$$\dot{a}_2 = \frac{\beta}{\sqrt{2^4 + p_{cr}^2}} b_2 - (a_3 b_4 - b_3 a_4), \tag{3.5.30}$$

$$\dot{b}_2 = -\frac{\beta}{\sqrt{2^4 + p_{cr}^2}} a_2 + a_3 a_4 + b_3 b_4, \tag{3.5.31}$$

$$\dot{a}_3 = \frac{\beta}{\sqrt{3^4 + p_{cr}^2}} b_3 + \frac{3}{32}\sqrt{65}\,(2 a_1 b_1) - (a_2 b_4 - b_2 a_4), \tag{3.5.32}$$

$$\dot{b}_3 = -\frac{\beta}{\sqrt{3^4 + p_{cr}^2}} a_3 - \frac{3}{32}\sqrt{65}\left(a_1^2 - b_1^2\right) + a_2 a_4 + b_2 b_4, \tag{3.5.33}$$

$$\dot{a}_4 = \frac{\beta}{\sqrt{4^4 + p_{cr}^2}} b_4 - (a_2 b_3 + b_2 a_3), \tag{3.5.34}$$

$$\dot{b}_4 = -\frac{\beta}{\sqrt{4^4 + p_{cr}^2}} a_4 + (a_2 a_3 - b_2 b_3), \tag{3.5.35}$$

where $\beta = \frac{135\pi}{192}\alpha$. (The dot represents differentiation with respect to τ.) By introducing polar coordinates for each mode, as done in the previous subsections, the system (3.5.28)-(3.5.35) can be brought back to a four-dimensional system of equations for two amplitudes (r_3, r_4) and two phase combinations ($\psi_1 = \phi_3 - 2\phi_1$, $\psi_2 = \phi_4 - (\phi_2 + \phi_3)$). For this four-dimensional system the exact solution cannot be given, as far as we know. However, the critical points in the $(r_3, r_4, \psi_1, \psi_2)$ phase space can be determined and the system can be linearized locally around these critical points to get an idea of the behavior of the solutions of (3.5.28)-(3.5.35). The system can also be integrated numerically. It can be seen directly from (3.5.28)-(3.5.35) that if mode 1 has zero initial energy, an energy transfer will occur between modes 2, 3, and 4 only (energy in mode 1 remains zero). In this case the analysis is similar to the case with mode interaction between three modes, as discussed in Section 3.5.2. If mode 1 has nonzero initial energy but modes 2 and 4 have zero initial energy, an energy transfer will occur between modes 1 and 3 only (the energy in modes 2 and 4 remains zero). In this case the analysis is similar to the case with mode interaction between two modes, as discussed in Section 3.5.1. It can be easily seen from (3.5.28)-(3.5.35) that if the initial energy in the subcases mentioned above is of $\mathcal{O}(\delta)$, with δ a small parameter, then the energy remains of $\mathcal{O}(\delta)$ on a ϵ^{-1} time-scale. This means these subcases are structurally stable. In all other cases a combined oscillation between modes 1, 2, 3, and 4 will occur.

3.6 Conclusions

In this chapter we consider an initial-boundary value problem for a weakly nonlinear beam equation. We have constructed asymptotic approximations of order ϵ and considered the interaction between different oscillation modes. We presented an asymptotic theory which states that the constructed approximation is asymptotically valid on an ϵ^{-1} time-scale. In general it is difficult to extend the asymptotic validity to longer time-scales (ϵ^{-n}, $n > 1$). For very specific cases, for example, if there is an energy conservation for longer time-scales, the extension of the asymptotic validity may be possible. However, this is not the case for this problem. We showed that for most p^2-values no mode interactions occur between different modes up to $\mathcal{O}(\epsilon)$, which means that there is no energy transfer between different modes up to $\mathcal{O}(\epsilon)$. We say the coupling between the modes is of $\mathcal{O}(\epsilon)$, and truncation is allowed to those modes that have nonzero initial energy. However, for some p^2-values interactions between different modes occur, which cause complicated internal resonances. Physically this means that in some cases (depending on the value of p^2, which depends on the stay-characteristics of the bridge and on certain properties of the beam), if the beam initially oscillates in a high vibration mode, lower vibration modes can be excited, and an energy transfer occurs between the different modes. For $p^2 \in (0, 100)$ there are seven critical values, listed in Table 3.1. For $p^2 \geq 100$ other internal resonances can be found in a similar way for special values of p^2. For three different values of p^2_{cr}, $p^2_{cr} = \frac{17}{3}$, $p^2_{cr} = -91 + \frac{10}{3}\sqrt{2457}$, and $p^2_{cr} = \frac{77}{3}$, the equation has been studied in more detail, including the detuning $p^2 = p^2_{cr} + \epsilon\alpha$.

For $p_{cr}^2 = \frac{17}{3}$ it is shown that an energy transfer occurs between modes 2 and 3, even if mode 3 has zero initial energy. We call this a coupling between the modes of $\mathcal{O}(1)$. Truncation to one mode will give loss of information, and approximations will not be valid. Both modes have to be taken into account. Examining the behavior of the oscillations in this case, we see that the system oscillates around an equilibrium state which is a combination of two modes (energy in both modes). There is no energy loss, as can be expected, since no other external and damping forces are considered. The detuning analysis shows that the system gradually changes from a combined oscillation of two modes to an oscillation of mode 2 only, as p^2 moves away from the critical value $\frac{17}{3}$ (as can be seen in Figure 3.2). All this holds up to $\mathcal{O}(\epsilon)$ on a time-scale of ϵ^{-1}.
For $p_{cr}^2 = -91 + \frac{10}{3}\sqrt{2457}$ it is shown that an energy transfer occurs between modes 1, 2, and 4, even if one of the modes has zero initial energy. Truncation to one or two modes will give loss of information in this case, and an approximation will not be valid. All three modes have to be taken into account. Examining the behavior of the oscillations in this case, we see that the system oscillates around an equilibrium state which is a combination of three modes (energy in all three modes) or reduces to a case which is similar to the case $p_{cr}^2 = \frac{17}{3}$. Also in this case detuning is considered. The system gradually changes from a combined oscillation of three modes to uncoupled oscillations of two of the three modes (one of the modes then has zero energy) as the value of p^2 moves away from $-91 + \frac{10}{3}\sqrt{2457}$ (as can be seen in Figure 3.3).
For $p_{cr}^2 = \frac{77}{3}$ it is shown that an energy transfer occurs between modes 1, 2, 3, and 4. It is shown that if mode 1 has zero initial energy, an energy transfer will occur between modes 2, 3, and 4 only (energy in mode 1 remains zero). In this case the analysis is similar to the case $p_{cr}^2 = -91 + \frac{10}{3}\sqrt{2457}$ (mode interaction between three modes). If mode 1 has nonzero initial energy but modes 2 and 4 have zero initial energy, an energy transfer will occur between modes 1 and 3 only (the energy in modes 2 and 4 remains zero). In this case the analysis is similar to the case $p_{cr}^2 = \frac{17}{3}$ (mode interaction between two modes). In all other cases a combined oscillation of all four modes occurs around an equilibrium state.
In this chapter an asymptotic theory is presented for an initial-boundary value problem for a nonlinear beam equation with a nonlinearity $f = f(x,t,w)$, using a fixed point theorem and certain properties of f. We expect that this theory can be extended to more complicated nonlinearities (due to external forces), such as a Rayleigh perturbation $(w_t - \frac{1}{3}w_t^3)$. Formal approximations for a nonlinear beam equation with this Rayleigh perturbation are constructed in Chapter 2. A similar fixed point theorem (in a different Banach space) can most likely be used for these approximations.

3.7 Appendix A - Green's function G and the integral equation

In this appendix we want to construct the Green's function G for the linear operator $\frac{\partial^4}{\partial x^4} + \frac{\partial^2}{\partial t^2} + p^2$ with simply supported boundary conditions. We will also derive the integral equation given in (3.2.8). The Green's function $G(\xi, \tau; x, t)$ is defined as the solution of the following problem:

$$G_{tt} + G_{xxxx} + p^2 G = \delta(x-\xi, t-\tau), \quad x, \xi \in]0, \pi[, t > 0, \tau > 0, \tag{3.7.1}$$

$$G(\xi, \tau; 0, t) = G(\xi, \tau; \pi, t) \equiv 0, \quad t > 0, \tau > 0, \tag{3.7.2}$$

$$G_{xx}(\xi, \tau; 0, t) = G_{xx}(\xi, \tau; \pi, t) \equiv 0, \quad t > 0, \tau > 0, \tag{3.7.3}$$

$$G(\xi, \tau; x, t) \equiv 0, \quad \tau \geq t. \tag{3.7.4}$$

The boundary conditions imply that G can be written as a Fourier sine series in x:

$$G(\xi, \tau; x, t) = \sum_{n=1}^{\infty} g_n(\xi, \tau; t) \sin(nx).$$

Substituting this series into (3.7.1) and using orthogonality properties of the sine functions, we obtain the following set of equations for g_n (where a dot represents differentiation with respect to t):

$$\ddot{g}_n + (n^4 + p^2) g_n = \frac{2}{\pi} \int_0^{\pi} \delta(x-\xi, t-\tau) \sin(nx) dx, \tag{3.7.5}$$

$$0 < \xi < \pi, t > 0, \tau > 0,$$

$$g_n(\xi, \tau; 0) = g_n(\xi, \tau; \tau) \equiv 0, \quad 0 < \xi < \pi, \tag{3.7.6}$$

for $n = 1, 2, 3, \ldots$. (3.7.5)-(3.7.6) can be solved using the method of "variation of constants." We find the following solutions:

$$g_n(\xi, \tau; t) = \frac{2}{\pi\sqrt{n^4 + p^2}} \sin\left[\sqrt{n^4 + p^2}(t-\tau)\right] H(t-\tau) \sin(n\xi),$$

for $n = 1, 2, 3, \ldots$, and therefore

$$G(\xi, \tau; x, t) = \tag{3.7.7}$$

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4 + p^2}} \sin\left[\sqrt{n^4 + p^2}(t-\tau)\right] H(t-\tau) \sin(n\xi) \sin(nx).$$

We find the integral equation (3.2.8) by multiplying $G_{\tau\tau} + G_{\xi\xi\xi\xi} + p^2 G = \delta(x-\xi, t-\tau)$, (which is equal to (3.7.1)) with $w(\xi, \tau)$ (where w satisfies the simply supported boundary conditions (3.1.5)-(3.1.6)) and integrating over $0 \leq \xi \leq \pi$, $0 \leq \tau \leq t$. Using integration by parts, the boundary conditions

(3.7.2)-(3.7.3) for G and the boundary conditions (3.1.5)-(3.1.6) for w, we obtain

$$w(x,t) = \epsilon \int_0^t \int_0^\pi G(\xi,\tau;x,t) f(\xi,\tau,w(\xi,\tau);\epsilon) d\xi d\tau \tag{3.7.8}$$
$$+ \int_0^\pi \{G(\xi,0;x,t)w_1(\xi;\epsilon) - G_\tau(\xi,0;x,t)w_0(\xi;\epsilon)\} d\xi = (Tw)(x,t).$$

3.8 Appendix B - Integral inequalities

In this appendix two integral inequalities are derived. These inequalities play an important role in the asymptotic theory presented in this chapter. Let f_1 be a four-times continuously differentiable function on $[0,\pi]$ and f_2 a twice continuously differentiable function on $[0,\pi]$. We will show that for all $x,t \in \Omega_L$ (defined in (3.2.11)) and $p \geq 0$ the following inequalities hold:

$$\left|\int_0^\pi G_\tau(\xi,0;x,t) f_1(\xi) d\xi\right| \leq \pi^2 \max_{0 \leq x \leq \pi} \left\{ p|f_1(x)| + \left|\frac{d^2 f_1(x)}{dx^2}\right| \right\}, \tag{3.8.1}$$

$$\left|\int_0^\pi G(\xi,0;x,t) f_2(\xi) d\xi\right| \leq \pi^2 \max_{0 \leq x \leq \pi} |f_2(x)|. \tag{3.8.2}$$

To prove these inequalities we consider the following linear initial-boundary value problem for a three-times continuously differentiable function $w(x,t)$, with w_{xxxx} continuous:

$$w_{xxxx} + w_{tt} + p^2 w = 0, \qquad 0 < x < \pi, t > 0, \tag{3.8.3}$$
$$w(0,t) = w(\pi,t) = 0, \qquad t \geq 0, \tag{3.8.4}$$
$$w_{xx}(0,t) = w_{xx}(\pi,t) = 0, \qquad t \geq 0, \tag{3.8.5}$$
$$w(x,0) = f_1(x), \quad w_t(x,0) = f_2(x), \qquad 0 < x < \pi. \tag{3.8.6}$$

Elementary calculations using Green's function G for the linear operator $\frac{\partial^4}{\partial x^4} + \frac{\partial^2}{\partial t^2} + p^2$ with simply supported boundary conditions show that the unique and three-times continuously differentiable solution of the initial-boundary value problem (3.8.3)-(3.8.6), which has a continuous fourth derivative with respect to x, is given by

$$w(x,t) = \int_0^\pi \{G(\xi,0;x,t)f_2(\xi) - G_\tau(\xi,0;x,t)f_1(\xi)\} d\xi. \tag{3.8.7}$$

To be able to estimate $|w(x,t)|$ and thus the integral given in (3.8.7) we will use the following energy equation related to the initial-boundary value problem (3.8.3)-(3.8.6):

$$\int_0^\pi \left\{ w_t^2(x,t_0) + p^2 w^2(x,t_0) + w_{xx}^2(x,t_0) \right\} dx$$
$$= \int_0^\pi \left\{ f_2^2(x) + p^2 f_1^2(x) + \left(\frac{d^2 f_1(x)}{dx^2}\right)^2 \right\} dx.$$

We obtain this energy equation by multiplying (3.8.3) with w_t and integrating with respect to x and t over $0 \leq x \leq \pi$, $0 \leq t \leq t_0$, using the initial and boundary conditions (3.8.4)-(3.8.6). On the other hand, we have, since w_x exists and is continuous,

$$w(x,t) = w(0,t) + \int_0^x w_\xi(\xi,t)d\xi = \int_0^x w_\xi(\xi,t)d\xi. \tag{3.8.8}$$

Since w_{xx} exists and is continuous, and since $w(0,t) = w(\pi,t) \equiv 0$ we know that there exists an $\eta \in]0,\pi[$ with $w_x(\eta,t) \equiv 0$, so

$$w_x(x,t) = w_x(\eta,t) + \int_\eta^x w_{\xi\xi}(\xi,t)d\xi = \int_\eta^x w_{\xi\xi}(\xi,t)d\xi. \tag{3.8.9}$$

Using (3.8.9) and Hölder's inequality we now have

$$\begin{aligned}
|w_x(x,t)| &\leq \left|\int_\eta^x |w_{\xi\xi}(\xi,t)|\,d\xi\right| \leq \int_0^\pi |w_{xx}(x,t)|\,dx \\
&\leq \left(\int_0^\pi 1^2 dx\right)^{\frac{1}{2}} \left(\int_0^\pi w_{xx}^2 dx\right)^{\frac{1}{2}} \leq \pi^{\frac{1}{2}} \left(\int_0^\pi \left\{w_t^2 + p^2 w^2 + w_{xx}^2\right\} dx\right)^{\frac{1}{2}} \\
&\leq \pi^{\frac{1}{2}} \left(\int_0^\pi \left\{f_2^2(x) + p^2 f_1^2(x) + \left(\frac{d^2 f_1(x)}{dx^2}\right)^2\right\} dx\right)^{\frac{1}{2}} \\
&\leq \pi^{\frac{1}{2}} \left(\int_0^\pi \max_{0\leq x\leq\pi} \left\{f_2^2(x) + p^2 f_1^2(x) + \left(\frac{d^2 f_1(x)}{dx^2}\right)^2\right\} dx\right)^{\frac{1}{2}} \\
&\leq \pi \max_{0\leq x\leq\pi} \left\{f_2^2(x) + p^2 f_1^2(x) + \left(\frac{d^2 f_1(x)}{dx^2}\right)^2\right\}^{\frac{1}{2}} \\
&\leq \pi \max_{0\leq x\leq\pi} \left\{|f_2(x)| + p\,|f_1(x)| + \left|\frac{d^2 f_1(x)}{dx^2}\right|\right\}.
\end{aligned}$$

Using (3.8.8) we now get the following inequality for $|w(x,t)|$:

$$\begin{aligned}
|w(x,t)| &\leq \int_0^x |w_\xi(\xi,t)|\,d\xi \leq \int_0^\pi |w_x(x,t)|\,dx \\
&\leq \int_0^\pi \left(\pi \max_{0\leq x\leq\pi} \left\{|f_2(x)| + p\,|f_1(x)| + \left|\frac{d^2 f_1(x)}{dx^2}\right|\right\}\right) dx \\
&\leq \pi^2 \max_{0\leq x\leq\pi} \left\{|f_2(x)| + p\,|f_1(x)| + \left|\frac{d^2 f_1(x)}{dx^2}\right|\right\}.
\end{aligned} \tag{3.8.10}$$

Hence it follows from (3.8.7) and (3.8.10) that if $f_2(x) \equiv 0$, then

$$\left|\int_0^\pi G_T(\xi,0;x,t) f_1(\xi)d\xi\right| \leq \pi^2 \max_{0\leq x\leq\pi} \left\{p|f_1(x)| + \left|\frac{d^2 f_1(x)}{dx^2}\right|\right\},$$

and similarly, if we take $f_1(x) \equiv 0$, we get

$$\left| \int_0^\pi G(\xi, 0; x, t) f_2(\xi) d\xi \right| \leq \pi^2 \max_{0 \leq x \leq \pi} |f_2(x)|,$$

which proves that the integral inequalities (3.8.1) and (3.8.2) hold.

3.9 Appendix C - Determination and elimination of secular terms

In this appendix we show which differential equations the functions q_n have to satisfy such that no secular terms occur in the approximations for the displacement function $w(x, t)$. In Section 3.4, (3.4.4) was given:

$$\sum_{m=1}^{\infty} \left(\ddot{q}_m + (m^4 + p^2) q_m \right) \sin(mx) \tag{3.9.1}$$
$$= \epsilon \frac{4}{\pi} \sum_{m,k=1}^{\infty} \sum_{j=0}^{\infty} \frac{1}{2j+1} q_m q_k \sin(mx) \sin(kx) \sin((2j+1)x).$$

The last summation on the right-hand side of (3.9.1) can be rewritten using the goniometric formula $\sin(mx)\sin(kx)\sin(lx) = \frac{1}{4}(\sin((m+k-l)x) - \sin((m-k-l)x) - \sin((m+k+l)x) + \sin((m-k+l)x))$, where $l = 2j+1$ in this case. We obtain the equations for q_n by multiplying (3.9.1) with $\frac{2}{\pi}\sin(nx)$ and then by integrating the so-obtained equation with respect to x from 0 to π. Using orthogonality relations and the symmetry in m, k we obtain the following equation for each q_n, for $n = 1, 2, 3, \ldots$:

$$\ddot{q}_n + (n^4 + p^2) q_n \tag{3.9.2}$$
$$= \frac{\epsilon}{\pi} \left(2 \sum_{n=m-k+2j+1} - 2 \sum_{n=m-k-2j-1} + \sum_{n=m+k-2j-1} - \sum_{n=m+k+2j+1} - \sum_{n=-m-k+2j+1} \right) \frac{1}{2j+1} q_m q_k.$$

To avoid secular terms in $q_n(t)$ a two time-scales perturbation method is introduced in Section 3.4 and $q_n(t)$ is expanded in $q_n(t) = q_{n,0}(t_0, t_1) + \epsilon q_{n,1}(t_0, t_1) + \epsilon^2 q_{n,2}(t_0, t_1) + \cdots$, where $t_0 = t$ and $t_1 = \epsilon t$. It has been shown in Section 3.4 that $q_{n,1}$ has to satisfy (3.4.12):

$$\frac{\partial^2}{\partial t_0^2} q_{n,1} + \omega_{n_p}^2 q_{n,1} = 2\omega_{n_p} \left(\frac{dA_{n,0}}{dt_1} \sin(\omega_{n_p} t_0) - \frac{dB_{n,0}}{dt_1} \cos(\omega_{n_p} t_0) \right) \tag{3.9.3}$$
$$+ \frac{1}{\pi} \left(2 \sum_{n=m-k+2j+1} - 2 \sum_{n=m-k-2j-1} + \sum_{n=m+k-2j-1} \right.$$

$$- \sum_{n=m+k+2j+1} - \sum_{n=-m-k+2j+1} \Bigg) \frac{1}{2j+1} \mathcal{H},$$

where

$$\mathcal{H} = (A_{m,0} \cos(\omega_{m_p} t_0) + B_{m,0} \sin(\omega_{m_p} t_0))(A_{k,0} \cos(\omega_{k_p} t_0) + B_{k,0} \sin(\omega_{k_p} t_0)),$$

with $\omega_{l_p} = \sqrt{l^4 + p^2}$, and where $A_{n,0}$ and $B_{n,0}$ are still arbitrary functions in t_1. The functions $A_{n,0}$ and $B_{n,0}$ will now be determined such that no secular terms occur in $q_{n,1}$. The last term in (3.9.3) can be expanded using goniometric formulas:

$$\begin{aligned}
\mathcal{H} = &\frac{1}{2} (A_{m,0} A_{k,0} - B_{m,0} B_{k,0}) \cos\left(\left[\sqrt{m^4 + p^2} + \sqrt{k^4 + p^2}\right] t_0\right) \\
&+ \frac{1}{2} (A_{m,0} B_{k,0} + B_{m,0} A_{k,0}) \sin\left(\left[\sqrt{m^4 + p^2} + \sqrt{k^4 + p^2}\right] t_0\right) \\
&+ \frac{1}{2} (A_{m,0} A_{k,0} + B_{m,0} B_{m,0}) \cos\left(\left[\sqrt{m^4 + p^2} - \sqrt{k^4 + p^2}\right] t_0\right) \\
&+ \frac{1}{2} (-A_{m,0} B_{k,0} + B_{m,0} A_{m,0}) \sin\left(\left[\sqrt{m^4 + p^2} - \sqrt{k^4 + p^2}\right] t_0\right).
\end{aligned} \tag{3.9.4}$$

As stated in Section 3.4, $\cos(\omega_{n_p} t_0)$ and $\sin(\omega_{n_p} t_0)$ are part of the homogeneous solution for $q_{n,1}$. We want the coefficients of $\cos(\omega_{n_p} t_0)$ and $\sin(\omega_{n_p} t_0)$ on the right-hand side of (3.9.3) to be equal to zero in order to eliminate secular terms. The terms given in (3.9.4) can cause secular terms if $\pm\sqrt{n^4 + p^2} = \pm\sqrt{m^4 + p^2} \pm \sqrt{k^4 + p^2}$. To determine the contribution of the summations in (3.9.3) to the coefficients of $\cos(\omega_{n_p} t_0)$ and $\sin(\omega_{n_p} t_0)$ on the right-hand side of (3.9.3), we have to examine the following Diophantine-like equations:

$$\mathrm{I} \; : \; \begin{cases} n = m + k + \lambda, \\ \pm\sqrt{n^4 + p^2} = \pm\sqrt{m^4 + p^2} \pm \sqrt{k^4 + p^2}, \end{cases} \tag{3.9.5}$$

$$\mathrm{II} \; : \; \begin{cases} n = m + k - \lambda, \\ \pm\sqrt{n^4 + p^2} = \pm\sqrt{m^4 + p^2} \pm \sqrt{k^4 + p^2}, \end{cases} \tag{3.9.6}$$

$$\mathrm{III} \; : \; \begin{cases} n = m - k + \lambda, \\ \pm\sqrt{n^4 + p^2} = \pm\sqrt{m^4 + p^2} \pm \sqrt{k^4 + p^2}, \end{cases} \tag{3.9.7}$$

$$\mathrm{IV} \; : \; \begin{cases} n = m - k - \lambda, \\ \pm\sqrt{n^4 + p^2} = \pm\sqrt{m^4 + p^2} \pm \sqrt{k^4 + p^2}, \end{cases} \tag{3.9.8}$$

$$\mathrm{V} \; : \; \begin{cases} n = -m - k + \lambda, \\ \pm\sqrt{n^4 + p^2} = \pm\sqrt{m^4 + p^2} \pm \sqrt{k^4 + p^2}, \end{cases} \tag{3.9.9}$$

where $\lambda = 2j + 1$ is odd and ≥ 1, $p^2 \in (0, 100)$, and $m, k \geq 1$. We want to find out when (3.9.5)-(3.9.9) have solutions and thus give rise to secular terms. In the following we will use the inequality

$$x^2 < \sqrt{x^4 + p^2} \leq x^2 - a^2 + \sqrt{a^4 + p^2} \tag{3.9.10}$$

for $x \geq a$. We will now determine the solutions of (3.9.5)-(3.9.9). We will show that only for specific values of p^2 solutions of (3.9.5)-(3.9.9) can be found. We can see directly that (3.9.5)-(3.9.9) do not hold if $\sqrt{n^4 + p^2} = -\sqrt{m^4 + p^2} - \sqrt{k^4 + p^2}$. This leaves us with the following possibilities.
Case I(i):

$$\begin{aligned} n &= m + k + \lambda, & (3.9.11) \\ \sqrt{n^4 + p^2} &= \sqrt{m^4 + p^2} + \sqrt{k^4 + p^2}. & (3.9.12) \end{aligned}$$

From (3.9.10), (3.9.12), and $n \geq 1$, $m \geq 1$, $k \geq 1$, it follows that $n^2 < \sqrt{n^4 + p^2} \leq m^2 + k^2 - 2 + 2\sqrt{1 + p^2}$. It also follows from (3.9.11) that $n^2 = m^2 + k^2 + \lambda^2 + 2(mk + (m + k)\lambda) = m^2 + k^2 + \mu$, where $\mu = \lambda^2 + 2(mk + (m + k)\lambda)$. This gives us

$$\mu < 2(\sqrt{1 + p^2} - 1). \tag{3.9.13}$$

On the other hand, since $\lambda \geq 1$ and odd and $m \geq 1$, $k \geq 1$ we can see that $\mu \geq 7$ and odd. When $p^2 \leq \frac{77}{4}$, (3.9.13) contradicts $\mu \geq 7$, so (3.9.11)-(3.9.12) have no solutions. For larger values of p^2, solutions can occur. For instance, if we take $m = k = 1$ and $\lambda = 1$, then it follows from (3.9.11) that $n = 3$, and (3.9.12) can then be satisfied for $p^2 = \frac{77}{3}$. For $m \neq 1$ we can improve the lower and upper bounds for μ. From the definition of μ it follows that for $m \geq 2$, $k \geq 1$, $\lambda \geq 1$ we have $\mu \geq 11$. However, using (3.9.10), we can show that $\mu < \sqrt{2^4 + p^2} + \sqrt{1 + p^2} - 5$, which contradicts $\mu \geq 11$ when $p^2 \leq 55.72$. So (3.9.11)-(3.9.12) have no solutions for $m \geq 2$, $k \geq 1$, $\lambda \geq 1$. For larger values of p^2, solutions can occur. For instance, if we take $m = 2$, $k = 1$, and $\lambda = 1$, it follows from (3.9.11) that $n = 4$, and (3.9.12) can then be satisfied for $p^2 = -91 + \frac{10}{3}\sqrt{2457} \approx 74.23$. Due to symmetry in m and k, $k = 2$, $m = \lambda = 1$, $n = 4$ is also a solution for $p^2 \approx 74.23$. For $m \neq 2$ we can improve the lower and upper bounds for μ. From the definition of μ it follows that for $m \geq 3$, $k \geq 1$, $\lambda \geq 1$ we have $\mu \geq 15$. Conversely, using (3.9.10), we can show that $\mu < \sqrt{3^4 + p^2} + \sqrt{1 + p^2} - 10$, which contradicts $\mu \geq 15$ when $p^2 \leq 100$. So (3.9.11)-(3.9.12) have no solutions for $m \geq 3$, $k \geq 1$, $\lambda \geq 1$. For $m \neq 1$ and $k \neq 1$ we can improve the lower and upper bounds for μ. From the definition of μ it follows that for $m \geq 2$, $k \geq 2$, $\lambda \geq 1$ we have $\mu \geq 17$. On the other hand, using (3.9.10), we can show that $\mu < 2(\sqrt{2^4 + p^2} - 4)$, which contradicts $\mu \geq 17$

when $p^2 \leq 100$. So (3.9.11)-(3.9.12) have no solutions for $m \geq 2$, $k \geq 2$, $\lambda \geq 1$. For $\lambda \neq 1$ we can improve the lower and upper bounds for μ. From the definition of μ it follows that for $m \geq 1$, $k \geq 1$, $\lambda \geq 3$ we have $\mu \geq 23$. However, using (3.9.10), we can show that $\mu < 2(\sqrt{1+p^2}-1)$, which contradicts $\mu \geq 23$ when $p^2 \leq 100$. So (3.9.11)-(3.9.12) have no solutions for $m \geq 1$, $k \geq 1$, $\lambda \geq 2$. This means that for $p^2 \in (0, 100)$, (3.9.11)-(3.9.12) have the following solutions:

$$\begin{aligned} &\lambda = 1, \quad k = 1, \quad m = 1, \quad n = 3, \quad p^2 = \frac{77}{3}, \\ &\lambda = 1, \quad k = 1, \quad m = 2, \quad n = 4, \quad p^2 \approx 74.23. \end{aligned}$$

Due to symmetry in m, k, m, and k can be switched. It can be shown that more solutions exist for $p^2 > 100$.

Case I(ii):

$$\begin{aligned} n &= m + k + \lambda, && (3.9.14) \\ \sqrt{n^4+p^2} &= \sqrt{m^4+p^2} - \sqrt{k^4+p^2}. && (3.9.15) \end{aligned}$$

We show that (3.9.14) and (3.9.15) do not have any solutions, by contradiction. Suppose both equations hold. From (3.9.14) we have $n^4 = (m+k+\lambda)^4 > m^4 + k^4$. On the other hand, if we square (3.9.15), we obtain $2\sqrt{m^4+p^2}\sqrt{k^4+p^2} = m^4 + k^4 - n^4 + p^2$. With $n^4 > m^4 + k^4$ we have $2p^2 < 2\sqrt{m^4+p^2}\sqrt{k^4+p^2} < p^2$. So (3.9.14)-(3.9.15) cannot hold.

Case I(iii): $n = m+k+\lambda$, $\sqrt{n^4+p^2} = -\sqrt{m^4+p^2} + \sqrt{k^4+p^2}$ is equivalent to Case I(ii) with m and k switched.

Case II(i):

$$\begin{aligned} n &= m + k - \lambda, && (3.9.16) \\ \sqrt{n^4+p^2} &= \sqrt{m^4+p^2} + \sqrt{k^4+p^2}. && (3.9.17) \end{aligned}$$

From (3.9.10) and (3.9.17) we know that $n^2 < \sqrt{n^4+p^2} \leq m^2 + k^2 - 2 + 2\sqrt{1+p^2}$, since n, m, $k \geq 1$. We also know from (3.9.16) that $n^2 = m^2 + k^2 + \lambda^2 + 2(mk - (m+k)\lambda) = m^2 + k^2 + \nu$, where $\nu = \lambda^2 + 2(mk - (m+k)\lambda)$. This gives us

$$\nu < 2(\sqrt{1+p^2} - 1). \tag{3.9.18}$$

From the definition of ν we see that ν is odd. We first show that for $\nu \leq 0$ (3.9.16)-(3.9.17) cannot hold. We introduce $N = n^2$, $M = m^2$, and $K = k^2$. This gives us the following:

$$N = M + K + \nu, \tag{3.9.19}$$

$$\sqrt{N^2+p^2} = \sqrt{M^2+p^2} + \sqrt{K^2+p^2}. \tag{3.9.20}$$

From (3.9.20) we see that $K < N$. This means (the proof is similar to a proof in [26])

$$\frac{1}{N+\sqrt{N^2+p^2}} < \frac{1}{K+\sqrt{K^2+p^2}} < \frac{1}{K+\sqrt{K^2+p^2}} + \frac{1}{M+\sqrt{M^2+p^2}},$$
$$\iff \frac{1}{p^2}\left(\sqrt{N^2+p^2}-N\right) < \frac{1}{p^2}\left(\sqrt{M^2+p^2}-M+\sqrt{K^2+p^2}-K\right),$$
$$\iff \sqrt{N^2+p^2} < \sqrt{M^2+p^2}+\sqrt{K^2+p^2}+\nu.$$

We see that when $\nu \leq 0$ (3.9.19)-(3.9.20) cannot hold. So only for $\nu > 0$ may solutions of (3.9.16)-(3.9.17) occur. From (3.9.18) we see that for $p^2 \leq 100$ we have $\nu \leq 18.10$.

We can improve this upper bound of ν. Since $n^2 = m^2 + k^2 + \nu$ with $\nu > 0$ we have $n > m$ and therefore $k > \lambda$ (from (3.9.16)). Due to symmetry in m and k we can assume $m \geq k$. This means $n > m \geq k > \lambda \geq 1$ so $\lambda \geq 1$, $k \geq 2$, $m \geq 2$, $n \geq 3$. Using (3.9.10) we now obtain $n^2 < \sqrt{n^4+p^2} \leq m^2 + k^2 - 8 + 2\sqrt{2^4+p^2}$, so $\nu < 2(\sqrt{2^4+p^2} - 4)$. For $p^2 \leq 100$ we then have $\nu \leq 13.54$. Since ν is odd and > 0 it suffices to examine the cases $\nu = 1, 3, 5, 7, 9, 11, 13$ for $\lambda \geq 1$, $m \geq k \geq 2$, $n \geq 3$.

Suppose $\lambda = 1$. From the definition of ν we can derive $m = 1 + \frac{1+\nu}{2(k-1)}$. For $\nu = 1$, $m = 1 + \frac{1}{k-1}$. The only possible solution is $k = m = 2$. Then it follows from (3.9.16) that $n = 3$, and (3.9.17) can be satisfied for $p^2 = \frac{17}{3}$. For $\nu = 3$, $m = 1 + \frac{2}{k-1}$. The two possible solutions are $k = 2$, $m = 3$ and $m = 2$, $k = 3$. Due to symmetry in m, k it suffices to examine the case $k = 2$, $m = 3$. Then it follows from (3.9.16) that $n = 4$, and (3.9.17) can be satisfied for $p^2 = \frac{77}{3}$. For $\nu = 5$, $m = 1 + \frac{3}{k-1}$. The two possible solutions are $k = 2$, $m = 4$ and $m = 2$, $k = 4$. Due to symmetry in m, k it suffices to examine the case $k = 2$, $m = 4$. Then it follows from (3.9.16) that $n = 5$, and (3.9.17) can be satisfied for $p^2 = -299 + 2\sqrt{31369} \approx 55.23$. For $\nu = 7$, $m = 1 + \frac{4}{k-1}$. The three possible solutions are $k = 2$, $m = 5$; $m = 2$, $k = 5$; and $k = m = 3$. Due to symmetry it suffices to examine the cases $k = 2$, $m = 5$, and $k = m = 3$. When $k = 2$, $m = 5$ it follows from (3.9.16) that $n = 6$, and (3.9.17) can be satisfied for $p^2 = -\frac{1937}{3} + \frac{2}{3}\sqrt{1229761} \approx 93.63$. When $k = m = 3$ it follows from (3.9.16) that $n = 5$, but in that case (3.9.17) can be satisfied only for $p^2 > 100$. This analysis can be continued for $\nu = 9, 11, 13$. No more solutions are found for $p^2 \leq 100$.

For $\lambda \neq 1$ we can improve the upper bound of ν. We now have $\lambda \geq 3$, $k \geq 4$, $m \geq 4$, $n \geq 5$. Using (3.9.10) we now obtain $\nu < 2(\sqrt{4^4+p^2} - 16)$, and for $p^2 \leq 100$ we then have $\nu < 5.74$. This upper bound can be improved even further since it can be shown that $m \neq k$ for $\nu \leq 5$. So we have $\lambda \geq 3$, $k \geq 4$, $m \geq 5$, $n \geq 6$, and $\nu < \sqrt{5^4+p^2} + \sqrt{4^4+p^2} - 41$, and for $p^2 \leq 100$ we then have $\nu \leq 4.79$. This means that for $\lambda = 3$ it suffices

to examine the cases $\nu = 1, 3$. From the definition of ν we can derive $m = 3 + \frac{9+\nu}{2(k-3)}$. For $\nu = 1$, $m = 3 + \frac{5}{k-3}$. The two possible solutions are $k = 4$, $m = 8$ and $m = 4$, $k = 8$. Due to symmetry it suffices to examine the case $k = 4$, $m = 8$. Then it follows from (3.9.16) that $n = 9$, and (3.9.17) can be satisfied for $p^2 = -\frac{10913}{3} + \frac{10}{3}\sqrt{1211497} \approx 31.27$. For $\nu = 3$ a similar analysis holds, but no solutions are found for $p^2 \leq 100$.
We continue with $\lambda = 5$. With a similar analysis as above the upper bound for ν can be improved. We obtain $\nu < 2(\sqrt{6^4 + p^2} - 36)$, and for $p^2 \leq 100$ we then have $\nu \leq 2.73$. This means that for $\lambda = 5$ it suffices to examine the case $\nu = 1$. From the definition of ν we can derive $m = 5 + \frac{25+\nu}{2(k-5)}$. For $\nu = 1$, $m = 5 + \frac{13}{k-5}$. The two possible solutions are $k = 6$, $m = 18$ and $m = 6$, $k = 18$. Due to symmetry it suffices to examine the case $k = 6$, $m = 18$. Then it follows from (3.9.16) that $n = 19$, and (3.9.17) can be satisfied for $p^2 = -\frac{236593}{3} + \frac{10}{3}\sqrt{560787241} \approx 72.16$. This can be continued for $\lambda \geq 7$. It can be shown that no more solutions exist for $p^2 \in (0, 100)$. So for $p^2 \in (0, 100)$, (3.9.16)-(3.9.17) have the following solutions:

$$
\begin{array}{lllll}
\lambda = 1, & k = 2, & m = 2, & n = 3 & p^2 = \dfrac{17}{3}, \\
\lambda = 1, & k = 2, & m = 3, & n = 4 & p^2 = \dfrac{77}{3}, \\
\lambda = 3, & k = 4, & m = 8, & n = 9 & p^2 \approx 31.27, \\
\lambda = 1, & k = 2, & m = 4, & n = 5 & p^2 \approx 55.23, \\
\lambda = 5, & k = 6, & m = 18, & n = 19 & p^2 \approx 72.16, \\
\lambda = 1, & k = 2, & m = 5, & n = 6 & p^2 \approx 93.63.
\end{array}
$$

Due to symmetry in m, k, m and k can be switched. It can be shown that more solutions occur for $p^2 > 100$.
Case II(ii): $n = m + k - \lambda$, $\sqrt{n^4 + p^2} = \sqrt{m^4 + p^2} - \sqrt{k^4 + p^2}$ is equivalent to Case I(i) with n and m switched and λ replaced by $\lambda + 2k$, and to Case II(i) with n and m switched and λ replaced by $-\lambda + 2k$.
Case II(iii): $n = m+k-\lambda$, $\sqrt{n^4 + p^2} = -\sqrt{m^4 + p^2} + \sqrt{k^4 + p^2}$ is equivalent to Case II(ii) with m and k switched.
Case III(i): $n = m - k + \lambda$, $\sqrt{n^4 + p^2} = \sqrt{m^4 + p^2} + \sqrt{k^4 + p^2}$ is equivalent to Case I(i) where λ is replaced by $\lambda + 2k$ and to Case II(i) where λ is replaced by $-\lambda + 2k$.
Case III(ii): $n = m-k+\lambda$, $\sqrt{n^4 + p^2} = \sqrt{m^4 + p^2} - \sqrt{k^4 + p^2}$ is equivalent to Case II(i) with n and m switched.
Case III(iii): $n = m - k + \lambda$, $\sqrt{n^4 + p^2} = -\sqrt{m^4 + p^2} + \sqrt{k^4 + p^2}$ is equivalent to Case III(ii) with m and k switched.
Case IV(i): $n = m - k - \lambda$, $\sqrt{n^4 + p^2} = \sqrt{m^4 + p^2} + \sqrt{k^4 + p^2}$ is equivalent to Case I(ii) with n and m switched.

Case IV(ii): $n = m - k - \lambda$, $\sqrt{n^4 + p^2} = \sqrt{m^4 + p^2} - \sqrt{k^4 + p^2}$ is equivalent to Case I(i) with n and m switched.
Case IV(iii): $n = m - k - \lambda$, $\sqrt{n^4 + p^2} = -\sqrt{m^4 + p^2} + \sqrt{k^4 + p^2}$ is equivalent to Case IV(ii) with m and k switched.
Case V(i): $n = -m - k + \lambda$, $\sqrt{n^4 + p^2} = \sqrt{m^4 + p^2} + \sqrt{k^4 + p^2}$ is equivalent to Case I(i) where λ is replaced by $\lambda + 2k + 2m$ and to Case II(i) where λ is replaced by $-\lambda + 2k + 2m$.
Case V(ii): $n = -m - k + \lambda$, $\sqrt{n^4 + p^2} = \sqrt{m^4 + p^2} - \sqrt{k^4 + p^2}$ is equivalent to Case V(i) with n and m switched.
Case V(iii): $n = -m - k + \lambda$, $\sqrt{n^4 + p^2} = -\sqrt{m^4 + p^2} + \sqrt{k^4 + p^2}$ is equivalent to Case V(ii) with m and k switched.
Using these results the secular terms in the right-hand side of (3.9.3) can be determined explicitly: In the general case the summations in (3.9.3) do not give any contributions and the only secular terms on the right-hand side of (3.9.3) are $2\omega_{n_p} \frac{dA_{n,0}}{dt_1} \sin(\omega_{n_p} t_0)$ and $-2\omega_{n_p} \frac{dB_{n,0}}{dt_1} \cos(\omega_{n_p} t_0)$. For $p^2 = p_{cr}^2 + \epsilon\alpha$, α a detuning parameter, and $p_{cr}^2 \in \{\frac{17}{3}, \frac{77}{3}, -\frac{10913}{3} + \frac{10}{3}\sqrt{1211497}, -299 + 2\sqrt{31369}, -\frac{236593}{3} + \frac{10}{3}\sqrt{560787241}, -91 + \frac{10}{3}\sqrt{2457}, -\frac{1937}{3} + \frac{2}{3}\sqrt{1229761}\}$ extra secular terms occur and the total of secular terms on the right-hand side of (3.9.3) is $\Big(2\omega_{n_{p_{cr}}} \frac{dA_{n,0}}{dt_1} + \alpha B_{n,0} + \mathcal{F}\Big)$ $\sin(\omega_{n_{p_{cr}}} t_0) + \Big(-2\omega_{n_{p_{cr}}} \frac{dB_{n,0}}{dt_1} + \alpha A_{n,0} + \mathcal{G}\Big) \cos(\omega_{n_{p_{cr}}} t_0)$, where $\mathcal{F}$ and $\mathcal{G}$ can be determined explicitly by substituting the possible combinations found above for m, k, j for every p_{cr}^2 into the last term on the right-hand side of (3.9.3). The equation for $A_{n,0}$ (resp., $B_{n,0}$) can then be easily determined by setting the coefficients of $\sin(\omega_{n_{p_{cr}}} t_0)$ (resp., $\cos(\omega_{n_{p_{cr}}} t_0)$) equal to zero.

Chapter 4

A Weakly Nonlinear Beam Equation with an Integral Nonlinearity†

Abstract In this chapter an initial-boundary value problem for the vertical displacement of a weakly nonlinear elastic beam with an harmonic excitation in horizontal direction at the ends of the beam will be studied. The initial-boundary value problem can be regarded as a simple model describing oscillations of flexible structures like suspension bridges or iced overhead transmission lines. Using a two time-scales perturbation method an approximation of the solution of the initial-boundary problem will be constructed. Interactions between different oscillation modes of the beam will be studied. It will be shown that for certain external excitations, depending on the phase of an oscillation mode, the amplitude of specific oscillation modes changes.

4.1 Introduction

Flexible structures, like tall buildings, suspension bridges or iced overhead transmission lines with bending stiffness, are subjected to oscillations due to different causes. Simple models which describe these oscillations can involve nonlinear second and fourth order PDE's, as can be seen for example in [1] or [9]. In many cases perturbation methods can be used to construct approximations for solutions of this type of second or fourth order equations. Initial-boundary value problems for second order PDE's have been considered for a long time, for instance in [26]-[30],[33] and [34]. These

†This chapter is a revised version of [42] *On Interactions of Oscillation Modes for a Weakly Nonlinear Undamped Elastic Beam with an External Force*, accepted for publication in J. Sound and Vibration and scheduled to appear in August 2000

problems have been studied in [1]-[3],[31],[32],[40] and [41], using a two time-scales perturbation method or a Galerkin-averaging method to construct approximations. For fourth order PDE's the analysis is more complex. In a number of papers ([9]-[11] and [43]) approximations for solutions of initial-boundary value problems for fourth order weakly nonlinear PDE's are constructed using perturbation methods. In most cases the solutions are approximated by a single mode representation, without justification whether truncation to one mode is valid. In this chapter approximations are constructed using a two time-scales perturbation method. The interaction between the different oscillation modes is studied and a justification is given in which cases mode truncation is valid. For fourth order strongly nonlinear PDE's numerical finite element methods can be used, as is done for example in [35].

In this chapter we will consider the following initial-boundary value problem, which describes, up to $\mathcal{O}(\epsilon)$, the vertical displacement of an elastic beam with a linear spring force and a constant gravity force acting on it, and with an external force $F(t)$ acting on the ends of the beam in horizontal direction:

$$w_{tt} + w_{xxxx} + p^2 w = \epsilon \left(F(t) + \frac{2}{\pi} \int_0^{\pi} w_x^2 dx \right) w_{xx}, 0 < x < \pi, \ t > 0, \quad (4.1.1)$$

$$w(0,t) = w(\pi,t) = 0, t \geq 0, \quad (4.1.2)$$

$$w_{xx}(0,t) = w_{xx}(\pi,t) = 0, t \geq 0, \quad (4.1.3)$$

$$w(x,0) = w_0(x), w_t(x,0) = w_1(x), 0 < x < \pi, \quad (4.1.4)$$

where $F(t) = u(\pi,t) - u(0,t)$ and ϵ a small dimensionless parameter. For the derivation of this problem we refer to Section 4.2.

In this chapter formal approximations, i.e. functions that satisfy the differential equation and the initial and boundary values up to some order in ϵ, will be constructed for the initial-boundary value problem (4.1.1)-(4.1.4), using a Fourier mode expansion and a two time-scales perturbation method. The interaction (energy exchange) between the different oscillation modes will be considered for the cases $F(t) = 0$ (no external forcing, that is, the ends of the beams are fixed in horizontal direction, i.e. the case of free vibrations) and $F(t) = C\cos(\omega t)$ (external forcing). It will be shown that in the case $F(t) = 0$ the amplitudes of the different modes are constant and the only interaction between the modes occurs in the phases of the different oscillations modes (for example mode n causes a phase shift of the phases of all other modes $m \neq n$). No internal resonances occur. In this chapter we mean by internal resonance that there is an energy transfer from one oscillation mode to another oscillation mode. So by no internal resonance we mean no energy transfer occurs between the different oscillation modes (up to $\mathcal{O}(\epsilon)$ on a time-scale of order ϵ^{-1}). The case $F(t) = C\cos(\omega t)$ is more complicated. It will be shown that for most values of ω the analysis is similar to the case $F(t) = 0$. The influence of $F(t)$ in that case is of $\mathcal{O}(\epsilon)$ on a time-scale of order ϵ^{-1}, and extra terms appear in the $\mathcal{O}(\epsilon)$-approximation. However, for specific values of ω, i.e. $\omega \approx 2\omega_{k_p}$, where ω_{k_p} is an eigenfrequency of the linearized system ($\epsilon = 0$), the influence of $F(t)$ is of $\mathcal{O}(1)$ on a time-scale of order ϵ^{-1}. The amplitude of mode k is no longer constant, but the amplitudes of all other modes remain constant. The mode interactions remain restricted to phase-shifts of the phases of the different oscillation modes. Similar mode interactions have been studied for example in [36],[40],[41] and [46], but to our knowledge these mode interactions for weakly nonlinear beam equations have not yet been studied thoroughly. The analysis presented in this chapter holds for all p^2-values, which is different from the analysis in Chapter 2 or Chapter 3, where mode interactions and internal resonances occur for specific p^2-values.

The outline of the chapter is as follows. In Section 4.2 the initial-boundary value problem (4.1.1)-(4.1.4) will be derived. In Section 4.3 we apply a two time-scales perturbation method to the initial-boundary value problem (4.1.1)-(4.1.4). We show that for most values of ω the amplitudes of the different oscillation modes remain constant. For specific ω-values the oscillation of specific modes changes and the amplitudes of certain modes is no longer constant. We construct a formal approximation of $\mathcal{O}(\epsilon)$ for solutions of the initial-boundary value problem for the cases $F(t) = 0$, $F(t) = C\cos(\omega t)$ with $\omega \neq 2\omega_{k_p} + \epsilon\alpha$ and $F(t) = C\cos(\omega t)$ with $\omega = 2\omega_{k_p} + \epsilon\alpha$, where $\alpha \in \mathbb{R}$ is a detuning parameter. In Section 4.4 the mode interactions between the different oscillation modes will be studied in detail for the three cases mentioned above. In Section 4.5 some conclusions and general remarks will be given.

4.2 Mathematical Formulation Of The Problem

To derive the equations of motion for an elastic beam we will follow part of the analysis given in [37]. We consider an elastic beam of length l, simply supported at the ends, in vertical direction. An external force is be applied at the ends of the beam such that no vertical displacement is possible. Oscillations are possible due to the strain of the beam. The x-axis is taken along the beam axis, such that the left end of the beam corresponds with $x = 0$. The z-axis is taken vertically. The y-axis is perpendicular to the (x, z)-plane. We assume that the beam can move in the x- and z-direction only. We introduce the following symbols: μ is the mass of the beam per unit length, ρ the mass density of the beam, A the area of the cross-section Q of the beam perpendicular to the x-axis (so $\mu = \rho A$), E the elasticity modulus (Young's modulus), I the axial moment of inertia of the cross-section. The inertial axes of the cross-section Q are the y- and z-axes, so $I = \int\int_Q z^2 dydz$. The vertical displacement of the beam from rest is $w = w(x,t)$, the horizontal displacement of the beam is $u = u(x,t)$. The curvature of the beam in the (x,z)-plane can be approximated by w_{xx} as follows. We consider an element of the beam of length $\triangle x$ in static state. In deformed state the arc length of this element is $\triangle s$, where $\triangle s \approx \mathcal{R}\triangle\varphi$, with $\triangle\varphi$ the arc angle, as can be seen in Figure 1.1 and $\mathcal{R}$ the radius of

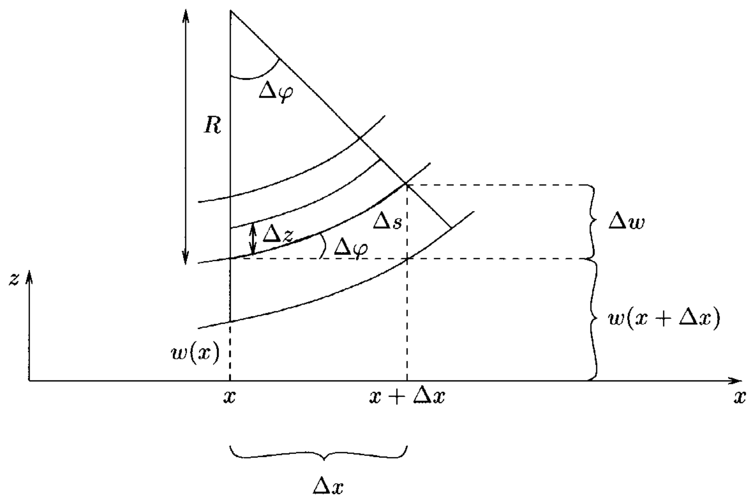


Figure 4.1: The bending of a line-element $\triangle x$.

curvature of the beam axis in deformed state at position x. Furthermore, $\triangle\varphi \approx \tan\triangle\varphi \approx \triangle w/\triangle x$ and $\triangle s \approx \sqrt{(\triangle x)^2 + (\triangle w)^2}$. For $\triangle x \to 0$ this gives us $\mathcal{R} = (1 + w_x^2)^{\frac{3}{2}}/w_{xx}$. Assuming that w_x is small with respect to 1, we can approximate the curvature, which is equal to $1/\mathcal{R}$, by w_{xx}. Using this, the strain ε_{xx} due to 'pure' bending of a line-element of the beam at a distance z from the line of centroids (the x-axis) is given by

$$\varepsilon_{xx} = \frac{(\mathcal{R} - z)\triangle\varphi - \mathcal{R}\triangle\varphi}{\mathcal{R}\triangle\varphi} = -\frac{z}{\mathcal{R}} \approx -zw_{xx}.$$

Furthermore, the strain ε_{x0} due to stretching of the line of centroids of a line-element of the beam can be approximated by $u_x + \frac{1}{2}w_x^2$ as follows. From Figure 4.2 and the definition of strain due to stretching, which can be found in any standard textbook on mechanics (see for example [38]) we have the following expression for ε_{x0}:

$$\varepsilon_{x0} = \frac{\sqrt{(\triangle x + \triangle u)^2 + (\triangle w)^2} - \triangle x}{\triangle x}.$$

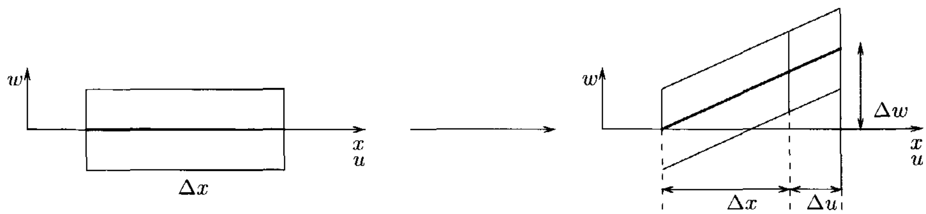


Figure 4.2: The stretching of a line-element Δx.

For $\Delta x \to 0$ this gives us $\varepsilon_{x0} = \sqrt{1 + 2u_x + u_x^2 + w_x^2} - 1$. By assuming that u_x^2 is small with respect to u_x, and by expanding the square-root as a Taylor series, we have $\varepsilon_{x0} \approx u_x + 1/2w_x^2$. The total strain of a line-element of the beam at a distance z from the x-axis is given by $\varepsilon_x = \varepsilon_{x0} + \varepsilon_{xx} = u_x + \frac{1}{2}w_x^2 - zw_{xx}$. It is shown in [37] that, using Hooke's Law, the work performed to deflect the beam from its initial position, is

$$\mathcal{A}(t) = \frac{1}{2}EA\int_0^l \left[u_x + \frac{1}{2}w_x^2\right]^2 dx + \frac{1}{2}EI\int_0^l (w_{xx})^2\, dx. \tag{4.2.1}$$

The kinetic energy of the beam is given by

$$\mathcal{E}_k(t) = \frac{1}{2}\mu\int_0^l \left[u_t^2 + w_t^2\right] dx. \tag{4.2.2}$$

Using (4.2.1) and (4.2.2) the Hamiltonian integral is

$$\begin{aligned} \mathcal{F} &= \mathcal{F}(t_2) - \mathcal{F}(t_1) = \int_{t_1}^{t_2} (\mathcal{A}(t) - \mathcal{E}_k(t))dt \qquad (4.2.3)\\ &= \frac{1}{2}\int_{t_1}^{t_2}\int_0^l \{EA[u_x + \frac{1}{2}w_x^2]^2 + EI(w_{xx})^2 - \mu[u_t^2 + w_t^2]\}dxdt. \end{aligned}$$

Using Hamilton's Principle, which states that the variation of $\mathcal{F}$ is equal to 0, the Euler equations for this problem are

$$\mu u_{tt} - EA\frac{\partial}{\partial x}\left[u_x + \frac{1}{2}w_x^2\right] = 0, \tag{4.2.4}$$

$$\mu w_{tt} + EIw_{xxxx} - EA\frac{\partial}{\partial x}\left\{w_x\left[u_x + \frac{1}{2}w_x^2\right]\right\} = 0. \tag{4.2.5}$$

The system given by (4.2.4)-(4.2.5) can be simplified by the following assumption, introduced by Kirchhoff (see [39]): the velocity of the beam in x-direction, u_t, is small compared to w_t and can be neglected in (4.2.3), so $\mathcal{F} = \frac{1}{2}\int_{t_1}^{t_2}\int_0^l\{EA[u_x + \frac{1}{2}w_x^2]^2 + EI(w_{xx})^2 - \mu[w_t^2]\}dxdt$. The system given by (4.2.4)-(4.2.5) can now be simplified to

$$EA\frac{\partial}{\partial x}\left[u_x + \frac{1}{2}w_x^2\right] = 0, \tag{4.2.6}$$

$$\mu w_{tt} + EIw_{xxxx} - EAw_{xx}\left[u_x + \frac{1}{2}w_x^2\right] = 0. \tag{4.2.7}$$

From (4.2.6) we get $u_x + \frac{1}{2}w_x^2 = \varepsilon_{x0}$, a function of t only. Integrating ε_{x0} with respect to x from 0 to l gives us $\int_0^l \left(u_x + \frac{1}{2}w_x^2\right) dx = \varepsilon_{x0}l$, which means $u(l,t) - u(0,t) + \frac{1}{2}\int_0^l w_x^2 dx = \varepsilon_{x0}l = \left(u_x + \frac{1}{2}w_x^2\right) l$. Substituting this into (4.2.7) gives us the following equation for the vertical displacement w:

$$\mu w_{tt} + EIw_{xxxx} - \frac{EA}{l}\left[u(l,t) - u(0,t) + \frac{1}{2}\int_0^l w_x^2 dx\right] w_{xx} = 0. \quad (4.2.8)$$

If other external forces are considered, the right-hand side of (4.2.8) becomes nonzero.

In [9] a survey of literature on oscillations of suspension bridges is given. Using a similar analysis, we will derive a simplified model for nonlinear oscillations in suspension bridges, where the vertical displacement of an elastic beam is given by (4.2.8). We model the suspension bridge as a beam of length l. In this chapter the stays of the bridge are modeled as two-sided springs, as sketched in Figure 4.3. In Chapter 3 the stays of the bridge are

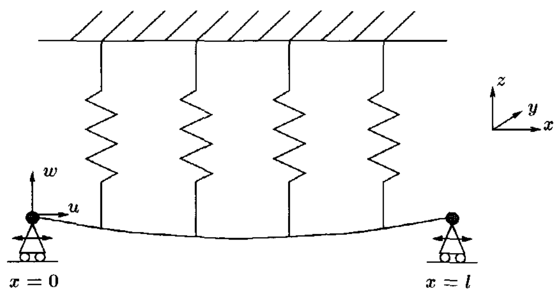


Figure 4.3: A simple model of a suspension bridge.

modeled as two-sided springs with a small nonlinearity (ϵw^2). A next step would be to model the springs using w^+ and w^-, as is done for example in [9]. The torsional vibration of the beam is not taken into account (that is, is considered to be small compared to the vertical vibration). We introduce the following symbols: κ, the spring constant of the stays of the bridge, and W, the weight of the bridge per unit length, which we consider to be constant, i.e. $W = \mu g$, with g the gravitational acceleration. The equation describing the vertical displacement of the beam then is

$$\begin{aligned} &\mu w_{tt} + EIw_{xxxx} + \kappa w && (4.2.9)\\ &\quad = -\mu g + \frac{EA}{l}\left[u(l,t) - u(0,t) + \frac{1}{2}\int_0^l w_x^2 dx\right] w_{xx}. \end{aligned}$$

Equation (4.2.9) will be simplified by eliminating the term $-\mu g$ using $w = \tilde{w} + \frac{\mu g}{\kappa}s(x)$, where $s(x)$ satisfies the following time-independent linear equation with boundary conditions:

$$s^{(4)}(x) + \frac{\kappa}{EI}s(x) = -\frac{\kappa}{EI}, \quad 0 < x < l,$$
$$s(0) = s(l) = 0, \; s^{(2)}(0) = s^{(2)}(l) = 0.$$

It can be shown that with $\beta = (\frac{\kappa}{4EI})^{\frac{1}{4}}$, $s(x) = \cos(\beta x)\cosh(\beta x) - 1 + (\sin(\beta l)\sin(\beta x)\cosh(\beta x) - \sinh(\beta l)\cos(\beta x)\sinh(\beta x))/(\cos(\beta l) + \cosh(\beta l))$. The term $\frac{\mu g}{\kappa}s(x)$ represents the deflection of the beam in static state due to gravity. Using the dimension-less variables

$$\bar{w} = \frac{l}{A}\tilde{w}, \; \bar{x} = \frac{\pi}{l}x, \; \bar{t} = \left(\frac{\pi}{l}\right)^2 \left(\frac{EI}{\mu}\right)^{\frac{1}{2}} t, \; \bar{u} = \frac{4}{\pi}\left(\frac{l}{A}\right)^2 \frac{l}{\pi}u,$$

(4.2.9) becomes

$$\begin{aligned} &\bar{w}_{\bar{t}\bar{t}} + \bar{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} + \left(\frac{l}{\pi}\right)^4 \frac{\kappa}{EI}\bar{w} \\ &= \frac{A}{I}\frac{A}{l}\left(\frac{1}{4}\frac{A}{l}\left(\bar{u}(\pi,\bar{t}) - \bar{u}(0,\bar{t}) + \frac{2}{\pi}\int_0^\pi \bar{w}_{\bar{x}}^2 d\bar{x}\right)\bar{w}_{\bar{x}\bar{x}} + \frac{1}{4}\frac{\mu g}{\kappa}\mathcal{H}\right), \end{aligned} \tag{4.2.10}$$

with

$$\begin{aligned} \mathcal{H} = &\left(\bar{u}(\pi,\bar{t}) - \bar{u}(0,\bar{t}) + \frac{2}{\pi}\int_0^\pi \bar{w}_{\bar{x}}^2 d\bar{x}\right) s^{(2)}(\frac{l}{\pi}\bar{x}) \\ &+\frac{4}{\pi}\left[\int_0^\pi \bar{w}_{\bar{x}} s^{(1)}(\frac{l}{\pi}\bar{x})d\bar{x}\left(\bar{w}_{\bar{x}\bar{x}} + \frac{\mu g}{\kappa}\frac{l}{A}s^{(2)}(\frac{l}{\pi}\bar{x})\right)\right] \\ &\quad +\frac{2}{\pi}\frac{\mu g}{\kappa}\frac{l}{A}\left[\int_0^\pi \left(s^{(1)}(\frac{l}{\pi}\bar{x})\right)^2 d\bar{x}\left(\bar{w}_{\bar{x}\bar{x}} + \frac{\mu g}{\kappa}\frac{l}{A}s^{(2)}(\frac{l}{\pi}\bar{x})\right)\right]. \end{aligned}$$

Assuming that the area A of the cross-section is small compared to the length l, we put $\tilde{\epsilon} = \frac{A}{l}$, with $\tilde{\epsilon}$ a small parameter. We assume w, and therefore $\tilde{w}$, to be of $\mathcal{O}(\tilde{\epsilon})$. Furthermore, we assume that the deflection of the beam in static state due to gravity, $\frac{\mu g}{\kappa}s$, is small with respect to the vertical displacement $\tilde{w}$, which is of order $\tilde{\epsilon}$. This means we assume $\frac{\mu g}{\kappa}$ is $\mathcal{O}(\tilde{\epsilon}^n)$, with $n > 1$, since $s(x)$ is of order 1, as can be seen from the expression for s which was given above (as well as $s^{(1)}(x)$ and $s^{(2)}(x)$). Since $\mathcal{H} = \mathcal{O}(1)$, (4.2.10) becomes

$$\begin{aligned} &\bar{w}_{\bar{t}\bar{t}} + \bar{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} + p^2\bar{w} \\ &= \frac{1}{4}\frac{A}{I}\left[\tilde{\epsilon}^2\left(\bar{u}(\pi,\bar{t}) - \bar{u}(0,\bar{t}) + \frac{2}{\pi}\int_0^\pi \bar{w}_{\bar{x}}^2 d\bar{x}\right)\bar{w}_{\bar{x}\bar{x}} + \mathcal{O}(\tilde{\epsilon}^m)\right], \end{aligned}$$

with $m > 2$ and $p^2 = (\frac{l}{\pi})^4\frac{\kappa}{EI}$. Setting $\epsilon = \frac{1}{4}\frac{A}{I}\tilde{\epsilon}^2$, we can now introduce the following initial-boundary value problem, which describes, up to $\mathcal{O}(\epsilon^n)$, $n>1$, the vertical displacement of an elastic beam with a linear spring force and a constant gravity force acting on it, and with an external force $F(t)$ acting on the ends of the beam in horizontal direction:

$$w_{tt} + w_{xxxx} + p^2 w = \epsilon \left(F(t) + \frac{2}{\pi} \int_0^\pi w_x^2 dx \right) w_{xx},\ 0 < x < \pi,\ t > 0, \tag{4.2.11}$$

$$w(0,t) = w(\pi,t) = 0,\ t \geq 0, \tag{4.2.12}$$

$$w_{xx}(0,t) = w_{xx}(\pi,t) = 0,\ t \geq 0, \tag{4.2.13}$$

$$w(x,0) = w_0(x), w_t(x,0) = w_1(x),\ 0 < x < \pi, \tag{4.2.14}$$

where $F(t) = u(\pi,t) - u(0,t)$ and ϵ a small dimension-less parameter. In this chapter we are interested in an harmonic excitation of the ends of the beam in horizontal direction, which means we take $F(t) = C\cos(\omega t)$, with C a constant $\neq 0$ which represents the amplitude of the external excitation and ω the frequency of the external excitation. Since we consider u_t small compared to w_t, it can be shown that ω must be of $\mathcal{O}(1)$. Furthermore, ω can be taken positive without loss of generality. Furthermore, ϵ and p are constants with $0 < \epsilon \ll 1$ and $p > 0$, $w = w(x,t)$ is the vertical displacement of the beam, x is the coordinate along the beam, $w_0(x)$ is the initial displacement of the beam in vertical direction and $w_1(x)$ is the initial velocity of the beam in vertical direction. All functions are assumed to be sufficiently smooth. The first two terms on the left-hand side of (4.2.11) are the linear part of the beam equation, $p^2 w$ represents the linear restoring force of the spring, $(\int_0^\pi w_x^2 dx) w_{xx}$ is due to the strain of the beam and $F(t) w_{xx}$ is due to an external force acting on the ends of the beam in horizontal direction. The boundary conditions describe a simply supported beam. As we showed above, the initial-boundary value problem (4.2.11)-(4.2.14) can be considered as a simple model for nonlinear oscillations in suspension bridges. In the next section a formal approximation will be constructed of the solution of (4.2.11)-(4.2.14).

4.3 Construction Of Formal Approximations - General Case

In this and the next section we construct a formal approximation of the solution of the initial-boundary value problem (4.2.11)-(4.2.14). When straightforward ϵ-expansions are used to approximate solutions, secular terms can occur in the approximations. To avoid these secular terms we use a two time-scales perturbation method.

The boundary conditions imply that w can be written as a Fourier sine-series in x: $w(x,t) = \sum_{m=1}^{\infty} q_m(t)\sin(mx)$. Substituting this series in

(4.2.11), we obtain the following system of equations:

$$\sum_{k=1}^{\infty}\left(\ddot{q}_k+(k^4+p^2)q_k\right)\sin(kx)$$
$$=-\epsilon\left(\sum_{k=1}^{\infty}\left(F(t)+\sum_{m=1}^{\infty}m^2q_m^2\right)k^2q_k\sin(kx)\right).$$

Using orthogonality properties of the sine functions on $[0,\pi]$ it can be shown easily that the equation for each q_n is

$$\ddot{q}_n+(n^4+p^2)q_n=-\epsilon\left(F(t)+\sum_{m=1}^{\infty}m^2q_m^2\right)n^2q_n, \tag{4.3.1}$$

for $n=1,2,3,\dots$, with $F(t)=C\cos(\omega t)$ and where q_n must satisfy the following initial conditions:

$$q_n(0)=\frac{2}{\pi}\int_0^{\pi}w_0(x)\sin(nx)dx,\ \dot{q}_n(0)=\frac{2}{\pi}\int_0^{\pi}w_1(x)\sin(nx)dx\ .$$

As stated above, terms that give rise to secular terms may occur in the right hand side of (4.3.1). To eliminate these terms we introduce two time-scales, $t_0=t$ and $t_1=\epsilon t$, and assume that q_n can be expanded in a formal power series in ϵ, i.e., $q_n(t)=q_{n,0}(t_0,t_1)+\epsilon q_{n,1}(t_0,t_1)+\epsilon^2q_{n,2}(t_0,t_1)+\cdots$. We substitute this into (4.3.1) and collect equal powers in ϵ. The $\mathcal{O}(\epsilon^0)$-problem becomes

$$\frac{\partial^2}{\partial t_0^2}q_{n,0}+\omega_{n_p}^2q_{n,0}=0,\ t>0, \tag{4.3.2}$$

$$q_{n,0}(0,0)=\frac{2}{\pi}\int_0^{\pi}w_0(x)\sin(nx)dx, \tag{4.3.3}$$

$$\frac{\partial}{\partial t_0}q_{n,0}(0,0)=\frac{2}{\pi}\int_0^{\pi}w_1(x)\sin(nx)dx, \tag{4.3.4}$$

for $n=1,2,3,\dots$, with $\omega_{n_p}=\sqrt{n^4+p^2}$. The general solution for (4.3.2)-(4.3.4) is

$$q_{n,0}(t_0,t_1)=A_{n,0}(t_1)\cos(\omega_{n_p}t_0)+B_{n,0}(t_1)\sin(\omega_{n_p}t_0), \tag{4.3.5}$$

where $A_{n,0},B_{n,0}$ satisfy the following initial conditions:

$$A_{n,0}(0)=q_{n,0}(0,0),\ B_{n,0}(0)=\frac{1}{\omega_{n_p}}\frac{\partial}{\partial t_0}q_{n,0}(0,0).$$

Next we consider the $\mathcal{O}(\epsilon^1)$-problem

$$\frac{\partial^2}{\partial t_0^2}q_{n,1} + \omega_{n_p}^2 q_{n,1} \qquad (4.3.6)$$

$$= -2\frac{\partial^2}{\partial t_0 \partial t_1}q_{n,0} - \left(C\cos(\omega t_0) + \sum_{m=1}^{\infty} m^2 q_{m,0}^2\right) n^2 q_{n,0},$$

$$q_{n,1}(0,0) = 0, \qquad \frac{\partial}{\partial t_0}q_{n,1}(0,0) = -\frac{\partial}{\partial t_1}q_{n,0}(0,0), \qquad (4.3.7)$$

for $n = 1,2,3,\ldots$. We substitute (4.3.5) into (4.3.6) and get

$$\frac{\partial^2}{\partial t_0^2}q_{n,1} + \omega_{n_p}^2 q_{n,1} = 2\omega_{n_p}\left(\frac{dA_{n,0}}{dt_1}\sin(\omega_{n_p}t_0) - \frac{dB_{n,0}}{dt_1}\cos(\omega_{n_p}t_0)\right) \qquad (4.3.8)$$

$$-\left(C\cos(\omega t_0) + \sum_{m=1}^{\infty} m^2 \mathcal{J}_m\right) n^2 \left(A_{n,0}\cos(\omega_{n_p}t_0) + B_{n,0}\sin(\omega_{n_p}t_0)\right),$$

with

$$\mathcal{J}_m = \frac{1}{2}\left(A_{m,0}^2 + B_{m,0}^2\right)$$

$$+\frac{1}{2}\left(A_{m,0}^2 - B_{m,0}^2\right)\cos(2\omega_{m_p}t_0) + A_{m,0}B_{m,0}\sin(2\omega_{m_p}t_0).$$

Since $\cos(\omega_{n_p}t_0)$ and $\sin(\omega_{n_p}t_0)$ are homogeneous solutions of $q_{n,1}$, we want the coefficients of $\cos(\omega_{n_p}t_0)$ and $\sin(\omega_{n_p}t_0)$ on the right-hand side of (4.3.8) to be equal to zero (elimination of secular terms). This gives us equations that $A_{n,0}$ and $B_{n,0}$ have to satisfy. In appendix 4.6 we show that for specific values of ω, the term $C\cos(\omega t_0)$ gives rise to secular terms. This means that for specific values of ω, i.e. $\omega = 2\omega_{k_p} + \mathcal{O}(\epsilon)$ ($k = 1,2,3,\ldots$), extra terms appear in the equations for $A_{k,0}$ and $B_{k,0}$ (for $n \neq k$ the equations remain the same).

For $F(t) \equiv 0$ it can be seen easily from (4.6) (see appendix 4.6) that the equations for $A_{n,0}$ and $B_{n,0}$ are

$$\frac{dA_{n,0}}{dt_1} = \frac{n^2}{4\omega_{n_p}}B_{n,0}\left[\frac{3}{2}n^2\left(A_{n,0}^2 + B_{n,0}^2\right) + \sum_{m\neq n} m^2\left(A_{m,0}^2 + B_{m,0}^2\right)\right], \qquad (4.3.9)$$

$$\frac{dB_{n,0}}{dt_1} = \frac{-n^2}{4\omega_{n_p}}A_{n,0}\left[\frac{3}{2}n^2\left(A_{n,0}^2 + B_{n,0}^2\right) + \sum_{m\neq n} m^2\left(A_{m,0}^2 + B_{m,0}^2\right)\right], \qquad (4.3.10)$$

for $n = 1,2,3,\ldots$. From (4.3.9)-(4.3.10) we see that if $A_{n,0}(0) = B_{n,0}(0) = 0$ then $\forall t_1 > 0$ $A_{n,0}(t_1) = B_{n,0}(t_1) \equiv 0$. So, if we start with zero initial energy in the n-th mode, there will be no energy present up to $\mathcal{O}(\epsilon)$ on a time-scale of order ϵ^{-1}. We say the coupling between the modes is of $\mathcal{O}(\epsilon)$.

This allows truncation to those modes that have non-zero initial energy. In this case there is an interaction between all modes with non-zero initial energy, but this interaction does not give rise to internal resonances. It will be shown in Section 4.4.1 that all modes oscillate with a constant amplitude and a linearly changing phase, depending on the initial amplitudes of the oscillation modes. We will discuss (4.3.9)-(4.3.10) in more detail in Section 4.4.1.

For $F(t) = C\cos(\omega t)$ with $\omega \neq 2\omega_{k_p} + \epsilon\alpha$ it can be seen easily from (4.6) that the equations for $A_{n,0}$ and $B_{n,0}$ are the same as for the case $F(t) = 0$, i.e the equations are given by (4.3.9)-(4.3.10). The only influence $F(t)$ has is of $\mathcal{O}(\epsilon)$ on a time-scale of order ϵ^{-1} in the inhomogeneous solution for $q_{n,1}$, as is shown at the end of this section.

For $F(t) = C\cos(\omega t)$ with $\omega = 2\omega_{k_p} + \epsilon\alpha$, where $\alpha \in \mathbb{R}$ of $\mathcal{O}(1)$, it can be seen from (4.6) that the equations for $A_{k,0}$ and $B_{k,0}$ are

$$\begin{aligned}
\frac{dA_{k,0}}{dt_1} &= \frac{1}{4}\frac{k^2}{\omega_{k_p}}B_{k,0}\left[\frac{3}{2}k^2\left(A_{k,0}^2 + B_{k,0}^2\right) + \sum_{m\neq k} m^2\left(A_{m,0}^2 + B_{m,0}^2\right)\right] \\
&\quad -\frac{1}{4}\frac{k^2}{\omega_{k_p}}C\left(A_{k,0}\sin(\alpha t_1) + B_{k,0}\cos(\alpha t_1)\right), \qquad (4.3.11) \\
\frac{dB_{k,0}}{dt_1} &= -\frac{1}{4}\frac{k^2}{\omega_{k_p}}A_{k,0}\left[\frac{3}{2}k^2\left(A_{k,0}^2 + B_{k,0}^2\right) + \sum_{m\neq k} m^2\left(A_{m,0}^2 + B_{m,0}^2\right)\right] \\
&\quad -\frac{1}{4}\frac{k^2}{\omega_{k_p}}C\left(A_{k,0}\cos(\alpha t_1) - B_{k,0}\sin(\alpha t_1)\right). \qquad (4.3.12)
\end{aligned}$$

For $n \neq k$ equations (4.3.9)-(4.3.10) still hold. We see that for $F(t) = C\cos(\omega t)$ with $\omega = 2\omega_{k_p} + \epsilon\alpha$ the influence of $F(t)$ is of $\mathcal{O}(1)$ on a time-scale of order ϵ^{-1} and extra terms appear in the equations for $A_{k,0}, B_{k,0}$. We see that if $A_{n,0}(0) = B_{n,0}(0) = 0$ then for all $t_1 > 0$ $A_{n,0}(t_1) = B_{n,0}(t_1) \equiv 0$, which holds for all n. So, if we start with zero initial energy in the n-th mode, there will be no energy present up to $\mathcal{O}(\epsilon)$ on a time-scale of order ϵ^{-1}. We say the coupling between the modes is of $\mathcal{O}(\epsilon)$. This again allows truncation to those modes that have non-zero initial energy. In this case there is an interaction between all modes with non-zero initial energy and this interaction does not give rise to internal resonances. It will be shown in Section 4.4.2 that for all modes $n \neq k$ the oscillation has a constant amplitude and a linearly changing phase, depending on the initial values of the oscillation modes. Mode k, however, oscillates with changing amplitude and phase, due to the influence of $F(t)$. We will discuss (4.3.11)-(4.3.12) in more detail in Section 4.4.2.

When $A_{n,0}$ and $B_{n,0}$ have been determined, and thus $q_{n,0}$, we have constructed an approximation v of the exact solution w of the initial-boundary value problem (4.2.11)-(4.2.14)

$$v(x,t;\epsilon) = \sum_{n=1}^{\infty} (q_{n,0}(t_0,t_1) + \epsilon q_{n,1}(t_0,t_1))\sin(nx), \tag{4.3.13}$$

with $q_{n,0}(t_0,t_1) = A_{n,0}(t_1)\cos(\omega_{n_p}t_0) + B_{n,0}(t_1)\sin(\omega_{n_p}t_0)$ and $q_{n,1}(t_0,t_1) = q_{n,1}^{inh}(t_0,t_1) + A_{n,1}(t_1)\cos(\omega_{n_p}t_0) + B_{n,1}(t_1)\sin(\omega_{n_p}t_0)$, with $q_{n,1}^{inh}$ an inhomogeneous solution of (4.3.8). $A_{n,1}(t_1)$ and $B_{n,1}(t_1)$ can be constructed such that secular term in the $\mathcal{O}(\epsilon^2)$ approximation are eliminated. Since we are interested in the $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ approximations, we consider $A_{n,1}$ and $B_{n,1}$ to be constant functions which depend on the initial values for $q_{n,1}$ which are given in (4.3.7). From equation (4.6) in appendix 4.6 it can be shown elementary that, for $\omega \neq 2\omega_{k_p} + \epsilon\alpha$, $q_{n,1}^{inh}$ is of the following form

$$\begin{aligned} q_{n,1}^{inh} &= D_1\cos(3\omega_{n_p}t_0) + D_2\sin(3\omega_{n_p}t_0) + \sum_{m\neq n} E_{1,m}\cos((2\omega_{m_p} - \omega_{n_p})t_0) \\ &\quad + \sum_{m\neq n} E_{2,m}\sin((2\omega_{m_p} - \omega_{n_p})t_0) + \sum_{m\neq n} F_{1,m}\cos((2\omega_{m_p} + \omega_{n_p})t_0) \\ &\quad + \sum_{m\neq n} F_{2,m}\sin((2\omega_{m_p} + \omega_{n_p})t_0) \\ &\quad + G_1C\cos((\omega - \omega_{n_p})t_0) + G_2C\sin((\omega - \omega_{n_p})t_0), \end{aligned} \tag{4.3.14}$$

where $D_1, D_2, E_{1,m}, E_{2,m}, F_{1,m}, F_{2,m}, G_1, G_2$ can be determined easily as functions of $A_{n,0}, B_{n,0}, A_{m,0}, B_{m,0}$. For $\omega = 2\omega_{k_p} + \epsilon\alpha$ $q_{n,1}^{inh}$ is given by (4.3.14) with $G_1 = G_2 = 0$. The approximation v given by (4.3.13) satisfies (4.2.11)-(4.2.14) up to order ϵ. In Chapter 3 an asymptotic theory for a similar problem has been presented. This asymptotic theory implies that approximations v as constructed above are $\mathcal{O}(\epsilon)$ approximations of the exact solution on a time-scale of order ϵ^{-1}.

In the next section we discuss the behavior of the solutions for $A_{n,0}, B_{n,0}$ for three different cases: $F(t) = 0$, $F(t) = C\cos(\omega t_0)$ with $\omega \neq 2\omega_{k_p} + \epsilon\alpha$ and $F(t) = C\cos(\omega t_0)$ with $\omega = 2\omega_{k_p} + \epsilon\alpha$, with $\alpha = 0$ and $\alpha \neq 0$ (detuning).

4.4 Modal interactions

4.4.1 The case $F(t) = 0$

In the previous section equations (4.3.9)-(4.3.10) were given for $A_{n,0}$ and $B_{n,0}$. We introduce polar coordinates to transform these equations

$$A_{n,0} = r_n\cos(\phi_n)\ , \quad B_{n,0} = r_n\sin(\phi_n)\ , \tag{4.4.1}$$

with the amplitude $r_n = r_n(t_1)$ and the phase of the oscillation $\phi_n = \phi_n(t_1)$. We get the following equations for r_n, ϕ_n, for $n = 1, 2, 3, \ldots$:

$$\dot{r}_n = 0, \qquad \dot{\phi}_n = -\frac{1}{4}\frac{n^2}{\omega_{n_p}}\left[\frac{3}{2}n^2 r_n^2 + \sum_{m \neq n} m^2 r_m^2\right], \tag{4.4.2}$$

where the dot represents differentiation with respect to t_1. The solution for (4.4.2) is

$$r_n = c_{1,n}, \qquad \phi_n = -\frac{1}{4}\frac{n^2}{\omega_{n_p}}\left[\frac{3}{2}n^2 c_{1,n}^2 + \sum_{m \neq n} m^2 c_{1,m}^2\right] t_1 + c_{2,n},$$

for $n = 1, 2, 3, \ldots$, where $c_{1,n}, c_{2,n}$ are constants of integration determined by the initial values $A_{n,0}(0)$ and $B_{n,0}(0)$. In the phase space (r_n, ϕ_n) we have the orbits given by $r_n = c_{1,n}$ and $\dot{\phi}_n < 0$. In this case the interaction between the oscillation modes is restricted to interaction between the phases of the modes. This interaction depends on the initial values. This means the following: if we increase the initial amplitude of mode n, then due to the interaction with for instance mode m the frequency of mode m becomes higher, and mode m then has a shorter period. There are no internal resonances and no oscillation modes with initial energy zero are excited (as was for instance the case in Chapter 2 or Chapter 3.

4.4.2 The case $F(t) = C\cos(\omega t_0)$ with $C \neq 0$

In appendix 4.6 it is shown that only for specific values of ω extra interactions occur between the different oscillation modes. These values are $\omega = 2\omega_{k_p}$ with $k = 1, 2, 3, \ldots$. We therefore consider the following cases separately.

The case $\omega \neq 2\omega_{k_p} + \epsilon\alpha$

As stated in the previous section, the equations for $A_{n,0}, B_{n,0}$ for all n, are equal to the equations for the case $F(t) = 0$. There is no extra interaction between the different oscillation modes due to the external force $F(t)$.

The case $\omega = 2\omega_{k_p} + \epsilon\alpha$

In the previous section equations (4.3.11)-(4.3.12) were given for $A_{k,0}$ and $B_{k,0}$. For $A_{n,0}, B_{n,0}$, $n \neq k$ (4.3.9)-(4.3.10) hold. We transform these equations using (4.4.1) and get the following equations for the oscillation modes k and $n(\neq k)$:

$$\dot{r}_k = -\frac{1}{4}\frac{k^2}{\omega_{k_p}} r_k C \sin(2\phi_k + \alpha t_1), \tag{4.4.3}$$

$$\dot{\phi}_k = -\frac{1}{4}\frac{k^2}{\omega_{k_p}} \left[\frac{3}{2}k^2 r_k^2 + \sum_{m\neq k} m^2 r_m^2 + C\cos(2\phi_k + \alpha t_1)\right], \tag{4.4.4}$$

$$\dot{r}_n = 0, \quad n \neq k, \tag{4.4.5}$$

$$\dot{\phi}_n = -\frac{1}{4}\frac{n^2}{\omega_{n_p}} \left[\frac{3}{2}n^2 r_n^2 + \sum_{m\neq n} m^2 r_m^2\right], \quad n \neq k. \tag{4.4.6}$$

We start our analysis of (4.4.3)-(4.4.6) by assuming there is initial energy present in mode k only, which means the initial conditions are such that initially the system oscillates in one mode only (mode k). That is $w(x,0) = q_k(0)\sin kx$, $w_t(x,0) = \dot{q}_k(0)\sin kx$, i.e. $r_n(0) = 0\ \forall n \neq k$. We introduce $\psi = 2\phi_k + \alpha t_1$. This means we get the following equations for r_k and ψ:

$$\dot{r}_k = -\frac{1}{4}\frac{k^2}{\omega_{k_p}} r_k C\sin(\psi), \qquad \dot{\psi} = \alpha - \frac{3}{4}\frac{k^4}{\omega_{k_p}} r_k^2 - \frac{1}{2}\frac{k^2}{\omega_{k_p}} C\cos(\psi), \tag{4.4.7}$$

We consider the cases $C > 0$ and $C < 0$ separately. We start with $C > 0$. The critical points of (4.4.7) are given in Table 4.1, where $\overline{\psi}, \widetilde{\psi}$ are

α-range	# cr. points	cr. points	behavior
$\alpha < -\frac{k^2}{2\omega_{k_p}}C$	0	–	–
$\alpha = -\frac{k^2}{2\omega_{k_p}}C$	1	$(0,\pi)$	h.o. singularity
$-\frac{k^2}{2\omega_{k_p}}C < \alpha < \frac{k^2}{2\omega_{k_p}}C$	3	$(0,\overline{\psi})$	saddle
		$(0,\widetilde{\psi})$	saddle
		$(\overline{r}_k,\pi)$	center
$\alpha = \frac{k^2}{2\omega_{k_p}}C$	2	$(0,0)$	h.o. singularity
		$(\overline{r}_k,\pi)$	center
$\alpha > \frac{k^2}{2\omega_{k_p}}C$	2	$(\widetilde{r}_k,0)$	saddle
		$(\overline{r}_k,\pi)$	center

Table 4.1: Critical points for C>0

solutions of $\cos(\psi) = \frac{2\omega_{k_p}}{k^2 C}\alpha$ and where $\overline{r}_k = \sqrt{\frac{4\omega_{k_p}}{3k^4}\left(\alpha + \frac{k^2}{2\omega_{k_p}}C\right)}$, $\widetilde{r}_k = \sqrt{\frac{4\omega_{k_p}}{3k^4}\left(\alpha - \frac{k^2}{2\omega_{k_p}}C\right)}$. The system is 2π-periodic in ψ, so we consider $\psi \in$

$[0, 2\pi[$. We see that it makes a difference (different bifurcation from critical value) whether ω approaches the critical value $2\omega_{k_p}$ from above or below, i.e. a different behaviour for $\alpha < 0$ and $\alpha > 0$. For $C < 0$ the analysis is similar, where ψ is shifted with a factor π. The behavior of solutions of (4.4.7) in the (r_k, ψ) phase space is given in Figure 4.4 for $-10 \leq \alpha \leq 0$. and Figure 4.5 for $0 < \alpha \leq 10$.

These phase spaces have been constructed using a numerical integration method. For the sake of convenience, we have taken $p^2 = 0$, $k = 1$, $C = 1$ (i.e. $\omega_{k_p} = 1$). A similar behaviour is obtained for $p^2 > 0$, $k \neq 1$ and $C \neq 1$. It can be shown that the larger C becomes, the larger the range of α in which interaction occurs.

It should be noted that a first integral can be obtained for (4.4.7)

$$\cos(2\phi_k + \alpha t_1) = \gamma\frac{1}{r_k^2} + \frac{2\omega_{k_p}}{k^2 C}\alpha - \frac{3k^2}{4C}r_k^2,$$

where γ is a constant of integration depending on $A_{k,0}(0), B_{k,0}(0), k^2, \omega_{k_p}, C$.

Next we consider the case with initial energy present in two modes, m and k, which means the initial conditions are such that the system initially oscillates in two modes only (modes m and k). That is $w(x, 0) = q_k(0)\sin kx + q_m(0)\sin mx$, $w_t(x, 0) = \dot{q}_k(0)\sin kx + \dot{q}_m(0)\sin mx$ and so $r_n(0) = 0\ \forall n \neq k, m$. We have the following equations (see (4.4.3)-(4.4.6)):

$$\dot{r}_k = -\frac{1}{4}\frac{k^2}{\omega_{k_p}}r_k C\sin(2\phi_k + \alpha t_1), \tag{4.4.8}$$

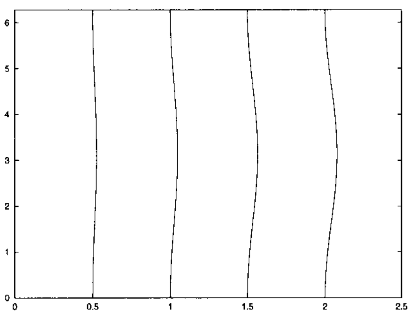
$$\dot{\phi}_k = -\frac{3}{8}\frac{k^4}{\omega_{k_p}}r_k^2 - \frac{1}{4}\frac{k^2}{\omega_{k_p}}m^2 r_m^2 - \frac{1}{4}\frac{k^2}{\omega_{k_p}}C\cos(2\phi_k + \alpha t_1), \tag{4.4.9}$$

$$\dot{r}_m = 0, \tag{4.4.10}$$

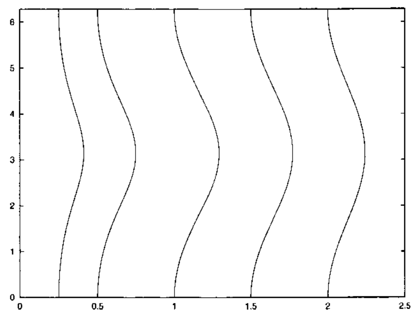
$$\dot{\phi}_m = -\frac{3}{8}\frac{n^4}{\omega_{m_p}}r_m^2 - \frac{1}{4}\frac{m^2}{\omega_{m_p}}k^2 r_k^2. \tag{4.4.11}$$

From (4.4.10) we can see that $r_m = c_m$ with c_m a constant. Furthermore ϕ_m does not appear in the equations for r_k, ϕ_k, so we can analyse the behaviour of solutions of (4.4.8)-(4.4.11) in the (r_k, ψ) phase space (with $\psi = 2\phi_k + \alpha t_1$). The analysis is similar to the analysis for one mode. The only difference is an extra constant term in the equation for ψ, $-\frac{1}{4}\frac{k^2}{\omega_{k_p}}m^2 c_m^2$, which means a phase shift for ψ, which depends on the initial values of mode m. We will not discuss these equations in more detail.

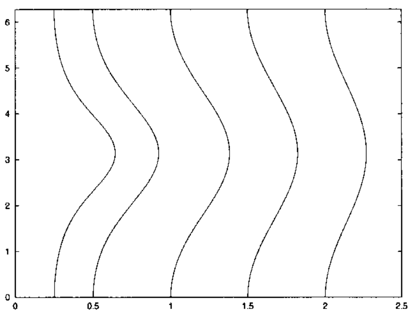
For initial energy present in more than two modes a similar analysis holds. The behaviour of solutions can again be analysed in the (r_k, ψ) phase space.



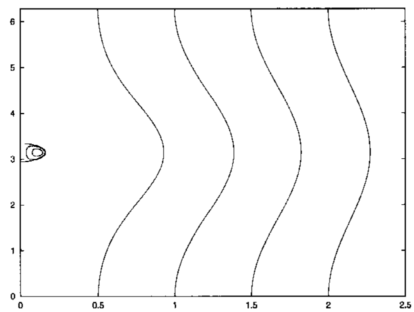
(a) $\alpha = -10$



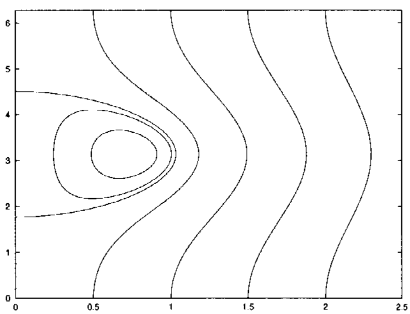
(b) $\alpha = -1$



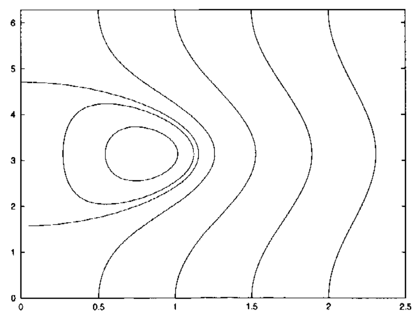
(c) $\alpha = -0.5$



(d) $\alpha = -0.49$

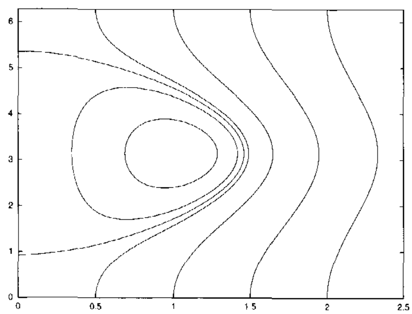


(e) $\alpha = -0.1$

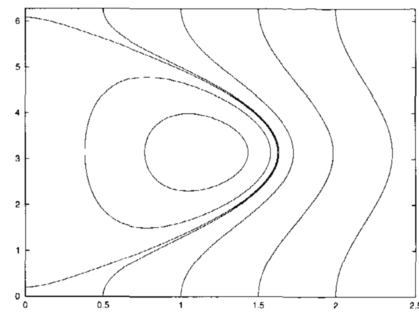


(f) $\alpha = 0$

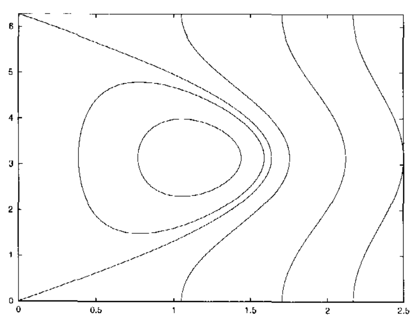
Figure 4.4: Phase space for $-10 \leq \alpha \leq 0$, with r_k (horizontal) from 0 to 2.5 and ψ (vertical) from 0 to 2π.



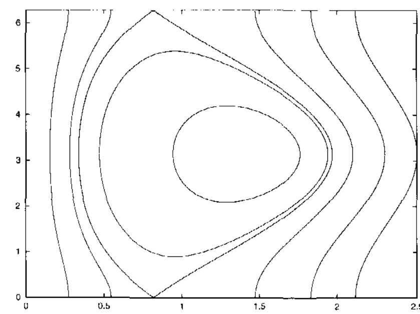
(g) $\alpha = 0.3$



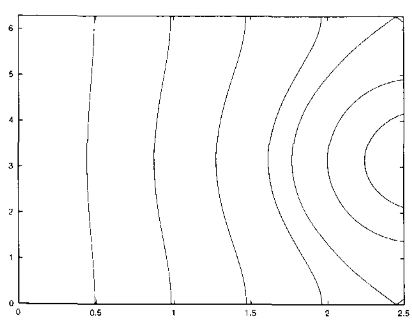
(h) $\alpha = 0.49$



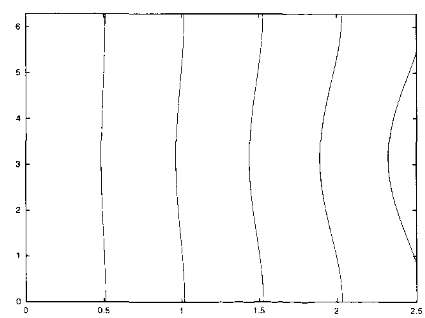
(i) $\alpha = 0.5$



(j) $\alpha = 1$



(k) $\alpha = 5$



(l) $\alpha = 10$

Figure 4.5: Phase space for $0 < \alpha \leq 10$, with r_k (horizontal) from 0 to 2.5 and ψ (vertical) from 0 to 2π.

4.5 Conclusions

In this chapter we consider an initial-boundary value problem for the vertical displacement of a weakly nonlinear elastic beam with an external force acting in horizontal direction on the ends of the beam. We have constructed formal approximations of order ϵ and considered the interaction between different oscillation modes. The analysis presented in this chapter holds for all $p \in \mathbb{R}$. In Chapter 2 and Chapter 3 it has been shown that certain values of p^2 can cause internal resonances. We have shown that in this case this does not occur. We showed that for all cases mode interactions occur only between modes with non-zero initial energy (up to $\mathcal{O}(\epsilon)$). That is, no modes with zero initial energy are excited up to $\mathcal{O}(\epsilon)$. We then say the coupling between the modes is of $\mathcal{O}(\epsilon)$ and truncation is allowed to those modes with non-zero initial energy.

We considered the case with no external forcing ($F(t)$=0) and the case with external forcing ($F(t)$=$C\cos(\omega t)$). For $F(t)$=0 and $F(t)$=$C\cos(\omega t)$ for most ω-values, the mode interaction between the modes with non-zero initial energy is restricted to an interaction between the different phases: phase shifts occur due to the interaction. The amplitudes of the oscillating modes remain constant and depend on the initial values only. We showed that for specific values of ω, i.e. $\omega = 2\omega_{k_p}$ special interactions occur. The mode interactions between the different oscillation modes is still restricted to an interaction between the different phases but the amplitude of mode k is no longer constant: the amplitude of mode k now oscillates around an equilibrium state. This also holds for $\omega = 2\omega_{k_p} + \epsilon\alpha$ where α a detuning parameter. The detuning is considered in Section 4.4.2. It has been shown how the system detunes from the case $\omega = 2\omega_{k_p}$ to the case $\omega \neq 2\omega_{k_p} + \epsilon\alpha$. In this chapter we considered an harmonic external force of the form $F(t) = C\cos(\omega t)$. This analysis can be extended to a more general form of $F(t)$, where F is a T-periodic force, $F(t) = \frac{a_0}{2} + \sum_n (a_n \cos(\nu_n t) + b_n \sin(\nu_n t))$ with $\nu_n = \frac{2\pi n}{T}$. This has been discussed in [46] for elastic beams or strings, where truncation to one or two oscillation modes is applied, without giving a justification. We have shown that in the cases discussed in this chapter truncation is valid up to $\mathcal{O}(\epsilon)$. As can be seen in Chapter 2 and Chapter 3, truncation to one or two oscillation modes is not valid for all cases. For some cases discussed in those papers, mode interactions occur and more modes have to be taken into account. In a way similar to the methods in Chapter 2 and Chapter 3 the problem with a more general form of $F(t)$ can be studied. The analysis will essentially be the same (depending on the function F), however, the equations will become a bit more complicated. A justification can be given whether truncation is allowed in those cases. This elementary straightforward analysis is beyond the scope of this chapter.

4.6 Appendix - Determination and Elimination of Secular Terms

In Section 4.3 we obtained the following equation for each q_n:

$$\ddot{q}_n + (n^4 + p^2)q_n = -\epsilon \left(F(t) + \sum_{k=1}^{\infty} k^2 q_k^2 \right) n^2 q_n,$$

for $n = 1, 2, 3, \dots$. To avoid secular terms in $q_n(t)$ a two time-scales perturbation method was introduced and $q_n(t)$ was expanded in $q_n(t) = q_{n,0}(t_0, t_1) + \epsilon q_{n,1}(t_0, t_1) + \dots$, where $t_0 = t$ and $t_1 = \epsilon t$. It has been shown that $q_{n,1}$ has to satisfy

$$\begin{aligned}\frac{\partial^2}{\partial t_0^2} q_{n,1} + \omega_{n_p}^2 q_{n,1} &= 2\omega_{n_p} \left(\frac{dA_{n,0}}{dt_1} \sin(\omega_{n_p} t_0) - \frac{dB_{n,0}}{dt_1} \cos(\omega_{n_p} t_0) \right) \quad (4.6.1)\\ &- \left(C \cos(\omega t_0) + \sum_{m=1}^{\infty} m^2 \mathcal{J}_m \right) n^2 \left(A_{n,0} \cos(\omega_{n_p} t_0) + B_{n,0} \sin(\omega_{n_p} t_0) \right),\end{aligned}$$

with

$$\begin{aligned}\mathcal{J}_m &= \frac{1}{2} \left(A_{m,0}^2 + B_{m,0}^2 \right) \\ &\quad + \frac{1}{2} \left(A_{m,0}^2 - B_{m,0}^2 \right) \cos(2\omega_{m_p} t_0) + A_{m,0} B_{m,0} \sin(2\omega_{m_p} t_0).\end{aligned}$$

and $\omega_{n_p} = \sqrt{n^4 + p^2}$. The equations for the functions $A_{n,0}$ and $B_{n,0}$ will now be determined such that no secular terms occur in $q_{n,1}$. The right-hand side of (4.6.1) can be expanded using goniometric formula's and becomes

$$\begin{aligned}&2\omega_{n_p} \frac{dA_{n,0}}{dt_1} \sin(\omega_{n_p} t_0) - 2\omega_{n_p} \frac{dB_{n,0}}{dt_1} \cos(\omega_{n_p} t_0) \\ &-\frac{1}{4} n^4 \Big[3A_{n,0} \left(A_{n,0}^2 + B_{n,0}^2 \right) \cos(\omega_{n_p} t_0) + 3B_{n,0} \left(A_{n,0}^2 + B_{n,0}^2 \right) sin(\omega_{n_p} t_0) \\ &+A_{n,0} \left(A_{n,0}^2 - 3B_{n,0}^2 \right) \cos(3\omega_{n_p} t_0) + B_{n,0} \left(3A_{n,0}^2 - B_{n,0}^2 \right) \sin(3\omega_{n_p} t_0) \Big] \\ &- \sum_{m \neq n} n^2 m^2 \frac{1}{4} \Big[\left(2B_{n,0} A_{m,0} B_{m,0} - A_{n,0} \left(B_{m,0}^2 - A_{m,0}^2 \right) \right) \cos((2\omega_{m_p} - \omega_{n_p}) t_0) \\ &- \left(B_{n,0} \left(A_{m,0}^2 - B_{m,0}^2 \right) - 2A_{n,0} A_{m,0} B_{m,0} \right) \sin((2\omega_{m_p} - \omega_{n_p}) t_0) \\ &+ \left(A_{n,0} \left(A_{m,0}^2 - B_{m,0}^2 \right) - 2B_{n,0} A_{m,0} B_{m,0} \right) \cos((2\omega_{m_p} + \omega_{n_p}) t_0) \\ &+ \left(2A_{n,0} A_{m,0} B_{m,0} + B_{n,0} \left(A_{m,0}^2 - B_{m,0}^2 \right) \right) \sin((2\omega_{m_p} + \omega_{n_p}) t_0) \\ &+2A_{n,0} \left(A_{m,0}^2 + B_{m,0}^2 \right) \cos(\omega_{n_p} t_0) + 2B_{n,0} \left(A_{m,0}^2 + B_{m,0}^2 \right) \sin(\omega_{n_p} t_0) \Big]\end{aligned}$$

$$-n^2\frac{1}{2}C\Bigg[A_{n,0}\cos((\omega-\omega_{n_p})t_0)-B_{n,0}\sin((\omega-\omega_{n_p})t_0)$$
$$+A_{n,0}\cos((\omega+\omega_{n_p})t_0)+B_{n,0}\sin((\omega+\omega_{n_p})t_0)\Bigg]. \tag{4.6.2}$$

As stated in Section 4.3, $\cos(\omega_{n_p}t_0)$ and $\sin(\omega_{n_p}t_0)$ are homogeneous solutions of $q_{n,1}$. We want the coefficients of $\cos(\omega_{n_p}t_0)$ and $\sin(\omega_{n_p}t_0)$ in (4.6) to be equal to zero in order to eliminate secular terms. This gives us equations for $A_{n,0}$ and $B_{n,0}$. From (4.6) it can be seen that we have to consider two cases for ω: $\omega \neq 2\omega_{n_p}$ and $\omega = 2\omega_{n_p}$, as is done in Section 4.3 and Section 4.4. When the secular terms on the right-hand side of (4.6.1) have been eliminated, the remaining terms are the inhomogeneous part of the equation for $q_{n,1}$ and an inhomogeneous solution for $q_{n,1}$ can be determined easily.

Chapter 5

A Weakly Nonlinear Beam Equation with a Rayleigh Perturbation and an Integral Nonlinearity

Abstract In this chapter an initial-boundary value problem for the vertical displacement of a weakly nonlinear elastic beam in a wind-field is studied. At the ends of the beam a harmonic force in horizontal direction acts on the beam. The initial-boundary value problem can be regarded as a simple model describing oscillations of flexible structures like suspension bridges or iced overhead transmission lines in a wind-field. Using a two time-scales perturbation method an approximation of the solution of the initial-boundary value problem is constructed. Interactions between different oscillation modes of the beam are studied. It is shown that for specific values of the parameters internal resonances occur. The aim of this chapter is to see whether the applied (harmonic) horizontal excitation can act as a damping force on the oscillations of the beam. In this chapter the effects of the external excitation are studied for a small range of the parameters involved. It is shown that for the cases considered in this chapter no damping effect, but rather the opposite effect occurs and that the ultimate amplitudes of the oscillation modes become larger.

5.1 Introduction

Flexible structures with bending stiffness, like tall buildings, suspension bridges or iced overhead transmission lines, are subjected to oscillations due to different causes. Simple models which describe these oscillations can involve nonlinear second and fourth order partial differential equations

(PDE's), as can be seen for example in [1] or [9]. In many cases perturbation methods can be used to construct approximations for solutions of this type of second or fourth order equations. Initial-boundary value problems for second order PDE's have been considered for a long time, for instance in [26]-[30],[33] and [34]. Only recently initial-boundary value problems for fourth order weakly nonlinear PDE's have been considered, for instance in [9]-[11] and [40]-[43]. These problems for second and fourth order PDE's have been studied in [1]-[3],[9]-[11],[31],[32] and [40]-[43], using a two time-scales perturbation method or a Galerkin-averaging method to construct approximations. In [9]-[11] and [43] the solutions for fourth order PDE's are approximated by a single mode representation, without justification as to whether truncation to one mode is valid or not. In this chapter approximations for solutions of an initial-boundary value problem for a fourth order PDE are constructed using a two time-scales perturbation method. The interaction between the different oscillation modes is studied and a justification is given in which cases mode truncation is valid. For fourth order strongly nonlinear PDE's numerical finite element methods can be used, as is done for example in [35].
The equations of motion for a linear beam can be found in standard text books, such as [37] or [38]. The derivation of the equations of motion in [37] is also discussed in Chapter 4. Oscillations are possible due to the strain of the beam. The x-axis is defined to be the horizontal axis. The z-axis is defined to be the vertical axis. The y-axis is perpendicular to the (x, z)-plane. We introduce the following symbols: μ is the mass of the beam per unit length, ρ the mass density of the beam, A the area of the cross-section Q of the beam perpendicular to the x-axis (so $\mu = \rho A$), E the elasticity modulus (Young's modulus) and I the axial moment of inertia of the cross-section. The inertial axes of the cross-section Q are the y- and z-axes, so $I = \int\int_Q z^2 dydz$. We assume that the beam can move in the x- and z-direction only. Furthermore, we assume that the vertical displacement of the beam from rest is given by $w = w(x,t)$ and that the horizontal displacement of the beam is given by $u = u(x,t)$. In Chapter 4 the following equation, describing the vertical displacement w of an elastic beam, is derived:

$$\mu w_{tt} + EIw_{xxxx} - \frac{EA}{l}\left[u(l,t) - u(0,t) + \frac{1}{2}\int_0^l w_x^2 dx\right] w_{xx} = 0. \qquad (5.1.1)$$

In [9] a survey of literature on oscillations of suspension bridges is given. Using a similar analysis, we will derive a simplified model for nonlinear oscillations of a suspension bridge in a wind-field, where the vertical displacement is described by an equation like (5.1.1). We model the suspension bridge as a beam of length l. In this chapter the stays of the bridge are

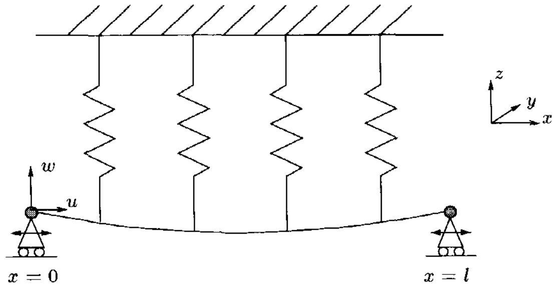


Figure 5.1: A simple model of a suspension bridge.

modeled as linear, two-sided springs, as sketched in Figure 5.1. In Chapter 3 the stays of the bridge were modeled as (almost linear) two-sided springs with a small nonlinearity (ϵw^2). The torsional vibrations of the beam are not taken into account (i.e. are considered to be small compared to the vertical vibrations). We neglect internal damping. Furthermore, we consider a uniform wind flow in y-direction, which causes nonlinear drag and lift forces, given by F_D and F_L respectively, and acting on the structure per unit length. We introduce the following symbols: κ, the spring constant of the stays of the bridge, and W, the weight of the bridge per unit length, which we consider to be constant, i.e. $W = \mu g$, where g is the gravitational acceleration. The following equation describes the vertical displacement of the beam:

$$\begin{aligned}\mu w_{tt} + EI w_{xxxx} + \kappa w = \\ -\mu g + F_D + F_L + \frac{EA}{l}\left[u(l,t) - u(0,t) + \frac{1}{2}\int_0^l w_x^2 dx\right] w_{xx}.\end{aligned} \tag{5.1.2}$$

In Section 4 of [1] it has been shown that $F_D + F_L$ can be approximated by

$$\frac{\rho_a d v_\infty^2}{2}\left(a_0 + \frac{a_1}{v_\infty}w_t + \frac{a_2}{v_\infty^2}w_t^2 + \frac{a_3}{v_\infty^3}w_t^3\right),$$

where ρ_a is the density of air, d is the diameter of the cross-section of the beam, v_∞ is the uniform wind flow velocity in y-direction and a_0, a_1, a_2, a_3 depend on certain drag and lift coefficients and are given explicitly in [1]. Equation (5.1.2) will be simplified by eliminating the term $-\mu g$ using $w = \tilde{w} + \frac{\mu g}{\kappa}s(x)$, where $s(x)$ satisfies the following time-independent linear equation with boundary conditions:

$$\begin{aligned}&s^{(4)}(x) + \frac{\kappa}{EI}s(x) = -\frac{\kappa}{EI}, \quad 0 < x < l,\\ &s(0) = s(l) = 0,\ s^{(2)}(0) = s^{(2)}(l) = 0.\end{aligned}$$

It can be shown that $s(x) = \cos(\beta x)\cosh(\beta x) + (\sin(\beta l)\sin(\beta x)\cosh(\beta x) - \sinh(\beta l)\cos(\beta x)\sinh(\beta x))/(\cos(\beta l) + \cosh(\beta l)) - 1$, with $\beta = (\frac{\kappa}{4EI})^{\frac{1}{4}}$. The term $\frac{\mu g}{\kappa}s(x)$ represents the deflection of the beam in static state due to gravity. Using the dimensionless variables

$$\bar{w} = \frac{\pi}{l}\frac{c}{v_\infty}\tilde{w},\ \bar{x} = \frac{\pi}{l}x,\ \bar{t} = \frac{\pi}{l}ct,\ \bar{u} = \frac{4}{l}\frac{c^2}{v_\infty^2}u,$$

with $c = \frac{\pi}{l}\sqrt{\frac{EI}{\mu}}$, (5.1.2) becomes

$$\bar{w}_{\bar{t}\bar{t}} + \bar{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} + \left(\frac{l}{\pi}\right)^4 \frac{\kappa}{EI}\bar{w} = \frac{\rho_a dl}{2\pi\mu}\frac{v_\infty}{c}\left(a_0 + a_1\bar{w}_{\bar{t}} + a_2\bar{w}_{\bar{t}}^2 + a_3\bar{w}_{\bar{t}}^3\right) \quad (5.1.3)$$
$$+\frac{1}{4}\frac{A}{I}\frac{v_\infty}{c}\frac{l}{\pi}\left(\frac{v_\infty}{c}\frac{l}{\pi}\left(\bar{u}(\pi,\bar{t}) - \bar{u}(0,\bar{t}) + \frac{2}{\pi}\int_0^\pi \bar{w}_{\bar{x}}^2 d\bar{x}\right)\bar{w}_{\bar{x}\bar{x}} + \frac{\mu g}{\kappa}\mathcal{H}\right),$$

with

$$\mathcal{H} = \left(\bar{u}(\pi,\bar{t}) - \bar{u}(0,\bar{t}) + \frac{2}{\pi}\int_0^\pi \bar{w}_{\bar{x}}^2 d\bar{x}\right)s^{(2)}(\frac{l}{\pi}\bar{x})$$
$$+\frac{4}{\pi}\left[\int_0^\pi \bar{w}_{\bar{x}}s^{(1)}(\frac{l}{\pi}\bar{x})d\bar{x}\left(\bar{w}_{\bar{x}\bar{x}} + \frac{\mu g}{\kappa}\frac{\pi}{l}\frac{c}{v_\infty}s^{(2)}(\frac{l}{\pi}\bar{x})\right)\right]$$
$$+\frac{2}{\pi}\frac{\mu g}{\kappa}\frac{\pi}{l}\frac{c}{v_\infty}\left[\int_0^\pi \left(s^{(1)}(\frac{l}{\pi}\bar{x})\right)^2 d\bar{x}\left(\bar{w}_{\bar{x}\bar{x}} + \frac{\mu g}{\kappa}\frac{\pi}{l}\frac{c}{v_\infty}s^{(2)}(\frac{l}{\pi}\bar{x})\right)\right].$$

Assuming that v_∞, the uniform wind velocity, is small with respect to $\frac{\pi}{l}c$, we put $\tilde{\epsilon}=\frac{v_\infty}{c}\frac{l}{\pi}$, where $\tilde{\epsilon}$ is a small parameter. Furthermore, we assume that the deflection of the beam in static state due to gravity, $\frac{\mu g}{\kappa}s$, is small with respect to the vertical displacement $\tilde{w}$, which is of order $\tilde{\epsilon}$. This means we assume that $\frac{\mu g}{\kappa}$ is $\mathcal{O}(\tilde{\epsilon}^n)$, with $n > 1$, since $s(x)$ is of order 1 (as well as $s^{(1)}(x)$ and $s^{(2)}(x)$), as can be seen from the expression for s which is given above. It can be shown easily that $\mathcal{H} = \mathcal{O}(1)$. Equation (5.1.3) now becomes

$$\bar{w}_{\bar{t}\bar{t}} + \bar{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} + \left(\frac{l}{\pi}\right)^4 \frac{\kappa}{EI}\bar{w} = \frac{\rho_a d}{2\mu}\tilde{\epsilon}\left(a_0 + a_1\bar{w}_{\bar{t}} + a_2\bar{w}_{\bar{t}}^2 + a_3\bar{w}_{\bar{t}}^3\right) \quad (5.1.4)$$
$$+\frac{1}{4}\frac{A}{I}\tilde{\epsilon}^2\left(\bar{u}(\pi,\bar{t}) - \bar{u}(0,\bar{t}) + \frac{2}{\pi}\int_0^\pi \bar{w}_{\bar{x}}^2 d\bar{x}\right)\bar{w}_{\bar{x}\bar{x}} + \frac{1}{4}\frac{A}{I}\mathcal{O}(\tilde{\epsilon}^{m_1}),\ m_1 > 2.$$

In Section 4 of [1] it has been shown that the first term in the right hand side of (5.1.4) can be approximated as follows

$$\frac{\rho_a d}{2\mu}\tilde{\epsilon}\left(a_0 + a_1\bar{w}_{\bar{t}} + a_2\bar{w}_{\bar{t}}^2 + a_3\bar{w}_{\bar{t}}^3\right) = \frac{\rho_a d}{2\mu}\tilde{\epsilon}\left(a\bar{w}_{\bar{t}} - b\bar{w}_{\bar{t}}^3\right) + \mathcal{O}(\tilde{\epsilon}^{m_2}),\ m_2 > 1$$

where a and b are specific combinations of drag and lift coefficients which are given explicitely in [1], and are of order 1. This means that (5.1.4) now becomes

$$\bar{w}_{\bar{t}\bar{t}} + \bar{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} + \left(\frac{l}{\pi}\right)^4 \frac{\kappa}{EI}\bar{w} = \frac{\rho_a d}{2\mu}\tilde{\epsilon}\left(a\bar{w}_{\bar{t}} - b\bar{w}_{\bar{t}}^3\right) + \frac{\rho_a d}{2\mu}\mathcal{O}(\tilde{\epsilon}^{m_2}) \tag{5.1.5}$$
$$+\frac{1}{4}\frac{A}{I}\tilde{\epsilon}^2\left(\bar{u}(\pi,\bar{t}) - \bar{u}(0,\bar{t}) + \frac{2}{\pi}\int_0^\pi \bar{w}_{\bar{x}}^2 d\bar{x}\right)\bar{w}_{\bar{x}\bar{x}} + \frac{1}{4}\frac{A}{I}\mathcal{O}(\tilde{\epsilon}^{m_1}),$$

with $m_1 > 2$ and $m_2 > 1$. Using the transformation $\hat{w} = \sqrt{\frac{3b}{a}}\bar{w}$ and $\hat{u} = \frac{3b}{a}\bar{u}$ (5.1.5) becomes

$$\hat{w}_{\bar{t}\bar{t}} + \hat{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} + \left(\frac{l}{\pi}\right)^4 \frac{\kappa}{EI}\hat{w} = \frac{\rho_a d}{2\mu}a\tilde{\epsilon}\left(\hat{w}_{\bar{t}} - \frac{1}{3}\hat{w}_{\bar{t}}^3\right) \tag{5.1.6}$$
$$+\frac{1}{4}\frac{A}{I}\frac{a}{3b}\tilde{\epsilon}^2\left(\hat{u}(\pi,\bar{t}) - \hat{u}(0,\bar{t}) + \frac{2}{\pi}\int_0^\pi \hat{w}_{\bar{x}}^2 d\bar{x}\right)\hat{w}_{\bar{x}\bar{x}}$$
$$+\sqrt{\frac{3b}{a}}\left(\frac{\rho_a d}{2\mu}\mathcal{O}(\tilde{\epsilon}^{m_2}) + \frac{1}{4}\frac{A}{I}\mathcal{O}(\tilde{\epsilon}^{m_1})\right),$$

with $m_1 > 2$ and $m_2 > 1$. Taking

$$\epsilon_1 = \frac{\rho_a d}{2\mu}a\tilde{\epsilon} = \frac{\rho_a d}{2\mu}\frac{v_\infty}{c}\frac{l}{\pi}a, \ \epsilon_2 = \frac{1}{4}\frac{A}{I}\frac{a}{3b}\tilde{\epsilon}^2 = \frac{1}{4}\frac{A}{I}\left(\frac{v_\infty}{c}\frac{l}{\pi}\right)^2\frac{3b}{a}, \tag{5.1.7}$$

(5.1.6) becomes

$$\hat{w}_{\bar{t}\bar{t}} + \hat{w}_{\bar{x}\bar{x}\bar{x}\bar{x}} + p^2\hat{w} = \epsilon_1\left(\hat{w}_{\bar{t}} - \frac{1}{3}\hat{w}_{\bar{t}}^3\right)$$
$$+\epsilon_2\left(\hat{u}(\pi,\bar{t}) - \hat{u}(0,\bar{t}) + \frac{2}{\pi}\int_0^\pi \hat{w}_{\bar{x}}^2 d\bar{x}\right)\hat{w}_{\bar{x}\bar{x}} + \epsilon_1 o(1) + \epsilon_2 o(1),$$

with $p^2 = \left(\frac{l}{\pi}\right)^4 \frac{k}{EI}$ of order 1. Three different cases can be considered for ϵ_1, ϵ_2 given by (5.1.7). The case $\epsilon_1 \ll \epsilon_2$ was considered (up to $\mathcal{O}(\epsilon_2^n)$, $n > 1$) in Chapter 4, using a slightly different scaling of w. The case $\epsilon_1 \gg \epsilon_2$ was considered (up to $\mathcal{O}(\epsilon_1^n)$, $n > 1$) in Chapter 2. In this chapter we will discuss the case $\epsilon_1 \approx \epsilon_2$, which means we assume ϵ_1 and ϵ_2 are of the same order of magnitude, i.e. $\epsilon_2 = \delta\epsilon_1$, with δ of order 1.

We can now introduce the following initial-boundary value problem, which describes, up to $\mathcal{O}(\epsilon^n)$, $n > 1$, the vertical displacement of an elastic beam with a linear spring force and a constant gravity force acting on it, with an external force $F(t)$ acting on the ends of the beam in horizontal direction and with a uniform windflow acting on the entire beam:

$$w_{tt} + w_{xxxx} + p^2 w \tag{5.1.8}$$
$$= \epsilon\left[\left(w_t - \frac{1}{3}w_t^3\right) + \delta\left(F(t) + \frac{2}{\pi}\int_0^\pi w_x^2 dx\right)w_{xx}\right], \ 0 < x < \pi, \ t > 0,$$

$$w(0,t) = w(\pi,t) = 0, \ \ t \geq 0, \tag{5.1.9}$$

$$w_{xx}(0,t) = w_{xx}(\pi,t) = 0, \ \ t \geq 0, \tag{5.1.10}$$

$$w(x,0) = w_0(x), \ \ w_t(x,0) = w_1(x), \ \ 0 < x < \pi, \tag{5.1.11}$$

where $F(t) = u(\pi,t) - u(0,t)$ and ϵ a small dimension-less parameter. In this chapter ϵ and p are constants with $0 < |\epsilon| \ll 1$ and $0 < p^2 < 10$, $\delta \in \mathbb{R}$ of order 1, $w = w(x,t)$ is the vertical displacement of the beam, x is the coordinate along the beam, $w_0(x)$ is the initial displacement of the beam in vertical direction and $w_1(x)$ is the initial velocity of the beam in vertical direction. All functions are assumed to be sufficiently smooth. The first two terms on the left hand side of the PDE (5.1.8) are the linear part of the beam equation, $p^2 w$ represents a linear restoring force, $w_t - \frac{1}{3}w_t^3$ is a so-called Rayleigh perturbation, which describes an external (wind)force (see [1]), $(\int_0^\pi w_x^2 dx)w_{xx}$ is due to the strain of the beam and $F(t)w_{xx}$ is due to an external force acting on the ends of the beam in horizontal direction. The boundary conditions describe a simply supported beam. As we showed above, the initial-boundary problem (5.1.8)-(5.1.11) can be considered as a simple model for nonlinear oscillations in suspension bridges. The aim of this chapter is to analyze if the forcing term $F(t)$ can be used as a damping term for the nonlinear oscillations due to the Rayleigh perturbation. We are interested in a harmonic excitation of the ends of the beam in horizontal direction, which means we take $F(t) = C\cos(\omega t)$, with C a non-zero constant which represents the amplitude of the external excitation and ω the frequency of the external excitation. To obtain (5.1.1) we assumed that u_t is small compared to w_t. It can be shown that this means that ω must be of $\mathcal{O}(1)$. Furthermore, ω can be taken positive without loss of generality.

In this chapter formal approximations, i.e. functions that satisfy the differential equation and the initial-boundary values up to some order in ϵ, will be constructed for the initial-boundary value problem (5.1.8)-(5.1.11), using a Fourier mode expansion and a two time-scales perturbation method. The interaction (energy exchange) between the different oscillation modes will be considered for different cases for p^2 and $F(t)$. As shown in Chapter 2, certain values of p^2 give rise to complicated interactions which cause internal resonances. In this chapter we mean by internal resonance that there is an energy transfer of order 1 on a time-scale of order ϵ^{-1} from one oscillation mode to another, even if one of the modes has zero initial energy. We will consider the cases $p^2 \in]0,10[\backslash\{\frac{693}{152}, 9\}$ and $p^2 \approx 9$. With $p^2 \in]0,10[\backslash\{\frac{693}{152}, 9\}$ we mean $p^2 \in]0,10] \wedge p^2 \neq \frac{693}{152} + \epsilon\beta \wedge p^2 \neq 9 + \epsilon\beta$, where β is a detuning parameter. This notation will be used throughout this chapter. For these values of p^2 we will consider the cases $F(t) = 0$ (no external forcing) and $F(t) = C\cos(\omega t)$ (external forcing). It will be shown

that for most values of ω the case $F(t) = C\cos(\omega t)$ is similar to the case $F(t) = 0$, which means $F(t)$ has no influence on the oscillations (up to $\mathcal{O}(\epsilon)$ on a time-scale of order ϵ^{-1}). For specific values of ω, $\omega \approx 2\omega_{k_p}$, where ω_{k_p} is an eigenfrequency of the linearized system ($\epsilon = 0$), the influence of $F(t)$ is of $\mathcal{O}(1)$ on a time-scale of order ϵ^{-1}. The amplitude of oscillation mode k changes due to $F(t)$. The mode interactions caused by $F(t)$ remain restricted between those modes that have non-zero initial energy (whereas due to specific values of p^2 also an energy transfer to modes with zero initial energy can occur). Similar mode interactions have been studied for example in [40],[41],[46] and [36], but to our knowledge these mode interactions for weakly nonlinear beam equations with a combination of external forces have not yet been studied thoroughly. Similar to the analysis in Chapter 2 and Chapter 3, it is shown that truncation to one or two modes as for example performed in [10]-[11], is not valid for all cases.
The outline of the chapter is as follows. In Section 5.2 we apply a two time-scales perturbation method to the initial-boundary value problem (5.1.8)-(5.1.11). We show that for most values of p^2 and ω mode interactions occur only between modes with non-zero initial energy (up to $\mathcal{O}(\epsilon)$). For some specific values of p^2, i.e. $p^2 \approx \frac{693}{152}$ and $p^2 \approx 9$ (in this chapter we only consider the latter case), also modes with zero initial energy are excited up to $\mathcal{O}(1)$ on a time-scale of order ϵ^{-1}. For some specific values of ω, i.e. $\omega \approx 2\omega_{k_p}$, the interaction between the different (non-zero) oscillation modes changes. We construct formal approximations of $\mathcal{O}(\epsilon)$ for solutions of the initial-boundary value problem for the cases

(i). $p^2 \in]0, 10[\setminus\{\frac{693}{152}, 9\}$, $F(t) = 0$ or $F(t) = C\cos(\omega t)$ with $\omega \neq 2\omega_{k_p} + \epsilon\alpha$,

(ii). $p^2 \in]0, 10[\setminus\{\frac{693}{152}, 9\}$, $F(t) = C\cos(\omega t)$ with $\omega = 2\omega_{k_p} + \epsilon\alpha$,

(iii). $p^2 = 9 + \epsilon\beta$, $F(t) = 0$ or $F(t) = C\cos(\omega t)$ with $\omega \neq 2\omega_{k_p} + \epsilon\alpha$,

(iv). $p^2 = 9 + \epsilon\beta$, $F(t) = C\cos(\omega t)$ with $\omega = 2\omega_{k_p} + \epsilon\alpha$,

where α and $\beta \in \mathbb{R}$ are detuning parameters. In Section 5.3 the mode interactions between the different oscillation modes will be studied in detail for the four cases mentioned above. In Section 5.4 some conclusions and general remarks will be given.

5.2 Construction of formal approximations - general case

In this section and in the next section we will construct formal approximations of the solution of the initial-boundary value problem (5.1.8)-(5.1.11).

When straightforward ϵ-expansions are used to approximate solutions, secular terms can occur in the approximations. To avoid these secular terms we use a two time-scales perturbation method.
The boundary conditions imply that w can be written as a Fourier sine-series in x: $w(x,t) = \sum_{m=1}^{\infty} q_m(t)\sin(mx)$. Substituting this series into (5.1.8), we obtain the following system of equations:

$$\sum_{k=1}^{\infty}\left(\ddot{q}_k + (k^4+p^2)q_k\right)\sin(kx) =$$
$$\epsilon\left[\left(\sum_{k=1}^{\infty}\dot{q}_k\sin(kx) - \frac{1}{3}\sum_{m,k,j=1}^{\infty}\dot{q}_m\dot{q}_k\dot{q}_j\sin(mx)\sin(kx)\sin(jx)\right)\right.$$
$$\left.-\delta\left(\sum_{k=1}^{\infty}\left(F(t)+\sum_{m=1}^{\infty}m^2q_m^2\right)k^2q_k\sin(kx)\right)\right].$$

Using orthogonality properties of the sine-functions on $[0,\pi]$ it can be seen easily from the results obtained in Chapter 2 and Chapter 4 that the equation for each q_n is

$$\ddot{q}_n + (n^4+p^2)q_n = \tag{5.2.1}$$
$$\epsilon\left[\dot{q}_n - \frac{1}{4}\left(\sum_{n=m+k-j} - \sum_{n=-m-k+j} - \frac{1}{3}\sum_{n=m+k+j}\right)\dot{q}_m\dot{q}_k\dot{q}_j\right.$$
$$\left.-\delta\left(F(t)+\sum_{m=1}^{\infty}m^2q_m^2\right)n^2q_n\right],$$

for $n = 1,2,3,\ldots$, with $F(t) = C\cos(\omega t)$ and where q_n must satisfy the following initial conditions:

$$q_n(0) = \frac{2}{\pi}\int_0^{\pi} w_0(x)\sin(nx)dx,\ \ \dot{q}_n(0) = \frac{2}{\pi}\int_0^{\pi} w_1(x)\sin(nx)dx\ .$$

As stated above, terms that give rise to secular terms may occur in the right hand side of (5.2.1). To eliminate these terms we introduce two time-scales, $t_0 = t$ and $t_1 = \epsilon t$, and assume that q_n can be expanded in a formal power series in ϵ, that is, $q_n(t) = q_{n,0}(t_0,t_1) + \epsilon q_{n,1}(t_0,t_1) + \epsilon^2 q_{n,2}(t_0,t_1) + \cdots$. We substitute this into (5.2.1) and collect equal powers in ϵ. The $\mathcal{O}(\epsilon^0)$-problem becomes

$$\frac{\partial^2}{\partial t_0^2}q_{n,0} + \omega_{n_p}^2 q_{n,0} = 0,\ t > 0, \tag{5.2.2}$$

$$q_{n,0}(0,0) = \frac{2}{\pi}\int_0^{\pi} w_0(x)\sin(nx)dx, \tag{5.2.3}$$

$$\frac{\partial}{\partial t_0}q_{n,0}(0,0) = \frac{2}{\pi}\int_0^{\pi} w_1(x)\sin(nx)dx, \tag{5.2.4}$$

for $n = 1, 2, 3, \ldots$, with $\omega_{n_p} = \sqrt{n^4 + p^2}$. The general solution for (5.2.2)-(5.2.4) is

$$q_{n,0}(t_0, t_1) = A_{n,0}(t_1)\cos(\omega_{n_p} t_0) + B_{n,0}(t_1)\sin(\omega_{n_p} t_0), \tag{5.2.5}$$

where $A_{n,0}, B_{n,0}$ satisfy the following initial conditions:

$$A_{n,0}(0) = q_{n,0}(0,0), \;\; B_{n,0}(0) = \frac{1}{\omega_{n_p}} \frac{\partial}{\partial t_0} q_{n,0}(0,0).$$

Next we consider the $\mathcal{O}(\epsilon^1)$-problem

$$\begin{aligned}
\frac{\partial^2}{\partial t_0^2} q_{n,1} + \omega_{n_p}^2 q_{n,1} = {} & -2\frac{\partial^2}{\partial t_0 \partial t_1} q_{n,0} + \frac{\partial}{\partial t_0} q_{n,0} \qquad (5.2.6)\\
& -\frac{1}{4}\left(\sum_{n=m+k-j} - \sum_{n=-m-k+j} - \frac{1}{3}\sum_{n=m+k+j}\right) \frac{\partial}{\partial t_0} q_{m,0} \frac{\partial}{\partial t_0} q_{k,0} \frac{\partial}{\partial t_0} q_{j,0}\\
& - \left(C\cos(\omega t_0) + \sum_{m=1}^{\infty} m^2 q_{m,0}^2\right) n^2 q_{n,0},
\end{aligned}$$

$$q_{n,1}(0,0) = 0, \qquad \frac{\partial}{\partial t_0} q_{n,1}(0,0) = -\frac{\partial}{\partial t_1} q_{n,0}(0,0), \tag{5.2.7}$$

for $n = 1, 2, 3, \ldots$. We substitute (5.2.5) into (5.2.6) and get

$$\begin{aligned}
\frac{\partial^2}{\partial t_0^2} q_{n,1} + \omega_{n_p}^2 q_{n,1} = {} & 2\omega_{n_p}\left(\frac{dA_{n,0}}{dt_1}\sin(\omega_{n_p} t_0) - \frac{dB_{n,0}}{dt_1}\cos(\omega_{n_p} t_0)\right) \qquad (5.2.8)\\
& + \mathcal{H}_n - \frac{1}{4}\left(\sum_{n=m+k-j} - \sum_{n=-m-k+j} - \frac{1}{3}\sum_{n=m+k+j}\right) \mathcal{H}_m \mathcal{H}_k \mathcal{H}_j,\\
& - \left(C\cos(\omega t_0) + \sum_{m=1}^{\infty} m^2 \mathcal{J}_m\right) n^2 \left(A_{n,0}\cos(\omega_{n_p} t_0) + B_{n,0}\sin(\omega_{n_p} t_0)\right),
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{H}_m &= \omega_{m_p}\left(B_{m,0}\cos(\omega_{m_p} t_0) - A_{m,0}\sin(\omega_{m_p} t_0)\right),\\
\mathcal{J}_m &= \frac{1}{2}\left(A_{m,0}^2 + B_{m,0}^2\right)\\
&\quad + \frac{1}{2}\left(A_{m,0}^2 - B_{m,0}^2\right)\cos(2\omega_{m_p} t_0) + A_{m,0} B_{m,0}\sin(2\omega_{m_p} t_0).
\end{aligned}$$

Since $\cos(\omega_{n_p} t_0)$ and $\sin(\omega_{n_p} t_0)$ are homogeneous solutions of $q_{n,1}$, we want the coefficients of $\cos(\omega_{n_p} t_0)$ and $\sin(\omega_{n_p} t_0)$ in the right-hand side of (5.2.8) to be equal to zero (elimination of secular terms). This gives us equations that $A_{n,0}$ and $B_{n,0}$ have to satisfy. In Chapter 2 we show that in order to find the equations for $A_{n,0}, B_{n,0}$ we first have to determine the secular terms due to the cubic term in (5.2.8), by solving the Diophantine-like equations

$$\begin{cases} n = m+k-j \ \vee \ n = -m-k+j \ \vee \ n = m+k+j, \\ \pm(n^4+p^2)^{\frac{1}{2}} = \pm(m^4+p^2)^{\frac{1}{2}} \pm (k^4+p^2)^{\frac{1}{2}} \pm (j^4+p^2)^{\frac{1}{2}} . \end{cases} \tag{5.2.9}$$

Only specific combinations of m, k, j will give solutions to (5.2.9). In Chapter 2 we show that for some values of $p^2 \in]0, 10[$ there are solutions to (5.2.9), which give additional contributions to the equations for $A_{n,0}, B_{n,0}$. These values are $p^2 = 0$, $p^2 = \frac{693}{152}$ and $p^2 = 9$. In this chapter we will only consider the cases $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$ and $p^2 \approx 9$. In Chapter 4 we show that for specific values of ω, the term $C\cos(\omega t_0)$ also gives rise to secular terms. This means that for specific values of ω, i.e. $\omega \approx 2\omega_{k_p}$ $(k = 1, 2, 3, \dots)$, extra terms occur in the equations for $A_{k,0}$ and $B_{k,0}$ (for $n \neq k$ the equations remain the same).

Case (i): For $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$ and $F(t) \equiv 0$ or $F(t) = C\cos(\omega t)$ with $\omega \neq 2\omega_{k_p} + \epsilon\alpha$ (where $\alpha \in \mathbb{R}$ of $\mathcal{O}(1)$), it can be seen easily from the results obtained in Chapter 2 and Chapter 4 that the equations for $\overline{A}_{n,0}, \overline{B}_{n,0}$ (with $\overline{A}_{n,0} = \omega_{n_p} A_{n,0}$ and $\overline{B}_{n,0} = \omega_{n_p} B_{n,0}$) are

$$\begin{aligned} \frac{d\overline{A}_{n,0}}{dt_1} &= \frac{1}{2}\overline{A}_{n,0}\left(1 - \frac{3}{16}\left(\overline{A}^2_{n,0} + \overline{B}^2_{n,0}\right) - \frac{1}{4}\sum_{m\neq n}^{\infty}\left(\overline{A}^2_{m,0} + \overline{B}^2_{m,0}\right)\right) \qquad (5.2.10) \\ &+\delta\frac{1}{4}\frac{n^2}{\omega_{n_p}}\overline{B}_{n,0}\left[\frac{3}{2}\frac{n^2}{\omega^2_{n_p}}\left(\overline{A}^2_{n,0} + \overline{B}^2_{n,0}\right) + \sum_{m\neq n}\frac{m^2}{\omega^2_{m_p}}\left(\overline{A}^2_{m,0} + \overline{B}^2_{m,0}\right)\right], \\ \frac{d\overline{B}_{n,0}}{dt_1} &= \frac{1}{2}\overline{B}_{n,0}\left(1 - \frac{3}{16}\left(\overline{A}^2_{n,0} + \overline{B}^2_{n,0}\right) - \frac{1}{4}\sum_{m\neq n}^{\infty}\left(\overline{A}^2_{m,0} + \overline{B}^2_{m,0}\right)\right) \qquad (5.2.11) \\ &-\delta\frac{1}{4}\frac{n^2}{\omega_{n_p}}\overline{A}_{n,0}\left[\frac{3}{2}\frac{n^2}{\omega^2_{n_p}}\left(\overline{A}^2_{n,0} + \overline{B}^2_{n,0}\right) + \sum_{m\neq n}\frac{m^2}{\omega^2_{m_p}}\left(\overline{A}^2_{m,0} + \overline{B}^2_{m,0}\right)\right], \end{aligned}$$

for $n = 1, 2, 3, \dots$. From (5.2.10)-(5.2.11) we see that if $A_{n,0}(0) = B_{n,0}(0) = 0$ then $\forall t_1 > 0$ $A_{n,0}(t_1) = B_{n,0}(t_1) \equiv 0$. So, if we start with zero initial energy in the n-th mode, there will be no energy present up to $\mathcal{O}(\epsilon)$ on a time-scale of order ϵ^{-1}. We say the coupling between the modes is of $\mathcal{O}(\epsilon)$. This allows truncation to those modes that have non-zero initial energy. In this case there is an interaction between all modes with non-zero initial energy, but this interaction does not give rise to internal resonances. We will discuss (5.2.10)-(5.2.11) in more detail in Section 5.3.1.

Case (ii): For $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$ and $F(t) = C\cos(\omega t)$ with $\omega = 2\omega_{k_p} + \epsilon\alpha$, where $\alpha \in \mathbb{R}$ of $\mathcal{O}(1)$ is a detuning parameter, it can be seen easily from the results obtained in Chapter 2 and Chapter 4 that the equations for $\overline{A}_{k,0}, \overline{B}_{k,0}$ are

$$\frac{d\overline{A}_{k,0}}{dt_1} = \frac{1}{2}\overline{A}_{k,0}\left(1 - \frac{3}{16}\left(\overline{A}^2_{k,0} + \overline{B}^2_{k,0}\right) - \frac{1}{4}\sum_{m\neq k}^{\infty}\left(\overline{A}^2_{m,0} + \overline{B}^2_{m,0}\right)\right) \tag{5.2.12}$$
$$+\delta\frac{1}{4}\frac{k^2}{\omega_{k_p}}\overline{B}_{k,0}\left[\frac{3}{2}\frac{k^2}{\omega^2_{k_p}}\left(\overline{A}^2_{k,0} + \overline{B}^2_{k,0}\right) + \sum_{m\neq k}\frac{m^2}{\omega^2_{m_p}}\left(\overline{A}^2_{m,0} + \overline{B}^2_{m,0}\right)\right]$$
$$-\frac{1}{4}\frac{k^2}{\omega_{k_p}}\delta C\left(\overline{A}_{k,0}\sin(\alpha t_1) + \overline{B}_{k,0}\cos(\alpha t_1)\right),$$

$$\frac{d\overline{B}_{k,0}}{dt_1} = \frac{1}{2}\overline{B}_{k,0}\left(1 - \frac{3}{16}\left(\overline{A}^2_{k,0} + \overline{B}^2_{k,0}\right) - \frac{1}{4}\sum_{m\neq k}^{\infty}\left(\overline{A}^2_{m,0} + \overline{B}^2_{m,0}\right)\right) \tag{5.2.13}$$
$$-\delta\frac{1}{4}\frac{k^2}{\omega_{k_p}}\overline{A}_{k,0}\left[\frac{3}{2}\frac{k^2}{\omega^2_{k_p}}\left(\overline{A}^2_{k,0} + \overline{B}^2_{k,0}\right) + \sum_{m\neq k}\frac{m^2}{\omega^2_{m_p}}\left(\overline{A}^2_{m,0} + \overline{B}^2_{m,0}\right)\right]$$
$$-\frac{1}{4}\frac{k^2}{\omega_{k_p}}\delta C\left(\overline{A}_{k,0}\cos(\alpha t_1) - \overline{B}_{k,0}\sin(\alpha t_1)\right).$$

For $n \neq k$ equations (5.2.10)-(5.2.11) still hold. We see that for $F(t) = C\cos(\omega t)$ with $\omega = 2\omega_{k_p} + \epsilon\alpha$ the influence of $F(t)$ is of $\mathcal{O}(1)$ on a time-scale of order ϵ^{-1} and extra terms occur in the equations for $A_{k,0}, B_{k,0}$. We see that if $A_{n,0}(0) = B_{n,0}(0) = 0$ then for all $t_1 > 0$ $A_{n,0}(t_1) = B_{n,0}(t_1) \equiv 0$, which holds for all n. So, if we start with zero initial energy in the n-th mode, there will be no energy present up to $\mathcal{O}(\epsilon)$ on a time-scale of order ϵ^{-1}. We say the coupling between the modes is of $\mathcal{O}(\epsilon)$. This again allows truncation to those modes that have non-zero initial energy. In this case there is an interaction between all modes with non-zero initial energy and this interaction does not give rise to internal resonances. We will discuss (5.2.12)-(5.2.13) in more detail in Section 5.3.1, for $k = 1, 3$.
Case (iii): For $p^2 = 9 + \epsilon\beta$ (where $\beta \in \mathbb{R}$ of $\mathcal{O}(1)$ is a detuning parameter) and $F(t) \equiv 0$ or $F(t) = C\cos(\omega t)$ with $\omega \neq 2\omega_{k_p} + \epsilon\alpha$, extra contributions occur in the equations for $A_{n,0}$ and $B_{n,0}$ for $n = 1$ and 3. It can be seen easily from the results obtained in Chapter 2 and Chapter 4 that the equations for $\overline{A}_{n,0}, \overline{B}_{n,0}$ now are:

$$\frac{d\overline{A}_{n,0}}{dt_1} = \frac{1}{2}\overline{A}_{n,0}\left(1 - \frac{3}{16}\left(\overline{A}^2_{n,0} + \overline{B}^2_{n,0}\right) - \frac{1}{4}\sum_{m\neq n}^{\infty}\left(\overline{A}^2_{m,0} + \overline{B}^2_{m,0}\right)\right)$$
$$+\delta\frac{1}{4}\frac{n^2}{\omega_{n_p}}\overline{B}_{n,0}\left[\frac{3}{2}\frac{n^2}{\omega^2_{n_p}}\left(\overline{A}^2_{n,0} + \overline{B}^2_{n,0}\right) + \sum_{m\neq n}\frac{m^2}{\omega^2_{m_p}}\left(\overline{A}^2_{m,0} + \overline{B}^2_{m,0}\right)\right]$$

$$+\mathcal{F}_n + \frac{\beta}{2\omega_{n_p}}\overline{B}_{n,0}, \tag{5.2.14}$$

$$\frac{d\overline{B}_{n,0}}{dt_1} = \frac{1}{2}\overline{B}_{n,0}\left(1 - \frac{3}{16}\left(\overline{A}^2_{n,0} + \overline{B}^2_{n,0}\right) - \frac{1}{4}\sum_{m\neq n}^{\infty}\left(\overline{A}^2_{m,0} + \overline{B}^2_{m,0}\right)\right)$$

$$-\delta\frac{1}{4}\frac{n^2}{\omega_{n_p}}\overline{A}_{n,0}\left[\frac{3}{2}\frac{n^2}{\omega^2_{n_p}}\left(\overline{A}^2_{n,0} + \overline{B}^2_{n,0}\right) + \sum_{m\neq n}\frac{m^2}{\omega^2_{m_p}}\left(\overline{A}^2_{m,0} + \overline{B}^2_{m,0}\right)\right]$$

$$+\mathcal{G}_n - \frac{\beta}{2\omega_{n_p}}\overline{B}_{n,0}, \tag{5.2.15}$$

for $n = 1, 2, 3, \dots$, where

$$\mathcal{F}_1 = -\frac{1}{32}\left((\overline{A}^2_{1,0} - \overline{B}^2_{1,0})\overline{A}_{3,0} + 2\overline{A}_{1,0}\overline{B}_{1,0}\overline{B}_{3,0}\right), \tag{5.2.16}$$

$$\mathcal{G}_1 = \frac{1}{32}\left(2\overline{A}_{1,0}\overline{B}_{1,0}\overline{A}_{3,0} + (\overline{B}^2_{1,0} - \overline{A}^2_{1,0})\overline{B}_{3,0}\right), \tag{5.2.17}$$

$$\mathcal{F}_3 = -\frac{1}{96}\overline{A}_{1,0}\left(\overline{A}^2_{1,0} - 3\overline{B}^2_{1,0}\right), \tag{5.2.18}$$

$$\mathcal{G}_3 = \frac{1}{96}\overline{B}_{1,0}\left(\overline{B}^2_{1,0} - 3\overline{A}^2_{1,0}\right), \tag{5.2.19}$$

$$\mathcal{F}_n = \mathcal{G}_n = 0, \qquad \text{for } n \neq 1, 3. \tag{5.2.20}$$

In this case there is an $\mathcal{O}(1)$ coupling between the modes 1 and 3 which indicates an internal resonance between these modes. This means that if there is initial energy present in mode 1 an energy transfer of $\mathcal{O}(1)$ on a time-scale of order ϵ^{-1} occurs between modes 1 and 3. Truncation to one mode is not valid, both mode 1 and 3 have to be taken into account. We will discuss (5.2.14)-(5.2.15) in more detail in Section 5.3.2 for $n = 1, 3$.
Case (iv): For $p^2 = 9 + \epsilon\beta$ (where $\beta \in \mathbb{R}$ of $\mathcal{O}(1)$ is a detuning parameter) and $F(t) = C\cos(\omega t)$ with $\omega = 2\omega_{k_p} + \epsilon\alpha$ (where $\alpha \in \mathbb{R}$ of $\mathcal{O}(1)$ is a detuning parameter), it can be seen easily from the results obtained in Chapter 2 and Chapter 4 that the equations for $\overline{A}_{k,0}, \overline{B}_{k,0}$ are

$$\frac{d\overline{A}_{k,0}}{dt_1} = \frac{1}{2}\overline{A}_{k,0}\left(1 - \frac{3}{16}\left(\overline{A}^2_{k,0} + \overline{B}^2_{k,0}\right) - \frac{1}{4}\sum_{m\neq k}^{\infty}\left(\overline{A}^2_{m,0} + \overline{B}^2_{m,0}\right)\right)$$

$$+\delta\frac{1}{4}\frac{k^2}{\omega_{k_p}}\overline{B}_{k,0}\left[\frac{3}{2}\frac{k^2}{\omega^2_{k_p}}\left(\overline{A}^2_{k,0} + \overline{B}^2_{k,0}\right) + \sum_{m\neq k}\frac{m^2}{\omega^2_{m_p}}\left(\overline{A}^2_{m,0} + \overline{B}^2_{m,0}\right)\right]$$

$$-\frac{1}{4}\frac{k^2}{\omega_{k_p}}\delta C\left(\overline{A}_{k,0}\sin(\alpha t_1) + \overline{B}_{k,0}\cos(\alpha t_1)\right) + \frac{\beta}{2\omega_{k_p}}\overline{B}_{k,0} + \mathcal{F}_k, \tag{5.2.21}$$

$$\frac{d\overline{B}_{k,0}}{dt_1} = \frac{1}{2}\overline{B}_{k,0}\left(1 - \frac{3}{16}\left(\overline{A}^2_{k,0} + \overline{B}^2_{k,0}\right) - \frac{1}{4}\sum_{m\neq k}^{\infty}\left(\overline{A}^2_{m,0} + \overline{B}^2_{m,0}\right)\right)$$

$$-\delta\frac{1}{4}\frac{k^2}{\omega_{k_p}}\overline{A}_{k,0}\left[\frac{3}{2}\frac{k^2}{\omega_{k_p}^2}\left(\overline{A}_{k,0}^2+\overline{B}_{k,0}^2\right)+\sum_{m\neq k}\frac{m^2}{\omega_{m_p}^2}\left(\overline{A}_{m,0}^2+\overline{B}_{m,0}^2\right)\right]$$
$$-\frac{1}{4}\frac{k^2}{\omega_{k_p}}\delta C\left(\overline{A}_{k,0}\cos(\alpha t_1)-\overline{B}_{k,0}\sin(\alpha t_1)\right)-\frac{\beta}{2\omega_{k_p}}\overline{B}_{k,0}+\mathcal{G}_k, \quad (5.2.22)$$

where $\mathcal{F}_k$ and $\mathcal{G}_k$ satisfy (5.2.16)-(5.2.20). For $n \neq k$ equations (5.2.14)-(5.2.15) still hold. We see that in this case the influence of $F(t)$ is of $\mathcal{O}(1)$ on a time-scale of order ϵ^{-1} and extra terms occur in the equations for $A_{k,0}, B_{k,0}$. Furthermore there is an $\mathcal{O}(1)$ coupling between modes 1 and 3 which indicates an internal resonance between these modes. This means that if there is initial energy present in mode 1 an energy transfer occurs between modes 1 and 3. Truncation to one mode is not valid, both mode 1 and 3 have to be taken into account. The interaction between the different oscillation modes due to the influence of $F(t)$ is restricted to the modes that have non-zero initial energy, i.e. this interaction does not give rise to internal resonances. This again allows truncation to those modes that have non-zero initial energy, with the restriction that even if only one of the modes 1 or 3 has non-zero initial energy, both modes have to be taken into account. The most interesting cases to examine are the cases $k = 1$ and $k = 3$. We will discuss (5.2.21)-(5.2.22) in more detail in Section 5.3.2. When $A_{n,0}$ and $B_{n,0}$ have been determined, and thus $q_{n,0}$, we have constructed a formal approximation v of the exact solution w of the initial-boundary value problem (5.1.8)-(5.1.11)

$$v(x,t;\epsilon)=\sum_{n=1}^{\infty}\left(q_{n,0}(t_0,t_1)+\epsilon q_{n,1}(t_0,t_1)\right)\sin(nx),$$

with $q_{n,0}(t_0,t_1) = A_{n,0}(t_1)\cos(\omega_{n_p}t_0)+B_{n,0}(t_1)\sin(\omega_{n_p}t_0)$ and $q_{n,1}(t_0,t_1) = q_{n,1}^{inh}(t_0,t_1)+A_{n,1}(t_1)\cos(\omega_{n_p}t_0)+B_{n,1}(t_1)\sin(\omega_{n_p}t_0)$, with $q_{n,1}^{inh}$ an inhomogeneous solution of (5.2.8). $A_{n,1}(t_1)$ and $B_{n,1}(t_1)$ can be constructed in such a way that secular terms in the $\mathcal{O}(\epsilon^2)$ approximation are eliminated. Since we are only interested in the $\mathcal{O}(\epsilon)$ approximations of the (exact) solution, we take $A_{n,1}$ and $B_{n,1}$ equal to their initial values $A_{n,1}(0)$ and $B_{n,1}(0)$ respectively.

In the next section some of the interesting features of the solutions for $A_{n,0}, B_{n,0}$ are given for four different cases as described before.

5.3 Modal interactions

In Section 5.2 it has been shown that for $p^2 \neq p_{cr}^2+\epsilon\beta$, where β is a detuning parameter of $\mathcal{O}(1)$ and $p_{cr}^2 \in \{\frac{693}{152}, 9\}$ the coupling between the different oscillation modes is of $\mathcal{O}(\epsilon)$, that is, if we start with zero initial energy in an oscillation mode, no energy will be present in that mode up to $\mathcal{O}(\epsilon)$ on a time-scale of order ϵ^{-1}. This allows truncation to those modes that have non-zero initial energy. For $p^2 = 9 + \epsilon\beta$ there is a coupling between modes 1 and 3, and both modes have to be taken into account. Therefore we assume initial energy present in modes 1 and 3 only, and for all other modes we have $A_{n,0}(0) = B_{n,0}(0) = 0$. In this section we consider the cases $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$ and $p^2 = 9 + \epsilon\beta$ in more detail.

5.3.1 The case $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$

As shown in Section 5.2, there is no extra coupling of $\mathcal{O}(1)$ between modes 1 and 3 due to the Rayleigh perturbation term in (5.1.8). However, due to the integral term in (5.1.8) interactions occur between those modes that have non-zero initial energy. These interactions are restricted to phase shifts of those oscillation modes that have non-zero initial energy. Due to the $F(t)$ term in (5.1.8) extra interactions can occur between the different oscillation modes for specific values of ω, $\omega \approx 2\omega_{k_p}$, $k = 1, 2, 3, \ldots$. Again these interactions can only occur for oscillation modes with non-zero initial energy. We consider the cases $F(t) = 0$, $F(t) = C\cos(\omega t_0)$ with $\omega \neq 2\omega_{k_p} + \epsilon\alpha$ or $\omega = 2\omega_{k_p} + \epsilon\alpha$, $k = 1$ and $k = 3$ (where α is a detuning parameter of $\mathcal{O}(1)$).

The case $F(t) = 0$ or $F(t) = C\cos(\omega t_0)$ with $\omega \neq 2\omega_{k_p} + \epsilon\alpha$

In Section 5.2 (5.2.10)-(5.2.11) have been given for $\overline{A}_{n,0}, \overline{B}_{n,0}$. As stated above we only consider the equations for $n = 1$ and $n = 3$. We introduce polar coordinates to transform (5.2.10)-(5.2.11)

$$\overline{A}_{n,0} = r_n \cos(\phi_n) \ , \quad \overline{B}_{n,0} = r_n \sin(\phi_n) \ , \tag{5.3.1}$$

with $r_n = r_n(t_1)$ the amplitude and $\phi_n = \phi_n(t_1)$ the phase of the oscillation and we obtain the following equations for r_1, r_3, ϕ_1, ϕ_3:

$$\dot{r}_1 = \frac{1}{2}r_1(1 - \frac{3}{16}r_1^2 - \frac{1}{4}r_3^2), \qquad \dot{r}_3 = \frac{1}{2}r_3(1 - \frac{1}{4}r_1^2 - \frac{3}{16}r_3^2), \tag{5.3.2}$$

$$\dot{\phi}_1 = -\frac{3}{8}\frac{\delta}{\omega_{1_p}^3}r_1^2 - \frac{9}{4}\frac{\delta}{\omega_{1_p}\omega_{3_p}^2}r_3^2, \ \dot{\phi}_3 = -\frac{243}{8}\frac{\delta}{\omega_{3_p}^3}r_3^2 - \frac{9}{4}\frac{\delta}{\omega_{1_p}^2\omega_{3_p}}r_1^2, \tag{5.3.3}$$

where a dot represents differentiation with respect to t_1. Equations (5.3.2)-(5.3.3) only have trivial critical points ($r_1 = 0$, $r_3 = 0$ and ϕ_1, ϕ_3 arbitrary). However, we can analyze (5.3.2)-(5.3.3) in the (r_1, r_3)-plane, since (5.3.3) are independent of ϕ_1, ϕ_3. In this space we have four 'critical points', which are listed in Table 5.1. Elementary analysis gives us the behavior of the solutions of (5.3.2)-(5.3.3) in the (r_1, r_3)-phase plane as shown in Figure

Critical point	Behavior
$(0,0)$	unstable $2d$-node
$(0,\sqrt{16/3})$	stable $2d$-node
$(\sqrt{16/3},0)$	stable $2d$-node
$(\sqrt{16/7},\sqrt{16/7})$	$2d$ saddle

Table 5.1: Critical points of (5.3.2).

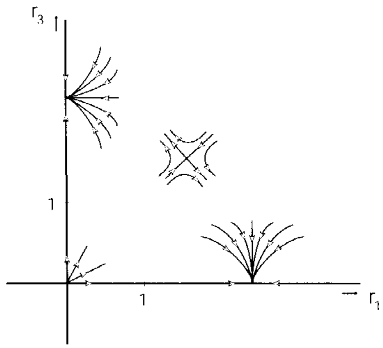


Figure 5.2: Phase plane of (5.3.2).

5.2. In the next subsections we will analyze most equations in a $3d$- or $4d$-space for r_1, r_3 and one or two phase angles.

The case $F(t) = C\cos(\omega t_0)$ with $\omega = 2\omega_{k_p} + \epsilon\alpha$

In Section 5.2 equations (5.2.12)-(5.2.13) and (5.2.10)-(5.2.11) have been given for $\overline{A}_{k,0}, \overline{B}_{k,0}$ and $\overline{A}_{n,0}, \overline{B}_{n,0}$, $n \neq k$ respectively. As stated above we only consider the equations for $n = 1$ and $n = 3$. We consider two cases in this subsection, $k = 1$ and $k = 3$.

The case $F(t) = C\cos(\omega t_0)$ with $\omega = 2\omega_{3_p} + \epsilon\alpha$
Introducing polar coordinates as defined in (5.3.1) for each mode, we transform (5.2.12)-(5.2.13) for $k = 3$ and (5.2.10)-(5.2.11) for $n = 1$:

$$\begin{aligned}
\dot{r}_1 &= \frac{1}{2}r_1\left(1 - \frac{3}{16}r_1^2 - \frac{1}{4}r_3^2\right),\\
\dot{r}_3 &= \frac{1}{2}r_3\left(1 - \frac{1}{4}r_1^2 - \frac{3}{16}r_3^2\right) - \frac{9}{4}\frac{\delta C}{\omega_{3_p}}r_3\sin(2\phi_3 + \alpha t_1),\\
\dot{\phi}_1 &= -\frac{3}{8}\frac{\delta}{\omega_{1_p}^3}r_1^2 - \frac{9}{4}\frac{\delta}{\omega_{1_p}\omega_{3_p}^2}r_3^2,
\end{aligned}$$

$$\dot{\phi}_3 = -\frac{243}{8}\frac{\delta}{\omega_{3_p}^3}r_3^2 - \frac{9}{4}\frac{\delta}{\omega_{1_p}^2\omega_{3_p}}r_1^2 - \frac{9}{4}\frac{\delta C}{\omega_{3_p}}\cos(2\phi_3 + \alpha t_1).$$

We introduce $\psi_3 = 2\phi_3 + \alpha t_1$ and obtain the following system for r_1, r_3, ψ_3:

$$\dot{r}_1 = \frac{1}{2}r_1(1 - \frac{3}{16}r_1^2 - \frac{1}{4}r_3^2), \tag{5.3.4}$$

$$\dot{r}_3 = \frac{1}{2}r_3(1 - \frac{1}{4}r_1^2 - \frac{3}{16}r_3^2) - \frac{9}{4}\frac{\delta C}{\omega_{3_p}}r_3\sin(\psi_3), \tag{5.3.5}$$

$$\dot{\psi}_3 = \alpha - \frac{243}{4}\frac{\delta}{\omega_{3_p}^3}r_3^2 - \frac{9}{2}\frac{\delta}{\omega_{1_p}^2\omega_{3_p}}r_1^2 - \frac{9}{2}\frac{\delta C}{\omega_{3_p}}\cos(\psi_3). \tag{5.3.6}$$

From a practical point of view it is impossible to consider all different cases for δ and $C \in \mathbb{R}$. To give an idea how the solutions of (5.3.4)-(5.3.6) can behave we will take $\delta = C = 1$ in this subsection. For $\delta \neq 1$ and $C \neq 1$ a similar analysis can be applied. Furthermore, we take $p^2 = 8$, since we consider $p^2 = 9$ in the next subsection and we are interested in detuning from this case. In the appendix some results for $p^2 \neq 8$ (with $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$) are given. The critical points of (5.3.4)-(5.3.6) for $\alpha \geq 0$ are listed in Table 5.2 and for $\alpha \leq 0$ in Table 5.3.

α-range	# points	critical points	behavior
$0 \leq \alpha < \alpha_{cr,1}^+$	6	$(0, 0, \chi_1)$	unstable $3d$-node
		$(0, 0, \chi_2)$	$3d$ saddle-node
		$(\sqrt{16/3}, 0, \chi_3)$	$3d$ saddle-node
		$(\sqrt{16/3}, 0, \chi_4)$	$3d$ saddle-node
		$(0, \hat{r}_3^+, \chi_6^+)$	$3d$ saddle-node
		$(0, \hat{r}_3^-, \chi_6^-)$	stable $3d$ node
$\alpha = \alpha_{cr,1}^+$	5	$(0, 0, 0)$	h.o. singularity
		$(\sqrt{16/3}, 0, \chi_3)$	$3d$ saddle-node
		$(\sqrt{16/3}, 0, \chi_4)$	$3d$ saddle-node
		$(0, \hat{r}_3^+, \chi_6^+)$	$3d$ saddle-node
		$(0, \hat{r}_3^-, \chi_6^-)$	stable $3d$ node
$\alpha_{cr,1}^+ < \alpha < \alpha_{cr,6}^+$	4	$(\sqrt{16/3}, 0, \chi_3)$	$3d$ saddle-node
		$(\sqrt{16/3}, 0, \chi_4)$	$3d$ saddle-node
		$(0, \hat{r}_3^+, \chi_6^+)$	$3d$ saddle-node
		$(0, \hat{r}_3^-, \chi_6^-)$	stable $3d$ node
$\alpha = \alpha_{cr,6}^+$	4	$(\sqrt{16/3}, 0, \chi_3)$	$3d$ saddle-node
		$(\sqrt{16/3}, 0, \chi_4) = (\tilde{r}_1^-, \tilde{r}_3^-, \chi_5^-)$	h.o. singularity
		$(0, \hat{r}_3^+, \chi_6^+)$	$3d$ saddle-node
		$(0, \hat{r}_3^-, \chi_6^-)$	stable $3d$ node

$\alpha^+_{cr,6}<\alpha<\alpha^+_{cr,5}$	5	$(\sqrt{16/3},0,\chi_3)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_4)$	stable $3d$-node
		$(\tilde{r}_1^-,\tilde{r}_3^-,\chi_5^-)$	$3d$ saddle-node
		$(0,\hat{r}_3^+,\chi_6^+)$	$3d$ saddle-node
		$(0,\hat{r}_3^-,\chi_6^-)$	stable $3d$ node
$\alpha=\alpha^+_{cr,5}$	5	$(\sqrt{16/3},0,\chi_3)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_4)$	stable $3d$-node
		$(\tilde{r}_1^-,\tilde{r}_3^-,\chi_5^-)$	$3d$ saddle-node
		$(0,\hat{r}_3^+,\chi_6^+)=(0,\tilde{r}_3^+,\chi_5^+)$	h.o. singularity
		$(0,\hat{r}_3^-,\chi_6^-)$	stable $3d$ node
$\alpha^+_{cr,5}<\alpha<\alpha^+_{cr,3}$	6	$(\sqrt{16/3},0,\chi_3)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_4)$	stable $3d$-node
		$(\tilde{r}_1^+,\tilde{r}_3^+,\chi_5^+)$	$3d$ saddle-node
		$(\tilde{r}_1^-,\tilde{r}_3^-,\chi_5^-)$	$3d$ saddle-node
		$(0,\hat{r}_3^+,\chi_6^+)$	$3d$ saddle-node
		$(0,\hat{r}_3^-,\chi_6^-)$	stable $3d$ node
$\alpha=\alpha^+_{cr,3}$	5	$(\sqrt{16/3},0,0)$	h.o. singularity
		$(\tilde{r}_1^+,\tilde{r}_3^+,\chi_5^+)$	$3d$ saddle-node
		$(\tilde{r}_1^-,\tilde{r}_3^-,\chi_5^-)$	$3d$ saddle-node
		$(0,\hat{r}_3^+,\chi_6^+)$	$3d$ saddle-node
		$(0,\hat{r}_3^-,\chi_6^-)$	stable $3d$ node
$\alpha^+_{cr,3}<\alpha<\alpha^+_{cr,4}$	4	$(\tilde{r}_1^+,\tilde{r}_3^+,\chi_5^+)$	$3d$ saddle-node
		$(\tilde{r}_1^-,\tilde{r}_3^-,\chi_5^-)$	$3d$ saddle-node
		$(0,\hat{r}_3^+,\chi_6^+)$	$3d$ saddle-node
		$(0,\hat{r}_3^-,\chi_6^-)$	stable $3d$ node
$\alpha=\alpha^+_{cr,4}$	3	$(\tilde{r}_1^+,\tilde{r}_3^+,\chi_5^+)=(\tilde{r}_1^-,\tilde{r}_3^-,\chi_5^-))$	h.o. singularity
		$(0,\hat{r}_3^+,\chi_6^+)$	$3d$ saddle-node
		$(0,\hat{r}_3^-,\chi_6^-)$	stable $3d$ node
$\alpha^+_{cr,4}<\alpha<\alpha^+_{cr,2}$	2	$(0,\hat{r}_3^+,\chi_6^+)$	$3d$ saddle-node
		$(0,\hat{r}_3^-,\chi_6^-)$	stable $3d$ node
$\alpha=\alpha^+_{cr,2}$	1	$(0,\hat{r}_3^+,\chi_6^+)=(0,\hat{r}_3^-,\chi_6^-)$	h.o. singularity
$\alpha^+_{cr,2}<\alpha$	0	–	–

Table 5.2: Critical points of (5.3.4)-(5.3.6) for $p^2 = 8$ and $F(t) = \cos(\omega t_0)$ with $\omega = 2\omega_{3_p} + \epsilon\alpha$, with $\alpha \geq 0$.

α-range	# points	critical points	behavior
$\alpha<\alpha^-_{cr,1}$	0	–	–
$\alpha=\alpha^-_{cr,1}$	1	$(0,0,\pi)$	h.o. singularity
$\alpha^-_{cr,1}<\alpha<\alpha^-_{cr,3}$	2	$(0,0,\chi_1)$	unstable $3d$-node
		$(0,0,\chi_2)$	$3d$ saddle-node
$\alpha=\alpha^-_{cr,3}$	3	$(0,0,\chi_1)$	unstable $3d$-node
		$(0,0,\chi_2)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\pi)$	h.o. singularity
$\alpha^-_{cr,3}<\alpha<\alpha^-_{cr,4}$	4	$(0,0,\chi_1)$	unstable $3d$-node
		$(0,0,\chi_2)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_3)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_4)$	stable $3d$-node
$\alpha=\alpha^-_{cr,4}$	5	$(0,0,\chi_1)$	unstable $3d$-node
		$(0,0,\chi_2)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_3)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_4)$	stable $3d$-node
		$(\tilde{r}_1^+,\tilde{r}_3^+,\chi_5^+)=(\tilde{r}_1^-,\tilde{r}_3^-,\chi_5^-)$	h.o. singularity
$\alpha^-_{cr,4}<\alpha<\alpha^-_{cr,2}$	6	$(0,0,\chi_1)$	unstable $3d$-node
		$(0,0,\chi_2)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_3)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_4)$	stable $3d$-node
		$(\tilde{r}_1^+,\tilde{r}_3^+,\chi_5^+)$	unst. $3d$ spiral-node
		$(\tilde{r}_1^-,\tilde{r}_3^-,\chi_5^-)$	$3d$ saddle-node
$\alpha=\alpha^-_{cr,2}$	7	$(0,0,\chi_1)$	unstable $3d$-node
		$(0,0,\chi_2)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_3)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_4)$	stable $3d$-node
		$(\tilde{r}_1^+,\tilde{r}_3^+,\chi_5^+)$	unst. $3d$ spiral-node
		$(\tilde{r}_1^+,\tilde{r}_3^+,\chi_5^-)$	$3d$ saddle-node
		$(0,\hat{r}_3^+,\chi_6^+)=(0,\hat{r}_3^-,\chi_6^-)$	h.o. singularity
$\alpha^-_{cr,2}<\alpha<\alpha^-_{cr,5}$	8	$(0,0,\chi_1)$	unstable $3d$-node
		$(0,0,\chi_2)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_3)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_4)$	stable $3d$-node
		$(\tilde{r}_1^+,\tilde{r}_3^+,\chi_5^+)$	$3d$ saddle-node
		$(\tilde{r}_1^-,\tilde{r}_3^-,\chi_5^-)$	$3d$ saddle-node
		$(0,\hat{r}_3^+,\chi_6^+)$	$3d$ saddle-node
		$(0,\hat{r}_3^-,\chi_6^-)$	stable $3d$ node

$\alpha=\alpha_{cr,5}^-$	7	$(0,0,\chi_1)$	unstable $3d$-node
		$(0,0,\chi_2)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_3)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_4)$	stable $3d$-node
		$(0,\hat{r}_3^+,\chi_6^+)=(\tilde{r}_1^+,\tilde{r}_3^+,\chi_5^+)$	h.o. singularity
		$(\tilde{r}_1^-,\tilde{r}_3^-,\chi_5^-)$	$3d$ saddle-node
		$(0,\hat{r}_3^-,\chi_6^-)$	stable $3d$ node
$\alpha_{cr,5}^-<\alpha<\alpha_{cr,6}^-$	7	$(0,0,\chi_1)$	unstable $3d$-node
		$(0,0,\chi_2)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_3)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_4)$	stable $3d$-node
		$(\tilde{r}_1^-,\tilde{r}_3^-,\chi_5^-)$	$3d$ saddle-node
		$(0,\hat{r}_3^+,\chi_6^+)$	$3d$ saddle-node
		$(0,\hat{r}_3^-,\chi_6^-)$	stable $3d$ node
$\alpha=\alpha_{cr,6}^-$	6	$(0,0,\chi_1)$	unstable $3d$-node
		$(0,0,\chi_2)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_3)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_4)=(\tilde{r}_1^-,\tilde{r}_3^-,\chi_5^-)$	h.o. singularity
		$(0,\hat{r}_3^+,\chi_6^+)$	$3d$ saddle-node
		$(0,\hat{r}_3^-,\chi_6^-)$	stable $3d$ node
$\alpha_{cr,6}^-<\alpha<0$	6	$(0,0,\chi_1)$	unstable $3d$-node
		$(0,0,\chi_2)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_3)$	$3d$ saddle-node
		$(\sqrt{16/3},0,\chi_4)$	$3d$ saddle-node
		$(0,\hat{r}_3^+,\chi_6^+)$	$3d$ saddle-node
		$(0,\hat{r}_3^-,\chi_6^-)$	stable $3d$ node

Table 5.3: Critical points of (5.3.4)-(5.3.6) for $p^2 = 8$ and $F(t) = \cos(\omega t_0)$ with $\omega = 2\omega_{3_p} + \epsilon\alpha$, with $\alpha < 0$.

In Table 5.2 and Table 5.3 the following parameters are defined:

$$\tilde{r}_1^\pm = \sqrt{\frac{16}{7}\left(1-\frac{18}{\sqrt{89}}\sin\chi_5^\pm\right)}, \qquad \tilde{r}_3^\pm = \sqrt{\frac{16}{7}\left(1+\frac{27}{2\sqrt{89}}\sin\chi_5^\pm\right)},$$

$$\hat{r}_3^\pm = \sqrt{\frac{16}{3}\left(1-\frac{9}{2\sqrt{89}}\sin\chi_6^\pm\right)},$$

$$\chi_1 = \arccos(\frac{2\sqrt{89}}{9}\alpha), \qquad \chi_2 = 2\pi - \chi_1,$$

$$\chi_3 = \arccos(\frac{2\sqrt{89}}{9}\alpha - \frac{16}{27}), \chi_4 = 2\pi - \chi_3,$$

$$\chi_5^+ = \begin{cases} \pi - \arcsin(\mathcal{S}_{1,\alpha}^+) & \alpha_{cr,4}^- \le \alpha \le \alpha_{cr,5}^-, \\ \arcsin(\mathcal{S}_{1,\alpha}^+) & \alpha_{cr,5}^+ \le \alpha \le \alpha_{cr,4}^+, \\ \text{not defined for all other } \alpha\text{-values,} & \end{cases}$$

$$\chi_5^- = \begin{cases} \pi - \arcsin(\mathcal{S}_{1,\alpha}^-) & \alpha_{cr,4}^- \le \alpha \le \alpha_{cr,6}^-, \\ 2\pi + \arcsin(\mathcal{S}_{1,\alpha}^-) & \alpha_{cr,6}^+ \le \alpha < \alpha_{cr,8}^-, \\ \arcsin(\mathcal{S}_{1,\alpha}^-) & \alpha_{cr,8}^- \le \alpha \le \alpha_{cr,4}^+, \\ \text{not defined for all other } \alpha\text{-values,} & \end{cases}$$

$$\chi_6^+ = \begin{cases} \pi - \arcsin(\mathcal{S}_{2,\alpha}^+) & \alpha_{cr,2}^- \le \alpha < \alpha_{cr,7}^-, \\ \arcsin(\mathcal{S}_{2,\alpha}^+) & \alpha_{cr,7}^- \le \alpha < \alpha_{cr,8}^+, \\ 2\pi + \arcsin(\mathcal{S}_{2,\alpha}^+) & \alpha_{cr,8}^+ \le \alpha \le \alpha_{cr,2}^+, \\ \text{not defined for all other } \alpha\text{-values,} & \end{cases}$$

$$\chi_6^- = \begin{cases} \pi - \arcsin(\mathcal{S}_{2,\alpha}^-) & \alpha_{cr,2}^- \le \alpha < \alpha_{cr,7}^+, \\ 2\pi + \arcsin(\mathcal{S}_{2,\alpha}^-) & \alpha_{cr,7}^+ \le \alpha \le \alpha_{cr,2}^+, \\ \text{not defined for all other } \alpha\text{-values,} & \end{cases}$$

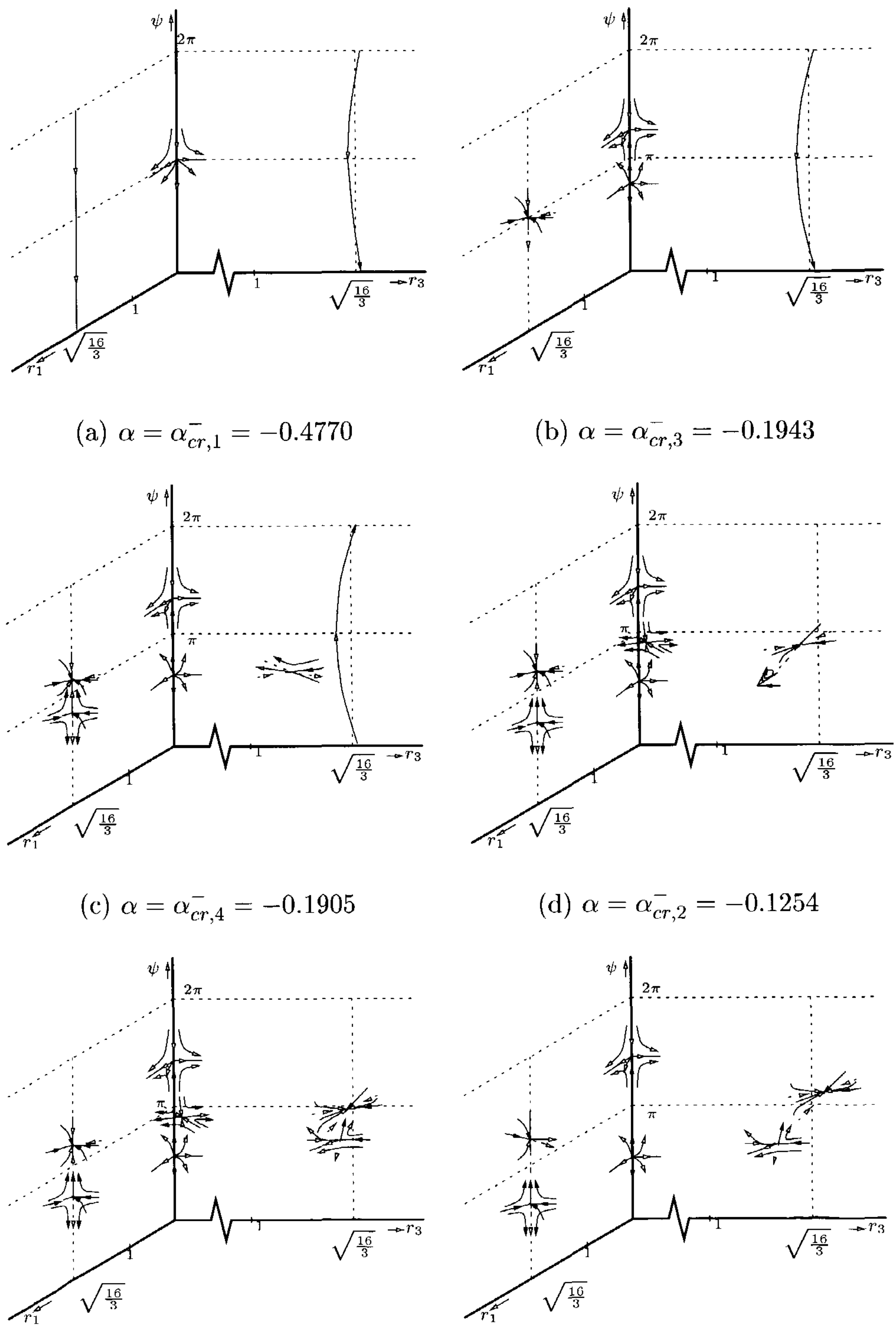
with

$$\begin{aligned} \mathcal{S}_{1,\alpha}^\pm &= \frac{136}{2798396505}\sqrt{89}(5607\sqrt{89} - 15156) \\ &\pm \frac{623}{310932945}\sqrt{89}\sqrt{-12297479236\alpha^2 + 746981984\sqrt{89}\alpha + 1788831769}, \\ \mathcal{S}_{2,\alpha}^\pm &= -\frac{72}{809945}\sqrt{89}(89\sqrt{89} - 324) \\ &\pm \frac{89}{7289505}\sqrt{89}\sqrt{-250968964\alpha^2 + 20531232\sqrt{89}\alpha + 28234089}, \end{aligned}$$

and

$$\begin{aligned} \alpha_{cr,1}^\pm &= \pm\frac{9}{178}\sqrt{89}, & \alpha_{cr,2}^\pm &= \frac{1}{15842}\left(648\sqrt{89} \pm 9\sqrt{809945}\right), \\ \alpha_{cr,3}^\pm &= \frac{1}{534}\sqrt{89}\,(16 \pm 27), & \alpha_{cr,4}^\pm &= \frac{1}{110894}\left(3368\sqrt{89} \pm 9\sqrt{34548105}\right), \\ \alpha_{cr,5}^\pm &= \frac{1}{31684}\sqrt{89}\left(972 \pm 89\sqrt{235}\right), & \alpha_{cr,6}^\pm &= \frac{1}{4806}\sqrt{89}\left(144 \pm 9\sqrt{373}\right), \\ \alpha_{cr,7}^\pm &= \frac{162}{7921}\left(2\sqrt{89} \pm 9\right), & \alpha_{cr,8}^\pm &= \frac{1}{110894}\sqrt{89}\,(9559 \pm 584). \end{aligned}$$

The critical points of (5.3.4)-(5.3.6) for $p^2 \neq 8$ (with $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$) can be found in a similar way, and are listed in Table 5.8 of the appendix. The system is 2π-periodic in ψ_3, so we consider $\psi_3 \in [0, 2\pi[$. The behavior of the solutions of (5.3.4)-(5.3.6) for $p^2 = 8$ (in a neighborhood of every critical point) in the (r_1, r_3, ψ_3) space is given in Figure 5.3, for $\alpha \le 0$. For $\alpha = 0$ system (5.3.4)-(5.3.6), with $p^2 = 8$, has six critical points: two critical points in the $r_1 = 0$ plane (one stable $3d$-node and one $3d$-saddle-node), two $3d$-saddle-nodes in the r_3=0 plane (on the line $r_1 = \sqrt{16/3}$)



(a) $\alpha = \alpha^-_{cr,1} = -0.4770$

(b) $\alpha = \alpha^-_{cr,3} = -0.1943$

(c) $\alpha = \alpha^-_{cr,4} = -0.1905$

(d) $\alpha = \alpha^-_{cr,2} = -0.1254$

(e) $\alpha = \alpha^-_{cr,5} = -0.1168$

(f) $\alpha = \alpha^-_{cr,6} = -0.0585$

Figure 5.3: Phase space of (5.3.4)-(5.3.6) for $p^2 = 8$, for several α-values.

and two critical points on the ψ_3-axis (one unstable $3d$-node and one $3d$-saddle-node). For $\alpha < 0$ there are two $3d$-saddle-nodes in the (r_1, r_3, ψ_3) space: one for $\alpha = \alpha_{cr,6}^-$, bifurcating from one of the saddle-nodes in the $r_3 = 0$ plane (which then becomes a stable $3d$-node for $\alpha < \alpha_{cr,6}^-$) and one for $\alpha = \alpha_{cr,5}^-$, bifurcating from the $3d$-saddle-node in the $r_1 = 0$ plane. As α decreases further one of the saddle-nodes in the (r_1, r_3, ψ_3) space changes into an unstable $3d$-spiral-node. Also, for decreasing α, the two critical points in the $r_1 = 0$ plane move closer to each other, coinciding for $\alpha = \alpha_{cr,2}^-$ and then disappearing. For $\alpha < \alpha_{cr,2}^-$ there exists an attracting curve in the $r_1 = 0$ plane, which moves closer to the line $(0, \sqrt{16/3}, \psi_3)$ as α decreases further. Also, for $\alpha < \alpha_{cr,2}^-$, the two unstable critical points in the (r_1, r_3, ψ_3) space move closer to each other, coinciding for $\alpha = \alpha_{cr,4}^-$ and then disappearing. As α decreases further the two critical points in the $r_3 = 0$ plane (on the line $r_1 = \sqrt{16/3}$) move closer to each other, coinciding for $\alpha = \alpha_{cr,3}^-$ and then disappearing. For $\alpha < \alpha_{cr,3}^-$ the two critical points on the ψ_3-axis move closer to each other, coinciding for $\alpha = \alpha_{cr,1}^-$ and disappearing. We see that for $\alpha < \alpha_{cr,1}^-$ (and for $\alpha > \alpha_{cr,2}^+$) there are no critical points for (5.3.4)-(5.3.6) (with $p^2 = 8$). However, we have two 'critical' lines ($(\sqrt{16/3}, 0, \psi_3)$ and $(0, 0, \psi_3)$), i.e. lines for which $\dot{r}_1 = \dot{r}_3 = 0$ and $\dot{\psi}_3 \neq$). The line $(\sqrt{16/3}, 0, \psi_3)$ is an attracting line. Furthermore, an attracting curve remains in the $r_1 = 0$ plane, which comes in a small neighborhood of (but does not coincide with) the line $(0, \sqrt{16/3}, \psi_3)$ for $\alpha \ll \alpha_{cr,1}^-$ (or $\alpha \gg \alpha_{cr,2}^+$). We see that for $\alpha \ll \alpha_{cr,1}^-$ (or $\alpha \gg \alpha_{cr,2}^+$) the phase space analysis connects to the case $\omega \neq \omega_{3,p} + \epsilon\alpha$, considered in the (r_1, r_3, ψ_3) space. In the (r_1, r_3, ψ_3) space system (5.3.2)-(5.3.3), which has been considered in Section 5.3.1, becomes

$$\dot{r}_1 = \frac{1}{2} r_1 (1 - \frac{3}{16} r_1^2 - \frac{1}{4} r_3^2), \tag{5.3.7}$$

$$\dot{r}_3 = \frac{1}{2} r_3 (1 - \frac{1}{4} r_1^2 - \frac{3}{16} r_3^2), \tag{5.3.8}$$

$$\dot{\psi}_3 = \alpha - \frac{243}{4} \frac{\delta}{\omega_{3_p}^3} r_3^2 - \frac{9}{2} \frac{\delta}{\omega_{1_p}^2 \omega_{3_p}} r_1^2. \tag{5.3.9}$$

In this case system (5.3.7)-(5.3.9) has no critical points, but we have four 'critical' lines, for which $\dot{r}_1 = \dot{r}_3 = 0$ and $\dot{\psi}_3 \neq 0$. These lines are: $(0, 0, \psi_3)$, $(\sqrt{16/3}, 0, \psi_3)$, $(0, \sqrt{16/3}, \psi_3)$ and $(\sqrt{16/7}, \sqrt{16/7}, \psi_3)$. This case connects to the case considered above for large values of $|\alpha|$.

The case $F(t) = C\cos(\omega t_0)$ with $\omega = 2\omega_{1_p} + \epsilon\alpha$

Introducing polar coordinates as defined in (5.3.1) for each mode, we transform (5.2.12)-(5.2.13) for $k = 1$ and (5.2.10)-(5.2.11) for $n = 3$

$$
\begin{aligned}
\dot{r}_1 &= \frac{1}{2}r_1(1 - \frac{3}{16}r_1^2 - \frac{1}{4}r_3^2) - \frac{1}{4}\frac{\delta C}{\omega_{1_p}}r_3\sin(2\phi_1 + \alpha t_1),\\
\dot{r}_3 &= \frac{1}{2}r_3(1 - \frac{1}{4}r_1^2 - \frac{3}{16}r_3^2),\\
\dot{\phi}_1 &= -\frac{3}{8}\frac{\delta}{\omega_{1_p}^3}r_1^2 - \frac{9}{4}\frac{\delta}{\omega_{1_p}\omega_{3_p}^2}r_3^2 - \frac{1}{4}\frac{\delta C}{\omega_{1_p}}\cos(2\phi_1 + \alpha t_1),\\
\dot{\phi}_3 &= -\frac{243}{8}\frac{\delta}{\omega_{3_p}^3}r_3^2 - \frac{9}{4}\frac{\delta}{\omega_{1_p}^2\omega_{3_p}}r_1^2.
\end{aligned}
$$

We introduce $\psi_2 = 2\phi_1 + \alpha t_1$ and obtain the following system for r_1, r_3, ψ_1:

$$\dot{r}_1 = \frac{1}{2}r_1(1 - \frac{3}{16}r_1^2 - \frac{1}{4}r_3^2) - \frac{1}{4}\frac{\delta C}{\omega_{1_p}}r_3\sin(\psi_2), \tag{5.3.10}$$

$$\dot{r}_3 = \frac{1}{2}r_3(1 - \frac{1}{4}r_1^2 - \frac{3}{16}r_3^2), \tag{5.3.11}$$

$$\dot{\psi}_2 = \alpha - \frac{3}{4}\frac{\delta}{\omega_{1_p}^3}r_1^2 - \frac{9}{2}\frac{\delta}{\omega_{1_p}\omega_{3_p}^2}r_3^2 - \frac{1}{2}\frac{\delta C}{\omega_{1_p}}\cos(\psi_2). \tag{5.3.12}$$

From a practical point of view it is impossible to consider all different cases for δ and $C \in \mathbb{R}$. Again, as done in the previous subsection, we will consider the case $\delta = C = 1$. Furthermore, we take $p^2 = 8$ as was done in the previous subsection. It can be seen easily from the symmetry in r_1 and r_3 that the results for this case are similar to the results given in the previous subsection. The only difference are the constants in the equations for r_1, r_3, ψ_3 which contain different powers of $\omega_{1_p}, \omega_{3_p}$. A numerical analysis of system (5.3.10)-(5.3.12) gives us results, which are similar to the results obtained for $\omega = 2\omega_{3_p} + \epsilon\alpha$ in the previous subsection (with r_1 and r_3 exchanged). Also, for $\delta \neq 1$ and $C \neq 1$ a similar analysis can be applied.

5.3.2 The case $p^2 = 9 + \epsilon\beta$

As shown in Section 5.2 an interaction occurs between oscillation modes 1 and 3 if $p^2 = 9$. Furthermore, due to the integral term in (5.1.8), interactions occur between those modes that have non-zero initial energy. Due to the $F(t)$ term in (5.1.8) extra interactions can occur between the different oscillation modes for specific values of ω, $\omega \approx 2\omega_{k_p}$, $k = 1, 2, 3, \ldots$. Again these interactions can only occur for oscillation modes with non-zero initial energy. We consider the cases $F(t) = 0$, $F(t) = C\cos(\omega t_0)$ with $\omega \neq 2\omega_{k_p} + \epsilon\alpha$ or $\omega = 2\omega_{k_p} + \epsilon\alpha$, $k = 1, 3$.

The case $F(t) = 0$ or $F(t) = C\cos(\omega t_0)$ with $\omega \neq 2\omega_{k_p} + \epsilon\alpha$

In Section 5.2 equations (5.2.14)-(5.2.15) have been given for $\overline{A}_{n,0}, \overline{B}_{n,0}$. As stated above we only consider the equations for $n = 1$ and $n = 3$. Introducing polar coordinates as defined in (5.3.1) for each mode, we transform (5.2.14)-(5.2.15) for $n = 1, 3$

$$
\begin{aligned}
\dot{r}_1 &= \frac{1}{2}r_1(1 - \frac{3}{16}r_1^2 - \frac{1}{4}r_3^2) - \frac{1}{32}r_1^2 r_3\cos(\phi_3 - 3\phi_1),\\
\dot{r}_3 &= \frac{1}{2}r_3(1 - \frac{1}{4}r_1^2 - \frac{3}{16}r_3^2) - \frac{1}{96}r_1^3\cos(\phi_3 - 3\phi_1),\\
\dot{\phi}_1 &= -\frac{3}{80}\frac{\delta}{\sqrt{10}}r_1^2 - \frac{1}{40}\frac{\delta}{\sqrt{10}}r_3^2 - \frac{1}{32}r_1 r_3\sin(\phi_3 - 3\phi_1) - \frac{\beta}{2\sqrt{10}},\\
\dot{\phi}_3 &= -\frac{9}{80}\frac{\delta}{\sqrt{10}}r_3^2 - \frac{3}{40}\frac{\delta}{\sqrt{10}}r_1^2 + \frac{1}{96}\frac{r_1^3}{r_3}\sin(\phi_3 - 3\phi_1) - \frac{\beta}{6\sqrt{10}}.
\end{aligned}
$$

We introduce $\psi_1 = \phi_3 - 3\phi_1$ and obtain the following system for r_1, r_3, ψ_1:

$$\dot{r}_1 = \frac{1}{2}r_1(1 - \frac{3}{16}r_1^2 - \frac{1}{4}r_3^2) - \frac{1}{32}r_1^2 r_3\cos(\psi_1), \tag{5.3.13}$$

$$\dot{r}_3 = \frac{1}{2}r_3(1 - \frac{1}{4}r_1^2 - \frac{3}{16}r_3^2) - \frac{1}{96}r_1^3\cos(\psi_1), \tag{5.3.14}$$

$$\dot{\psi}_1 = \frac{4}{3}\frac{\beta}{\sqrt{10}} + \frac{1}{96}\frac{r_1}{r_3}\left(r_1^2 + 9r_3^2\right)\sin(\psi_1) + \frac{3}{80}\frac{\delta}{\sqrt{10}}(r_1^2 - r_3^2). \tag{5.3.15}$$

For $r_3 = 0$ (5.3.13)-(5.3.15) do not hold. In that case we have to analyze the original differential equations (5.2.14)-(5.2.15) for $n = 1, 3$. We analyze (5.3.13)-(5.3.15) in the (r_1, r_3, ψ_1) space. We assume $\delta = 1$. For $\delta \neq 1$ a similar analysis can be applied.
We start with $\beta = 0$. The critical points of (5.3.13)-(5.3.15) with $\beta = 0$ are listed in Table 5.4. For a point on the line $(0, \sqrt{16/3}, \psi_1)$ we have $\dot{r}_1 = \dot{r}_3 = 0$ and $\dot{\psi}_1 > 0$, which means we have an (attracting) line with flow along it. The behavior of the solutions of (5.3.13)-(5.3.15) for $\beta = 0$ in the (r_1, r_3, ψ_1) space is sketched in Figure 5.4. We next examine the behavior of solutions of (5.3.13)-(5.3.15) in the (r_1, r_3, ψ_1) space for several values of $\beta > 0$. For $\beta < 0$ a similar analysis holds.
For $\beta > 0$ the two saddle nodes in the (r_1, r_3, ψ_1) space move closer to each other. Furthermore, the stable $3d$-node in the (r_1, r_3, ψ_1) space moves closer to the $r_3 = 0$ plane. At some point, for $\beta \approx 0.03715$, one of the saddle-nodes in the (r_1, r_3, ψ_1) space changes singularity and becomes an unstable $3d$ spiral-node. The stability analysis in the (r_1, r_3, ψ_1) space does not change, since this critical point remains unstable. Numerical analysis indicates that for $\beta \geq 1/10$ the effects of the α-term in system (5.3.13)-(5.3.15) are significant. We therefore start our analysis with $\beta = 1/10$. The

Critical point	Behavior
$(0, 0, \psi_1)$	unstable $2d$-node for each ψ_1
$(1.5930, 1.2627, 6.2328)$	$3d$-saddle-node
$(1.2267, 1.8538, 3.0389)$	$3d$-saddle-node
$(2.2975, 0.6312, 3.3146)$	stable $3d$-node

Table 5.4: Critical points of (5.3.13)-(5.3.15) for β=0.

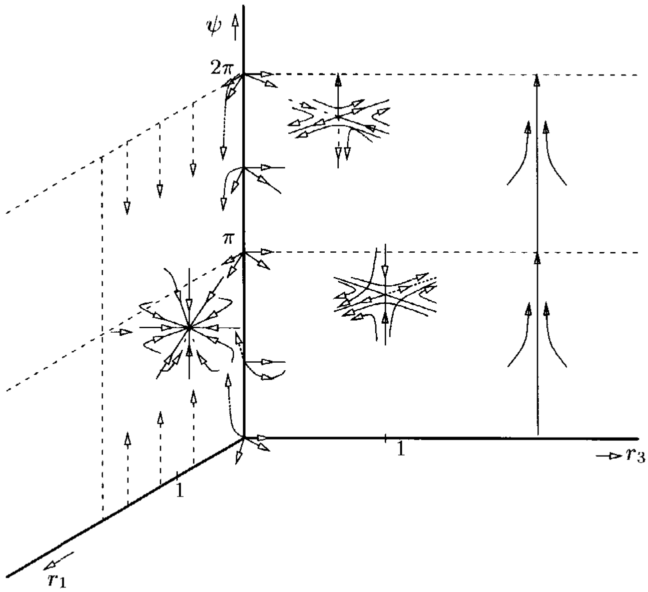


Figure 5.4: Phase space of (5.3.13)-(5.3.15) for $\beta = 0$.

critical points are listed in Table 5.5. For a point on the line $(0,\sqrt{16/3},\psi_1)$ we still have $\dot{r}_1 = \dot{r}_3 = 0$ and $\dot{\psi}_1 > 0$. The behavior of the solutions of (5.3.13)-(5.3.15) for $\beta = 1/10$ in the (r_1, r_3, ψ_1) space is sketched in Figure 5.5. As β increases further the two unstable points in the (r_1, r_3, ψ_1) space move closer to each other. Furthermore the stable $3d$-node moves closer ot the $r_3 = 0$ plane. At some point, for $\beta \approx 0.1860$, the $3d$-node changes sigularity to a $3d$ spiral-node. The stability analysis in the (r_1, r_3, ψ_1) space does not change, since this critical point remains stable. As β increases further the two unstable points in the (r_1, r_3, ψ_1) space move closer to each other, coinciding for $\beta \approx 0.5675$ and then disappering. The critical points for $\beta \approx 0.5675$ are listed in Table 5.6. The behavior of the solutions of (5.3.13)-(5.3.15) for $\beta \approx 0.5675$ in the (r_1, r_3, ψ_1) space is sketched in Figure 5.6. The line $(0,\sqrt{16/3},\psi_1)$ remains a 'critical' line, i.e. we have $\dot{r}_1 = \dot{r}_3 = 0$ and $\dot{\psi}_1 > 0$. For $\beta > 0.5675$ a separating surface remains in

Critical point	Behavior
$(0, 0, \psi_1)$	unstable $2d$-node for each ψ_1
$(1.5927, 1.2678, 6.0418)$	$3d$-saddle-node
$(1.2262, 1.8544, 3.2277)$	unstable $3d$ spiral-node
$(2.2971, 0.6173, 3.4431)$	stable $3d$-node

Table 5.5: Critical points of (5.3.13)-(5.3.15) for $\beta = 1/10$.

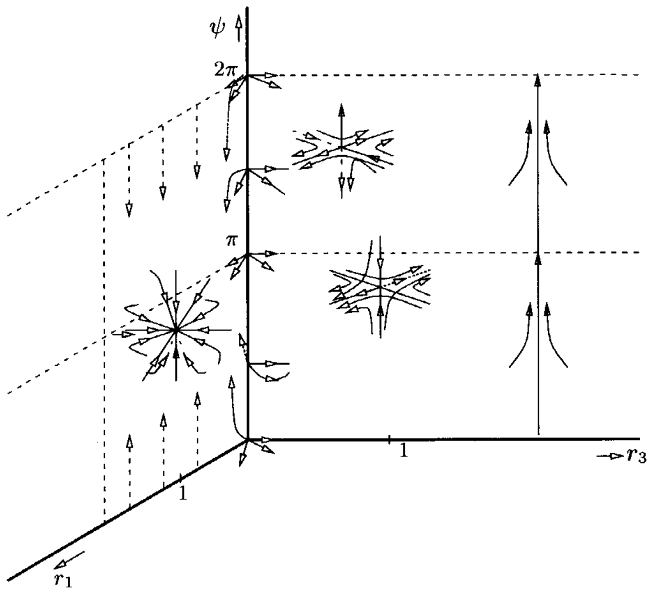


Figure 5.5: Phase space of (5.3.13)-(5.3.15) for $\beta = \frac{1}{10}$.

the (r_1, r_3, ψ_1). The stable point in the (r_1, r_3, ψ_1) space move closer to the $r_3 = 0$ plane as β increases further. For $\beta = 1$ the critical points are listed in Table 5.7. The line $(0, \sqrt{16/3}, \psi_1)$ remains a 'critical' line. The behavior of the solutions of (5.3.13)-(5.3.15) for $\beta = 1$ in the (r_1, r_3, ψ_1) space is sketched in Figure 5.7. For $\beta \gg 1$ the stable point in the (r_1, r_3, ψ_1) space is in a small neighborhood of the $r_3 = 0$ plane. It can be easily seen that for $\beta \gg 1$ the phase space behavior connects to the behavior in the case $p^2 \neq 9$ and $\omega \neq 2\omega_{k,p}$, considered in the (r_1, r_3, ψ_1) space (see also Section 5.3.1). In the (r_1, r_3, ψ_3) space system (5.3.2)-(5.3.3), which has been considered in Section 5.3.1, becomes

$$\dot{r}_1 = \frac{1}{2} r_1 \left(1 - \frac{3}{16} r_1^2 - \frac{1}{4} r_3^2\right), \tag{5.3.16}$$

$$\dot{r}_3 = \frac{1}{2} r_3 \left(1 - \frac{1}{4} r_1^2 - \frac{3}{16} r_3^2\right), \tag{5.3.17}$$

Critical point	Behavior
$(0,0,\psi_1)$	unstable $2d$-node for each ψ_1
$(1.4935, 1.5435, 4.6165)$	h.o. singularity
$(2.2989, 0.4399, 4.0419)$	stable $3d$ spiral-node

Table 5.6: Critical points of (5.3.13)-(5.3.15) for $\beta \approx 0.5675$.

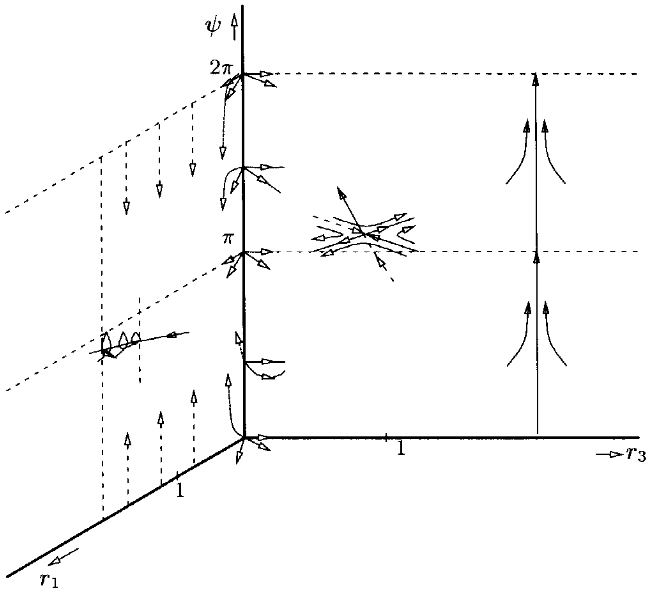


Figure 5.6: Phase space of (5.3.13)-(5.3.15) for $\beta \approx 0.5675$.

$$\dot{\psi}_1 \;=\; \frac{27}{8}\frac{\delta}{\omega_{3_p}^3\omega_{1_p}}\left(2\omega_{3_p} - 9\omega_{1_p}\right) r_3^2 + \frac{9}{8}\frac{\delta}{\omega_{1_p}^3\omega_{3_p}}\left(\omega_{3_p} - 2\omega_{1_p}\right) r_1^2. \quad (5.3.18)$$

In this case system (5.3.16)-(5.3.18) only has trivial critical points ($r_1 = 0$, $r_3 = 0$, ψ_1 arbitrary). However, we have three 'critical' lines, for which $\dot{r}_1 = \dot{r}_3 = 0$ and $\dot{\psi}_3 \neq 0$. These lines are: $(\sqrt{16/3}, 0, \psi_1)$, $(0, \sqrt{16/3}, \psi_1)$ and $(\sqrt{16/7}, \sqrt{16/7}, \psi_1)$. This case connects to the case considered above for large values of $|\beta|$.

The case $F(t) = C\cos(\omega t_0)$ with $\omega = 2\omega_{k_p} + \epsilon\alpha$

In Section 5.2 equations (5.2.21)-(5.2.22) and (5.2.14)-(5.2.15) have been given for $\overline{A}_{k,0}, \overline{B}_{k,0}$ and $\overline{A}_{n,0}, \overline{B}_{n,0}$, $n \neq k$, respectively. As stated above we only consider the equations for $n = 1$ and $n = 3$. We consider two cases in this subsection, $k = 1$ and $k = 3$.

Critical point	Behavior
$(0, 0, \psi_1)$	unstable $2d$-node for each ψ_1
$(2.3044, 0.2764, 4.3326)$	stable $3d$ spiral-node

Table 5.7: Critical points of (5.3.13)-(5.3.15) for $\beta = 1$.

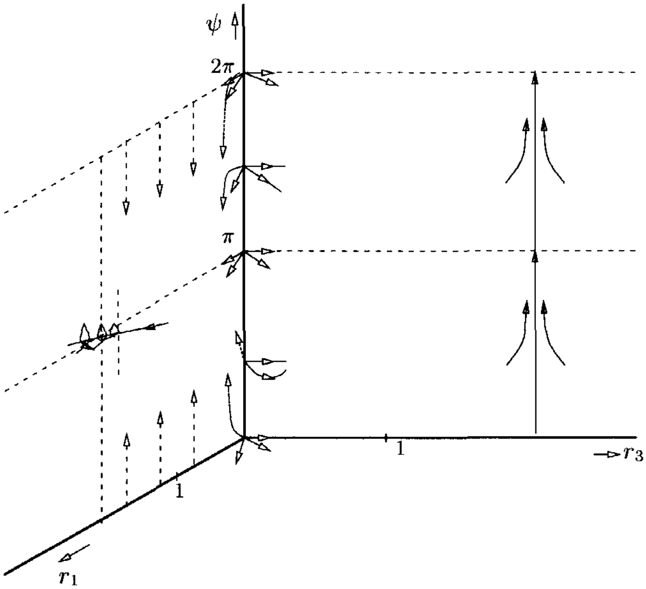


Figure 5.7: Phase space of (5.3.13)-(5.3.15) for $\beta = 1$.

The case $F(t) = C\cos(\omega t_0)$ *with* $\omega = 2\omega_{3_p} + \epsilon\alpha$
Introducing polar coordinates as defined in (5.3.1) for each mode, we transform (5.2.21)-(5.2.22) for $k = 3$ and (5.2.14)-(5.2.15) for $n = 1$

$$\begin{aligned}
\dot{r}_1 &= \frac{1}{2}r_1(1 - \frac{3}{16}r_1^2 - \frac{1}{4}r_3^2) - \frac{1}{32}r_1^2 r_3\cos(\phi_3 - 3\phi_1),\\
\dot{r}_3 &= \frac{1}{2}r_3(1 - \frac{1}{4}r_1^2 - \frac{3}{16}r_3^2) - \frac{1}{96}r_1^3\cos(\phi_3 - 3\phi_1) - \frac{3}{4}\frac{\delta C}{\sqrt{10}}r_3\sin(2\phi_3 + \alpha t_1),\\
\dot{\phi}_1 &= -\frac{3}{80}\frac{\delta}{\sqrt{10}}r_1^2 - \frac{1}{40}\frac{\delta}{\sqrt{10}}r_3^2 - \frac{1}{32}r_1 r_3\sin(\phi_3 - 3\phi_1) - \frac{\beta}{2\sqrt{10}},\\
\dot{\phi}_3 &= -\frac{9}{80}\frac{\delta}{\sqrt{10}}r_3^2 - \frac{3}{40}\frac{\delta}{\sqrt{10}}r_1^2 + \frac{1}{96}\frac{r_1^3}{r_3}\sin(\phi_3 - 3\phi_1)\\
&\qquad - \frac{\beta}{6\sqrt{10}} - \frac{3}{4}\frac{\delta C}{\sqrt{10}}\cos(2\phi_3 + \alpha t_1).
\end{aligned}$$

We introduce $\psi_1 = \phi_3 - 3\phi_1$ and $\psi_3 = 2\phi_3 + \alpha t_1$, and obtain the following (autonomous) system for r_1, r_3, ψ_1, ψ_3:

$$\dot{r}_1 = \frac{1}{2}r_1(1 - \frac{3}{16}r_1^2 - \frac{1}{4}r_3^2) - \frac{1}{32}r_1^2 r_3 \cos(\psi_1), \tag{5.3.19}$$

$$\dot{r}_3 = \frac{1}{2}r_3(1 - \frac{1}{4}r_1^2 - \frac{3}{16}r_3^2) - \frac{1}{96}r_1^3 \cos(\psi_1) - \frac{3}{4}\frac{\delta C}{\sqrt{10}} r_3 \sin(\psi_3), \tag{5.3.20}$$

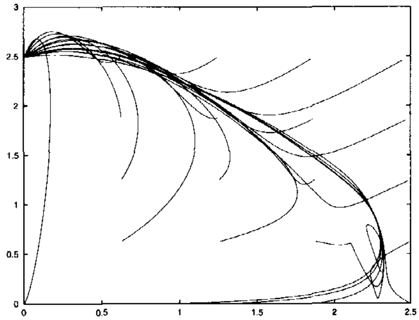
$$\dot{\psi}_1 = \frac{4}{3}\frac{\beta}{\sqrt{10}} + \frac{1}{96}\frac{r_1}{r_3}\left(r_1^2 + 9r_3^2\right)\sin(\psi_1) + \frac{3}{80}\frac{\delta}{\sqrt{10}}(r_1^2 - r_3^2) - \frac{3}{4}\frac{\delta C}{\sqrt{10}}\cos(\psi_3), \tag{5.3.21}$$

$$\dot{\psi}_3 = \alpha - \frac{\beta}{3\sqrt{10}} - \frac{3}{40}\frac{\delta}{\sqrt{10}}\left(3r_3^2 + 2r_1^2\right) + \frac{1}{48}\frac{r_1^3}{r_3}\sin(\psi_1) - \frac{3}{2}\frac{\delta C}{\sqrt{10}}\cos(\psi_3). \tag{5.3.22}$$

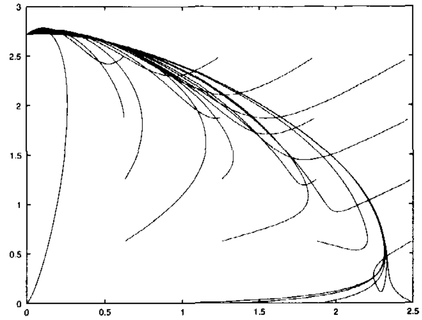
For $r_3 = 0$ (5.3.19)-(5.3.22) do not hold. In that case we have to analyze the original differential equations (5.2.14)-(5.2.15) for $n = 1$ and (5.2.21)-(5.2.22) for $k = 3$. From a practical point of view it is impossible to consider all different cases for δ and $C \in \mathbb{R}$. To give an idea how the solutions of (5.3.19)-(5.3.22) can behave we will take $\delta = C = 1$ in this subsection. For $\delta \neq 1$ and $C \neq 1$ a similar analysis can be applied.

We see that in this subsection system (5.3.19)-(5.3.22) has to be analyzed in a 4d phase space, a reduction to a lower dimension seems to be impossible. Furthermore, we have two detuning parameters in (5.3.19)-(5.3.22), α and β. In this section we only consider the cases $\beta = 0, \alpha \neq 0$ and $\beta \neq 0$, $\alpha = 0$. The (numerical) analysis can be extended to the case $\beta \neq 0$, $\alpha \neq 0$, but is much more complicated and beyond the scope of this chapter.

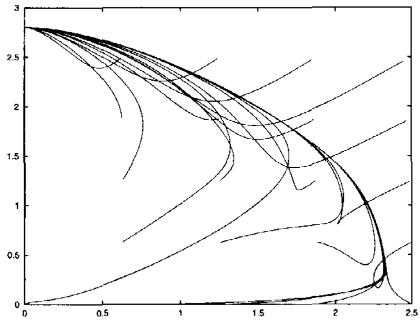
We start with $\beta = 0$. We consider $\alpha = 0$ and $\alpha > 0$ (the case $\alpha < 0$ is similar). The behavior of some solutions of (5.3.19)-(5.3.22), projected on the (r_1, r_3) plane, is given in Figure 5.8 for some values of $\alpha \in [0, 10]$. For $\alpha = 0$ we have two critical points, $(r_1, r_3, \psi_1, \psi_3) = (2.0667, 1.0133, 4.2568, 3.5320)$ and $(1.9999, 0.91953, 5.0557, 3.6226)$, which are $4d$ saddle-nodes. Furthermore, we have two 'critical' lines, for which $\dot{r}_1 = \dot{r}_3 = \dot{\psi}_3 = 0$ and $\dot{\psi}_1 > 0$: the lines $(0, 2.4964, \psi_1, 3.5047)$ and $(0, 1.7584, \psi_1, 2.0531)$. The first line is an attracting line, the second one is 'unstable'. Also, we have two lines of unstable critical points, $(0, 0, \psi_1, \pi/2)$ and $(0, 0, \psi_1, 3\pi/2)$. For $\alpha > 0$ the two $4d$ saddle-nodes move closer to each other. The two lines in the $r_1 = r_3 = 0$ plane are no longer lines of critical points, but are 'critical' lines (i.e. $\dot{r}_1 = 0$, $\dot{r}_3 = 0$, $\dot{\psi}_3 = 0$ and $\dot{\psi}_1 \neq 0$). The ψ_3-values of the lines in the $r_1 = r_3 = 0$ plane change with α, and the two lines move closer to each other as α increases. Furthermore, for $\alpha > 0$, the r_3-values of the lines in the $r_1 = 0$ space change and the lines move away from each other.



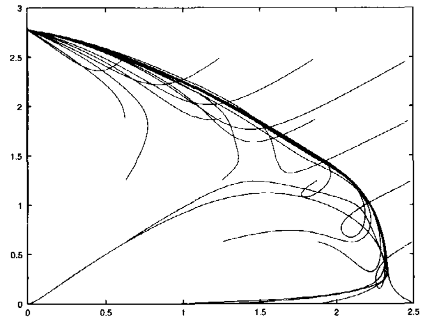
(a) $\alpha = 0$



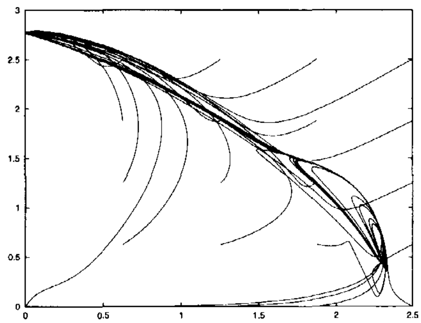
(b) $\alpha = 0.25$



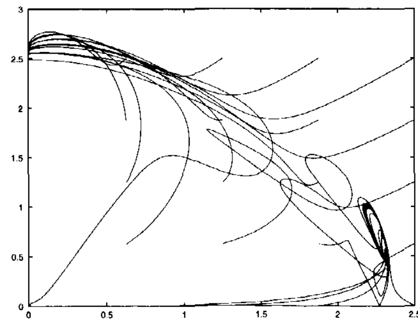
(c) $\alpha = 0.5595$



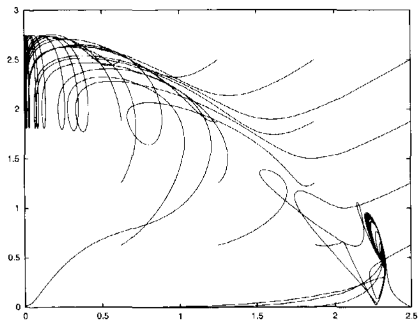
(d) $\alpha = 0.7$



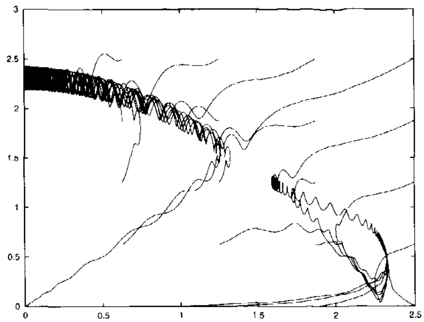
(e) $\alpha = 0.7233$



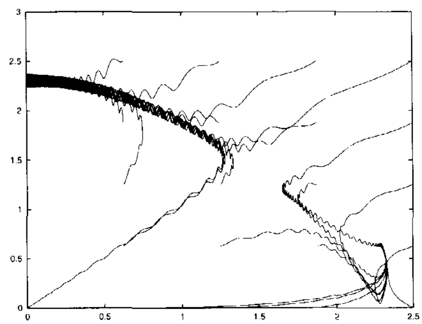
(f) $\alpha = 0.8868$



g) $\alpha = 1.0$



(h) $\alpha = 5.0$



(i) $\alpha = 10.0$

Figure 5.8: Phase space of (5.3.19)-(5.3.22) for $\beta = 0$, for several values of α, with r_1 (horizontal) from 0 to 2.5, r_3 (vertical) from 0 to 2.5.

At some point, for $\alpha \approx 0.1995$, the r_3-value of the 'unstable' line reaches a minimum, $(0, 1.6744, \psi_1, \pi/2)$. As α increases further, the r_3-value of the unstable line in the $r_1 = 0$ space becomes larger again. Furthermore, the two $4d$ saddle-nodes keep moving closer to each other, as do the lines in the $r_1 = r_3 = 0$ plane. At some point, for $\alpha \approx 0.4743$, the two lines in the $r_1 = r_3 = 0$ plane coincide to $(0, 0, \psi_1, 0)$ and then disappear. However, an unstable surface, $(0, 0, \psi_1, \psi_3)$, for which $\dot{r}_1 = \dot{r}_3 = 0$ and $\dot{\psi}_1 \neq 0$, $\dot{\psi}_3 \neq 0$, occurs for $\alpha > 0.4743$. As α increases further the two $4d$ saddle-nodes keep moving closer to each other. Furthermore, the r_3-values of the lines in the $r_1 = 0$ space become larger. At some point, for $\alpha \approx 0.5595$ the r_3-value of the attracting line in the $r_1 = 0$ space reaches a maximum, $(0, 2.8041, \psi_1, 3\pi/2)$. As α increases further, the r_3-value of the attracting line in the $r_1 = 0$ space becomes smaller again and the two lines in the $r_1 = 0$ space move closer to each other. Also, as α increases further the two $4d$ saddle-nodes move closer to each other, coinciding for $\alpha \approx 0.7098$ and

then disappearing. A separating surface remains. As α increases further some kind of periodicity occurs, for $\alpha \approx 0.7233$, with r_1-values between 1.59 and 2.31, r_3-values between 0.43 and 1.44, ψ_1-values between 2.23 and 3.45 and with $\dot{\psi}_3 > 0$. This periodic phenomenon becomes smaller as α increases. As α increases further the two lines in the $r_1 = 0$ space move closer to each other, coinciding for $\alpha \approx 0.8868$, and then disappearing. However, some kind of periodicity occurs with $r_1 = 0$ and values of r_3 between 1.78 and 2.76. This periodic phenomenon becomes smaller as α increases, and the r_3-values come in a small neighborhood of $\sqrt{16/3}$ for large values of α. Furthermore, the other periodic phenomenenon becomes smaller as α increases, and the r_1, r_3, ψ_1 values come in a small neighborhood of $2.29, 0.63, 3.31$ respectively. Also a separating surface remains in the $(r_1, r_3, \psi_1, \psi_3)$ space.
We see that for $\alpha \gg 1$ (or $\alpha \ll 1$) the phase space analysis connects to the behavior in the case $p^2 = 9$, $\omega \neq \omega_{3,p} + \epsilon\alpha$, considered in the $(r_1, r_3, \psi_1, \psi_3)$ space (see also Section 5.3.2). In the $(r_1, r_3, \psi_1, \psi_3)$ space system (5.3.13)-(5.3.15), which has been considered in Section 5.3.2, becomes

$$\dot{r}_1 = \frac{1}{2}r_1(1 - \frac{3}{16}r_1^2 - \frac{1}{4}r_3^2) - \frac{1}{32}r_1^2 r_3 \cos(\psi_1), \tag{5.3.23}$$

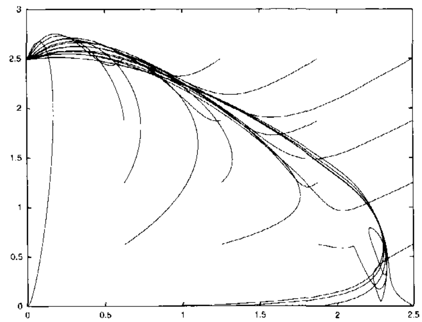
$$\dot{r}_3 = \frac{1}{2}r_3(1 - \frac{1}{4}r_1^2 - \frac{3}{16}r_3^2) - \frac{1}{96}r_1^3 \cos(\psi_1), \tag{5.3.24}$$

$$\dot{\psi}_1 = \frac{4}{3}\frac{\beta}{\sqrt{10}} + \frac{1}{96}\frac{r_1}{r_3}\left(r_1^2 + 9r_3^2\right)\sin(\psi_1) + \frac{3}{80}\frac{\delta}{\sqrt{10}}(r_1^2 - r_3^2), \tag{5.3.25}$$

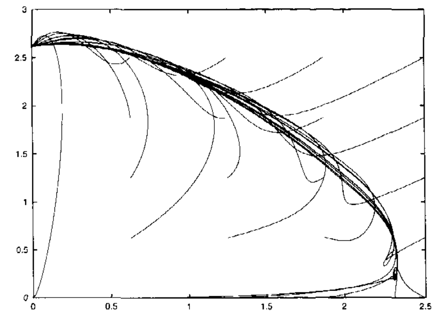
$$\dot{\psi}_3 = \alpha - \frac{\beta}{3\sqrt{10}} - \frac{3}{40}\frac{\delta}{\sqrt{10}}\left(3r_3^2 + 2r_1^2\right) + \frac{1}{48}\frac{r_1^3}{r_3}\sin(\psi_1). \tag{5.3.26}$$

In this case system (5.3.23)-(5.3.26) only has trivial critical points (for $\alpha = \beta = 0$). However, we have an attracting line $(2.2975, 0.6312, 3.31, \psi_3)$, for $\beta = 0$, for which $\dot{r}_1 = \dot{r}_3 = \dot{\psi}_1 = 0$ and $\dot{\psi}_3 \neq 0$, and an attracting surface $(0, \sqrt{16/3}, \psi_1, \psi_3)$, for which $\dot{r}_1 = \dot{r}_3 = 0$, $\dot{\psi}_1 \neq 0$ and $\dot{\psi}_3 \neq 0$. Furthermore, we have two 'unstable' lines (for $\beta = 0$), $(1.5930, 1.2627, 6.2328, \psi_3)$ and $(1.2267, 1.8538, 3.0389, \psi_3)$. This case connects to the case considered above for large values of $|\alpha|$.

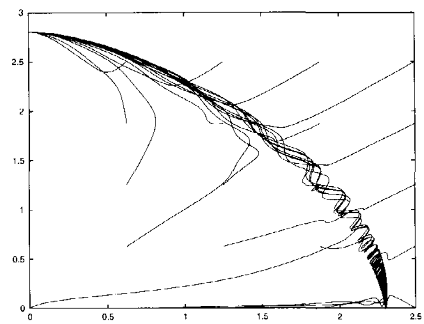
Next we consider $\beta < 0$ and $\alpha = 0$, since we want to analyze detuning from the case $p^2 = 9$ to the $p^2 = 8$, which was discussed in Section 5.3.1. For $\beta > 0$ a similar analysis holds. The behavior of some solutions of (5.3.19)-(5.3.22) projected on the (r_1, r_3) plane is given in Figure 5.9, for $-100 \leq \beta < 0$. For $\beta < 0$ system (5.3.19)-(5.3.22) has no critical points. However, a separating surface occurs in the $(r_1, r_3, \psi_1 \psi_3)$ space. Also, for $\beta < 0$ the r_3-value of the attracting line in the $r_1 = 0$ space increases. The r_3-value of the attracting line has a maximum for $\beta \approx -5.3076$. As β decreases further,



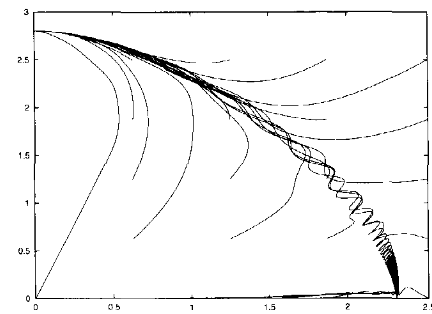
(a) $\beta = 0$



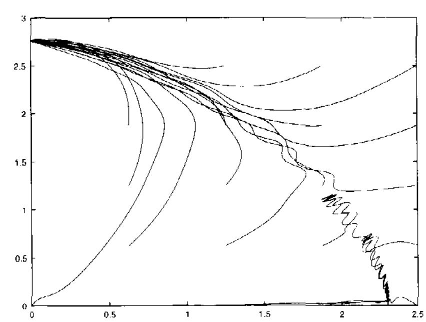
(b) $\beta = -1.0$



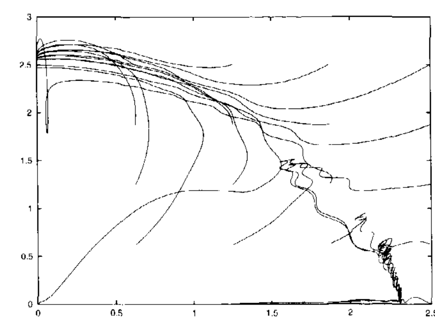
(c) $\beta = -5.0$



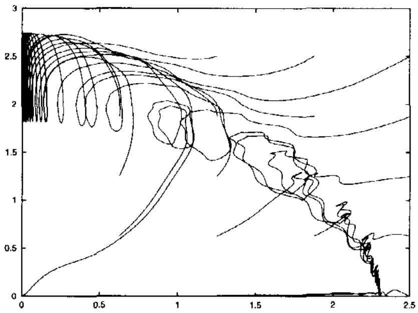
(d) $\beta = -5.3076$



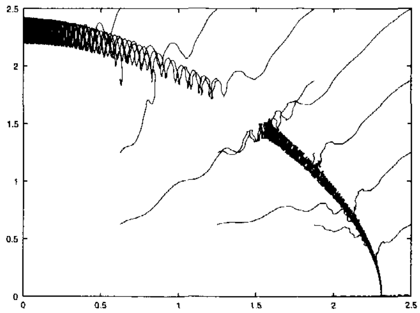
(e) $\beta = -7.0$



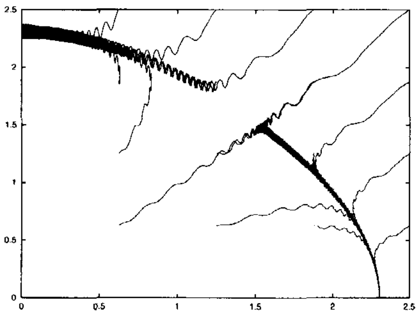
(f) $\beta = -8.4131$



(g) $\beta = -10.0$



(h) $\beta = -50.0$



(i) $\beta = -100.0$

Figure 5.9: Phase space of (5.3.19)-(5.3.22) for $\alpha = 0$, for several values of β, with r_1 (horizontal) from 0 to 2.5, r_3 (vertical) from 0 to 2.5.

some kind of attracting periodicity occurs (for $\beta \approx -7.0$), with values of r_1, r_3 in the neighborhood of $2.3, 0.5$ respectively. As β decreases further this periodic phenonmenon becomes smaller and moves closer to the $r_3 = 0$ space. For $\beta \approx -8.4131$ the two lines in the $r_1 = 0$ space coincide and disappear for $\beta > -8.4131$, but an attracting periodic phenomenon occurs with values of r_3 'oscillate' around $r_3 = \sqrt{16/3}$. As β decreases further this periodic phenomenon becomes smaller and the r_3-values come in a small neighborhood of $\sqrt{16/3}$. The periodic phenomenon with values of r_1, r_3 which initially were in a neighborhood of $2.3, 0.5$ respectively, comes in a small neighborhood of $r_1 = \sqrt{16/3}$ and $r_3 = 0$.

We see that for $\beta \ll -8.4131$ the phase space analysis connects to the case $p^2 \neq 9 + \epsilon\beta$, $\omega = \omega_{3,p} + \epsilon\alpha$, considered in Section 5.3.1. $\omega_{3,p}$ is taken $= \sqrt{90}$ in this subsection, whereas the $\omega_{3,p}$ considered in Section 5.3.1 is based upon the assumption that $p^2 = 8$, i.e. $\omega_{3,p} = \sqrt{89}$. In order to compare both cases, we have to take $\alpha \gg 0$ in the case considered in Section 5.3.1.

The case $F(t) = C\cos(\omega t_0)$ *with* $\omega = 2\omega_{1_p} + \epsilon\alpha$
Introducing polar coordinates for each mode we transform (5.2.21)-(5.2.22) for $k = 1$ and (5.2.14)-(5.2.15) for $n = 3$, using (5.3.1)

$$\begin{aligned}
\dot{r}_1 &= \frac{1}{2}r_1(1-\frac{3}{16}r_1^2-\frac{1}{4}r_3^2)-\frac{1}{32}r_1^2r_3\cos(\phi_3-3\phi_1)\\
&\qquad -\frac{1}{4}\frac{\delta C}{\sqrt{10}}r_1\sin(2\phi_1+\alpha t_1),\\
\dot{r}_3 &= \frac{1}{2}r_3(1-\frac{1}{4}r_1^2-\frac{3}{16}r_3^2)-\frac{1}{96}r_1^3\cos(\phi_3-3\phi_1),\\
\dot{\phi}_1 &= -\frac{3}{80}\frac{\delta}{\sqrt{10}}r_1^2-\frac{1}{40}\frac{\delta}{\sqrt{10}}r_3^2-\frac{1}{32}r_1r_3\sin(\phi_3-3\phi_1)\\
&\qquad -\frac{\beta}{2\sqrt{10}}-\frac{1}{4}\frac{\delta C}{\sqrt{10}}\cos(2\phi_1+\alpha t_1),\\
\dot{\phi}_3 &= -\frac{9}{80}\frac{\delta}{\sqrt{10}}r_3^2-\frac{3}{40}\frac{\delta}{\sqrt{10}}r_1^2+\frac{1}{96}\frac{r_1^3}{r_3}\sin(\phi_3-3\phi_1)-\frac{\beta}{6\sqrt{10}}.
\end{aligned}$$

We introduce $\psi_1 = \phi_3 - 3\phi_1$ and $\psi_2 = 2\phi_1 + \alpha t_1$, and obtain the following system for r_1, r_3, ψ_1, ψ_2:

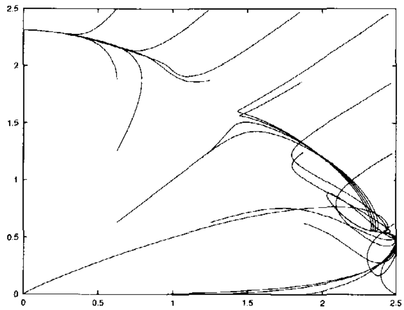
$$\dot{r}_1 = \frac{1}{2}r_1(1-\frac{3}{16}r_1^2-\frac{1}{4}r_3^2)-\frac{1}{32}r_1^2r_3\cos(\psi_1)-\frac{1}{4}\frac{\delta C}{\sqrt{10}}r_1\sin(\psi_2), \tag{5.3.27}$$

$$\dot{r}_3 = \frac{1}{2}r_3(1-\frac{1}{4}r_1^2-\frac{3}{16}r_3^2)-\frac{1}{96}r_1^3\cos(\psi_1), \tag{5.3.28}$$

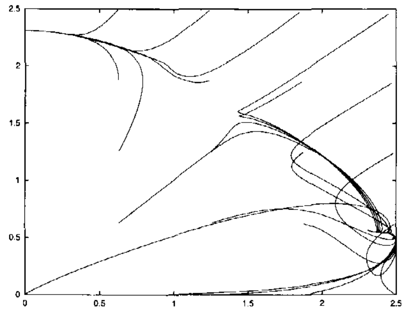
$$\begin{aligned}\dot{\psi}_1 &= \frac{4}{3}\frac{\beta}{\sqrt{10}}+\frac{1}{96}\frac{r_1}{r_3}\left(r_1^2+9r_3^2\right)\sin(\psi_1)\\
&\qquad +\frac{3}{80}\frac{\delta}{\sqrt{10}}(r_1^2-r_3^2)+\frac{3}{4}\frac{\delta C}{\sqrt{10}}\cos(\psi_2),\end{aligned} \tag{5.3.29}$$

$$\begin{aligned}\dot{\psi}_2 &= \alpha-\frac{\beta}{\sqrt{10}}-\frac{1}{40}\frac{\delta}{\sqrt{10}}\left(3r_1^2+2r_3^2\right)\\
&\qquad -\frac{1}{16}r_1r_3\sin(\psi_1)-\frac{1}{2}\frac{\delta}{\sqrt{10}}C\cos(\psi_2).\end{aligned} \tag{5.3.30}$$

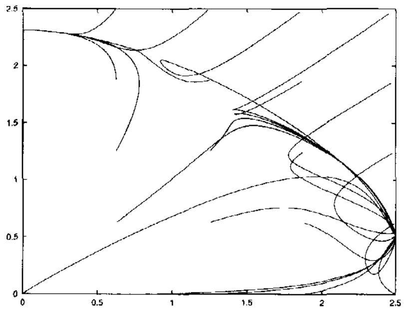
For $r_3 = 0$ (5.3.27)-(5.3.30) do not hold. In that case we have to analyze the original differential equations (5.2.14)-(5.2.15) for $n = 3$ and (5.2.21)-(5.2.22) for $k = 1$. From a practical point of view it is impossible to consider all different cases for δ and $C \in \mathbb{R}$. To give an idea how the solutions of (5.3.27)-(5.3.30) can behave we will take $\delta = C = 1$ in this subsection. For $\delta \neq 1$ and $C \neq 1$ a similar analysis can be applied. We see that in this subsection system (5.3.27)-(5.3.30) has to be analyzed in a 4d phase space, a reduction to a lower dimension seems to be impossible. Furthermore, we have two detuning parameters in (5.3.19)-(5.3.22), α and β. In this chapter we only consider the cases $\beta = 0, \alpha \neq 0$ and $\beta \neq 0, \alpha = 0$. The (numerical) analysis can be extended to the case $\beta \neq 0, \alpha \neq 0$, but is much more complicated and beyond the scope of this chapter. We start with $\beta = 0$. We consider $\alpha = 0$ and $\alpha > 0$ (the case $\alpha < 0$ is similar). The behavior of some solutions of (5.3.27)-(5.3.21) projected on the (r_1, r_3) plane is given in Figure 5.10 for some values of $\alpha \in [-10, 10]$.



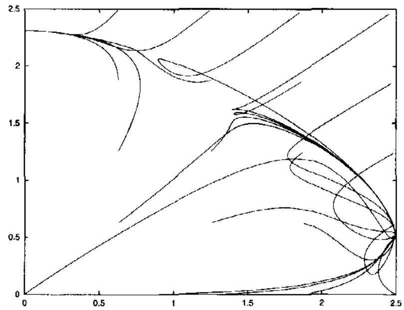
(a) $\alpha = 0$



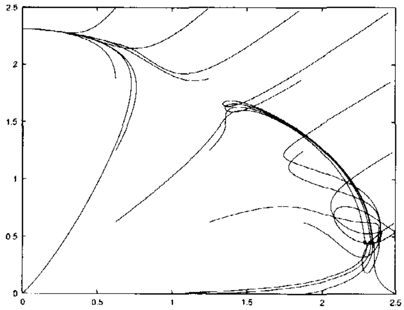
(b) $\alpha = 0.0144$



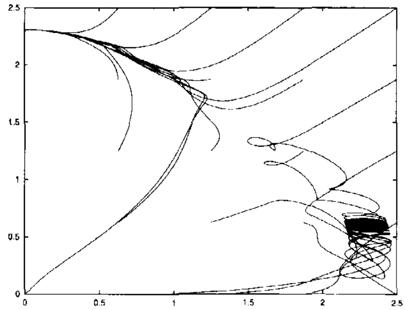
(c) $\alpha = 0.1$



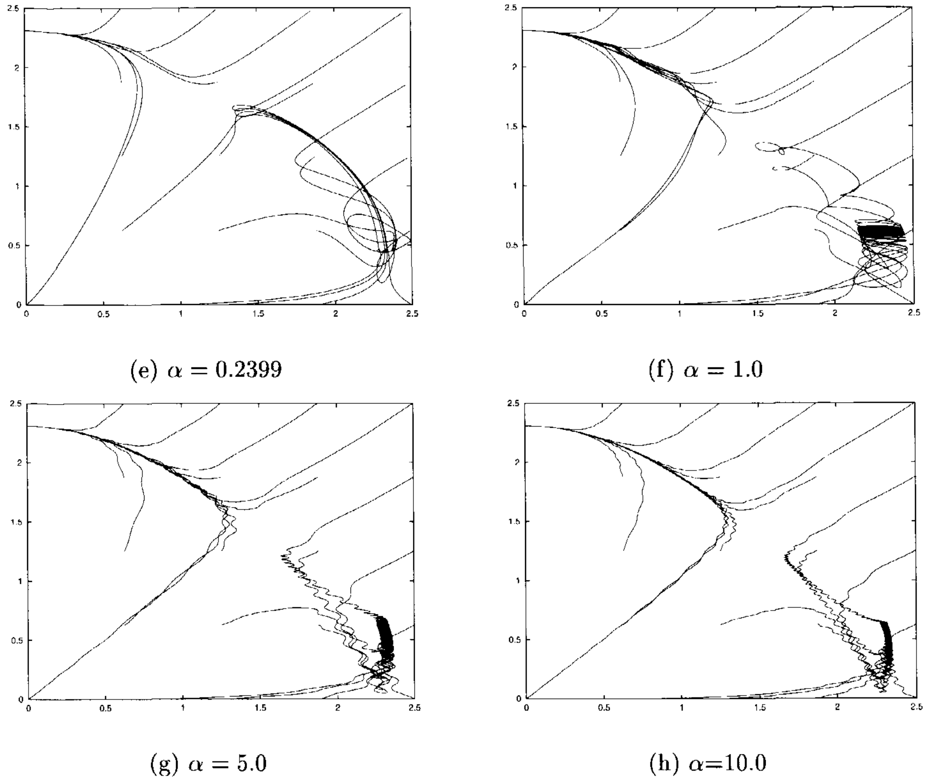
(d) $\alpha = 0.1318$



(e) $\alpha = 0.2399$



(f) $\alpha = 1.0$



(e) $\alpha = 0.2399$

(f) $\alpha = 1.0$

(g) $\alpha = 5.0$

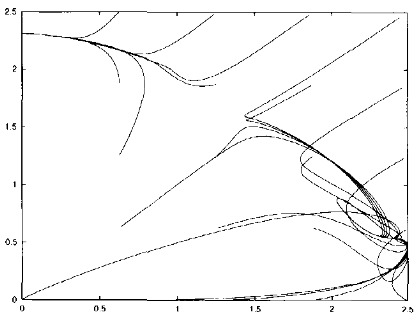
(h) α=10.0

Figure 5.10: Phase space of (5.3.27)-(5.3.30) for $\beta = 0$, for several values of α, with r_1 (horizontal) from 0 to 2.5, r_3 (vertical) from 0 to 2.5.

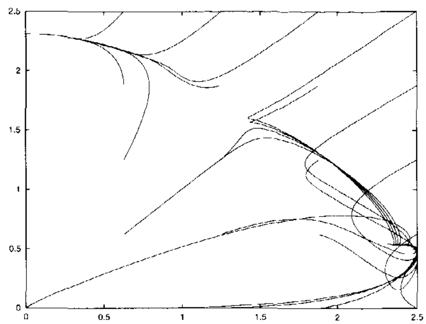
For $\alpha = 0$ system (5.3.27)-(5.3.30) has no critical points. However, we have two 'critical' lines, for which $\dot{r}_1 = \dot{r}_3 = \dot{\psi}_2 = 0$ and $\dot{\psi}_1 > 0$: the line $(0, \sqrt{16/3}, \psi_1, 2.1333)$ and $(0, \sqrt{16/3}, \psi_1, 4.1499)$. The second line is an attracting line, the first one is 'unstable'. Also some kind of periodicity exists, with r_1-values between 2.03 and 2.49, r_3-values between 0.49 and 0.88, ψ_1-values between 2.57 and 4.03 and with $\dot{\psi}_2 > 0$. Furthermore, we have two lines of unstable critical points, $(0, 0, \psi_1, \pi/2)$ and $(0, 0, \psi_1, 3\pi/2)$. Also, a separating surface exists in the $(r_1, r_3, \psi_1, \psi_2)$ space. For $\alpha > 0$ the two lines in the $r_1 = r_3 = 0$ plane are no longer lines of critical points, but are 'critical' lines (i.e. $\dot{r}_1 = 0$, $\dot{r}_3 = 0$, $\dot{\psi}_2 = 0$ and $\dot{\psi}_1 \neq 0$). The ψ_2-values of the lines in the $r_1 = r_3 = 0$ plane change with α, and the two lines move closer to each other as α increases. Furthermore, for $\alpha > 0$, the two lines in the $r_1 = 0$, $r_3 = \sqrt{16/3}$ plane move closer to each other. Also, for $\alpha > 0$, the periodic phenomenon seems to collapse to a point, for $\alpha \approx 0.0144$. As α increases further, the r_1-value of the point which arose from the periodic phenomenon becomes larger, reaches a maximum for $\alpha \approx 0.1318$ and then becomes smaller again. As α increases further the two lines in the $r_1 = r_3 = 0$ plane move closer to each other, coinciding for $\alpha \approx 0.1581$ to $(0, 0, \psi_1, 0)$ and then disappearing. However, an unstable surface, $(0, 0, \psi_1, \psi_2)$, for which $\dot{r}_1 = \dot{r}_3 = 0$ and $\dot{\psi}_1 \neq 0$, $\dot{\psi}_2 \neq 0$, occurs for $\alpha > 0.1581$. At some point, for $\alpha \approx 0.2399$ the periodic phenomenon reappears, with r_1-values between 2.08 and 2.49, r_3-values between 0.44 and 0.76, ψ_1-values between 2.71 and 4.00 and with $\dot{\psi}_2 > 0$. As α increases further the two lines in the $r_1 = 0, r_3 = \sqrt{16/3}$ plane move closer to each other, coinciding for $\alpha \approx 0.2424$ to $(0, \sqrt{16/3}, \psi_1, 0)$ and then disappearing. However, an attracting surface, $(0, \sqrt{16/3}, \psi_1, \psi_2)$, for which $\dot{r}_1 = \dot{r}_3 = 0$ and $\dot{\psi}_1 \neq 0$, $\dot{\psi}_2 \neq 0$ occurs for $\alpha > 0.2424$. As α increases further the periodic phenomenon becomes smaller and for large values of α the r_1, r_3, ψ_1 values come in a small neighborhood of 2.29,0.63,3.31 respectively. Furthermore, the attracting surface $(0, \sqrt{16/3}, \psi_1, \psi_2)$ still exists. Also a separating surface remains in the $(r_1, r_3, \psi_1, \psi_2)$ space.

We see that for $\alpha \gg 1$ (or $\alpha \ll 1$) the phase space analysis connects to the behavior in the case $p^2 = 9$, $\omega \neq \omega_{1,p} + \epsilon\alpha$, considered in the $(r_1, r_3, \psi_1, \psi_2)$ space in a similar way as considered in (5.3.23)-(5.3.26) (see also Section 5.3.2).

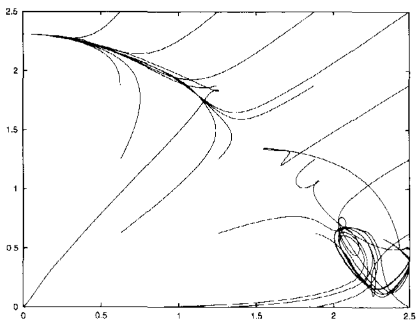
Next we consider $\beta < 0$ and $\alpha = 0$, since we want to analyze detuning from the case $p^2 = 9$ to the $p^2 = 8$, which was discussed in Section 5.3.1. For $\beta > 0$ a similar analysis holds. The behavior of some solutions of (5.3.27)-(5.3.30) projected on the (r_1, r_3) plane is given in Figure 5.11, for $-100 \leq \beta < 0$. For $\beta > 0$ the two lines in the $r_1 = 0$, $r_3 = \sqrt{16/3}$ plane move closer to each other. Furthermore, for $\beta < 0$, the periodic phenomenon seems to collapse to a point, for $\beta \approx -0.0545$. As β decreases further, the r_1-value of the point which arose from the periodic phenomenon becomes smaller. Also, the two lines in the $r_1 = 0, r_3 = \sqrt{16/3}$ plane move closer to each other, coinciding for $\beta \approx -0.7667$, and then disappearing. However, for $\beta < -0.7667$ we have an attracting plane $r_1 = 0, r_3 = \sqrt{16/3}$ with $\dot{r}_1 = \dot{r}_3 = 0$ and $\dot{\psi}_1 \neq 0$, $\dot{\psi}_2 \neq 0$. As β decreases further, the periodic phenomenon reappears, for $\beta \approx -1.0226$. For $\beta < -1.0226$ this periodic phenomenon becomes smaller, moves to the $r_3 = 0$ space and comes in a small neighborhood of $r_1 = \sqrt{16/3}$ and $r_3 = 0$. We see that for $\beta \ll -1.0226$ the phase space analysis connects to the case $p^2 \neq 9 + \epsilon\beta$, $\omega = \omega_{1,p} + \epsilon\alpha$, as considered in Section 5.3.1. $\omega_{1,p}$ is taken $= \sqrt{10}$ in this subsection, whereas the $\omega_{1,p}$ considered in Section 5.3.1 is based upon the assumption that $p^2 = 8$, i.e. $\omega_{1,p}$ is taken $= \sqrt{9}$. In order to compare both cases, we have to take $\alpha \gg 0$ in the case considered in Section 5.3.1.



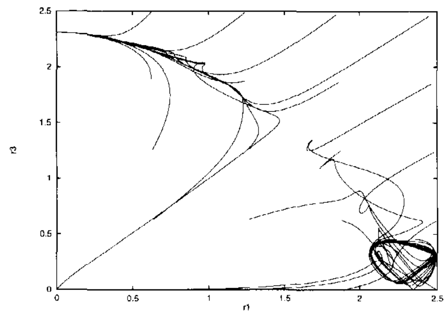
(a) $\beta = 0$



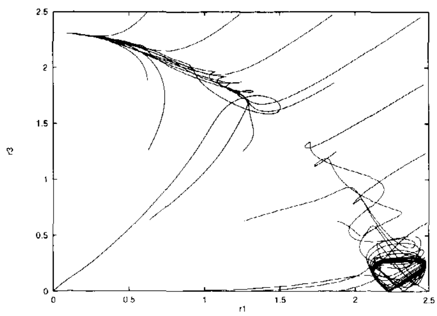
(b) $\beta = -0.0545$



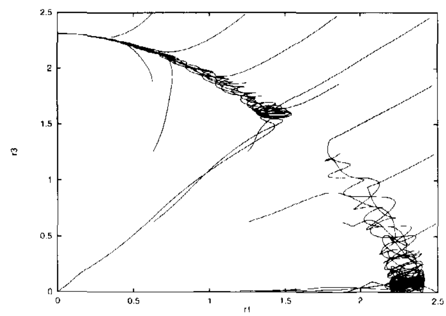
(c) $\beta = -1.0226$



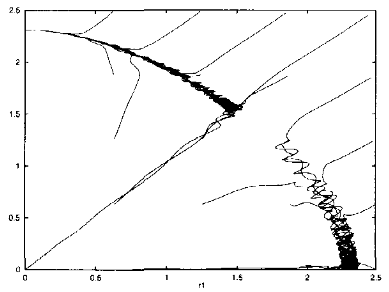
(d) $\beta = -1.5$



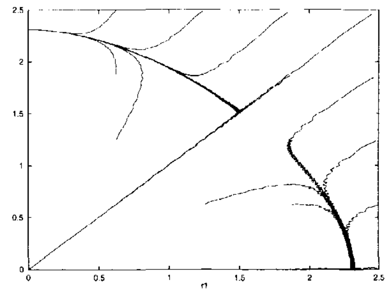
(e) $\beta = -2.0$



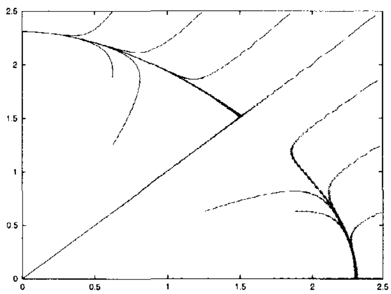
(f) $\beta = -5.0$



(g) $\beta = -10.0$



(h) $\beta = -50.0$



(i) $\beta = -100.0$

Figure 5.11: Phase space of (5.3.27)-(5.3.30) for $\alpha = 0$, for several values of β, with r_1 (horizontal) from 0 to 2.5, r_3 (vertical) from 0 to 2.5.

5.4 Conclusions

In this chapter we considered an initial-boundary value problem for the vertical displacement of a weakly nonlinear elastic beam with a Rayleigh perturbation due to a wind force and a harmonic excitation in horizontal direction at the ends of the beam. The PDE considered contains the parameters p^2, C, δ, ω and ϵ $(0 < \epsilon \ll 1)$.
For some specific values of the parameters $(0 < p^2 < 10,\ C = \delta = 1)$ we have constructed formal approximations of order ϵ and considered the interaction between different oscillation modes. The aim of this chapter was to see whether the applied (harmonic) external excitation can act as a damping force on the oscillations of the beam. We considered the cases $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$ (with $p^2 = 8$ as an example) and $p^2 = 9$ (with detuning). With $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$ we mean $p^2 \in]0, 10] \wedge p^2 \neq \frac{693}{152} + \epsilon\beta \wedge p^2 \neq 9 + \epsilon\beta$, where β is a detuning parameter of order 1. For these cases of p^2 we considered three possible cases for $F(t)$: $F(t) = 0$ or $F(t) = C\cos(\omega t_0)$ (with $\omega \neq 2\omega_{k_p} + \epsilon\alpha$), $F(t) = C\cos(2\omega_{3,p}t_0)$, $F(t) = C\cos(2\omega_{1,p}t_0)$, with

detuning.
For $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$ we have shown that for the case $F(t) = 0 \wedge F(t) = C\cos(\omega t_0)$ $(\omega \neq 2\omega_{k_p} + \epsilon\alpha)$ no significant changes occur due to the external excitation, and the analysis is similar to the case with no external excitation as considered in Chapter 2. For the cases $F(t) = C\cos(2\omega_{3,p}t_0)$ and $F(t) = C\cos(2\omega_{1,p}t_0)$ we have shown that, with $\delta = C = 1$, the behavior of the oscillation modes changes due to the external excitation. The system will still oscillate in one mode only for large times (on a time scale of order ϵ), as was the case with no external excitation, but the amplitude of oscillation has changed, i.e. is larger. Numerical analysis indicates that this effect occurs for all values of $C \in \mathbb{R}$. Also it is expected that for $F(t) = C\cos(\omega t_0 + \varphi)$ the effect is the same for all values of φ. The case $\delta \neq 1$ is expected to be more complicated to analyze, and is beyond the scope of this chapter. This means that for $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$ and $\delta = C = 1$, the external excitation does not work as a damping force but the amplitude of oscillation becomes larger. To understand the effect of the external excitation better we have analyzed detuning from the specific values of ω considered, i.e. $F(t) = C\cos(\omega t_0)$ with $\omega = 2\omega_{3_p} + \epsilon\alpha$ (or $\omega = 2\omega_{1_p} + \epsilon\alpha$), $\alpha \in \mathbb{R}$.
For $p^2 = 9$ we have shown that for the case $F(t) = 0 \wedge F(t) = C\cos(\omega t_0)$ $(\omega \neq 2\omega_{k_p})$ no significant changes occur due to the external excitation, and the analysis is similar to the case with no external excitation as considered in Chapter 2. For the case $F(t) = C\cos(2\omega_{3,p}t_0)$, with $\delta = C = 1$, we have shown that the behavior of the oscillation modes changes due to the external excitation. The combined oscillation of the two oscillation modes considered (modes 1 and 3), which was found in Chapter 2, has disappeared. The amplitude of the (stable) oscillation of the third mode has become larger, due to the external excitation. Again it can be shown numerically that this effect occurs for all values of $C \in \mathbb{R}$. Also it is expected that for $F(t) = C\cos(\omega t_0 + \varphi)$ the effect is the same for all values of φ. The case $\delta \neq 1$ is expected to be more complicated to analyze, and is beyond the scope of this chapter. This means that for $p^2 = 9$ and $\delta = C = 1$ the external excitation does not work as a damping force but the amplitude of oscillation becomes larger. We have shown that if we consider detuning from $\omega = 2\omega_{3_p}$ (that is $\omega = 2\omega_{3_p} + \epsilon\alpha$, $\alpha \in \mathbb{R}$), the combined oscillation appears again for α, the detuning parameter, $\neq 0$. For the case $F(t) = C\cos(2\omega_{1,p}t_0)$, with $\delta = C = 1$, we have shown that the behavior changes due to the external excitation. The combined oscillation of the two oscillation modes considered (modes 1 and 3), which was found in Chapter 2, still exists, but the amplitudes of oscillation are no longer fixed: the stable combined oscillation has changed into some kind of periodic phenomenon in the phase space considered. The amplitude of the (stable) oscillation of the third

mode remains unchanged. For this case the external excitation has no significant effects on the amplitudes of oscillation and certainly no damping effects occur (for $\delta = C = 1$). We have shown that if we consider detuning from $\omega = 2\omega_{1_p}$ (that is $\omega = 2\omega_{1_p} + \epsilon\alpha$, $\alpha \in \mathbb{R}$), the periodic phenomenon collapses to a point as α moves away from 0, but reappears again for larger as α changes further. To compare the analysis for the case $p^2 = 9$ with the analysis for the case $p^2 = 8$ we considered detuning, i.e. $p^2 = 9 + \epsilon\beta$, $\beta < 0$. We have shown that for the combination of the two effects considered in this chapter (a Rayleigh perturbation due to a wind force and a harmonic excitation in horizontal direction at the ends of the beam) the phase space analysis of the approximations of solutions of the PDE considered is much more complicated than the analysis performed in the separate cases, as considered in Chapter 2 and Chapter 4, and many bifurcations occur when detuning is considered.

As was done in Chapter 2, Chapter 3 and Chapter 4, we have shown for which cases truncation to one or more modes is valid. Again, similar to the results found in Chapter 2, we have shown that in specific cases internal resonances occur, truncation to one mode gives loss of information and approximations are no longer valid.

The analysis in this chapter can be easily extended to larger values of p^2, i.e. $p^2 \geq 10$. Furthermore, the influence of the different nonlinear terms can be changed by taking the parameters δ and $C \neq 1$ in the PDE. The analysis can be performed in the same way as presented in this chapter.

5.5 Appendix - Critical points for $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$ and $F(t) = C\cos(\omega t_0)$ with $\omega = 2\omega_{3_p} + \epsilon\alpha$ - general case.

In Section 5.3.1, the case $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$ and $F(t) = C\cos(\omega t_0)$ with $\omega = 2\omega_{3_p} + \epsilon\alpha$ is discussed, analyzing (5.3.4)-(5.3.6) for r_1, r_3, ψ_3. In Section 5.3.1 the critical points of (5.3.4)-(5.3.6) are given for $p^2 = 8$, in Table 5.2 (for $\alpha \geq 0$) and Table 5.3 (for $\alpha < 0$). In this appendix we give a more general analysis of the critical points of (5.3.4)-(5.3.6), for $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$.

For $p^2 \neq 8$ the critical points and the range of α in which they are occur, are listed in Table 5.8.

In Table 5.8 the following parameters were defined

$$\hat{r}_3^{\pm} = \sqrt{\frac{16}{3}\left(1 - \frac{9}{2\omega_{3,p}}\sin\chi_6^{\pm}\right)},$$

Critical points	α-range
$(0, 0, \chi_{1,2})$	$\alpha_{cr,1}^- \le \alpha \le \alpha_{cr,1}^+$
$(\sqrt{16/3}, 0, \chi_{3,4})$	$\alpha_{cr,3}^- \le \alpha \le \alpha_{cr,3}^+$
$(0, \hat{r}_3^\pm, \chi_6^\pm)$	$\alpha_{cr,2}^- \le \alpha \le \alpha_{cr,2}^+$
$(\tilde{r}_1^+, \tilde{r}_3^+, \chi_5^+)$	$\alpha_{cr,4}^- \le \alpha \le \alpha_{cr,5}^- \wedge \alpha_{cr,5}^+ \le \alpha \le \alpha_{cr,4}^+$
$(\tilde{r}_1^-, \tilde{r}_3^-, \chi_5^-)$	$\alpha_{cr,4}^- \le \alpha \le \alpha_{cr,6}^- \wedge \alpha_{cr,6}^+ \le \alpha \le \alpha_{cr,4}^+$

Table 5.8: Critical points for $p^2 \in]0, 10[\backslash \{\frac{693}{152}, 9\}$ and $F(t) = C\cos(\omega t_0)$ with $\omega = 2\omega_{3_p} + \epsilon\alpha$.

$$\tilde{r}_1^\pm = \sqrt{\frac{16}{7}\left(1 - \frac{18}{\omega_{3,p}}\sin\chi_5^\pm\right)}, \qquad \tilde{r}_3^\pm = \sqrt{\frac{16}{7}\left(1 + \frac{27}{2\omega_{3,p}}\sin\chi_5^\pm\right)},$$

and

$$\cos\chi_{1,2} = \frac{2\omega_{3,p}}{9}\alpha, \qquad \cos\chi_{3,4} = \frac{2\omega_{3,p}}{9}\alpha - \frac{16}{3\omega_{1,p}^2},$$

$$\begin{aligned}
\sin\chi_5^\pm & \\
&= \frac{-72\omega_{3,p}(8\omega_{3,p}^2 - 81\omega_{1,p}^2)(7\omega_{1,p}^2\omega_{3,p}^3\alpha - 36(2\omega_{3,p}^2 + 27\omega_{1,p}^2)) \pm 7\omega_{1,p}^2\omega_{3,p}^3\,(\ )^{\frac{1}{2}}}{9(1296(8\omega_{3,p}^2 - 81\omega_{1,p}^2)^2 + 49\omega_{1,p}^4\omega_{3,p}^6},\\
&\quad \text{with } (\ ) = -4\omega_{3,p}^2\left(7\omega_{1,p}^2\omega_{3,p}^3\alpha - 36(2\omega_{3,p}^2 + 27\omega_{1,p}^2)\right)^2\\
&\qquad\qquad +81\left(1296(8\omega_{3,p}^2 - 81\omega_{1,p}^2)^2 + 49\omega_{1,p}^4\omega_{3,p}^6\right),\\
\sin\chi_6^\pm &= \frac{-648\omega_{3,p}(\omega_{3,p}^3\alpha - 324) \pm \omega_{3,p}^3\,(\ )^{\frac{1}{2}}}{9(104976 + \omega_{3,p}^6)},\\
&\quad \text{with } (\ ) = -4\omega_{3,p}^8\alpha^2 + 2592\omega_{3,p}^5\alpha + 8503056 - 419904\omega_{3,p}^2 + 81\omega_{3,p}^6,
\end{aligned}$$

and

$$\begin{aligned}
\alpha_{cr,1}^\pm &= \pm\frac{9}{2\omega_{3,p}}, \qquad \alpha_{cr,2}^\pm = \frac{324}{\omega_{3,p}^3} \pm \frac{9}{2\omega_{3,p}^4}\sqrt{\omega_{3,p}^6 + 104976},\\
\alpha_{cr,3}^\pm &= \frac{24}{\omega_{1,p}^2\omega_{3,p}} \pm \frac{9}{2\omega_{3,p}},\\
\alpha_{cr,4}^\pm &= \frac{9}{14\omega_{1,p}^2\omega_{3,p}^4}\left(8\omega_{3,p}(2\omega_{3,p}^2 + 27\omega_{1,p}^2)\right.\\
&\qquad\qquad \left.\pm\sqrt{1296(8\omega_{3,p}^2 - 81\omega_{1,p}^2)^2 + 49\omega_{1,p}^4\omega_{3,p}^6}\right),\\
\alpha_{cr,5}^\pm &= \frac{243}{\omega_{3,p}^3} \pm \frac{1}{4\omega_{3,p}}\sqrt{324 - \omega_{3,p}^2}, \quad \alpha_{cr,6}^\pm = \frac{24}{\omega_{1,p}^2\omega_{3,p}} \pm \frac{1}{6\omega_{3,p}}\sqrt{729 - 4\omega_{3,p}^2}.
\end{aligned}$$

The order in which the critical points from Table 5.8 (dis)appear depends on the value of p^2. The values of p^2 for which the order of appearance of the critical points of (5.3.4)-(5.3.6) changes can be found easily when analyzing the critical values of α as functions of p^2 and are listed in Table 5.9.

p^2-value	α-values	p^2-value	α-values
0.2751	$\alpha^-_{cr,1} = \alpha^-_{cr,4}$,	3.4302	$\alpha^+_{cr,2} = \alpha^+_{cr,6}$,
0.5259	$\alpha^+_{cr,3} = \alpha^+_{cr,4}$,	3.7059	$\alpha^+_{cr,4} = \alpha^+_{cr,2}$
0.7945	$\alpha^+_{cr,2} = \alpha^-_{cr,3}$,		$\wedge\ \alpha^-_{cr,4} = \alpha^-_{cr,2}$,
0.9670	$\alpha^+_{cr,2} = \alpha^-_{cr,6}$,	4.3333	$\alpha^-_{cr,3} = 0$,
1.1150	$\alpha^+_{cr,5} = \alpha^-_{cr,3}$,	4.8032	$\alpha^+_{cr,2} = \alpha^+_{cr,3}$,
1.3603	$\alpha^+_{cr,5} = \alpha^-_{cr,6}$,	5.9662	$\alpha^-_{cr,5} = \alpha^-_{cr,3}$,
1.6667	$\alpha^+_{cr,1} = \alpha^-_{cr,3}$,	6.0868	$\alpha^+_{cr,5} = \alpha^+_{cr,6}$,
2.0692	$\alpha^+_{cr,1} = \alpha^-_{cr,6}$,	6.1106	$\alpha^-_{cr,2} = \alpha^-_{cr,3}$,
2.0803	$\alpha^+_{cr,4} = \alpha^+_{cr,6}$,	6.3926	$\alpha^-_{cr,6} = 0$,
2.4335	$\alpha^-_{cr,4} = \alpha^-_{cr,2}$,	7.7671	$\alpha^+_{cr,4} = \alpha^+_{cr,3}$
2.9634	$\alpha^-_{cr,4} = \alpha^-_{cr,5}$,		$\wedge\ \alpha^-_{cr,4} = \alpha^-_{cr,3}$,

Table 5.9: Values of p^2 for which order of appearance of critical points changes.

Chapter 6

Conclusions

The purpose of this chapter is to give some concluding remarks, and to present some possibilities for future research by extending the analysis as presented in this thesis.

In this thesis we considered initial boundary value problems for four cases of weakly nonlinear beam equations (with four different kinds of nonlinearities). We constructed formal approximations of order ϵ for the four cases involved in this thesis, and considered the possible interactions between the different oscillation modes. Furthermore, in all cases a justification is given whether truncation of the infinite series for the formal approximations of the solution is valid or not. We showed that for some specific values of the parameters involved complicated mode interactions can occur.

In Chapter 2 we considered an initial-boundary value problem for a weakly nonlinear beam equation with a Rayleigh perturbation. We showed that for most p^2-values the behavior of solutions of the Rayleigh beam equation is similar to that of solutions of the Rayleigh wave equation as presented in [26]. We showed that truncation is allowed for most p^2-values, restricted to those modes that have non-zero initial energy. However, for some p^2-values extra mode interactions occur, which cause complicated internal resonances. For $0 < p^2 < 10$ these values are $p^2 \approx 9$ and $p^2 \approx \frac{693}{152}$. For $p^2 \geq 10$ other internal resonances can be found in a similar way for special values of p^2. A so-called coupling of $\mathcal{O}(1)$ occurs and truncation to a few modes can give loss of information, and approximations may possibly not be valid on large time-scales. The analysis performed in this chapter can be extended to other values of p^2.

In Chapter 3 we considered an initial-boundary value problem for a weakly nonlinear beam equation with a quadratic nonlinearity. We presented an asymptotic theory which states that the constructed approximation is asymptotically valid on an ϵ^{-1} time-scale. We showed that, up to $\mathcal{O}(\epsilon)$, for most p^2-values no mode interactions occur between different modes. Truncation in that case is allowed to those modes that have non-zero initial energy. However, for some p^2-values interactions between different modes occur, which cause complicated internal resonances. For $p^2 \in]0, 100[$ seven critical values are given. For $p^2 \geq 100$ other internal resonances can be found in a similar way for special values of p^2. A so-called coupling of $\mathcal{O}(1)$ occurs and truncation to one mode will give loss of information, and approximations will not be valid. Also in Chapter 3 the stays of the bridge are modeled as two-sided springs with a small nonlinearity (ϵw^2). A next step would be to model the springs using w^+ and w^-, as is done for example in [9].

In Chapter 4 we considered an initial-boundary value problem for the vertical displacement of a weakly nonlinear elastic beam with an harmonic excitation in horizontal direction at the ends of the beam. The analysis presented in this chapter holds for all $p \in \mathbb{R}$. In Chapter 2 and Chapter 3 it was shown that certain values of p^2 can cause internal resonances. We showed that in Chapter 4 this does not occur. We showed that for all cases mode interactions occur only between modes with non-zero initial energy (up to $\mathcal{O}(\epsilon)$). These mode interactions are restricted to an interaction between the different phases. The amplitudes of all oscillation modes are constant. Truncation in that case is allowed to those modes with non-zero initial energy. We considered the case with no external forcing ($F(t) = 0$) and the case with external forcing ($F(t) = C\cos(\omega t)$). For $F(t) = 0$ and $F(t) = C\cos(\omega t)$ for most ω-values, the mode interaction between the modes with non-zero initial energy is restricted to an interaction between the different phases. We showed that for specific values of ω, i.e. $\omega \approx 2\omega_{k_p}$ special interactions occur. The mode interactions between the different oscillation modes are still restricted to an interaction between the different phases but the amplitude of mode k is no longer constant. Also, in Chapter 4 we considered an harmonic external force of the form $F(t) = C\cos(\omega t)$. This analysis can be extended to a more general form of $F(t)$, where F is a T-periodic force, $F(t) = \frac{a_0}{2} + \sum_n (a_n \cos(\nu_n t) + b_n \sin(\nu_n t))$ with $\nu_n = \frac{2\pi n}{T}$. This has been discussed in [46] for elastic beams or strings. In [46], truncation to one or two oscillation modes is applied, without giving a justification. We showed that in the cases discussed in this chapter truncation is valid up to $\mathcal{O}(\epsilon)$. In a way similar to the methods in Chapter 2 and Chapter 3 the problem with a more general form of $F(t)$ can be studied. The analysis will essentially be the same (depending on the function F), however, the equations will become a bit more complicated. A justification can be given whether truncation is allowed in those cases or not.

In Chapter 5 we considered an initial-boundary value problem for the vertical displacement of a weakly nonlinear elastic beam with a Rayleigh perturbation due to a wind force and a harmonic excitation in horizontal direction at the ends of the beam. The PDE considered contains the parameters p^2, C, δ, ω and ϵ $(0 < \epsilon \ll 1)$. For some specific values of the parameters $(0 < p^2 < 10,\ C = \delta = 1)$ we constructed formal approximations of order ϵ and considered the interaction between different oscillation modes. The aim of this chapter was to see whether the applied (harmonic) external excitation can act as a damping force on the oscillations of the beam or not. We considered the cases $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$ (with $p^2 = 8$ as an example) and $p^2 \approx 9$ (with detuning). For these cases of p^2 we considered three possible cases for $F(t)$: $F(t) = 0$ or $F(t) = C\cos(\omega t_0)$ (with $\omega \neq 2\omega_{k_p} + \epsilon\alpha$), $F(t) = C\cos(2\omega_{3,p}t_0)$, $F(t) = C\cos(2\omega_{1,p}t_0)$, with detuning. For $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$ we showed that for the case $F(t) = 0 \wedge F(t) = C\cos(\omega t_0)$ $(\omega \neq 2\omega_{k_p} + \epsilon\alpha)$ no significant changes occur due to the external excitation, and the analysis is similar to the case with no external excitation as considered in Chapter 2. For the cases $F(t) = C\cos(2\omega_{3,p}t_0)$ and $F(t) = C\cos(2\omega_{1,p}t_0)$ we showed that, with $\delta = C = 1$, the behavior of the oscillation modes changes due to the external excitation. The amplitude of oscillation has changed, i.e. is larger. This means that for $p^2 \in]0, 10[\backslash\{\frac{693}{152}, 9\}$ and $\delta = C = 1$, the external excitation does not work as a damping force. For $p^2 \approx 9$ we showed that for the case $F(t) = 0 \wedge F(t) = C\cos(\omega t_0)$ $(\omega \neq 2\omega_{k_p})$ no significant changes occur due to the external excitation, and the analysis is similar to the case with no external excitation as considered in Chapter 2. For the case $F(t) = C\cos(2\omega_{3,p}t_0)$, with $\delta = C = 1$, we showed that the behavior of the oscillation modes changes due to the external excitation. The amplitude of the (stable) oscillation of the third mode has become larger, due to the external excitation. This means that for $p^2 \approx 9$ and $\delta = C = 1$ the external excitation does not work as a damping force. For the case $F(t) = C\cos(2\omega_{1,p}t_0)$, with $\delta = C = 1$, we showed that the behavior changes slightly due to the external excitation. For this case the external excitation has no significant effects on the amplitudes of oscillation and certainly no damping effects occur (for $\delta = C = 1$). We showed that for the combination of the two effects considered in this chapter (a Rayleigh perturbation due to a wind force and a harmonic excitation in horizontal direction at the ends of the beam) the phase space analysis of the approximations of solutions of the PDE considered is much more complicated than the analysis performed in the separate cases, as considered in Chapter 2 and Chapter 4, and many bifurcations occur when detuning is considered. As was done in Chapters 2, 3 and 4, we showed for which cases truncation to one or more modes is valid. Again, similar to the results found in Chapter 2, we showed that in specific cases internal resonances occur, truncation to one mode gives loss of information and approximations are no longer valid.

The analysis in Chapter 5 can be easily extended to larger values of p^2, i.e. $p^2 \geq 10$. Furthermore, the influence of the different nonlinear terms can be changed by taking the parameters δ and $C \neq 1$ in the PDE. The analysis can be performed in the same way as presented in this chapter, but is expected to be more difficult for the case $\delta \neq 1$.
We expect that the asymptotic theory, presented in Chapter 3, can be extended to the other equations considered in this thesis. For example the Rayleigh perturbation, $w_t - \frac{1}{3}w_t^3$. Formal approximations for a nonlinear beam equation with this Rayleigh perturbation were constructed in Chapter 2. A similar fixed point theorem (in a different Banach space), where the vector (w, w_t) is considered in stead of w, can most likely be used in this case.
In the derivation of the beam equation as given in the introduction of this thesis, the so-called Kirchhoff assumption was applied (the velocity of the beam in x-direction, u_t, is small compared to w_t and can be neglected). This assumption reduced the coupled system of PDE's to a single PDE. A next step would be to consider the case where the Kirchhoff assumption does not hold, and the coupled system of two PDE's in u and w must be analyzed.
Also in the derivation of the beam equation as given in the introduction of this thesis, the rotation of the beam is not taken into account, i.e. the effects of rotation are taken to be small compared to the vertical displacement of the beam. The analysis of the problems considered in this thesis can be extended to the case where rotation is also taken into account.

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Samenvatting

Bepaalde typen buigzame bouwwerken, zoals hoge gebouwen, hangbruggen of met ijs bedekte elektriciteitsdraden, staan, door verschillende oorzaken, zoals (sterke) wind of aardbevingen, bloot aan trillingen. Een klassiek voorbeeld is de Tacoma narrows hangbrug. Wie heeft niet de video-film gezien over de grootschalige trillingen en de ineenstortingen van deze brug, getoond tijdens natuurkundelessen op de middelbare school. Een meer recent voorbeeld is het trillen van de tuien van de Erasmus Brug in Rotterdam, tijdens stormachtig en regenachtig weer.
In dit proefschrift bekijken we een (vereenvoudigd) model voor niet-lineaire trillingen van een hangbrug. Voor verschillende (externe) krachten uitgeoefend op het bouwwerk is een begin-randwaarden probleem gedefinieerd, dat de verticale uitwijking van de hangbrug beschrijft. Voor elk geval is het begin-randwaarden probleem bestudeerd, door een meer-tijdschalen storingsmethode te gebruiken. Formele benaderingen, d.w.z. benaderingen die voldoen aan de differentiaalvergelijking en de begin- en randwaarden tot op zeker orde in ϵ, worden voor elk geval geconstrueerd in de vorm van een machtreeks. Verder wordt aangetoond of er interne resonanties ontstaan of niet. Voor alle gevallen wordt een rechtvaardiging gegeven van de (zogeheten Galerkin) truncatie van de oneindige reeks van de formele benaderingen van de oplossing.
Voor één klasse van begin-randwaarden problemen wordt de existentie en eenduidigheid van oplossingen en de asymptotische geldigheid van benaderingen voor bepaalde goed-gedefinieerde tijdschalen bewezen met behulp van de dekpuntstelling van Banach.

De resultaten van dit proefschrift kunnen onder andere dienen om te bekijken of Galerkin truncatie, die vaak gebruikt wordt als dit type problemen (numeriek) bekeken wordt, toepasbaar is of niet. Ook kan, met behulp van de resultaten van dit proefschrift, een rechtvaardiging gegeven worden van de asymptotische geldigheid van benaderingen, en kan er iets gezegd worden over het verschil tussen benaderingen en exacte oplossingen.

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