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Weakly Primary Elements in Multiplicative Lattices

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Abstract

Let L be a multiplicative lattice. We define a proper element p of L to be weakly primary if $0 \neq ab \leq p$ implies $a \leq p$ or $b \leq \sqrt{p}$. Our objective is to investigate the properties of weakly primary elements in multiplicative lattices.

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1 Introduction

A multiplicative lattice is a complete lattice L, with least element 0_L and compact greatest element 1_L, on which there is defined a commutative, associative, completely join distributive product for which 1_L is a multiplicative identity. An element $a \in L$ is said to be proper if $a < 1_L$. An element $p < 1_L$ in L is said to be prime if $ab \leq p$ implies either $a \leq p$ or $b \leq p$. An element $p < 1_L$ in L is said to be weakly prime if $0_L \neq ab \leq p$ implies $a \leq p$ or $b \leq p$ (See [5]). For $a \in L$, we define $\sqrt{a} = \bigvee \{p \in L : p \textit{ is prime and } a \leq p\}$(See [2]). For any $a \in L$, $L/a = \{b \in L : a \leq b\}$ is a multiplicative lattice with multiplication $c \circ d = cd \vee a$.

An element $p < 1_L$ in L is said to be primary if $ab \leq p$ implies $a \leq p$ or $b \leq \sqrt{p}$ (See[1]). If $a, b \in L$, $(a : b)$ is the join of all $c \in L$ such that $cb \leq a$. An element a of a multiplicative lattice L is called compact if $a \leq \vee_{\alpha \in I} b_\alpha$ implies;

$$a \leq b_{\alpha_1} \vee b_{\alpha_2} \vee ... \vee b_{\alpha_n}$$

for some finite subset as $I=\{\alpha_1, \alpha_2, ..., \alpha_n\}$ (See [4]). L_* denotes the set of all compact elements of a multiplicative lattice L.

A complete multiplicative lattice (*not necessarily modular*) with the least element 0_L and compact greatest element 1_L (*a multiplicative identity*) which is generated under joins by a multiplicatively closed subset C of compact elements is called $C-$lattice. Like the ideal lattice of a ring, any $C-$lattice can be localized at a multiplicatively closed set.

If S is a multiplicatively closed subset of L_* in a $C-$lattice L, then for $a \in L$, $a_{(s)} = \bigvee \{x \in L_* : xs \leq a \ \ for \ \ some \ \ s \in S\}$ and $L_{(s)} = \left\{x_{(s)} : x \in L\right\}$ (See [2]).

For $p \in L$, we denote $V(p) = \{p_1 : p \leq p_1 \ such \ that \ p_1 \ is \ prime \ in \ L\}$. For various characterizations of prime and primary elements of multiplicative lattices the reader is referred to [1-5].

2 Weakly primary elements in multiplicative lattices

In this section we study weakly primary elements in multiplicative lattices. These concepts have been studied in [6] in the case of commutative rings and we shall begin with the following definition.

Definition 1 *An element $p < 1_L$ in L is said to be weakly primary if $0_L \neq ab \leq p$ implies $a \leq \sqrt{p}$ or $b \leq p$.*

A prime element is weakly prime, a weakly prime element is weakly primary. 0_L is weakly primary by the definition but it is not a primary element. Thus, a weakly primary element is not necessarily a primary element.

Example 1 *Let $a, p \in L$ such that $a \leq p$ and p be a weakly primary element of L. Then, $\bar{p}$ is a weakly primary element in L/a.*

Lemma 1 *Let L be a $C-$lattice $a_1, a_2 \in L$. Suppose that $b \in L$ satisfies the following property:*

- *If $h \in L$ is compact with $h \leq b$ and either $h \leq a_1$ or $h \leq a_2$*

then $b \leq a_1$ or $b \leq a_2$.

Proof *See* $[5, Lemma 1]$.

Proposition 1 *Let L be a $C-$lattice and p be a proper element of L. Then the following assertions are equivalent:*

1. *p is a weakly primary element of L.*
2. *Either $(p : y) = p$ or $(p : y) = (0_L : y)$ for every $y \nleq \sqrt{p}$.*
3. *For every $x, y \in L_*$, $0_L \neq xy \leq p$ implies either $x \leq p$ or $y \leq \sqrt{p}$.*

Proof $(1) \Longrightarrow (2)$. *Suppose (1) holds. Let h be a compact element of L such that $h \leq (p : y)$ and $y \nleq \sqrt{p}$. Then $hy \leq p$. If $0_L = hy \leq p$, then $h \leq (0_L : y)$. Let $0_L \neq hy$. Since $hy \leq p$, $y \nleq \sqrt{p}$ and p is a weakly primary element, it follows that $h \leq p$. Hence by Lemma 1, either $(p : y) \leq (0_L : y)$ or $(p : y) \leq p$. Consequently, either $(p : y) = (0_L : y)$ or $(p : y) = p$.*
$(2) \Longrightarrow (3)$. *Suppose (2) holds. Let $0_L \neq xy \leq p$ and $y \nleq \sqrt{p}$ for some $x, y \in L_*$. We show that $x \leq p$. Since $xy \leq p$, it follows that $x \leq (p : y)$. If $(p : y) = p$, then $x \leq p$. If $(p : y) = (0_L : y)$, then $xy = 0_L$. This is a contradiction. Consequently, $x \leq p$ and so p is a weakly primary element.*
$(3) \Longrightarrow (1)$. *Suppose (3) holds. Let $0_L \neq ab \leq p$, $a \nleq p$ and $b \nleq \sqrt{p}$ for some $a, b \in L$. Since L is a $C-$lattice, L is compactly generated. Choose $x, y \in L_*$ such that $x \leq a$, $y \leq b$, $x \nleq p$ and $y \nleq \sqrt{p}$. Let $\acute{x} \leq a$ and $\acute{y} \leq b$ be any two compact elements of L. Then, $(\acute{x} \vee x)(\acute{y} \vee y) \leq ab \leq p$. Since $(\acute{x} \vee x) \nleq p$ and $(\acute{y} \vee y) \nleq \sqrt{p}$, it follows that $(\acute{x} \vee x)(\acute{y} \vee y) = 0_L$ and so $\acute{x}\acute{y} = 0_L$, by (3). Therefore, $ab = 0_L$. This shows that p is a weakly primary element of L.*

Theorem 1 *Let L be a multiplicative lattice and $p \in L$. If p is a weakly primary element that is not primary, then $p^2 = 0_L$.*

Proof *Suppose that $p^2 \neq 0_L$. We show that p is primary. Let $xy \leq p$. If $0_L \neq xy \leq p$, then by the definition of a weakly primary element, either $x \leq p$ or $y \leq \sqrt{p}$. So assume that $0_L = xy$. First suppose that $0_L \neq xp$. Then $0_L \neq xp = x(y \vee p) \leq p$, so either $x \leq p$ or $y \leq \sqrt{p}$. So we can assume that $0_L = xp$. Likewise, we can assume that $0_L = yp$. Since $0_L \neq p^2$ and $0_L \neq p^2 = (x \vee p)(y \vee p) \leq p$, it follows that either $(x \bigvee p) \leq p$ or $(y \bigvee p) \leq \sqrt{p}$. Hence either $x \leq p$ or $y \leq \sqrt{p}$. Thus p is primary.*

Theorem 2 *Let L be a multiplicative lattice and $\{p_i\}_{i \in I}$ be a family of weakly primary elements of L that are not primary. Then $p = \bigwedge_{i \in I} p_i$ is a weakly primary element of L.*

Proof We show that $\sqrt{\bigwedge_{i\in I} p_i} = \bigwedge_{i\in I} \sqrt{p_i}$.
It is easy to see that $\sqrt{\bigwedge_{i\in I} p_i} \leq \sqrt{p_i}$ is hold for each $i \in I$. Thus,

$$\sqrt{\bigwedge_{i\in I} p_i} \leq \bigwedge_{i\in I} \sqrt{p_i}.$$

Let $x \leq \bigwedge_{i\in I} \sqrt{p_i}$. Then $x \leq \sqrt{p_i}$ for each $i \in I$. We know that $\sqrt{p_i} = \sqrt{0_L}$, for all $i \in I$, by Theorem 1. If $x \leq \sqrt{0_L}$, then there is at least one $n \in Z^+$ such that $x^n = 0_L \leq p_i$, for all $i \in I$. Thus, $x^n \leq \bigwedge_{i\in I} p_i$ for some $n \in Z^+$. Therefore, $x \leq \sqrt{\bigwedge_{i\in I} p_i}$. So, $\bigwedge_{i\in I} \sqrt{p_i} \leq \sqrt{\bigwedge_{i\in I} p_i}$. Since $p = \bigwedge_{i\in I} p_i$ and $\sqrt{p_i} = \sqrt{0_L}$ for each $i \in I$; $\sqrt{p} = \sqrt{\bigwedge p_i} = \bigwedge \sqrt{p_i} = \sqrt{0_L}$. We show that p is a weakly primary element of L. Let $0_L \neq ab \leq p$. Assume that $a \nleq p$. Since $a \nleq p$, $a \nleq p_i$ for at least one $i \in I$. Since each p_i is a weakly primary element, $b \leq \sqrt{p_i} = \sqrt{0_L} = \sqrt{p}$ for $i \in I$. Thus, p is a weakly primary element of L.

Lemma 2 *Let L_1 and L_2 be multiplicative lattices and let $L = L_1 \times L_2$. Then the following hold:*

1. *If $p_1 \in L_1$ then $\sqrt{(p_1, 1_{L_2})} = (\sqrt{p_1}, 1_{L_2})$*

2. *If $q_2 \in L_2$ then $\sqrt{(1_{L_1}, q_2)} = (1_{L_1}, \sqrt{q_2})$*

Proof For the proof of the first assertion assume,

$$\sqrt{(p_1, 1_{L_2})} = \bigwedge \{y = (y_1, y_2) \in L \ : y \text{ is prime such that } (p_1, 1_{L_2}) \leq (y_1, y_2)\}$$

By [5, Lemma 2], $y = (y_1, y_2)$ is a prime element of $L = L_1 \times L_2$ if and only if y has one of the following forms:

1. $y = (p, 1_{L_2})$, where p is a prime element of L_1

2. $y = (1_{L_1}, q)$, where q is a prime element of L_2

Therefore,

$$\begin{aligned}
\sqrt{(p_1, 1_{L_2})} &= \bigwedge \{y = (y_1, 1_{L_2}) \in L : y \text{ is prime such that } (p_1, 1_{L_2}) \leq (y_1, 1_{L_2})\} \\
&= \bigwedge_{y_1 \in V(p_1)} \{y = (y_1, 1_{L_2}) \in L : (p_1, 1_{L_2}) \leq (y_1, 1_{L_2})\}. \\
&= (\sqrt{p_1}, 1_{L_2})
\end{aligned}$$

The second assertion is proved similarly.

Lemma 3 *Let L_1 and L_2 be multiplicative lattices and let $L = L_1 \times L_2$. Then an element of $L = L_1 \times L_2$ is primary if it has one of the following two forms.*

1. $(p, 1_{L_2})$, *where p is a primary element of L_1*
2. $(1_{L_1}, q)$, *where q is a primary element of L_2*

Proof *We proved the first assertion here. The proof for the second assertion is similar and therefore it is omitted.*

Let $(a,b)(c,d) \leq (p, 1_{L_2})$ where $(a,b),(c,d) \in L$, so either $a \leq p$ or $c \leq \sqrt{p}$ since p is primary. It follows that either $(a,b) \leq (p, 1_{L_2})$ or $(c,d) \leq (\sqrt{p}, 1_{L_2}) = \sqrt{(p, 1_{L_2})}$ by Lemma 2. Thus $(p, 1_{L_2})$ is primary.

Theorem 3 *Let L_1 and L_2 be multiplicative lattices and let $L = L_1 \times L_2$. If p is a weakly primary element of L, then either $p = (0_{L_1}, 0_{L_2})$ or p is a primary element of L.*

Proof *Let $p \neq 0_L$ be a weakly primary element. Then there is an element such that $(0_{L_1}, 0_{L_2}) \neq (a,b) = (a, 1_{L_2})(1_{L_1}, b) \leq p$, where $a \in L_1$ and $b \in L_2$. Therefore, $(a, 1_{L_2}) \leq p$(Case 1) or $(1_{L_1}, b) \leq \sqrt{p}$(Case 2).*

- *Case 1. If $(a, 1_{L_2}) \leq p$, then $p = (p_1, 1_{L_2})$ where p_1 is an element of L_1. We show that p_1 is primary. Let $cd \leq p_1$, where $c, d \in L_1$. Then $(0_{L_1}, 0_{L_2}) \neq (cd, 1_{L_2}) = (c, 1_{L_2})(d, 1_{L_2}) \leq (p_1, 1_{L_2}) = p$, either $(c, 1_{L_2}) \leq (p_1, 1_{L_2})$ or $(d, 1_{L_2}) \leq \sqrt{(p_1, 1_{L_2})} = (\sqrt{p_1}, 1_{L_2})$ by Lemma 2. Hence either $c \leq p_1$ or $d \leq \sqrt{p_1}$. Therefore, p_1 is a primary element of L_1. Thus, p is a primary element of L by Lemma 3.*

- *Case 2. If $(1_{L_1}, b) \leq \sqrt{p}$, then $(1_{L_1}, b^n) \leq p$ for some $n \in Z^+$. Therefore, $p = (1_{L_2}, p_2)$ where p_2 is an element of L_2. We show that p_2 is primary. Let $cd \leq p_2$, where $c, d \in L_2$. Then $(0_{L_1}, 0_{L_2}) \neq (1_{L_1}, cd) = (1_{L_1}, c)(1_{L_1}, d) \leq (1_{L_1}, p_2) = p$, either $(1_{L_1}, c) \leq (1_{L_1}, p_2)$ or $(1_{L_1}, d) \leq \sqrt{(1_{L_1}, p_2)} = (1_{L_1}, (\sqrt{p_2}))$ by Lemma 2. Hence either $c \leq p_2$ or $d \leq \sqrt{p_2}$. Therefore, p_2 is a primary element of L_2. Thus, p is a primary element of L by Lemma 3.*

Corollary 1 *Let L_1 and L_2 be multiplicative lattices and let $L = L_1 \times L_2$. Then an element of L is weakly primary if it has one of the following three forms.*

1. $p = (0_{L_1}, 0_{L_2})$.
2. $(p, 1_{L_2})$, *where p is a primary element of L_1.*

3. $(1_{L_1}, p)$, *where* p *is a primary element of* L_2.

Proposition 2 *Let* L *be a* $C-$*lattice and* p *be an element of* L. *Suppose* m *is a maximal element of* L. *If* p *is weakly primary, then* $p_{(m)}$ *is a weakly element of* $L_{(m)}$.

Proof *Suppose* p *is a weakly primary element of* L. *Let* $0_{(m)} \neq a_{(m)}b_{(m)} \leq p_{(m)}$ *for some* $a, b \in L_*$. *Then* $ab \leq p_{(m)}$, *so* $abu \leq p$ *for some compact element* $u \nleq m$. *Since* $0_{(m)} \neq a_{(m)}b_{(m)}$, *it follows that* $abu \neq 0_L$. *As* p *is a weakly primary, we have either* $a \leq p$ *or* $bu \leq \sqrt{p}$ *so either* $a_{(m)} \leq p_{(m)}$ *or* $b_{(m)} \leq \sqrt{p}_{(m)} = \sqrt{p_{(m)}}$, *since* $u_{(m)} = 1_{(m)}$. *Therefore* $p_{(m)}$ *is a weakly primary element of* $L_{(m)}$.

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