Weakly Smooth Nonselfadjoint Spectral Elliptic Boundary Problems

M. S. Agranovich, R. Denk and M. Faierman

Weakly Smooth Nonselfadjoint Spectral Elliptic Boundary Problems

Mikhail Agranovich, Robert Denk, and Melvin Faierman[1]

Abstract. The paper is devoted to general elliptic boundary problems

$$(A-\lambda)u = f \quad \text{in} \quad G, \quad B_j u = 0 \quad (j = 1, \ldots, m) \quad \text{on} \quad \Gamma = \partial G, \tag{1}$$

generally nonselfadjoint, where G is a bounded domain in $\mathbb{R}^n$. The main goal is to minimize, to some extent, the smoothness assumptions under which the known spectral results are true. The main results concern the asymptotics of the trace of $R(\lambda)^q$ with $q > n/2m$, where $R(\lambda)$ is the resolvent, in an angle of ellipticity with parameter. For example, for the Dirichlet problem these asymptotics are obtained in the case of bounded and measurable coefficients in A and continuous coefficients in the principal part of A, while the boundary is assumed to belong to $C^{2m-1,1}$. The asymptotics of the moduli of the eigenvalues are investigated. The last section is devoted to indefinite spectral problems, with a real-valued multiplier $\omega(x)$ before λ changing the sign. The references contain 44 items.

1991 Mathematics Subject Classification. 35Pxx, 35J40, 47F05, 58G03.

1 Introduction

1.1. Let G be a bounded domain in $\mathbb{R}^n$ with $(n-1)$-dimensional boundary Γ. Consider the boundary problem

$$\begin{aligned} A(x,\mathcal{D})\,u(x) - \lambda\,u(x) &= f(x) \quad \text{in } G\,, && (1.1)\\ B_j(x,\mathcal{D})\,u(x) &= g_j(x) \quad (j=1,\ldots,m) \ \text{ on } \Gamma\,. && (1.2)\end{aligned}$$

[1]The first author is partially supported by INTAS, Grant No. 94-2187, and RFFI, Grant No. 95-01-00549. The third author is partially supported by a grant from the FRD of South Africa.

Here

$$A = A(x, \mathcal{D}) = \sum_{|\alpha| \le 2m} a_\alpha(x) \mathcal{D}^\alpha \tag{1.3}$$

is a partial differential operator in G of order $2m$, and

$$B_j = B_j(x, \mathcal{D}) = \sum_{|\beta| \le m_j} b_{j\beta}(x) \mathcal{D}^\beta \tag{1.4}$$

are partial differential operators of orders $m_j < 2m$ with coefficients defined only on Γ; in (1.2) the derivatives $\mathcal{D}^\beta u(x)$ are assumed to be restricted to Γ. All the functions in (1.1)–(1.4) are scalar and, in general, complex-valued. As usual,

$$\mathcal{D}^\alpha = \mathcal{D}_1^{\alpha_1} \ldots \mathcal{D}_n^{\alpha_n}, \quad \mathcal{D}_j = -i \frac{\partial}{\partial x_j}, \quad |\alpha| = \alpha_1 + \ldots + \alpha_n,$$

and below $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$. The main part of this paper is devoted to the spectral problem obtained from (1.1)–(1.2), roughly speaking, by setting $f = 0$ in G and $g_j = 0$ on Γ.

This problem is in general far from being selfadjoint. We only assume that it has an angle, or angles, of ellipticity with parameter. We recall the definition. Denote by $a_0(x, \xi)$ and $b_{j0}(x, \xi)$ the principal symbols of the operators $A(x, \mathcal{D})$ and $B_j(x, \mathcal{D})$:

$$a_0(x, \xi) = \sum_{|\alpha| = 2m} a_\alpha(x) \xi^\alpha, \quad b_{j0}(x, \xi) = \sum_{|\beta| = m_j} b_{j\beta}(x) \xi^\beta. \tag{1.5}$$

Definition 1.1. Let $\mathcal{L}$ be a closed angle (sector) in the complex plane with vertex at the origin. Then the boundary problem (1.1)–(1.2) is called *elliptic with parameter in* $\mathcal{L}$ if the following two conditions are satisfied.

(1) $a_0(x, \xi) - \lambda \ne 0$ for $(x, \xi) \in \overline{G} \times \mathbb{R}^n$ and $\lambda \in \mathcal{L}$ if $|\xi| + |\lambda| \ne 0$.

(2) Let x_0 be any point on Γ. Assume that the boundary problem (1.1)–(1.2) is rewritten in the coordinate system associated with x_0: it is obtained from the original one by a rotation after which the positive x_n-axis has the direction of the interior normal to Γ at x_0. Then the boundary problem on the half-line

$$\begin{gathered} a_0(x_0, \xi', \mathcal{D}_n)\, v(t) - \lambda\, v(t) = 0 \quad (t = x_n > 0), \\ b_{j0}(x_0, \xi', \mathcal{D}_n)\, v(t) = 0 \quad (j = 1, \ldots, m) \;\text{ at } t = 0, \\ v(t) \to 0 \quad (t \to +\infty) \end{gathered} \tag{1.6}$$

has only the trivial solution for $\xi' \in \mathbb{R}^{n-1}$ and $\lambda \in \mathcal{L}$ if $|\xi'| + |\lambda| \ne 0$.

Other terms are “Agmon’s condition”(Seeley, see e.g. (Seeley 1969)) and “parameter-ellipticity”(Grubb, see e.g. (Grubb 1986)). Conditions (1) and (2) were introduced by Agmon (see (Agmon 1962)). In particular, $\mathcal{L}$ can be a ray issuing from the origin. However, the set of rays of ellipticity with parameter is open.

In Section 4 we will mention some generalizations of Definition 1.1 to boundary problems that depend upon λ polynomially and to boundary problems with vector-valued $u(x)$.

Now let us fix our attention upon smoothness assumptions. Recall that for $k \in \mathbb{Z}_+$ and $0 < \gamma \leq 1$ the space $C^{k,\gamma}$ consists of all functions which are continuous with their derivatives of order up to k and whose derivatives of order k are Hölder continuous with exponent γ. In particular, the derivatives of order k of the functions in $C^{k,1}$ satisfy the Lipschitz condition. We will distinguish three possibilities.

Definition 1.2. (a) **Minimal smoothness.** The boundary problem (1.1)–(1.2) will be called *minimally smooth* if 1) Γ is a submanifold in $\mathbb{R}^n$ of class $C^{2m-1,1}$; 2) all the coefficients $a_\alpha(x)$ are measurable and bounded, while the top order coefficients $a_\alpha(x)$ ($|\alpha| = 2m$) are continuous in $\overline{G}$; 3) the coefficients $b_{j\beta}(x)$ belong to the space $C^{2m-m_j-1,1}(\Gamma)$.

(b) **Weak smoothness.** The boundary problem (1.1)–(1.2) will be called *weakly smooth* if the formally adjoint to (1.1)–(1.2) boundary problem

$$\begin{aligned} A^*(x,\mathcal{D})v(x) - \lambda v(x) &= \tilde{f}(x) \quad \text{in } G\,, && (1.7)\\ \tilde{B}_j(x,\mathcal{D})v(x) &= \tilde{g}_j(x) \quad (j = 1,\dots,m) \text{ on } \Gamma && (1.8)\end{aligned}$$

is well-defined and both boundary problems, (1.1)–(1.2) and (1.7)–(1.8), are at least minimally smooth.

Here $A^*(x,\mathcal{D})$ is the operator formally adjoint to $A(x,\mathcal{D})$:

$$A^*(x,\mathcal{D})v(x) = \sum_{|\alpha|\leq 2m} \mathcal{D}^\alpha\big(\overline{a_\alpha(x)}v(x)\big)\,. \tag{1.9}$$

Recall that the boundary problems (1.1)–(1.2) and (1.7)–(1.8) are called formally adjoint if

$$(Au,v)_G = (u,A^*v)_G \tag{1.10}$$

for any functions $u, v \in C^{2m}(G)$ satisfying the boundary conditions $B_j u = 0$ and $\tilde{B}_j v = 0$ on Γ ($j = 1,\dots,m$), respectively. Here $(\ ,\ )_G$ is the standard scalar product in $L_2(G)$. Of course, in a weakly smooth boundary problem (1.1)–(1.2) the coefficients $a_\alpha(x)$ and $b_{j\beta}(x)$ have to possess some additional smoothness. The sufficient conditions will be indicated in Subsection 2.5.

(c) **Smooth problems.** The boundary problem (1.1)–(1.2) will be called *smooth* if Γ is a C^∞ submanifold in $\mathbb{R}^n$, $a_\alpha(x) \in C^\infty(\overline{G})$ for all α, and $b_{j\beta}(x) \in C^\infty(\Gamma)$ for all j and β.

1.2. In Section 2 we will formulate the Basic Theorem; according to it the boundary problem (1.1)–(1.2) elliptic with parameter in some angle $\mathcal{L}$ has a unique solution in the corresponding Sobolev L_p-spaces ($1 < p < \infty$) for $\lambda \in \mathcal{L}$ with sufficiently large modulus. This theorem completes the corresponding result in (Agmon 1962), see Subsection 2.2 for details and other references. For the completeness of our presentation, we sketch the proof of this theorem in Subsection 2.4.

In particular, this theorem makes it possible to introduce the operator $A_B = A_{B,2}$ in $L_2(G)$ acting as $A(x, \mathcal{D})$, with domain

$$\mathcal{D}(A_B) = \{u \in W_2^{2m}(G) : B_j u = 0 \quad (j = 1, \dots, m) \quad \text{on } \Gamma\}\,. \tag{1.11}$$

It is closed and densely defined; its resolvent set is nonvoid (contains all $\lambda \in \mathcal{L}$ with large $|\lambda|$), and the resolvent

$$R(\lambda) = R_{A_B}(\lambda) = (A_B - \lambda)^{-1} \tag{1.12}$$

is compact. Hence the spectrum of A_B is discrete. Similar operators $A_{B,p}$ can be introduced in $L_p(G)$ $(1 < p < \infty)$, but they all are "spectrally equivalent" (see Section 3). The nontrivial solutions of (1.1)–(1.2) with $f = 0$ and $g_j = 0$ are the eigenfunctions that correspond to an eigenvalue λ; in general they form a subset of the set of all generalized eigenfunctions (or root functions), the nontrivial solutions of the equations $(A_B - \lambda)^k u = 0$ with $k \in \mathbb{N}$.

Some spectral properties of A_B were intensively discussed in the literature and are well known. Our aim in this paper is to minimize, to some extent, the smoothness assumptions under which A_B has these properties.

1.3. In Section 3 we collect the spectral properties of A_B that are the direct consequences of the Basic Theorem. The first of them is Agmon's completeness theorem (see (Agmon 1962)): the set of all generalized eigenfunctions is complete in $L_2(G)$ if for (1.1)–(1.2) there exist some rays of ellipticity with parameter with angles between the adjacent rays not greater than $2\pi m/n$. Moreover, in this case the Fourier series of any $f \in L_2(G)$ with respect to a minimal complete system of the generalized eigenfunctions admits the summation to f by the so-called Abel–Lidskiĭ method. We also present the statements on "angular distribution of eigenvalues" and the presence of "the rays of condensation of eigenvalues" from (Agmon 1962). See Section 3 for the details.

1.4. More deep is the problem of the regular behaviour of the moduli of the eigenvalues $\lambda_j = \lambda_j(A_B)$ as $j \to \infty$. We use the resolvent approach to this problem. Let $q \in \mathbb{N}$ be such that $2mq > n$ ($q = 1$ if $2m > n$). Then $R(\lambda)^q$ is a trace class operator. If the boundary problem (1.1)–(1.2) is smooth, then the following asymptotic formula is well-known for the trace of $R(\lambda)^q$ in the angle $\mathcal{L}$ of ellipticity with parameter; for simplicity we assume that $\mathcal{L}$ is symmetric with respect to the negative real half-axis $\mathbb{R}_-$:

$$\operatorname{tr} R(\lambda)^q = c_q\,(-\lambda)^{\frac{n}{2m}-q} + o\big(|\lambda|^{\frac{n}{2m}-q}\big) \quad (\mathcal{L} \ni \lambda \to \infty) \tag{1.13}$$

uniformly in $\arg\lambda$, where

$$c_q = \int_G c_q(x)\,dx \quad \text{and} \quad c_q(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{d\xi}{[a_0(x,\xi)+1]^q}\,, \tag{1.14}$$

see e.g. (Agranovich 1990). Here the holomorphic function $(-\lambda)^{\cdots}$ is defined outside $\overline{\mathbb{R}}_+$ and is equal to $|\lambda|^{\cdots}$ when $\lambda \in \mathbb{R}_-$.

Our main results in this paper are essentially Theorems 4.1 and 5.1. The first of them concerns the Dirichlet boundary problem (1.1),

$$\partial_\nu^{j-1} u(x) = 0 \quad (j = 1, \dots, m) \quad \text{on } \Gamma\,. \tag{1.15}$$

Here $\partial_\nu = \partial/\partial\nu$ is the derivative in the direction of the inner normal to Γ at x. Note that the Dirichlet problem is absolutely elliptic in the sense of (Hörmander 1958), i.e. elliptic with respect to any elliptic equation. In particular, Condition (2) in the Definition 1.1 is satisfied for (1.15) automatically if Condition (1) is satisfied.

Theorem 4.1 states that formula (1.13) is true for the Dirichlet problem if it is elliptic with parameter in $\mathcal{L}$ and satisfies the minimal smoothness assumptions and if q in the inequality $2mq > n$ is even.

The proof consists of the following steps. At first we approximate the domain G by a domain $\tilde{G}$ with C^∞ boundary such that we can use a $C^{2m-1,1}$-diffeomorphism of G onto $\tilde{G}$. Let us write A_D instead of A_B to indicate that we are considering the Dirichlet problem. The diffeomorphism defines a similarity transform $A_D \mapsto \tilde{A}_D = T^{-1}A_D T$, where $\tilde{A}_D$ corresponds again to the Dirichlet problem but in the new smooth domain. This permits us to assume, without loss of generality, that Γ is smooth. Now we construct an approximation $A^{(h)}(x,\mathcal{D})$ for $A(x,\mathcal{D})$ with C^∞ top order coefficients $a_\alpha^{(h)}(x)$ ($|\alpha| = 2m$) tending to $a_\alpha(x)$ uniformly as $h \to 0$ and zero lower order coefficients. We prove that for even q

$$|\operatorname{tr} R_{A_D}(\lambda)^q - \operatorname{tr} R_{A_D^{(h)}}(\lambda)^q| \le \varepsilon(h)\,|\lambda|^{\frac{n}{2m}-q} \tag{1.16}$$

in $\mathcal{L}$ for large $|\lambda|$, where $\varepsilon(h) \to 0$ as $h \to 0$. Here it is essential that

$$\mathcal{D}(A_D) = \mathcal{D}(A_D^{(h)})\,. \tag{1.17}$$

The Dirichlet problem for a smooth operator $A^{(h)}(x,\mathcal{D})$ in a smooth domain G is smooth; therefore we have a formula of the form (1.13) for $\operatorname{tr} R_{A_D^{(h)}}(\lambda)^q$ in (1.16). Obviously this leads to the desired result.

Of course, the idea to use smooth approximations for nonsmooth boundary problems is not new, see e.g. (Beals 1967). However, apparently a “jump” from a not very smooth domain to a smooth one by a similarity transform is a new element of considerations. We also note that to obtain the estimate (1.16) we derive an asymptotic formula

$$|R_{A_D^{(h)}}(\lambda)|_q^q \sim c(h, \arg\lambda)\,|\lambda|^{\frac{n}{2m}-q} \quad (\lambda \to \infty \text{ in } \mathcal{L})\,, \tag{1.18}$$

where $|\cdot|_q$ is the Neumann–Schatten norm of order q. See Section 4 for further details.

1.5. In Section 5 we prove that formula (1.13) is true for general weakly smooth problems (1.1)–(1.2) elliptic with parameter in $\mathcal{L}$ (Theorem 5.1). Again, in this section

q is assumed to be even, $q = 2k$, and for the proof we represent $R(\lambda)^q$ in the form

$$R(\lambda)^q = R_1(\lambda)R_2(\lambda)^*\,, \text{ where } R_1(\lambda) = R(\lambda)^k \text{ and } R_2(\lambda) = R_1(\lambda)^*\,. \tag{1.19}$$

We prove that for a fixed $\lambda \in \mathcal{L}$ with sufficiently large modulus

$$R_1(\lambda)f(x) = \int_G K_\lambda(x,y)f(y)\,dy\,, \tag{1.20}$$

where the kernel $K_\lambda(x,y)$ is a continuous function of $x \in \overline{G}$ with values in $L_2(G)$ and

$$\left(\int_G |K_\lambda(x,y)|^2\,dy\right)^{\frac{1}{2}} \leq \text{ Const } |\lambda|^{\frac{n}{4m}-\frac{q}{2}}\,. \tag{1.21}$$

For this we prove that $R_1(\lambda) = R(\lambda)^k$ is a bounded operator from $L_2(G)$ to $C(\overline{G})$. Since the same is true for $R_2(\lambda)$ (here we use the formally adjoint to (1.1)–(1.2) boundary problem), for $R(\lambda)$ we have

$$R(\lambda)^q f(x) = \int_G K(x,y,\lambda)f(y)\,dy\,, \tag{1.22}$$

where for $\lambda \in \mathcal{L}$ with sufficiently large modulus $K(x,y,\lambda)$ is a continuous function on $\overline{G} \times \overline{G}$ and

$$|K(x,y,\lambda)| \leq \text{ Const } |\lambda|^{\frac{n}{2m}-q} \tag{1.23}$$

uniformly in x and y. Moreover, we obtain a pointwise asymptotic formula

$$K(x,x,\lambda) = c(x)\,(-\lambda)^{\frac{n}{2m}-q} + o\Big(|\lambda|^{\frac{n}{2m}-q}\Big) \quad (\lambda \to \infty \text{ in } \mathcal{L}) \tag{1.24}$$

uniformly in x on compact subsets of G. To obtain (1.13) it remains to integrate (1.24) over G.

This approach is a modification of that in (Agmon 1965a) and (Beals 1970). Agmon at first considered the case $2m > n$, and he obtained some results (now well-known) concerning the kernels in the integral representation of bounded operators from $L_2(G)$ to $W_2^l(G)$ with $l > n/2$ or $l > n$. We actually generalize these results to bounded operators from $L_2(G)$ to $W_p^l(G)$, where $p \geq 2$ and $lp > n$ (see Subsection 5.1). Implicitly these generalizations were contained in (Beals 1970).

Beals considered selfadjoint nonsmooth boundary problems. Though for nonsmooth boundary problems the operator A_B^q is in general not defined, Beals defined the resolvent $(A_B^q - \lambda)^{-1}$ as

$$(A_B^q - \lambda)^{-1} = (A_B - \mu_q)^{-1} \ldots (A_B - \mu_1)^{-1}\,, \tag{1.25}$$

where μ_j are the pairwise different roots $\lambda^{1/q}$. Using a variant of the Basic Theorem, he considered $(A_B - \mu_j)^{-1}$ as acting from $L_{p_j}(G)$ to $W_{p_j}^l(G)$ and inserted the Sobolev embedding operators $S_j : W^l_{p_j}(G) \to L_{p_{j+1}}(G)$ to the left of $(A_B - \mu_j)^{-1}$. Here p_j are appropriate numbers and $2 = p_1 < p_2 < \dots$. In (Faierman 1995b) this approach was applied to some nonselfadjoint boundary problems with indefinite weight. Instead of (1.25), in the present paper we consider $R(\lambda)^q$ and $R(\lambda)^{q/2}$ in the same manner; this permits us to minimize the assumptions about the presence of the rays of ellipticity with parameter.

Finally we use operators of the form $\varphi(A_0(x_0, \mathcal{D}) - \lambda)\psi\,\cdot$ to deduce (1.24), where the supports of the functions φ and ψ lie in a neighbourhood of $x_0 \in G$. Here we again follow (Agmon 1965a). Note, however, that in the case $2m \le n$ the boundary problems are assumed in that paper to be sufficiently smooth.

1.6. In Section 6 we discuss the spectral consequences of the main Theorems 4.1 and 5.1 following essentially (Agranovich and Markus 1989), where smooth spectral problems were considered. We use the Hardy–Littlewood Tauberian Theorem and its rough analogue presented in that paper. Let $\{\lambda_j\}_1^\infty$ be the sequence of all eigenvalues of A_B enumerated in such a way that

$$|\lambda_1| \le |\lambda_2| \le \dots \tag{1.26}$$

and each eigenvalue is repeated according to its multiplicity. Set

$$d = \frac{1}{(2\pi)^n n} \int_G dx \int_{|\xi|=1} [a_0(x,\xi)]^{-\frac{n}{2m}}\, dS_\xi\,. \tag{1.27}$$

Here the values of $a_0(x,\xi)$ do not belong to $\mathcal{L}$, and we define $a_0^{-n/2m}$ using a cut along the bisectrix $\overline{\mathbb{R}}_-$ of $\mathcal{L}$. The number d is of course independent of q, and $c_q = \beta_q d$, where the coefficient β_q does not depend upon $a_0(x,\xi)$. Under the assumptions of Theorem 4.1 or 5.1, we obtain the relation

$$|\lambda_j| \asymp j^{\frac{2m}{n}} \tag{1.28}$$

if $d \ne 0$; this means that the ratio $|\lambda_j|/j^{2m/n}$ lies between positive constants for large j. If, in addition, the boundary problem is elliptic with parameter along each ray except, say, $\overline{\mathbb{R}}_+$, then $\lambda_j/j^{2m/n}$ has a positive limit which is calculated in terms of $a_0(x,\xi)$ by the same formula as for smooth positive selfadjoint boundary problems. The last result was obtained in (Agmon 1965b) and (Mizohata 1965) for sufficiently smooth nonselfadjoint boundary problems.

Moreover, the results are strengthened in the following situation. Assume that the boundary problem is elliptic with parameter in two closed angles $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ that have only the point $\lambda = 0$ in common, and let Λ_1 and Λ_2 be two open angles that constitute the complement of $\mathcal{L}^{(1)} \cup \mathcal{L}^{(2)}$. Assume that Λ_1 contains all the values of $a_0(x,\xi)$, $\xi \ne 0$. Then both results mentioned above remain true for the eigenvalues of A_B lying in Λ_1. Here we keep in mind the scalar case; more interesting results are valid in the matrix case, in which both Λ_1 and Λ_2 can contain the eigenvalues of

$a_0(x, \xi)$, see Remark 6.6, and in the case of indefinite problems, see Theorem 8.13. To obtain these results, we use the procedure of separating the asymptotics of the part of $\operatorname{tr} R(\lambda)^q$ corresponding to the eigenvalues of A_B in Λ_1, as in (Agranovich 1987); see also (Agranovich and Markus 1989). However, we carry out this procedure anew since now we do not have the estimate of the remainder in (1.13) used in those papers. For this we extend (1.13) to real $q > n/2m$ using integral formulas for noninteger powers of operators (see Theorem 6.4).

1.7. In Section 7 we extend the results to boundary problems (1.1)–(1.2) with additional transmission conditions along some closed surfaces $\Gamma_1, \dots, \Gamma_N$. These surfaces lie inside G and have no common points pairwise and with Γ. They divide their complement in G into subdomains $G_0, \dots, G_N$. If Γ_k separates G_l and $G_{l'}$, then the transmission conditions on Γ_k connect the boundary values of the solution and its derivatives from the side of G_l and those from the side of $G_{l'}$. In the conditions of minimal smoothness, the top order coefficients in $A(x, \mathcal{D})$ are assumed to be continuous in each G_l up to the boundary, i.e. they have to have continuous extensions from G_l to $\overline{G}_l$. Accordingly, the solution must belong to $W_p^{2m}(G_l)$ in each G_l separately. The theory of such problems is very close to that of usual elliptic problems (cf. e.g. (Schechter 1960)); because of this the extensions of our results to these problems are straightforward, and in Section 7 we only indicate necessary alterations in the definitions.

Especially important are the transmission conditions

$$\partial_\nu^{j-1} u^{(l)}(x) = \partial_\nu^{j-1} u^{(l')}(x) \quad (j = 1, \dots, 2m) \text{ on } \Gamma_k\,, \tag{1.29}$$

which will be used in Section 8. Here ∂_ν is the derivative along the normal to Γ_k, and by $u^{(l)}$ and $u^{(l')}$ we denote the solution in G_l and $G_{l'}$, respectively. The following fact is well known: if $u^{(l)} \in W_p^{2m}(G_l)$ and $u^{(l')} \in W_p^{2m}(G_{l'})$, then these conditions are equivalent to the inclusion $u \in W_p^{2m}(G_l \cup \Gamma_k \cup G_{l'})$. They are absolutely elliptic and smooth if Γ_k is smooth. These properties of (1.29) are similar to those of the Dirichlet conditions on Γ.

1.8. Section 8 is devoted to elliptic boundary problems with indefinite weight. The equation (1.1) is replaced by

$$A(x, \mathcal{D})u(x) - \lambda\omega(x)u(x) = f(x) \text{ in } G\,. \tag{1.30}$$

We preserve the notation introduced in Section 1.7. The weight function $\omega(x)$ is real-valued (actually we could consider the case of a complex-valued $\omega(x)$) and continuous in each G_l up to the boundary but can have a jump and change sign when we cross any Γ_k. There is an extensive literature devoted to such problems; see (Faierman 1988, 1990a,b, 1995a,b) and the references therein. We assume that $\omega(x)$ is separated from zero:

$$|\omega(x)| \ge c > 0\,, \tag{1.31}$$

which is, of course, a restrictive condition. Under this assumption, we generalize the results obtained in the previous sections. In the case of minimal smoothness, it suffices to divide (1.30) by $\omega(x)$ and to impose conditions (1.29) on each Γ_k, which leads to a transmission problem. In the case of weak smoothness, this reduction is only formal (if $\omega(x)$ is not smooth), but we can generalize the proof of Theorem 5.1.

1.9. Note that some of our results, beginning with the Basic Theorem, can be somewhat strengthened. Namely, we can assume that in (1.1)–(1.2) the boundary Γ and the coefficients of $A(x, \mathcal{D})$ satisfy the minimal smoothness assumptions while the coefficients of $B_j(x, \mathcal{D})$ belong to the Hölder class $C^{2m-m_j-1,\gamma}(\Gamma)$ with a fixed γ, $0 < \gamma \le 1$. If $\gamma < 1$, then the Basic Theorem remains true for any fixed p with $1 < p < (1-\gamma)^{-1}$. If $1/2 < \gamma < 1$, then we can extend the results in Section 3 and Theorem 4.1 for the corresponding p, as well as the consequences of Theorem 4.1. We will indicate these and some other generalizations in Section 9.

Our results were reported at the International Conference "Partial Differential Equations" held in Potsdam, July 29 – August 3, 1996. We would like to thank Professor Schulze for his kind attention to our work.

2 Basic Theorem and Smoothness Assumptions

2.1. Recall that for an arbitrary domain G in $\mathbb{R}^n$ the Sobolev space $W_p^s(G)$ ($s \in \mathbb{Z}_+$, $1 < p < \infty$) can be defined as the space of distributions $f \in \mathcal{D}'(G)$ such that f and the distributional derivatives $\mathcal{D}^\alpha f$ ($|\alpha| \le s$) are functions from $L_p(G)$. The norm in $W_p^s(G)$ is defined by the formula

$$\|u\|_{s,p,G} = \Big(\int_G \sum_{|\alpha| \le s} |\mathcal{D}^\alpha u(x)|^p \, dx \Big)^{\frac{1}{p}}, \tag{2.1}$$

and $W_p^s(G)$ is a Banach space with this norm. Under very general assumptions concerning the boundary (in particular, if the boundary is of class $C^{0,1}$, i.e. Lipschitz), this definition is equivalent to the following one: $W_p^s(G)$ is the completion of the space of the restrictions of functions from $C^\infty(\mathbb{R}^n)$ to G with respect to the norm (2.1). The space $W_p^0(G)$ coincides with $L_p(G)$. The space $W_2^s(G)$ is usually denoted by $H^s(G)$.

Here and below in this subsection we refer the reader to (Adams 1975), (Grisvard 1985), (Maz'ya 1985) and (Triebel 1978).

Since $\mathcal{D}^\alpha$ is (for an arbitrary domain G) a bounded operator from $W_p^s(G)$ to $W_p^{s-|\alpha|}(G)$ if $s \ge |\alpha|$ and since the multiplication by a bounded measurable function is a bounded operator in $L_p(G)$, the operator $A(x, \mathcal{D})$ in (1.1) with bounded measurable coefficients is a bounded operator from $W_p^{2m}(G)$ to $L_p(G)$.

In the following we will use Sobolev Embedding Theorems. More precisely, we will need the Gagliardo–Nirenberg inequalities, see e.g. (Maz'ya 1985, Section 1.4) and also (Gagliardo 1959) and (Nirenberg 1959).

I. Let $ps > n$. Then $W_p^s(G) \hookrightarrow C(\overline{G})$, i.e. $W_p^s(G)$ is continuously embedded into $C(\overline{G})$. More precisely, any function $u(x) \in W_p^s(G)$ can be changed on a set of zero Lebesgue measure in such a way that it becomes a continuous function in $\overline{G}$. Moreover, we then have

$$\max |u(x)| \le C \, \|u\|_{s,p,G}^{\frac{n}{ps}} \, \|u\|_{0,p,G}^{1-\frac{n}{ps}} , \tag{2.2}$$

and there exists a constant $\gamma \in (0,1)$ such that $u(x)$ satisfies the Hölder condition

$$|u(x) - u(\tilde{x})| \le C' \, \|u\|_{s,p,G} \, |x - \tilde{x}|^{\gamma} , \tag{2.3}$$

which means a continuous embedding of $W_p^s(G)$ into $C^{0,\gamma}(\overline{G})$. Here the constants $C = C(n,p,s,G)$ and $C' = C'(n,p,s,\gamma,G)$ do not depend upon u.

II. Let

$$0 < \tau = \frac{n}{s} \left(\frac{1}{p} - \frac{1}{p_1} \right) < 1 . \tag{2.4}$$

Then $W_p^s(G) \hookrightarrow L_{p_1}(G)$, and

$$\|u\|_{0,p_1,G} \le C_1 \, \|u\|_{s,p,G}^{\tau} \, \|u\|_{0,p,G}^{1-\tau} , \tag{2.5}$$

where the constant $C_1 = C_1(n,p,p_1,s,G)$ does not depend upon u.

We will also use the following interpolation inequality.

III. Let k be an integer with $0 < k < s$, $\tau = k/s$, and put

$$|u|_{k,p,G} = \Big(\int_G \sum_{|\alpha|=k} |D^{\alpha} u(x)|^p \, dx \Big)^{\frac{1}{p}}$$

for $u \in W_p^s(G)$. Then

$$|u|_{k,p,G} \le C_2 \, \|u\|_{s,p,G}^{\tau} \, \|u\|_{0,p,G}^{1-\tau},$$

where $C_2 = C_2(n,p,s,k,G)$ does not depend upon u.

IV. The results I, II, and III hold in full force if G is replaced by $\mathbb{R}^n$.

Moreover, Theorems I–III for functions in G are obtained from the corresponding results for $\mathbb{R}^n$ using an extension operator that preserves Sobolev spaces. Such an operator for Lipschitz domains, i.e. for domains with $C^{0,1}$ boundary, was constructed by Calderón, see its version in (Stein 1970, Chapter VI, Section 3). Cf. the second definition of Sobolev spaces at the beginning of this subsection.

Now we need a short discussion of the boundary values of functions $u \in W_p^s(G)$. We assume that the boundary Γ of the domain is of class $C^{2m-1,1}$.

Let s be an integer with $1 \le s \le 2m$. Then the functions $u \in W_p^s(G)$ have boundary values $v = u|_{\Gamma}$; the space of these boundary values is denoted by $B_{p,p}^{s-\frac{1}{p}}(\Gamma)$

or $W_p^{s-\frac{1}{p}}(\Gamma)$. The norm $\|v\|_{s-\frac{1}{p},p,\Gamma}$ in this space can be defined by the formula

$$\|v\|_{s-\frac{1}{p},p,\Gamma} = \inf_{\substack{u|_\Gamma = v \\ u \in W_p^s(G)}} \|u\|_{s,p,G} \,. \tag{2.6}$$

An equivalent norm can be defined using a sufficiently fine partition of unity and the following norm in the space $W_p^{s-\frac{1}{p}}(\mathbb{R}^{n-1})$:

$$\begin{aligned} \|v(x')\|_{s-\frac{1}{p},p,\mathbb{R}^{n-1}} = \Bigg\{ & \sum_{|\alpha'|\le s-1} \int_{\mathbb{R}^{n-1}} |\mathcal{D}^{\alpha'} v(x')|^p \, dx' \\ + & \sum_{|\alpha'|=s-1} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|\mathcal{D}^{\alpha'} v(x') - \mathcal{D}^{\alpha'} v(y')|^p}{|x'-y'|^{n-2+p}} \, dx' \, dy' \Bigg\}^{\frac{1}{p}} . \end{aligned} \tag{2.7}$$

See e.g. (Grisvard 1985, Sections 1.3 and 1.5).

The following result is stated, e.g., in (Grisvard 1985, Theorem 1.4.1.1). For the convenience of the reader, we include a proof of it in the Appendix.

V. The operator of multiplication by a function from $C^{s-1,1}(\Gamma)$ is continuous in the space $W_p^{s-\frac{1}{p}}(\Gamma)$.

It follows that the operators B_j in (1.2) with coefficients $b_{j\alpha} \in C^{2m-m_j-1,1}(\Gamma)$ are bounded operators from $W_p^{2m}(G)$ to $W_p^{2m-m_j-\frac{1}{p}}(\Gamma)$.

2.2. In this subsection we formulate the Basic Theorem following essentially (Agmon 1962) and (Agranovich and Vishik 1964, Chapter I) and comment on some slight contributions contained in our formulation.

We will use the norms depending on a parameter:

$$|\!|\!| u |\!|\!|_{s,p,G} = \|u\|_{s,p,G} + |\lambda|^{\frac{s}{2m}} \|u\|_{0,p,G} \tag{2.8}$$

and

$$|\!|\!| v |\!|\!|_{s-\frac{1}{p},p,\Gamma} = \|v\|_{s-\frac{1}{p},p,\Gamma} + |\lambda|^{\frac{s-\frac{1}{p}}{2m}} \|v\|_{0,p,\Gamma} \,. \tag{2.9}$$

The norms $|\!|\!| u |\!|\!|_{s,p,\mathbb{R}^n}$, $|\!|\!| u |\!|\!|_{s,p,\mathbb{R}^n_+}$ and $|\!|\!| v |\!|\!|_{s-\frac{1}{p},p,\mathbb{R}^{n-1}}$ are defined analogously. For $p = 2$ these norms were introduced in (Agranovich and Vishik 1964). For a fixed λ, these norms are equivalent to $\|u\|_{s,p,G}$ and $\|v\|_{s-\frac{1}{p},p,\Gamma}$, respectively.

Theorem 2.1 (The Basic Theorem). *Assume that the boundary problem* (1.1)–(1.2) *is elliptic with parameter in an angle* $\mathcal{L}$ *and satisfies the minimal smoothness assumptions. Let* $1 < p < \infty$. *Then there exists a* $\lambda_0 = \lambda_0(p) > 0$ *such that for* $\lambda \in \mathcal{L}$, $|\lambda| \ge \lambda_0$ *the boundary problem has a unique solution* $u \in W_p^{2m}(G)$ *for any*

$f \in L_p(G)$ and $g_j \in W_p^{2m-m_j-\frac{1}{p}}(\Gamma)$, and the a priori estimate

$$|||u|||_{2m,p,G} \leq C\left(\|f\|_{0,p,G} + \sum_{j=1}^{m} |||g_j|||_{2m-m_j-\frac{1}{p},p,\Gamma}\right) \tag{2.10}$$

holds, where the constant C does not depend upon f, g_j and λ.

This theorem was stated in (Agmon 1962) for the case $g_j = 0$ under slightly stronger smoothness assumptions. Namely, Agmon assumed that the boundary is of class C^{2m} and the coefficients of $B_j(x, \mathcal{D})$ belong to $C^{2m-m_j}(\Gamma)$. Let us call these smoothness assumptions *almost minimal*. Agmon obtained the a priori estimate in the following way: he introduced an additional variable t and applied the a priori estimate for elliptic boundary problems without parameter in the cylindrical domain $G \times \mathbb{R}$ to functions of the form $w(x,t) = \varphi(t)e^{i\lambda t}u(x)$, where $\varphi(t)$ has finite support. Of course, the a priori estimate implies the uniqueness.

Agmon mentioned that he wanted to publish a paper devoted to existence theorems. However, as far as we know, this paper has not appeared. Agmon also mentioned that the existence can be proved using the formally adjoint problem and assuming additional smoothness.

Concerning the further evolution of this approach by means of dual estimates, we refer to (Geymonat and Grisvard 1967), where the smoothness assumptions are almost minimal for homogeneous boundary conditions and are stronger for nonhomogeneous boundary conditions.

In (Agranovich and Vishik 1964) another approach to these problems was proposed. It is a direct method similar to that used for elliptic boundary problems without parameter and is based on a localization and the consideration, at the beginning, of a boundary problem in $\mathbb{R}^n_+$ with constant coefficients and without lower order terms. For elliptic boundary problems without parameter this method leads to the Fredholm property (see e.g. (Agranovich 1965)). For boundary problems elliptic with parameter this method leads to the existence theorem (as well as to the a priori estimate). Agranovich and Vishik assumed that $p = 2$ and the boundary problem is smooth; however, as we show, this direct approach works for any $p > 1$ under the minimal smoothness assumptions (and for nonhomogeneous boundary conditions). Note that the direct approach was also used in (Roĭtberg and Sheftel' 1967), where the smoothness were not specified, and in (Roĭtberg 1991), where the smoothness assumptions were somewhat stronger than the almost minimal smoothness.

In Subsection 2.4 we sketch the complete direct proof of the Basic Theorem. There we use some technical results which are formulated in Subsection 2.3, including the variants of the Basic Theorem for operators in $\mathbb{R}^n$ and $\mathbb{R}^n_+$, respectively, with constant coefficients and without lower order terms. For the convenience of the reader, we prove these results in the Appendix.

2.3. Here we at first summarize some trace and interpolation results.

Throughout this subsection, we will use the notation $\rho = |\lambda|^{1/2m}$ and for $s \in \mathbb{N}$ denote the trace of $u \in W_p^s(G)$ or $u \in W_p^s(\mathbb{R}^n_+)$ on Γ or $\mathbb{R}^{n-1}$, respectively, by γu.

Proposition 2.2. *Let Γ be of class $C^{2m-1,1}$, s an integer with $1 \le s \le 2m$, and $1 < p < \infty$. Let $\rho \ge 1$.*
a) *For all $u \in W_p^1(G)$ we have*

$$\rho^{1-\frac{1}{p}} \|\gamma u\|_{0,p,\Gamma} \le C_1 \Big(\|u\|_{1,p,G} + \rho \|u\|_{0,p,G} \Big) , \tag{2.11}$$

where the constant C_1 does not depend upon u and ρ. The same is true if G is replaced by $\mathbb{R}^n_+$ and Γ is replaced by $\mathbb{R}^{n-1}$.
b) *For all $u \in W_p^s(G)$ and all integers k with $1 \le k \le s-1$ we have*

$$\rho^{s-k} \|u\|_{k,p,G} \le C_2 \Big(\|u\|_{s,p,G} + \rho^s \|u\|_{0,p,G} \Big) \tag{2.12}$$

with a constant C_2 not depending upon u and ρ. The same is true if G is replaced by $\mathbb{R}^n$ or $\mathbb{R}^n_+$.
c) *For all $u \in W_p^s(G)$ we have*

$$\|\gamma u\|_{s-\frac{1}{p},p,\Gamma} \le C_3 \|u\|_{s,p,G} , \tag{2.13}$$

where the constant C_3 does not depend upon u and ρ. The same is true if G is replaced by $\mathbb{R}^n_+$ and Γ is replaced by $\mathbb{R}^{n-1}$.

Denote by F' the Fourier transform with respect to the first $n-1$ variables.

Proposition 2.3. *Let s be an integer with $s \ge 1$ and $1 < p < \infty$. Then for every $v \in W_p^{s-\frac{1}{p}}(\mathbb{R}^{n-1})$ and for every $\rho \ge \rho_0 > 0$ the function $u = {F'}^{-1} \Omega F' v$ with $\Omega(\xi', x_n, \rho) = \exp(-(|\xi'| + \rho) x_n)$ is an element of $W_p^s(\mathbb{R}^n_+)$ with $\gamma u = v$, and there exists a constant C_4, not depending upon v and ρ, such that*

$$\|u\|_{s,p,\mathbb{R}^n_+} \le C_4 \, \|v\|_{s-\frac{1}{p},p,\mathbb{R}^{n-1}} . \tag{2.14}$$

Proposition 2.4. *Assume that the boundary problem (1.1)–(1.2) satisfies the minimal smoothness assumptions. Let $1 < p < \infty$. Then for any $u \in W_p^{2m}(G)$ we have*

$$\|A(x, \mathcal{D}) u\|_{0,p,G} + \sum_{j=1}^{m} \|B_j(x, \mathcal{D}) u\|_{2m-m_j-\frac{1}{p},p,\Gamma} \le C_5 \, \|u\|_{2m,p,G} \tag{2.15}$$

for $\rho > 0$, where the constant C_5 is independent of u and λ.

Now we formulate the analogues of the Basic Theorem for operators in $\mathbb{R}^n$ and $\mathbb{R}^n_+$ with constant coefficients and without lower-order terms. For a proof of these analogues one can use Michlin's multiplier theorem as in (Volevich 1965) instead of using Plancherel's theorem which is possible for $p = 2$; see the Appendix for details. We do not formulate the obvious definitions of ellipticity with parameter for these situations.

Proposition 2.5. *Let $A(\mathcal{D}) = \sum_{|\alpha|=2m} a_\alpha \mathcal{D}^\alpha$ be elliptic with parameter in $\mathcal{L}$. Let $\lambda_0 > 0$ and $1 < p < \infty$. Then for any $f \in L_p(\mathbb{R}^n)$ and any $\lambda \in \mathcal{L}$, $\lambda \neq 0$, there exists a unique solution $u \in W_p^{2m}(\mathbb{R}^n)$ of $(A(\mathcal{D}) - \lambda)u = f$, and for $\lambda \in \mathcal{L}$, $|\lambda| \geq \lambda_0$, the a priori estimate*

$$|\!|\!| u |\!|\!|_{2m,p,\mathbb{R}^n} \leq C_6 \, \|f\|_{0,p,\mathbb{R}^n} \tag{2.16}$$

holds, where C_6 does not depend upon f and λ.

Proposition 2.6. *Consider the operators $A(\mathcal{D}) = \sum_{|\alpha|=2m} a_\alpha \mathcal{D}^\alpha$ and $B_j(\mathcal{D}) = \sum_{|\beta|=m_j} b_{j\beta} \mathcal{D}^\beta$ $(j = 1, \dots, m)$. Let the boundary problem*

$$\begin{array}{rcll} (A(\mathcal{D}) - \lambda)\, u & = & f & \text{in } \mathbb{R}^n_+, \\ B_j(\mathcal{D})\, u & = & g_j & (j = 1, \dots, m) \ \text{ on } \mathbb{R}^{n-1} \end{array} \tag{2.17}$$

be elliptic with parameter in $\mathcal{L}$. Let $\lambda_0 > 0$ and $1 < p < \infty$. Then for any $f \in L_p(\mathbb{R}^n_+)$, any $g_j \in W_p^{2m-m_j-\frac{1}{p}}(\mathbb{R}^{n-1})$, and any $\lambda \in \mathcal{L}$, $\lambda \neq 0$, the boundary problem (2.17) has a unique solution $u \in W_p^{2m}(\mathbb{R}^n_+)$, and for $\lambda \in \mathcal{L}$, $|\lambda| \geq \lambda_0$, the a priori estimate

$$|\!|\!| u |\!|\!|_{2m,p,\mathbb{R}^n_+} \leq C_7 \left[|\!|\!| f |\!|\!|_{0,p,\mathbb{R}^n_+} + \sum_{j=1}^m |\!|\!| g_j |\!|\!|_{2m-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}} \right] \tag{2.18}$$

holds, where the constant C_7 does not depend upon f, g_j and λ.

For the proofs, see the Appendix.

2.4. Now we are going to sketch the proof of the Basic Theorem (Theorem 2.1).

a) First we show that for every $x_0 \in \overline{G}$ there is a neighbourhood U of x_0 and a $\lambda_0 > 0$ such that (2.10) holds if $\operatorname{supp} u \subset U \cap \overline{G}$ and $\lambda \in \mathcal{L}$, $|\lambda| \geq \lambda_0$. We separately consider the cases $x_0 \in G$ and $x_0 \in \Gamma$.

If $x_0 \in G$, then, by freezing the coefficients of the operator $A(x, \mathcal{D})$ and taking only the principal part, we can apply the a priori estimate for homogeneous operators in $\mathbb{R}^n$ with constant coefficients, see Proposition 2.5. We do not dwell on details.

Now let $x_0 \in \Gamma$. We assume U to lie inside a coordinate neighbourhood and write the boundary problem (1.1)–(1.2) in local coordinates. Again we take only the principal parts $A_0(x, \mathcal{D})$ and $B_{j0}(x, \mathcal{D})$ of the operators $A(x, \mathcal{D})$ and $B_j(x, \mathcal{D})$, respectively, and freeze the coefficients at $x = x_0$. From the a priori estimate for operators in the half-space (Proposition 2.6) we obtain

$$\begin{aligned} |\!|\!| u |\!|\!|_{2m,p,\mathbb{R}^n_+} \ \leq \ & C_7 \Big[|\!|\!| (A_0(x_0, \mathcal{D}) - \lambda) u |\!|\!|_{0,p,\mathbb{R}^n_+} \\ & + \sum_{j=1}^m |\!|\!| B_{j0}(x_0, \mathcal{D}) u |\!|\!|_{2m-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}} \Big] \end{aligned} \tag{2.19}$$

with a constant C_7 independent of u and λ. Using the continuity of the coefficients $a_\alpha(x)$ for $|\alpha| = 2m$ and (2.12), it is easily seen that in the estimation

$$\|(A(x,\mathcal{D}) - A_0(x_0,\mathcal{D}))u\|_{0,p,\mathbb{R}^n_+} \leq C_8 \|\!|u|\!\|_{2m,p,\mathbb{R}^n_+} \tag{2.20}$$

the constant C_8 can be made arbitrarily small if U is chosen small enough and $|\lambda|$ is large enough. From the proof of Theorem V of Subsection 2.1 (see the Appendix for details) we see that the same is true for

$$\|\!|(B_{j0}(x,\mathcal{D}) - B_{j0}(x_0,\mathcal{D}))u|\!\|_{2m-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}} .$$

The operator $B_j(x,\mathcal{D}) - B_{j0}(x,\mathcal{D})$ contains no terms of the highest order, and therefore (cf. Proposition 2.4) the inequality

$$\|\!|(B_j(x,\mathcal{D}) - B_{j0}(x,\mathcal{D}))u|\!\|_{2m-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}} \leq C_9 \|\!|u|\!\|_{2m-1,p,\mathbb{R}^n_+} \tag{2.21}$$

holds for some C_9 independent of u and λ. As the right-hand side of (2.21) can be estimated by a constant times $|\lambda|^{-1/2m}\ \|\!|u|\!\|_{2m,p,\mathbb{R}^n_+}$ (Proposition 2.2 b), we see that for every $\varepsilon > 0$ there exists a neighbourhood U of x_0 such that for all solutions $u \in W_p^{2m}(G)$ with $\operatorname{supp} u \subset U$ and for $|\lambda|$ large enough we have

$$\begin{aligned}\|(A(x,\mathcal{D}) - A_0(x_0,\mathcal{D}))u\|_{0,p,\mathbb{R}^n_+} + \sum_{j=1}^{m} \|\!|(B_j(x,\mathcal{D}) - B_{j0}(x_0,\mathcal{D}))u|\!\|_{2m-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}} \\ \leq \quad \varepsilon \|\!|u|\!\|_{2m,p,\mathbb{R}^n_+} .\end{aligned} \tag{2.22}$$

From (2.19) and (2.22) the a priori estimate follows for u with $\operatorname{supp} u \subset U$.

b) To obtain (2.10) for general u, we use a C^∞ partition of unity. We only note that for a function $\varphi \in C^\infty(\overline{G})$ the trace of φ on Γ belongs to $C^{2m-1,1}(\Gamma)$ and therefore the multiplication by the trace of φ is continuous in $W_p^{2m-m_j-\frac{1}{p}}(\Gamma)$ by Theorem V of Subsection 2.1.

c) As from the a priori estimate (2.10) the uniqueness follows, it remains to prove the existence of the solution.

We fix $\varepsilon > 0$ and choose a finite covering $\overline{G} \subset \bigcup_{k=1}^N U_k$ of $\overline{G}$ and $x_k \in U_k$ such that for every $x \in U_k$ the estimate (2.22) holds with x_0 replaced by x_k. Consider a C^∞ partition of unity $\sum_{k=1}^N \varphi_k(x) \equiv 1$ subordinated to this covering and C^∞ functions ψ_k with $\psi_k(x) \equiv 1$ in a neighbourhood of $\operatorname{supp} \varphi_k$ and $\operatorname{supp} \psi_k \subset U_k$.

Let

$$\mathcal{A} = \mathcal{A}(x) : W_p^{2m}(G) \to L_p(G) \times \prod_{j=1}^{m} W_p^{2m-m_j-\frac{1}{p}}(\Gamma)$$

be defined by

$$\mathcal{A}u = \Big((A(x,\mathcal{D}) - \lambda)u, B_1(x,\mathcal{D})u, \ldots, B_m(x,\mathcal{D})u\Big) . \tag{2.23}$$

We write

$$\mathcal{A}u = \sum_{k=1}^{N} \varphi_k \mathcal{A}(\psi_k u) = \sum_{k=1}^{N} \varphi_k \mathcal{A}_k(\psi_k u)\,, \tag{2.24}$$

where $\mathcal{A}_k$ is the corresponding operator written in local coordinates and acting in $\mathbb{R}^n$ or $\mathbb{R}^n_+$.

Freezing the coefficients of $\mathcal{A}_k(x)$ at x_k and taking only the principal parts, we obtain the operator $\mathcal{A}_{k0}(x_k)$ whose inverse $\mathcal{R}_k$ exists by the results on operators in $\mathbb{R}^n$ or $\mathbb{R}^n_+$, respectively. Now we set

$$\mathcal{R}(f, g_1, \dots, g_m) = \sum_{k=1}^{N} \psi_k \mathcal{R}_k(\varphi_k f, \varphi_k g_1, \dots, \varphi_k g_m)\,. \tag{2.25}$$

Making use of the a priori estimate for $\mathcal{R}_k$ and (2.22), it can be seen that

$$\mathcal{A}\mathcal{R}(f, g_1, \dots, g_m) = (f, g_1, \dots, g_m) + \mathcal{T}(f, g_1, \dots, g_m)\,, \tag{2.26}$$

where for $|\lambda|$ large enough the norm of $\mathcal{T}$ as an operator in $L_p(G) \times \prod W_p^{2m-m_j-\frac{1}{p}}(\Gamma)$, where we use norms of the form (2.9) on the boundary, is not greater than a constant times ε. With ε small enough we see that $\mathcal{A}$ is invertible and therefore the boundary problem (1.1)–(1.2) has a solution. □

2.5. In (Faierman 1990b) it was proved that the following conditions are sufficient for the existence of the boundary problem (1.7)–(1.8) formally adjoint to (1.1)–(1.2) also satisfying the minimal smoothness assumptions:

$$\Gamma \in C^{2m,1},\ a_\alpha \in C^{|\alpha|-1,1}(\overline{G}),\ b_{j\alpha} \in C^{2m-1-m_j,1}(\Gamma) \cap C^{|\alpha|,1}(\Gamma)\,. \tag{2.27}$$

Here the notation $a_\alpha \in C^{|\alpha|-1,1}(\overline{G})$ means in the case $|\alpha| = 0$ that the function a_α is measurable and essentially bounded in $\overline{G}$.

2.6. One can find in the literature many works devoted to elliptic boundary problems in nonsmooth domains and/or with nonsmooth coefficients. However, they are either concerned with problems in a variational form or with non-variational problems of a very special kind.

3 Simplest Spectral Consequences

3.1. Here we assume that the assumptions of the Basic Theorem are satisfied. Consider the operator $A_{B,p}$ in $L_p(G)$, $1 < p < \infty$, that acts as $A(x, \mathcal{D})$ and has the domain

$$\mathcal{D}(A_{B,p}) = \{u \in W_p^{2m}(G) : B_j(x, \mathcal{D})\, u(x) = 0 \quad (j = 1, \dots, m) \text{ on } \Gamma\}\,. \tag{3.1}$$

Obviously $\mathcal{D}(A_{B,p})$ is dense in $L_p(G)$. From the a priori estimate (2.10) it follows that $A_{B,p}$ is closed. Its resolvent set is nonvoid, and the resolvent is compact since $W_p^{2m}(G)$ is compactly embedded in $L_p(G)$. Thus, $A_{B,p}$ has a discrete spectrum: it consists of isolated eigenvalues of finite multiplicity, with possible accumulation only at infinity. From the Sobolev Embedding Theorem II it follows that the generalized eigenfunctions belong to $\bigcap_{1<p<\infty} W_p^{2m}(G)$ and that the spectrum $\sigma(A_{B,p})$ does not depend upon p (see (Agmon 1962)). Because of this we will mainly consider $A_B = A_{B,2}$ in $L_2(G)$.

In the following we will summarize some spectral properties of the operator A_B which are consequences of the Basic Theorem. We will refer to these properties in Sections 7 and 8, where more general situations are considered.

Recall that the subset X_1 in a Banach space X is called *complete in* X if the set of all finite linear combinations of elements of X_1 is dense in X.

Theorem 3.1 (Agmon 1962). *Assume that the boundary problem* (1.1)–(1.2) *is elliptic with parameter along some rays* $\mathcal{L}^{(k)}$ $(k = 1, \dots, N)$ *and the angles between any two adjacent rays are not greater than* $2m\pi/n$. *Then* A_B *has an infinite number of eigenvalues and the set of all generalized eigenfunctions of* A_B *is complete in* $L_2(G)$ (*and in* $L_p(G)$).

In the proof of this theorem and Theorem 3.2 below an extension operator of functions in G to functions on a torus containing G is used that preserves Sobolev spaces (see (Agmon 1962, Appendix I) or (Beals 1967)). As we mentioned above, such an operator exists for Lipschitz domains.

Let $\{u_j\}_{j\geq 1}$ be the system of generalized eigenfunctions of A_B composed of bases in each generalized eigenspace in such a way that u_j belongs to the generalized eigenspace that corresponds to the eigenvalue λ_j. Under the assumptions of Theorem 3.1, the u_j's form an infinite complete system. Besides, this system is minimal, and hence there exists a system $\{w_k\}_1^\infty$ biorthogonal to $\{u_j\}_1^\infty$:

$$(u_j, w_k)_G = \delta_{jk} \,. \tag{3.2}$$

The system $\{w_k\}_1^\infty$ consists of the generalized eigenfunctions of the operator $(A_B)^*$ adjoint to A_B. To each function $f \in L_2(G)$ we can associate its formal Fourier series with respect to $\{u_j\}_1^\infty$:

$$f \sim \sum_{j=1}^{\infty} c_j u_j \,, \text{ where } c_j = (f, w_j)_G \,. \tag{3.3}$$

Theorem 3.2. *Under the assumptions of Theorem* 3.1, *the series in* (3.3) *admits the summability to* f *by the Abel–Lidskiĭ method of order* $\dfrac{n}{2m} + \varepsilon$ *if* $\varepsilon > 0$ *is sufficiently small.*

This method was defined in (Lidskiĭ 1962) and was called there *Abel's method.* In the simplest case, when all the generalized eigenfunctions are actually eigenfunctions and all the eigenvalues lie, say, in the angle $\{\lambda : |\arg\lambda| \le \theta\}$ with some $\theta \in (0,\pi)$, the definition of this method of order γ $(0 < \gamma\theta < \pi/2)$ is as follows: there exists an increasing sequence $\{\nu_k\}_1^\infty$ of nonnegative integers (independent of f) with $\nu_1 = 0$ such that the series

$$\sum_{k=1}^{\infty} \sum_{j=\nu_k+1}^{\nu_{k+1}} e^{-\lambda_j^\gamma t} c_j u_j \tag{3.4}$$

converges in $L_2(G)$ for $t > 0$ and its sum $f(t)$ tends to f in $L_2(G)$ as $t \searrow 0$. Here $\lambda_j^\gamma = |\lambda_j|^\gamma e^{i\gamma\arg\lambda_j}$. The definition for the general case can be found in (Lidskiĭ 1962) or in the survey (Agranovich 1990). Theorem 3.2 follows from a variant of Lidskiĭ's theorem indicated in (Agranovich 1977).

We now set

$$\mathcal{L}(\theta) = \overline{\{\lambda : \arg\lambda = \theta\}}\,. \tag{3.5}$$

Theorem 3.3 (Agmon 1962)**.** *Let the boundary problem* (1.1)–(1.2) *be elliptic with parameter along the rays* $\mathcal{L}(\theta_1)$ *and* $\mathcal{L}(\theta_2)$, *where* $0 < \theta_2 - \theta_1 < \min\{2m\pi/n, 2\pi\}$, *and not elliptic with parameter along some ray* $\mathcal{L}(\theta_0)$, $\theta_1 < \theta_0 < \theta_2$. *Then the angle* $\{\lambda : \theta_1 < \arg\lambda < \theta_2\}$ *contains infinitely many eigenvalues of* A_B.

This result was called in (Agmon 1962) the statement on the *angular distribution of eigenvalues.*

Theorem 3.4 (Agmon 1962)**.** *In particular, let the boundary problem* (1.1)–(1.2) *be elliptic with parameter along all the rays* $\mathcal{L}(\theta)$ *with* $\theta_0 - \varepsilon < \theta < \theta_0$ *and* $\theta_0 < \theta < \theta_0 + \varepsilon$ *for some* $\varepsilon > 0$ *and not elliptic with parameter along* $\mathcal{L}(\theta_0)$. *Then any angular neighbourhood of* $\mathcal{L}(\theta_0)$ *contains infinitely many eigenvalues of* A_B.

Such a ray $\mathcal{L}(\theta_0)$ is called the *ray of condensation of eigenvalues* in (Agmon 1962).

3.2. Remark 3.5. From the Basic Theorem it can be easily seen that under the condition of weak smoothness the Banach space adjoint $(A_{B,p})^*$ of the operator $A_{B,p}$ corresponds to the formally adjoint boundary problem.

Indeed, let v be in the domain of $(A_{B,p})^*$. Then, by definition, v and $h = (A_{B,p})^* v$ are elements of $L_{p'}(G)$ and

$$(Au, v)_G = (u, h)_G \quad \text{for all } u \in D(A_{B,p})\,,$$

where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Assume for simplicity that the boundary problem (1.1)–(1.2) and its formally adjoint (1.7)–(1.8) are both uniquely solvable for $\lambda = 0$. Let $w \in W_{p'}^{2m}(G)$ be the solution of the formally adjoint problem with homogeneous boundary conditions and $\tilde{f}$ in (1.7) replaced by h. Then we have, by the definition of the formally adjoint problem,

$$(Au, v - w)_G = 0$$

for all $u \in C^{2m}(G)$ satisfying the boundary conditions $B_j u = 0$ $(j = 1, \ldots, m)$, and therefore for all $u \in D(A_{B,p})$. But the range of $A_{B,p}$ is the whole space $L_p(G)$, and we obtain $v = w$ in $L_{p'}(G)$ which shows that v lies in the domain of the operator corresponding to the formally adjoint problem. On the other hand, it is clear that every function in this domain lies in $D((A_{B,p})^*)$ and that the two operators coincide for these functions.

We will use this result only for $p = 2$. Note that under the conditions indicated in Subsection 2.5, this result was proved for $p = 2$ in (Faierman 1990b).

4 Trace Asymptotics of Powers of the Resolvent

4.1. Let T be a compact operator in a Hilbert space H. Recall that the s-numbers $s_j(T)$ $(j = 1, 2, \dots)$ are the nonzero eigenvalues of the nonnegative operator $(TT^*)^{1/2}$ (or, which is the same, of $(T^*T)^{1/2}$) arranged so that

$$s_1(T) \geq s_2(T) \geq \dots \tag{4.1}$$

and each eigenvalue is repeated according to its multiplicity. As usual, we set

$$|T|_q = \Big(\sum_{j=1}^{\infty} [s_j(T)]^q \Big)^{\frac{1}{q}} \tag{4.2}$$

for $0 < q < \infty$. All operators T with $|T|_q < \infty$ form the Neumann–Schatten space $\mathcal{S}_q$; for $q \geq 1$ it is a Banach space with the norm $|\cdot|_q$. Obviously $\mathcal{S}_{q_1} \subset \mathcal{S}_{q_2}$ if $q_1 < q_2$. The operators from $\mathcal{S}_2$ are the Hilbert–Schmidt operators. If $H = L_2(G)$, where G is a domain in $\mathbb{R}^n$, then $\mathcal{S}_2$ coincides with the class of integral operators

$$Tf(x) = \int_G K(x, y) f(y)\, dy \tag{4.3}$$

with kernels $K(x, y) \in L_2(G \times G)$. The operators from $\mathcal{S}_1$ are the trace class operators; they have the trace

$$\operatorname{tr} T = \sum_{j=1}^{\infty} \lambda_j(T)\,, \tag{4.4}$$

where the series is absolutely convergent; here each eigenvalue of T is repeated according to its multiplicity, and

$$|\operatorname{tr} T| \leq |T|_1\,. \tag{4.5}$$

If $H = L_2(G)$ and T is a trace class operator with kernel $K(x, y)$ in (4.3) continuous in $\overline{G} \times \overline{G}$, then

$$\operatorname{tr} T = \int_G K(x, x)\, dx\,. \tag{4.6}$$

We also note that if B is a bounded operator in H, then

$$s_j(TB) \leq \|B\|\, s_j(T)\,, \tag{4.7}$$

so that

$$|TB|_q \leq \|B\| \, |T|_q \, . \tag{4.8}$$

Finally, we note that if $T = T_1 \dots T_s$, where $T_j \in \mathcal{S}_q$ and $s \leq q$, then

$$|T|_{\frac{q}{s}} \leq |T_1|_q \cdot \ldots \cdot |T_s|_q \, . \tag{4.9}$$

See (Gohberg and Kreĭn 1965, Chapters II and III) and also the survey (Agranovich 1990) and references therein.

4.2. Assume that the assumptions of the Basic Theorem 2.1 are satisfied. Then for $\lambda \in \mathcal{L}$ with large $|\lambda|$ the resolvent $R(\lambda)$ is a bounded operator from $L_2(G)$ to $W_2^{2m}(G)$. It follows that

$$s_j(R(\lambda)) \leq C(\lambda) \, j^{-\frac{2m}{n}} \, . \tag{4.10}$$

This estimate and the estimate (5.4) below are valid for Lipschitz domains and follow from similar estimates on a torus, cf. (Agmon 1962), (Beals 1967) or (Triebel 1978, Section 4.10.1).

Let q be a natural number such that $2mq > n$. Then $R(\lambda) \in \mathcal{S}_q$ in $L_2(G)$, and hence $R(\lambda)^q \in \mathcal{S}_1$. If the boundary problem is smooth, then we have the asymptotic formula (1.13) (even with the remainder $O(|\lambda|^{\frac{n-1}{2m}-q})$).

The facts mentioned in these two subsections will also be used in Section 5.

4.3. Now we intend to prove Theorem 4.1, the main theorem of this section.

Theorem 4.1. *Assume that* $|\arg a_0(x,\xi)| \leq \theta_1 < \theta \leq \pi$ *and that the minimal smoothness assumptions are satisfied for* G *and* $A(x,\mathcal{D})$*. Let* $2mq > n$*, and let* q *be even. Then for the Dirichlet boundary problem* (1.1), (1.15) *the formula* (1.13) *is true in*

$$\mathcal{L}_\theta = \{\lambda : |\arg \lambda| \geq \theta\} \cup \{0\} \tag{4.11}$$

with c_q *indicated in* (1.14).

Proof. At first we need

Lemma 4.2. *There exists a bounded domain* $\tilde{G}$ *with* C^∞ *boundary* $\tilde{\Gamma}$ *such that* G *and* $\tilde{G}$ *are connected by a diffeomorphism of class* $C^{2m-1,1}$.

This is almost obvious. Nevertheless, we give a complete proof.

Proof of the Lemma. We first consider a small part of the boundary. Using a rotation and a shift of the coordinate system in $\mathbb{R}^n$, we assume that this part is the graph of a function

$$x_n = \varphi(x') \in C^{2m-1,1}(\mathcal{O}'_r) \, , \tag{4.12}$$

where $\mathcal{O}'_r$ is the ball $\{x' \in \mathbb{R}^{n-1} : |x'| < r\}$. Moreover, we assume that for some $s > 0$ the sets

$$\{x \in \mathbb{R}^n : x' \in \mathcal{O}'_r, \; \varphi(x') < x_n < \varphi(x') + s\} \tag{4.13}$$

and

$$\{x \in \mathbb{R}^n : x' \in \mathcal{O}'_r,\ \varphi(x') - s < x_n < \varphi(x')\} \tag{4.14}$$

lie in G and outside $\overline{G}$, respectively. Let $\theta(x')$ be a C_0^∞ function, i.e. a C^∞ function with compact support, such that

$$\int \theta(x')\, dx' = 1\,. \tag{4.15}$$

We set $\theta_h(x') = h^{1-n}\theta(x'/h)$ for small h and choose a function $\alpha(x') \in C^\infty(\overline{\mathcal{O}'_r})$ such that

$$\alpha(x') \geq 0,\ \alpha(x') = 1 \text{ in } \mathcal{O}'_{r/3}, \text{ and } \alpha(x') = 0 \text{ outside } \mathcal{O}'_{2r/3}\,. \tag{4.16}$$

Now we set

$$\varphi_h(x') = \alpha(x') \int \theta_h(x' - y')\varphi(y')\, dy' + [1 - \alpha(x')]\varphi(x')\,. \tag{4.17}$$

Obviously $\varphi_h(x')$ is a C^∞ function in $\mathcal{O}'_{r/3}$, it uniformly tends to $\varphi(x')$ as $h \to 0$, and

$$\varphi_h(x') = \varphi(x') \ \text{ for } |x'| \in \left(\frac{2r}{3}, r\right). \tag{4.18}$$

We fix an $\varepsilon \in (0, s/3)$ (it will be chosen later) and assume h to be so small that

$$|\varphi_h(x') - \varphi(x')| \leq \varepsilon \ \text{ for } x' \in \mathcal{O}'_r\,. \tag{4.19}$$

Now we fix $h = h(\varepsilon)$. The local repairing of the boundary consists in the replacement of $\varphi(x')$ by $\varphi_h(x')$. The new domain is obtained from the original one by the replacement of the set (4.13) by the set

$$\{x \in \mathbb{R}^n : x' \in \mathcal{O}'_r,\ \varphi_h(x') < x_n < \varphi(x') + s\}\,. \tag{4.20}$$

To define a $C^{2m-1,1}$-diffeomorphism of the original domain onto the new one, it suffices to define a $C^{2m-1,1}$-diffeomorphism $x \mapsto \tilde{x}$ of the set (4.13) onto the set (4.20) with $\tilde{x}$ instead of x in (4.20) in such a way that the points x near those with $|x'| = r$ and/or $x_n = \varphi(x') + s$ do not move. For this we fix a non-increasing function $\beta(t) \in C^\infty([0, s])$ such that

$$\beta(0) = 1 \ \text{ and } \beta(t) = 0 \text{ near } s$$

and set

$$\left.\begin{array}{rcl} \tilde{x}' & = & x'\,, \\ \tilde{x}_n & = & x_n + [\varphi_h(x') - \varphi(x')]\,\beta(x_n - \varphi(x'))\,. \end{array}\right\} \tag{4.21}$$

Obviously $\tilde{x}_n = x_n$ for $|x'| > 2r/3$ (see (4.18)) and near $x_n = \varphi(x') + s$, and we only need to make the function $\tilde{x}_n(x_n)$ strongly monotonic. Since

$$\frac{\partial \tilde{x}_n}{\partial x_n} = 1 + [\varphi_h(x') - \varphi(x')]\,\beta'(x_n - \varphi(x'))\,,$$

it suffices to choose ε in (4.19) so that

$$\varepsilon \max|\beta'(t)| < 1\,.$$

Thus, we constructed the desired diffeomorphism. Repeating this procedure sufficiently many times, we repair Γ completely and obtain the desired domain $\tilde{G}$. $\square$

Now we will use the notation $x \mapsto \tilde{x}$ for the diffeomorphism of G onto $\tilde{G}$. For a function $v(\tilde{x})$ on $\tilde{G}$, we set

$$u(x) = v(\tilde{x}(x)) = (Tv)(x)\,.$$

Our diffeomorphism transforms the equation (1.1) into the equation

$$T^{-1}A(x,\mathcal{D})\,Tv - \lambda v = T^{-1}f\,. \tag{4.22}$$

Here $\tilde{A}(\tilde{x},\mathcal{D}_{\tilde{x}}) = T^{-1}A(x,\mathcal{D}_x)\,T$ is a partial differential operator of order $2m$ with the principal symbol

$$\tilde{a}_0(\tilde{x},\tilde{\xi}) = a_0\Big(x(\tilde{x}),\Big(\frac{\partial x}{\partial \tilde{x}}\Big)^{\prime -1}\tilde{\xi}\Big)\,, \tag{4.23}$$

where $\frac{\partial x}{\partial \tilde{x}}$ is the Jacobian matrix and the dash denotes the transposed matrix (see e.g. (Agranovich 1990, Section 1.4)). Obviously the set of values of the principal symbol remains the same. The minimal smoothness conditions for $\tilde{A}$ follow from those for A. Finally, our transform preserves the Dirichlet boundary conditions. Indeed, they can be interpreted as the inclusion

$$u \in W_2^{2m}(G) \cap \overset{\circ}{W}{}_2^m(G)\,, \tag{4.24}$$

where $\overset{\circ}{W}{}_2^m(G)$ is the closure of $C_0^\infty(G)$ in $W_2^m(G)$, and our transform preserves these Sobolev spaces. Thus, our diffeomorphism defines the similarity transform

$$A_D \mapsto \tilde{A}_D = T^{-1}A_D\,T \tag{4.25}$$

that preserves all the assumptions of the theorem. Since the spectrum is preserved under a similarity transform, we now assume, without loss of generality, that G is smooth, i.e. $\Gamma \in C^\infty$.

4.4. The next step is to construct an operator $A^{(h)}(x,\mathcal{D})$ with C^∞ top order coefficients $a_\alpha^{(h)}(x)$ ($|\alpha| = 2m$) that uniformly converge to $a_\alpha(x)$ as $h \to 0$. The procedure is routine: we prolongate $a_\alpha(x)$ ($|\alpha| = 2m$) to continuous functions in $\mathbb{R}^n$, take a C_0^∞ function $\theta(x)$ with property similar to (4.15), $\int \theta(x)\,dx = 1$, and set $\theta_h(x) = h^{-n}\theta(x/h)$ ($h > 0$). Finally, we set

$$a_\alpha^{(h)}(x) = \int \theta_h(x-y)a_\alpha(y)\,dy \quad (|\alpha| = 2m,\ x \in \overline{G})\,. \tag{4.26}$$

Obviously $a_\alpha^{(h)}(x) \in C^\infty(\overline{G})$ and $a_\alpha^{(h)}(x) \to a_\alpha(x)$ $(h \to 0)$ uniformly in $\overline{G}$.

For $|\alpha| < 2m$ we take $a_\alpha^{(h)}(x) \equiv 0$. Thus we set

$$A^{(h)}(x, \mathcal{D}) = \sum_{|\alpha|=2m} a_\alpha^{(h)}(x)\mathcal{D}^\alpha \tag{4.27}$$

and consider the Dirichlet problem for this operator. It is elliptic with parameter in $\mathcal{L}$ if h is sufficiently small, $0 < h < h_0$, and smooth. We can assume that the resolvents

$$R(\lambda) = R_{A_D}(\lambda) \quad\text{and}\quad R^{(h)}(\lambda) = R_{A_D^{(h)}}(\lambda) \tag{4.28}$$

with any $h < h_0$ exist for $\lambda \in \mathcal{L}$ with $|\lambda| > r_0$, where r_0 is independent of h, and we now consider the difference of the powers of these resolvents.

4.5. We have

$$R(\lambda)^q - R^{(h)}(\lambda)^q = \sum_{\substack{q_1+q_2=q-1\\ q_1,\, q_2 \in \mathbb{Z}_+}} R(\lambda)^{q_1}[R(\lambda) - R^{(h)}(\lambda)]R^{(h)}(\lambda)^{q_2} \tag{4.29}$$

for any positive integer q. Here

$$R(\lambda) - R^{(h)}(\lambda) = R^{(h)}(\lambda)[A_D^{(h)} - A_D]R(\lambda)\,. \tag{4.30}$$

This formula is correct since

$$\mathcal{D}(A_D) = \mathcal{D}(A_D^{(h)})\,. \tag{4.31}$$

Proposition 4.3. *For any $\varepsilon > 0$ there exist positive $h_1 = h_1(\varepsilon)$ and $r_1 = r_1(\varepsilon)$ such that*

$$\|\,[A_D - A_D^{(h)}]R(\lambda)\| < \varepsilon \tag{4.32}$$

for any $h \in (0, h_1)$ and $\lambda \in \mathcal{L}$ with $|\lambda| \geq r_1$.

Proof. For $f \in L_2(G)$ set

$$u = R(\lambda)f \quad\text{and}\quad w = [A_D - A_D^{(h)}]u\,.$$

According to the Basic Theorem,

$$\|u\|_{2m,2,G} + |\lambda|\,\|u\|_{0,2,G} \leq C\,\|f\|_{0,2,G}\,,$$

where C does not depend upon f and λ. Further, $w = w_1 + w_2$, where

$$w_1(x) = \sum_{|\alpha|=2m} [a_\alpha(x) - a_\alpha^{(h)}(x)]\mathcal{D}^\alpha u(x) \quad\text{and}\quad w_2(x) = \sum_{|\alpha|<2m} a_\alpha(x)\mathcal{D}^\alpha u(x)\,.$$

Here

$$\|w_1\|_{0,2,G} \le \eta(h)\,\|u\|_{2m,2,G}\,,$$

where $\eta(h) \to 0$ as $h \to 0$, and

$$\|w_2\|_{0,2,G} \le C'\,\|u\|_{2m-1,2,G}\,.$$

In view of the interpolation inequality (see (2.12)),

$$|\lambda|^{\frac{1}{2m}}\,\|u\|_{2m-1,2,G} \le C''\left(\|u\|_{2m,2,G} + |\lambda|\,\|u\|_{0,2,G}\right),$$

where C'' does not depend upon λ and u. Thus

$$\|\,[A_D - A_D^{(h)}]R(\lambda)\| \le C\,\eta(h) + C\,C'\,C''\,|\lambda|^{-\frac{1}{2m}}\,, \tag{4.33}$$

and from here we obtain the desired result. □

Combining Proposition 4.3 with (4.8) and (4.30), we obtain

Corollary 4.4. *For any $\varepsilon > 0$ there exist positive $h_1 = h_1(\varepsilon)$ and $r_1 = r_1(\varepsilon)$ such that*

$$|R(\lambda) - R^{(h)}(\lambda)|_q < \varepsilon\,|R^{(h)}(\lambda)|_q \tag{4.34}$$

and

$$|R(\lambda)|_q \le (1+\varepsilon)\,|R^{(h)}(\lambda)|_q \tag{4.35}$$

for $h \in (0, h_1)$ and $\lambda \in \mathcal{L}$ with $|\lambda| \ge r_1$.

4.6. Now, assuming that $2mq > n$, we deduce from (4.5), (4.9), (4.29), (4.34) and (4.35) that

$$|\mathrm{tr}\,R(\lambda)^q - \mathrm{tr}\,R^{(h)}(\lambda)^q| \le C_1\varepsilon\,|R^{(h)}(\lambda)|_q^q \tag{4.36}$$

for the same h and λ, where C_1 does not depend upon h and λ. We repeat that since $A_D^{(h)}$ corresponds to a smooth boundary problem, we have a formula of the form (1.13) for $\mathrm{tr}\,R^{(h)}(\lambda)^q$ with $c_q^{(h)}$ instead of c_q, where

$$c_q^{(h)} = \frac{1}{(2\pi)^n}\int_G dx \int_{\mathbb{R}^n} \frac{d\xi}{[a_0^{(h)}(x,\xi)+1]^q}$$

and $a_0^{(h)}$ is the (principal) symbol of $A^{(h)}$; obviously $c_q^{(h)} \to c_q$ as $h \to 0$. It remains to prove that

$$|R^{(h)}(\lambda)|_q^q \le C\,|\lambda|^{\frac{n}{2m}-q}\,. \tag{4.37}$$

Up to now q was any natural number such that $2mq > n$. Now we assume that q is even and set

$$q = 2k\,. \tag{4.38}$$

Instead of (4.37) we will indicate an asymptotic formula for the left-hand side of (4.37). We have

$$|R^{(h)}(\lambda)|_q^q = \sum_{j=1}^{\infty} s_j^q(R^{(h)}(\lambda)) = \sum_{j=1}^{\infty} \lambda_j^k [R^{(h)}(\lambda) R^{(h)}(\lambda)^*] \,. \tag{4.39}$$

Here the operator $R^{(h)}(\lambda)^*$ corresponds to the Dirichlet problem for the equation

$$A^{(h)}(x, \mathcal{D})^* v(x) - \overline{\lambda} v(x) = \tilde{f}(x) \ \text{ in } G \,. \tag{4.40}$$

Since the operator $R^{(h)}(\lambda) R^{(h)}(\lambda)^*$ is selfadjoint, from (4.39) it follows that

$$|R^{(h)}(\lambda)|_q^q = \sum_{j=1}^{\infty} \lambda_j [(R^{(h)}(\lambda) R^{(h)}(\lambda)^*)^k] \,. \tag{4.41}$$

The operator in the square brackets in (4.41) corresponds to a composition of q Dirichlet boundary problems. This composition is the boundary problem

$$(A^{(h)*} - \overline{\lambda})(A^{(h)} - \lambda) \dots (A^{(h)*} - \overline{\lambda})(A^{(h)} - \lambda) u = f \text{ in } G \tag{4.42}$$

(q factors) with the boundary conditions

$$\begin{gathered}
\partial_\nu^{j-1} u = 0 \,, \\
\partial_\nu^{j-1} (A^{(h)} - \lambda) u = 0 \,, \\
\partial_\nu^{j-1} (A^{(h)*} - \overline{\lambda})(A^{(h)} - \lambda) u = 0 \,, \\
\vdots \\
\partial_\nu^{j-1} (A^{(h)} - \lambda) \dots (A^{(h)*} - \overline{\lambda})(A^{(h)} - \lambda) u = 0
\end{gathered} \tag{4.43}$$

on Γ ($j = 1, \dots, m$; q rows). To simplify the notation, we temporarily assume that $\mathcal{L} = \overline{\mathbb{R}}_-$ and hence $\lambda = \overline{\lambda}$. Then this boundary problem is a particular case of boundary problems polynomially depending on a parameter and having the form

$$A(x, \mathcal{D}, \lambda) u \ \equiv \sum_{|\alpha| + \gamma l \le 2\mu} \lambda^l a_{\alpha l}(x) \mathcal{D}^\alpha u = f \quad \text{in } G \,, \tag{4.44}$$

$$B_j(x, \mathcal{D}, \lambda) u \ \equiv \sum_{|\beta| + \gamma l \le \mu_j} \lambda^l b_{j\beta l}(x) \mathcal{D}^\beta u = 0 \quad (j = 1, \dots, \mu) \text{ on } \Gamma \,. \tag{4.45}$$

In (4.44) the operator has even order 2μ; in (4.45) we assume for simplicity that $\mu_j < 2\mu$. The parameter λ has the weight γ with respect to the differentiation. In (4.42)–(4.43) $\gamma = 2m$ and $\mu = mq$.

The definition of ellipticity with parameter of the boundary problem (4.44)–(4.45) is a natural generalization of Definition 1.1. Let $\mathcal{L}$ be a closed angle in the complex plane with vertex at the origin. Denote by $a_0(x,\xi,\lambda)$ and $b_{j0}(x,\xi,\lambda)$ the principal symbols of $A(x,\mathcal{D},\lambda)$ and $B_j(x,\mathcal{D},\lambda)$:

$$a_0(x,\xi,\lambda)=\sum_{|\alpha|+\gamma l=2\mu}\lambda^l a_{\alpha l}(x)\xi^\alpha,\quad b_{j0}(x,\xi,\lambda)=\sum_{|\beta|+\gamma l=\mu_j}\lambda^l b_{j\beta l}(x)\xi^\beta\,. \tag{4.46}$$

The boundary problem (4.44)–(4.45) is called *elliptic with parameter in* $\mathcal{L}$ if

(1) $a_0(x,\xi,\lambda)\neq 0$ for $x\in\overline{G}$, $0\neq(\xi,\lambda)\in\mathbb{R}^n\times\mathcal{L}$;

(2) for any $x_0\in\Gamma$ in the coordinate system associated with this point the boundary problem

$$\begin{gathered}a_0(x_0,\xi',\mathcal{D}_n,\lambda)v(t)=0\quad(t=x_n>0)\,,\\ b_{j0}(x_0,\xi',\mathcal{D}_n,\lambda)v(t)=0\quad(j=1,\dots,\mu)\ \text{ at } t=0\,,\\ v(t)\to 0\ \text{ as } t\to\infty\end{gathered} \tag{4.47}$$

has only the trivial solution.

For such boundary problems a corresponding variant of the Basic Theorem is true; cf. (Agmon and Nirenberg 1963), (Agranovich and Vishik 1964) and further papers indicated in Section 2. In particular, for $\lambda\in\mathcal{L}$ with large modulus the boundary problem (4.44)–(4.45) with any $f\in L_2(G)$ has a unique solution $u\in W_2^{2\mu}(G)$, and the a priori estimate

$$\|u\|_{2\mu,2,G}+|\lambda|^{\frac{2\mu}{\gamma}}\,\|u\|_{0,2,G}\le C\,\|f\|_{0,2,G} \tag{4.48}$$

is true. Let $R(\lambda)f=u$ and $2\mu>n$; then $R(\lambda)$ is a trace class operator. Assume that the boundary problem is smooth; then an asymptotic formula for $\operatorname{tr}R(\lambda)$ is known, see e.g. (Grubb 1986, Section 3.4) and (Boĭmatov and Kostjuchenko 1991). For us it is convenient to write it in the form

$$\operatorname{tr}R(\lambda)=\tilde{c}\,(-\lambda)^{\frac{n}{\gamma}-\frac{2\mu}{\gamma}}+o\left(|\lambda|^{\frac{n}{\gamma}-\frac{2\mu}{\gamma}}\right)\quad(\lambda\to\infty\text{ in }\mathcal{L})\,, \tag{4.49}$$

where

$$\tilde{c}=\frac{1}{(2\pi)^n}\int_G dx\int_{\mathbb{R}^n}a_0^{-1}(x,\eta,-1)\,d\eta\,, \tag{4.50}$$

so that the main term in (4.49) is defined by the principal symbol, as in (1.13).[2]

In our case,

$$a_0(x,\xi,\lambda)=|a_0^{(h)}(x,\xi)-\lambda|^q$$

[2] In (Grubb 1986) the main term is written in the form

$$\frac{1}{(2\pi)^n}\int_G dx\int_{\mathbb{R}^n}a_0^{-1}(x,\xi,\lambda)\,d\xi\,.$$

If $\mathcal{L}=\mathbb{R}_-$, we make the substitution $\xi=|\lambda|^{1/\gamma}\eta$. If $\mathcal{L}$ is larger, we can additionally use the analytic continuation. Cf. (Agranovich 1990, Section 5.7).

and Condition (1) is obviously satisfied. To check Condition (2), it is convenient to linearize the corresponding boundary problem (4.47). We set

$$
\begin{aligned}
v_1(t) &= v(t)\,, \\
v_2(t) &= \left(a_0^{(h)}(x_0, \xi', \mathcal{D}_n) - \lambda\right) v_1(t)\,, \\
v_3(t) &= \left(\overline{a}_0^{(h)}(x_0, \xi', \mathcal{D}_n) - \lambda\right) v_2(t)\,, \\
&\ \vdots \\
v_q(t) &= \left(a_0^{(h)}(x_0, \xi', \mathcal{D}_n) - \lambda\right) v_{q-1}(t)\,.
\end{aligned}
$$

Then we successively obtain that $v_q(t) \equiv 0, \ldots, v_1(t) \equiv 0$ as solutions of the boundary problem

$$
\begin{aligned}
\left(\overline{a}_0^{(h)}(x_0, \xi', \mathcal{D}_n) - \lambda\right) w(t) = 0 \quad (t > 0)\,, \\
\mathcal{D}_n^{j-1} w(t) = 0 \quad (j = 1, \ldots, m) \text{ at } t = 0\,, \\
w(t) \to 0 \quad (t \to +\infty)
\end{aligned}
$$

or of the similar problem with $a_0^{(h)}$ instead of $\overline{a}_0^{(h)}$. Our problem is smooth, and using the result (4.49), we obtain

$$
|R^{(h)}(\lambda)|_q^q = \tilde{c}\, |\lambda|^{\frac{n}{2m} - q} + o\Big(|\lambda|^{\frac{n}{2m} - q}\Big) \tag{4.51}
$$

as $\lambda \to -\infty$ along $\mathbb{R}_-$, where

$$
\tilde{c} = \frac{1}{(2\pi)^n} \int_G dx \int_{\mathbb{R}^n} |a_0^{(h)}(x, \xi) + 1|^{-q}\, d\xi\,. \tag{4.52}
$$

To obtain a similar result if $\mathcal{L} \neq \mathbb{R}_-$, we consider the boundary problem (4.42)–(4.43) as depending polynomially on $|\lambda|$ with coefficients continuously depending upon $\arg \lambda$. The final result is the formula (4.51) with

$$
\tilde{c} = \tilde{c}(h, \arg \lambda) = \frac{1}{(2\pi)^n} \int_G dx \int_{\mathbb{R}^n} \left| a_0^{(h)}(x, \xi) - \frac{\lambda}{|\lambda|} \right|^{-q} d\xi\,. \tag{4.53}
$$

This coefficient is a continuous and hence a bounded function. As to the remainder estimate in (4.51), it is at least uniform in $\arg \lambda$ for a fixed h. This result together with (4.36) is sufficient to finish the proof of the theorem. $\square$

Remark 4.5. In the proof of (4.51) it is actually unessential that the boundary conditions are those of the Dirichlet problem. We see that the following proposition is true:

Proposition 4.6. *Let the boundary problem* (1.1)–(1.2) *be smooth and elliptic with parameter in* $\mathcal{L}$. *Let* q *be an even number such that* $2mq > n$. *Then*

$$
|R_{A_B}(\lambda)|_q^q = \tilde{c}(\arg \lambda)\, |\lambda|^{\frac{n}{2m} - q} + o\Big(|\lambda|^{\frac{n}{2m} - q}\Big) \quad (\lambda \to \infty \textit{ in } \mathcal{L}) \tag{4.54}
$$

uniformly in $\arg\lambda$*, where*

$$\tilde{c}(\arg\lambda) = \frac{1}{(2\pi)^n}\int_G dx \int_{\mathbb{R}^n} \Big| a_0(x,\xi) - \frac{\lambda}{|\lambda|}\Big|^{-q} d\xi\,. \tag{4.55}$$

Actually it is possible to improve the remainder estimate; moreover, it is possible to indicate an asymptotic expansion for the left-hand side. We do not dwell on this.

Remark 4.7. There is another way to obtain results of the form (4.54). Namely, we can linearize the boundary problem with respect to λ. In particular, in (4.42)–(4.43) we can set

$$u_1 = u,\ u_2 = (A^{(h)} - \lambda)u,\ \ldots, u_q = (A^{(h)} - \lambda)\ldots(A^{(h)*} - \overline{\lambda})(A^{(h)} - \lambda)u\,,$$

and then we obtain a matrix Dirichlet problem

$$\begin{aligned} \mathcal{A}\mathcal{U} - \mathcal{I}(\lambda)\mathcal{U} &= \mathcal{F} \quad \text{in } G\,,\\ \partial_\nu^{j-1}\mathcal{U} &= 0 \quad (j = 1,\ldots,m) \text{ on } \Gamma\,, \end{aligned}$$

where $\mathcal{U} = (u_1,\ldots,u_q)'$, $\mathcal{F} = (0,\ldots,0,f)'$,

$$\mathcal{A} = \begin{pmatrix} A^{(h)} & -1 & 0 & 0 & \ldots & 0 & 0\\ 0 & A^{(h)*} & -1 & 0 & \ldots & 0 & 0\\ \vdots & & & \ldots & & & \vdots\\ 0 & 0 & 0 & 0 & \ldots & A^{(h)} & -1\\ 0 & 0 & 0 & 0 & \ldots & 0 & A^{(h)*} \end{pmatrix},$$

and $\mathcal{I}(\lambda) = \operatorname{diag}(\lambda, \overline{\lambda}, \ldots, \lambda, \overline{\lambda})$. The matrix $\mathcal{A} = (\mathcal{A}_{jk})$ has a Douglis–Nirenberg structure: ord $\mathcal{A}_{jk} \le s_j + t_k$, where $t_k = 2m(q-k+1)$ and $s_j = -2m(q-j)$. Note that $t_j + s_j = 2m$ is independent of j. This matrix boundary problem is elliptic with parameter $|\lambda|$ if $\lambda \in \mathcal{L}$, in the sense of the corresponding generalization of Definition 1.1. In the case of a usual elliptic system (with s_k and t_k independent of k) the structure of the resolvent $\mathcal{R}(\lambda) = (\mathcal{R}_{jk}(\lambda))$ was investigated in (Seeley 1969). As was pointed out in (Agranovich 1992), this analysis can be carried over to Douglis–Nirenberg systems. In particular, we can find the structure of $\mathcal{R}_{1q}(\lambda)$, it again is defined by the principal symbol of the system, and this again yields (4.54).

The linearization with respect to the parameter can be convenient in more complicated situations, including matrix boundary problems that depend upon λ polynomially.

5 Trace Asymptotics of Powers of the Resolvent for Weakly Smooth Problems

Our main goal in this section is to prove Theorem 5.1:

Theorem 5.1. *Let the boundary problem* (1.1)–(1.2) *be weakly smooth and elliptic with parameter in* $\mathcal{L}_\theta$ *(see* (4.11)*). Let* $2mq > n$ *and* q *be even. Then formula* (1.13) *holds with* c_q *defined in* (1.14)*.*

As was indicated in the Introduction, we will obtain stronger results, the uniform estimate and the pointwise asymptotics of the kernel uniform on compact subsets in G (see (5.12) and (5.17)).

5.1. We begin with some preparations. Let T be a bounded operator in $L_2(G)$. If X and Y are normed linear submanifolds in $L_2(G)$ and if the restriction of T to X is a continuous operator from X to Y, then we denote its norm by $\|T\|_{X\to Y}$.

Lemma 5.2. *Let* T *be a bounded operator in* $L_2(G)$*. Assume that its range is contained in* $W_p^l(G)$*, where* $l \in \mathbb{N}$*,* $p \geq 2$*, and* $lp > n$*. Then* T *is an integral operator* (4.3)*, where the kernel* $K(x,y)$ *has the following properties:* $K(x_0,y) \in L_2(G)$ *for any fixed* $x_0 \in \overline{G}$*;* $K(x,\cdot)$ *is a continuous function of* x *with values in* $L_2(G)$*, and*

$$\left(\int_G |K(x,y)|^2\,dy\right)^{\frac{1}{2}} \leq c\,\|T\|^{\tau}_{L_2\to W_p^l}\,\|T\|^{1-\tau}_{L_2\to L_p}\,, \tag{5.1}$$

where $\tau = n/lp$ *and the constant* c *depends only upon* l*,* n*,* p *and* G*. In particular,* T *is a Hilbert–Schmidt operator.*

This lemma and its proof are the direct generalizations of those for $p = 2$ in (Agmon 1965a). The boundedness $T : L_2(G) \to W_p^l(G)$ follows from the continuity of the embedding $W_p^l(G) \hookrightarrow L_2(G)$ and the closed graph theorem. Since $lp > n$, the space $W_p^l(G)$ is continuously embedded in $C(\overline{G})$, and moreover, the Hölder inequality (2.3) holds for $u = Tf$, $f \in L_2(G)$. The mapping $L_2(G) \ni f \mapsto (Tf)(x)$ is a bounded linear functional; this yields the representation (4.3) with $K(x,y) \in L_2(G)$ for any fixed x. Taking $f_x = f_x(y) = \overline{K(x,y)}$ with a fixed x, we obtain (5.1) from (2.2). Taking $f = f_{x\tilde{x}} = f_x - f_{\tilde{x}}$ and using (2.3), we justify the continuity of $K(x,y)$ as a function of x with values in $L_2(G)$. □

Here for simplicity we restrict ourselves to bounded G and $l \in \mathbb{N}$. Note that the assumption that $\Gamma \in C^{0,1}$ is sufficient for the validity of the Lemma.

Lemma 5.3. *Let* $T = T_1T_2^*$*, where the operators* T_1 *and* T_2 *in* $L_2(G)$ *satisfy the assumptions of Lemma* 5.2*. Then* T *is a trace class operator, and in its integral representation* (4.3) *the kernel* $K(x,y)$ *is continuous in* $\overline{G}\times\overline{G}$*. In addition,*

$$|K(x,y)| \leq \tilde{c}\,\|T_1\|^{\tau}_{L_2\to W_p^l}\,\|T_1\|^{1-\tau}_{L_2\to L_p}\,\|T_2\|^{\tau}_{L_2\to W_p^l}\,\|T_2\|^{1-\tau}_{L_2\to L_p}\,, \tag{5.2}$$

where $\tau = n/lp$ *and the constant* $\tilde{c}$ *depends only upon* l*,* n*,* p*, and* G*.*

Proof. The operator T belongs to the trace class as the product of two Hilbert–Schmidt operators. Let $K_1(x,y)$ and $K_2(x,y)$ be the kernels of T_1 and T_2, respectively.

Then T is the integral operator with the kernel

$$K(x,y)=\int_G K_1(x,z)\overline{K_2(y,z)}\,dz\,. \tag{5.3}$$

Using the Schwartz inequality, it is easy to check that $K(x,y)$ is continuous in x in $\overline{G}$ uniformly with respect to y and continuous in y in $\overline{G}$ uniformly with respect to x. Thus $K(x,y)$ is continuous in $\overline{G}\times\overline{G}$. The estimate (5.2) is also obtained by means of the Schwartz inequality and (5.1). □

Now we consider an operator T in $L_2(G)$ with the following property: for any p, $2\le p<\infty$, its restriction to $L_p(G)$ is a bounded operator from this space to $W_p^l(G)$ with some fixed $l\in\mathbb{N}$. Obviously T is a compact operator in $L_2(G)$, and

$$s_j(T)\le C\,j^{-\frac{l}{n}}\,. \tag{5.4}$$

Let $k\in\mathbb{N}$ be such that

$$kl>\frac{n}{2}\,. \tag{5.5}$$

From (4.9) it follows that T^k is a Hilbert–Schmidt operator. Let us check that T^k satisfies the assumptions of Lemma 5.2. For this we choose the numbers $p_1,\dots,p_k$ such that

$$2=p_1<\ldots<p_k,\quad \tau_i=\frac{n}{l}\left(\frac{1}{p_i}-\frac{1}{p_{i+1}}\right)<1\quad (i=1,\dots,k-1) \tag{5.6}$$

and $p_k>n/l$. It is easy to check that this is possible (noting that the sum of the differences $\frac{1}{p_i}-\frac{1}{p_{i+1}}$ is equal to $\frac{1}{2}-\frac{1}{p_k}$). Now we consider T^k as the product of the operators $T=T_i: L_{p_i}(G)\to W_{p_i}^l(G)$ and the Sobolev embedding operators $S_i: W_{p_i}^l(G)\hookrightarrow L_{p_{i+1}}(G)$:

$$L_2(G)\xrightarrow{T_1}W_2^l(G)\xrightarrow{S_1}L_{p_2}(G)\xrightarrow{T_2}W_{p_2}^l(G)\xrightarrow{S_2}L_{p_3}(G)\to\ldots\xrightarrow{T_k}W_{p_k}^l(G)\,.$$

We see that T^k is a bounded operator from $L_2(G)$ to $W_p^l(G)$ with $p=p_k$. Moreover, using the estimate (2.5) and an easy induction with respect to k, we obtain

$$\|T^k\|_{L_2\to W_{p_k}^l}\le c_k\,\|T\|_{L_{p_k}\to W_{p_k}^l}\prod_{i=1}^{k-1}\left(\|T\|_{L_{p_i}\to W_{p_i}^l}^{\tau_i}\;\|T\|_{L_{p_i}\to L_{p_i}}^{1-\tau_i}\right) \tag{5.7}$$

and

$$\|T^k\|_{L_2\to L_{p_k}}\le c_k\,\|T\|_{L_{p_k}\to L_{p_k}}\prod_{i=1}^{k-1}\left(\|T\|_{L_{p_i}\to W_{p_i}^l}^{\tau_i}\;\|T\|_{L_{p_i}\to L_{p_i}}^{1-\tau_i}\right)\,, \tag{5.8}$$

where c_k is a constant depending only upon l, n, p_i, and G.

Now we apply Lemma 5.2 to T^k and obtain the following result.

Proposition 5.4. *Let T be a bounded operator in $L_2(G)$, and let its restriction to $L_p(G)$ with any $p \in [2, \infty)$ be a bounded operator from this space to $W_p^l(G)$ with some fixed $l \in \mathbb{N}$. Let $k \in \mathbb{N}$ be such that condition* (5.5) *holds, and let $p_1, \dots, p_k$ be chosen so that conditions* (5.6) *and $p_k > n/l$ hold. Then T^k is a Hilbert–Schmidt operator, and in its integral representation of the form* (4.3) *the kernel $K(x, y)$ belongs to $L_2(G)$ for any fixed $x = x_0$, $K(x, \cdot\,)$ is a continuous function on $\overline{G}$ with values in $L_2(G)$, and*

$$\left(\int_G |K(x,y)|^2 \, dy \right)^{1/2} \le c' \prod_{i=1}^{k} \left(\|T\|^{\tau_i}_{L_{p_i} \to W^l_{p_i}} \, \|T\|^{1-\tau_i}_{L_{p_i} \to L_{p_i}} \right) , \tag{5.9}$$

where the constant c' depends only upon l, n, p_i, and G, and $\tau_k = n/lp_k$.

Next we apply Lemma 5.3 to $T^{2k} = T^k[(T^*)^k]^*$ and obtain

Proposition 5.5. *Assume that not only T but also T^* satisfies the assumptions of Proposition* 5.4*. Then T^{2k} is a trace class operator, and in its integral representation* (4.3) *the kernel $K(x, y)$ is continuous on $\overline{G} \times \overline{G}$ and satisfies the estimate*

$$|K(x,y)| \le c'' \prod_{i=1}^{k} \left(\|T\|^{\tau_i}_{L_{p_i} \to W^l_{p_i}} \, \|T\|^{1-\tau_i}_{L_{p_i} \to L_{p_i}} \, \|T^*\|^{\tau_i}_{L_{p_i} \to W^l_{p_i}} \, \|T^*\|^{1-\tau_j}_{L_{p_i} \to L_{p_i}} \right) \tag{5.10}$$

in the notation of Proposition 5.4*, where the constant c'' depends only upon l, n, p_i, and G.*

Remark 5.6. Note that the conclusions of Proposition 5.4 remain true also for operators of the form $T^k V$, where V is a bounded operator in $L_2(G)$, and the conclusions of Proposition 5.5 remain true for operators of the form $[T_1^k V] T_2^k$ if the operators T_1 and T_2^* satisfy the assumptions of Proposition 5.4.

This remark will be used in Section 8.

5.2. Now we can return to the consideration of $R(\lambda)^q$. We apply Proposition 5.5 to $T = R(\lambda)$ with $l = 2m$ and $k = q/2$. Since $\|R(\lambda)\|_{L_p \to L_p} \le C_p |\lambda|^{-1}$ and the same is true for $R(\lambda)^*$, and since

$$-\sum_{i=1}^{k} (1 - \tau_i) = \frac{n}{4m} - \frac{q}{2} \, ,$$

we obtain

Proposition 5.7. *Let the assumptions of Theorem* 5.1 *be satisfied. Then*

$$R(\lambda)^q f(x) = \int_G K(x, y, \lambda) f(y) \, dy \, , \tag{5.11}$$

where for $\lambda \in \mathcal{L}_\theta$ with sufficiently large modulus the kernel $K(x,y,\lambda)$ is a function continuous in $\overline{G} \times \overline{G}$ for any fixed λ, and uniformly

$$|K(x,y,\lambda)| \le C\,|\lambda|^{\frac{n}{2m}-q}\,, \tag{5.12}$$

where C does not depend upon x, y and λ.

5.3. We are now going to use these results to obtain pointwise asymptotics for $K(x,x,\lambda)$, see the relation (5.17) below. As was indicated in the Introduction, these pointwise asymptotics, together with (5.12), lead directly to the proof of Theorem 5.1. However, before proceeding with this endeavour, let us briefly indicate the problems involved by comparing our work with that of Agmon (Agmon 1965a). In his work Agmon considers at first the case where $2m > n$, $q = 1$, and $\mathcal{L}_\theta$ is just a ray; and using methods different from ours, he is able to establish Proposition 5.7. In order to determine the asymptotic behaviour of $K(x,x,\lambda)$ at a point $x_0 \in G$, Agmon considers the resolvent $S(\lambda)$ of the operator induced in $L_2(\mathbb{R}^n)$ by the constant coefficient differential operator

$$A_0(x_0,\mathcal{D}) = \sum_{|\alpha|=2m} a_\alpha(x_0)\mathcal{D}^\alpha, \tag{5.13}$$

and shows that $S(\lambda)$ is an integral operator; denote its kernel by $F(x,y,\lambda)$. Then employing the arguments he used in establishing Proposition 5.7, he proves that for any ε, $0 < \varepsilon < \operatorname{dist}\{x_0,\Gamma\}$, there is a neighbourhood U_ε of x_0 such that for $\lambda \in \mathcal{L}_\theta$ with $|\lambda|$ sufficiently large and for $\zeta(x) \in C^\infty$ with compact support lying in U_ε and $\zeta(x_0) = 1$, the operator $\zeta\big(R(\lambda)-S(\lambda)\big)\zeta$ is an integral operator in $L_2(G)$ whose kernel $\zeta(x)\big(K(x,y,\lambda)-F(x,y,\lambda)\big)\zeta(y)$ is continuous in $G\times G$ for any fixed λ and is bounded in modulus by $C\varepsilon^{\frac{n}{2m}}|\lambda|^{\frac{n}{2m}-1}$, where the constant C does not depend upon x, y, λ, and ε, and where we have also used ζ to denote the operator of multiplication by ζ. The desired asymptotic formula for $K(x_0,x_0,\lambda)$ follows immediately from this last result.

Returning again to the problem under our consideration, we could try, like Agmon, to determine the asymptotic behaviour of $K(x_0,x_0,\lambda)$ by a consideration of the kernel of the operator $\zeta\big(R(\lambda)^q - S(\lambda)^q\big)\zeta$. Unfortunately, since we are dealing with products of operators, the arguments of Agmon cannot be used directly to obtain an estimate for the modulus of this kernel. However we shall show that by introducing a sequence of C_0^∞ functions $\{\zeta_j(x)\}_0^k$, where $\zeta_0(x) = \zeta(x)$ and $\zeta_{j+1}(x)\zeta_j(x) = \zeta_j(x)$, and by considering products of operators involving terms of the form $\zeta_{j-1}R(\lambda)\zeta_j$, $\zeta_{j-1}S(\lambda)\zeta_j$, $\zeta_{j-1}R^*(\overline{\lambda})\zeta_j$, and $\zeta_{j-1}S^*(\overline{\lambda})\zeta_j$, the arguments we used in proving Proposition 5.7 apply in full force in allowing us to establish the required estimate. As in the problem treated by Agmon, this estimate gives the desired asymptotic formula for $K(x_0,x_0,\lambda)$.

Let $A_0(x_0,\mathcal{D})$ be defined as above (see (5.13)), let $x_0 \in F$, where F is a compact subset of G, and for $1 < p < \infty$ let $\mathcal{A}_p$ denote the realization of $A_0(x_0,\mathcal{D})$ as an operator in $L_p(\mathbb{R}^n)$, with $\mathcal{D}(\mathcal{A}_p) = W_p^{2m}(\mathbb{R}^n)$. Then it follows from Proposition 2.5

that if $\lambda \in \mathcal{L}_\theta$ and $|\lambda| \geq a > 0$, then λ belongs to the resolvent set of $\mathcal{A}_p$ and

$$\|S_p(\lambda)f\|_{2m,p,\mathbb{R}^n} \leq c_p \|f\|_{0,p,\mathbb{R}^n} \tag{5.14}$$

for $f \in L_p(\mathbb{R}^n)$, where

$$S_p(\lambda) = (\mathcal{A}_p - \lambda I)^{-1} \tag{5.15}$$

and the constant c_p does not depend upon x_0, f, and λ. We shall suppose from now on that $\lambda \in \mathcal{L}_\theta$ with $|\lambda| \geq a$ and henceforth write $S(\lambda)$ for $S_2(\lambda)$; note for later use that $S_p(\lambda)f = S(\lambda)f$ for $f \in L_p(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$. Note also that $S(\lambda)^q$ is an integral operator in $L_2(\mathbb{R}^n)$ with the kernel

$$F^{(q)}(x,y,\lambda) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{i(x-y)\cdot\xi}}{[a_0(x_0,\xi) - \lambda]^q}\, d\xi\,, \tag{5.16}$$

which is continuous in $\mathbb{R}^n \times \mathbb{R}^n$ for any fixed λ and is bounded in modulus by the expression on the right side of (5.12), where now the constant C does not depend upon x, y, x_0, and λ.

Let $\varphi(x)$, $\psi(x)$, $\chi(x)$, and $\{\varphi_j(x)\}_1^k$ be functions in $C^\infty(\mathbb{R}^n)$ such that for $0 \leq j \leq k+2$, $0 \leq \varphi_j(x) \leq 1$, $\varphi_j(x) = 1$ in a neighbourhood of $x = 0$, $\operatorname{supp}\varphi_j$ is contained in the ball $|x| < 1$, and $\varphi_j(x)\varphi_{j+1}(x) = \varphi_j(x)$ for $j = 0,\ldots,k+1$, where we have written $\varphi_0(x)$ for $\varphi(x)$, $\varphi_{k+1}(x)$ for $\psi(x)$, and $\varphi_{k+2}(x)$ for $\chi(x)$. For $0 < \delta <$ dist $\{x_0, \Gamma\}$ let $\varphi^\delta(x) = \varphi\big(\delta^{-1}(x - x_0)\big)$, define $\psi^\delta(x)$, $\chi^\delta(x)$, and $\varphi_j^\delta(x)$, $j = 1,\ldots,k$, analogously, and put $R_{j\delta}(\lambda) = \varphi_{j-1}^\delta R(\lambda)\varphi_j^\delta$, $R_{j\delta}^\dagger(\overline{\lambda}) = \varphi_{j-1}^\delta R^*(\overline{\lambda})\varphi_j^\delta$ for $j = 1,\ldots,k$, where $\varphi_0^\delta = \varphi^\delta$ and where we also use the φ_j^δ to denote the operators of multiplication by the φ_j^δ. Now let us observe that if we write T for $R(\lambda)$ or $R^*(\overline{\lambda})$, then $\varphi_{j-1}^\delta T = \varphi_{j-1}^\delta \varphi_j^\delta T = \varphi_{j-1}^\delta T \varphi_j^\delta - \varphi_{j-1}^\delta [T, \varphi_j^\delta]$, where $[T,\varphi_j^\delta]$ denotes the commutator $T\varphi_j^\delta - \varphi_j^\delta T$ for $1 \leq j \leq k$. Hence if we apply this observation to the expression

$$\varphi^\delta R(\lambda)^q \varphi^\delta = \big(\varphi^\delta R(\lambda)^k\big)\big(\varphi^\delta R^*(\overline{\lambda})^k\big)^*$$

and pass through every term, except the last, in each of the products $R(\lambda)^k$ and $R^*(\overline{\lambda})^k$ twice by a φ_j^δ with corresponding j, proceeding from left to right in a successive manner, then it is not difficult to verify that

$$P_\delta(\lambda) = \varphi^\delta R(\lambda)^q \varphi^\delta - \left(\prod_{j=1}^k R_{j\delta}(\lambda)\right)\left(\prod_{j=1}^k R_{j\delta}^\dagger(\overline{\lambda})\right)^*$$

is a finite sum of operators of the form

$$-\left(\prod_{j=1}^{r-1} R_{j\delta}(\lambda)\right)\left(\varphi_{r-1}^\delta [R(\lambda), \varphi_r^\delta]\eta_r^\delta\right) R(\lambda)^{k-r}\left(\varphi^\delta R^*(\overline{\lambda})^k\right)^*$$

and

$$-\left(\prod_{j=1}^{k} R_{j\delta}(\lambda)\right)\left(\left(\prod_{j=1}^{r-1} R_{j\delta}^{\dagger}(\overline{\lambda})\right)\left(\varphi_{r-1}^{\delta}[R^*(\overline{\lambda}),\varphi_r^{\delta}]\eta_r^{\delta}\right)R^*(\overline{\lambda})^{k-r}\right)^*,$$

where $1 \le r \le k$, $\eta_r^\delta = I$ or φ_r^δ, and $\prod_{j=1}^{r-1}\cdots = 1$ if $r=1$. Bearing in mind that $\left[R(\lambda),\varphi_r^\delta\right] = R(\lambda)\left[\varphi_r^\delta, A(x,\mathcal{D})\right]R(\lambda)$, let us note that (for brevity we omit showing the Sobolev embedding operator S_{k+1-r} of Subsection 5.1)

$$\begin{aligned}&\|\varphi_{r-1}^{\delta}[R(\lambda),\varphi_r^{\delta}]\eta_r^{\delta}\|_{L_{p_{k+1-r}}\to L_{p_{k+2-r}}}\\ &\qquad\le \|R(\lambda)\|_{L_{p_{k+1-r}}\to L_{p_{k+2-r}}}\,\|\,[\varphi_r^{\delta},A(x,\mathcal{D})]R(\lambda)\,\|_{L_{p_{k+1-r}}\to L_{p_{k+1-r}}},\end{aligned}$$

and that (see Subsection 2.1 and Theorem 2.1)

$$\|\,[\varphi_r^{\delta},A(x,\mathcal{D})]R(\lambda)\,\|_{L_{p_{k+1-r}}\to L_{p_{k+1-r}}} \le c_1\sum_{j=0}^{2m-1}\delta^{j-2m}|\lambda|^{\frac{j}{2m}-1} \le c_2\delta^{-1}|\lambda|^{-\frac{1}{2m}}$$

for $|\lambda| \ge \max\{\delta^{-2m},\lambda_0\}$, where $p_{k+1}=\infty$ and the constants c_j do not depend upon x_0, λ, and δ. Hence since an analogous result also holds for $\varphi_{r-1}^{\delta}[R^*(\overline{\lambda}),\varphi_r^{\delta}]\eta_r^{\delta}$, we can now argue as we did above in establishing Proposition 5.7 to show that for $|\lambda|$ sufficiently large (and in particular we require that $|\lambda| \ge \delta^{-2m}$), $P_\delta(\lambda)$ is an integral operator in $L_2(G)$ with a kernel which is continuous in $\overline{G}\times\overline{G}$ for any fixed λ and which is bounded in modulus by $C\delta^{-1}|\lambda|^{\frac{n}{2m}-q-\frac{1}{2m}}$, where the constant C does not depend upon x, y, λ, x_0, and δ.

Next let

$$Q_\delta(\lambda) = \varphi^\delta S(\lambda)^q\varphi^\delta - \left(\prod_{j=1}^{k} S_{j\delta}(\lambda)\right)\left(\prod_{j=1}^{k} S_{j\delta}^{\dagger}(\overline{\lambda})\right)^*,$$

where $S_{j\delta}(\lambda) = \varphi_{j-1}^\delta S(\lambda)\varphi_j^\delta$, $S_{j\delta}^\dagger(\overline{\lambda}) = \varphi_{j-1}^\delta S^*(\overline{\lambda})\varphi_j^\delta$, $S^*(\lambda)$ denotes the resolvent of the operator induced in $L_2(\mathbb{R}^n)$ by the formal adjoint of $A_0(x_0,\mathcal{D})$, and here φ_{j-1}^δ and φ_j^δ are to be interpreted as $r_G\circ\varphi_{j-1}^\delta$ and $i_G\circ\varphi_j^\delta$, respectively, where φ_{j-1}^δ is used as a multiplication operator over $\mathbb{R}^n$ and r_G denotes the natural restriction: $\mathbb{R}^n\to G$, while φ_j^δ is used as a multiplication operator over G and i_G denotes the natural extension: $G\to\mathbb{R}^n$ (i.e., the extension by 0 outside G). Then by appealing to the results of Subsection 2.1 and to (5.14), we can argue with $Q_\delta(\lambda)$ as we argued with $P_\delta(\lambda)$ to show that all the assertions made above concerning $P_\delta(\lambda)$ are also valid for $Q_\delta(\lambda)$. Furthermore, observe that

$$\begin{aligned}R_{j\delta}(\lambda) &= S_{j\delta}(\lambda) + \varphi_{j-1}^\delta R(\lambda)\psi^\delta\big(A_0(x_0,\mathcal{D}) - A(x,\mathcal{D})\big)\chi^\delta S(\lambda)\varphi_j^\delta\\ &\quad+\varphi_{j-1}^\delta R(\lambda)\chi^\delta\left[\psi^\delta, A(x,\mathcal{D})\right]\chi^\delta S(\lambda)\varphi_j^\delta,\end{aligned}$$

and that an analogous result holds for $R^*_{j,\delta}(\overline{\lambda})$. Now we can argue as we did with $P_\delta(\lambda)$ above to show that for $|\lambda|$ sufficiently large (and in particular, we require that $|\lambda| \ge \delta^{-2m}$), $\varphi^\delta R(\lambda)^q \varphi^\delta - \varphi^\delta S(\lambda)^q \varphi^\delta$ is an integral operator in $L_2(G)$ with a kernel which is continuous in $\overline{G} \times \overline{G}$ for any fixed λ and which is bounded in modulus by $C\big(\Phi(\delta)+\delta^{-1}|\lambda|^{-\frac{1}{2m}}\big)|\lambda|^{\frac{n}{2m}-q}$, where the constant C does not depend upon x, y, λ, x_0, and δ, and $\Phi(\delta) \to 0$ as $\delta \to 0$.

Finally, if we observe from (5.16) that $F^{(q)}(x_0, x_0, \lambda) = c_q(x_0)(-\lambda)^{\frac{n}{2m}-q}$ (see (1.14)), then as a consequence of the foregoing results we see that for $|\lambda|$ sufficiently large, the estimate

$$|K(x_0, x_0, \lambda) - c_q(x_0)(-\lambda)^{\frac{n}{2m}-q}| \le C\left(\Phi(\delta) + \delta^{-1}|\lambda|^{-\frac{1}{2m}}\right)|\lambda|^{\frac{n}{2m}-q}$$

holds, where the constant C does not depend upon λ, x_0, and δ. Hence, since x_0 was an arbitrary point of F, we conclude from this last result that

$$K(x, x, \lambda) = c_q(x)(-\lambda)^{\frac{n}{2m}-q} + o\left(|\lambda|^{\frac{n}{2m}-q}\right) \quad \text{as} \quad |\lambda| \to \infty \tag{5.17}$$

uniformly in λ and x for $\lambda \in \mathcal{L}_\theta$ and x belonging to any compact subset of G. It is a simple matter to deduce from (5.12) and (5.17) that

$$\int_G K(x, x, \lambda)dx = c_q(-\lambda)^{\frac{n}{2m}-q} + o\left(|\lambda|^{\frac{n}{2m}-q}\right) \quad \text{as } |\lambda| \to \infty$$

uniformly in $\mathcal{L}_\theta$, and thus the proof of Theorem 5.1 is complete. □

6 Rough and Precise Asymptotics of Eigenvalues

6.1. In this section *we assume that we have a formula of the form* (1.13) *for* $\operatorname{tr} R(\lambda)^q$ *with an even q such that $2mq > n$ in an angle $\mathcal{L}$, or angles, of ellipticity with parameter.* Because of this the conclusions will hold under the assumptions of Theorem 4.1 or Theorem 5.1.

Besides d (see formula (1.27)), we define the number

$$\Delta = \frac{1}{(2\pi)^n n}\int_G dx \int_{|\xi|=1} |a_0(x,\xi)|^{-\frac{n}{2m}}\, dS_\xi\,. \tag{6.1}$$

We also introduce the counting function

$$N_\lambda(t) = \max\{j : |\lambda_j| \le t\} \tag{6.2}$$

for the *moduli* of the eigenvalues $\lambda_j(A_B)$ and the counting function

$$N_s(t) = \max\{j : s_j^{-1} \le t\}\,. \tag{6.3}$$

Here we assume for simplicity, and without loss of generality, that the operator A_B is invertible, and define s_j as the s-numbers of A_B^{-1}.

It is well known that if the boundary problem is elliptic, normal and smooth, then

$$N_s(t) = \Delta \cdot t^{\frac{n}{2m}} + o\left(t^{\frac{n}{2m}}\right) \quad \text{as} \quad t \to \infty\,. \tag{6.4}$$

This follows from the fact that if A_B is the operator corresponding to a smooth elliptic normal boundary problem, then $A_B(A_B)^*$ corresponds to a smooth selfadjoint elliptic normal boundary problem.

Proposition 6.1. *Let the boundary problem be elliptic with parameter in some angle $\mathcal{L}$, and assume that it satisfies the minimal smoothness assumptions. Then the formula* (6.4) *is true.*

This statement is close to some results in (Beals 1967), see Theorems 5.2 (about an inequality) and 5.3 (in the case $2m > n$) in that paper. See also the paper (Beals 1970) devoted to selfadjoint boundary problems.

We give a complete proof of Proposition 6.1 at the end of the Appendix.

It follows (see (Agranovich and Markus 1989)) that

$$\varlimsup_{t\to\infty} N_\lambda(t)\, t^{-\frac{n}{2m}} \leq \Delta e \tag{6.5}$$

and

$$\varliminf_{t\to\infty} N_\lambda(t)\, t^{-\frac{n}{2m}} \leq \Delta\,. \tag{6.6}$$

Moreover,

$$\varliminf_{t\to\infty} N_\lambda(t)\, t^{-\frac{n}{2m}} > 0 \tag{6.7}$$

if $d \neq 0$, so that the following theorem is true:

Theorem 6.2. *Let $d \neq 0$. Then*

$$N_\lambda(t) \asymp t^{\frac{n}{2m}} \quad (t \to \infty)\,. \tag{6.8}$$

This means that the ratio $N_\lambda(t)/t^{n/2m}$ lies between positive constants for large t. The relations (6.8) and (1.28) are equivalent. Recall that $d \neq 0$ if

$$|\arg a_0(x,\xi)| \leq \frac{\pi m}{n} \tag{6.9}$$

(see (Agranovich and Markus 1989)), and that the condition of ellipticity with parameter along all the rays outside the angle $\{\lambda : |\arg \lambda| < \pi m/n\}$ is sufficient for the completeness of the generalized eigenfunctions (see Section 3).

Theorem 6.3. *Let the boundary problem* (1.1)–(1.2) *be elliptic with parameter along any ray except $\mathbb{R}_+$. Then*

$$N_\lambda(t) = d \cdot t^{\frac{n}{2m}} + o\left(t^{\frac{n}{2m}}\right) \quad \textit{as} \quad t \to \infty\,, \tag{6.10}$$

where

$$d = \Delta = \frac{1}{(2\pi)^n} \int_G dx \int_{a_0(x,\xi)<1} d\xi \tag{6.11}$$

(*since* $a_0(x,\xi) > 0$).

We sketch one of the possible proofs (it was used e.g. in (Agranovich 1987)). Set $\lambda_j = \lambda_j' + i\lambda_j''$, where λ_j' and λ_j'' are real. From our assumptions it follows that

$$\lambda_j' \to +\infty \quad \text{and} \quad \frac{\lambda_j''}{\lambda_j'} \to 0 \quad (j \to \infty)\,. \tag{6.12}$$

Moreover, from (6.4) it follows that $s_j = O(j^{-2m/n})$, and hence $|\lambda_j|^{-1} = O(j^{-2m/n})$ (see (Gohberg and Kreĭn 1965, Chapter II, §3)). Therefore

$$\lambda_j' \geq C_1\, j^{\frac{2m}{n}} \tag{6.13}$$

with positive C_1 for sufficiently large j. Let us prove that $N_{\lambda'}(t) = \max\{j : \lambda_j' \leq t\}$ has the asymptotics (6.10). For this, in turn, it suffices to check that

$$\sum_{j=1}^{\infty} (\lambda_j + \mu)^{-q} - \sum_{j=1}^{\infty} (\lambda_j' + \mu)^{-q} = o\left(\mu^{\frac{n}{2m} - q}\right) \quad (\mu \to +\infty) \tag{6.14}$$

and then to combine (1.13) with the Hardy–Littlewood Tauberian theorem.[3]

In the left-hand side of (6.14) we may drop any finite number of terms. Let us fix an $\varepsilon > 0$; assume that we have (6.13) and $|\lambda_j''|/\lambda_j' < \varepsilon$ for $j \geq j_1(\varepsilon)$. Then for these j and $\mu > 0$

$$\begin{aligned} |(\lambda_j + \mu)^{-q} - (\lambda_j' + \mu)^{-q}| &\leq \sum_{\substack{q_1+q_2=q+1 \\ q_1,q_2 \in \mathbb{N}}} |\lambda_j + \mu|^{-q_1}\, |\lambda_j''|\, |\lambda_j' + \mu|^{-q_2} \\ &\leq C_2\, \varepsilon\, |\lambda_j' + \mu|^{-q} \leq C_2\, \varepsilon\, |C_1\, j^{\frac{2m}{n}} + \mu|^{-q} \end{aligned}$$

since $|\lambda_j + \mu|^{-1} \leq |\lambda_j' + \mu|^{-1}$, $\frac{|\lambda_j''|}{|\lambda_j' + \mu|} < \varepsilon$, and we have (6.13). Now (6.14) follows from this fact and the inequality

$$\sum_{j=1}^{\infty} \left(C_1\, j^{\frac{2m}{n}} + \mu\right)^{-q} \leq C_3\, \mu^{\frac{n}{2m} - q}\,,$$

which is a consequence of an Abelian theorem. See the formulations of the Tauberian and Abelian theorems e.g. in (Agranovich and Markus 1989). Since under our assumptions the moduli of the eigenvalues have the same asymptotics as their real parts, the theorem is proved. □

[3] Note that $c_q = b_{n/2m,q}d$, where $b_{t,q} = tB(t, q-t)$ and $B(t,s)$ is the Euler Beta function.

6.2. To go further, we need to generalize the asymptotic formula (1.13) to real $q > n/2m$.

Let T be a closed densely defined operator in a Hilbert space H. Assume that the resolvent set of T contains the angle $\mathcal{L}_\theta$ (see (4.11)), where $0 < \theta < \pi$, and that

$$(1 + |\lambda|)\, \|(T - \lambda)^{-1}\| \leq \text{ Const} \tag{6.15}$$

in this angle. (In Agmon's terminology, all the rays $\{\lambda : \arg \lambda = \text{ const}\}$ in $\mathcal{L}_\theta$ are the *rays of minimal growth* for the norm of the resolvent $R_T(\lambda)$.) Let $\Phi(\mu)$ be a function holomorphic in the angle $\overline{\mathbb{C}\backslash\mathcal{L}_\theta}$ (it contains the spectrum of T) and such that $|\Phi(\mu)| \leq C|\mu|^{-\delta}$ for large $|\mu|$, where $\delta > 0$. Then the operator $\Phi(T)$ is defined by the formula

$$\Phi(T) = \frac{1}{2\pi i} \int_{\mathfrak{S}} \Phi(\mu)(T - \mu)^{-1}\, d\mu\,, \tag{6.16}$$

where $\mathfrak{S}$ is the boundary of $\mathcal{L}_\theta$ oriented from below to above (see e.g. (Pattisier 1977) or the survey (Agranovich 1990) and references therein). In particular, if $q > 0$ and λ is an interior point of $\mathcal{L}_\theta$, then

$$(T - \lambda)^{-q} = \frac{1}{2\pi i} \int_{\mathfrak{S}} (\mu - \lambda)^{-q}(T - \mu)^{-1}\, d\mu\,. \tag{6.17}$$

Here $(\mu - \lambda)^{-q}$ is defined as $|\mu - \lambda|^{-q} e^{-iq \arg(\mu-\lambda)}$, $|\arg z| < \pi$, i.e. using a cut of the complex plane along the ray

$$R_-^{(\lambda)} = \{\mu = \sigma + \lambda,\ \sigma \in \mathbb{R}_-\}\,. \tag{6.18}$$

Integrating by parts s times ($s \in \mathbb{N}$), we transform this formula into the following one

$$(T - \lambda)^{-q} = \frac{1}{2\pi i}\ \frac{(-1)^s s!}{(1 - q)\dots(s - q)} \int_{\mathfrak{S}} (\mu - \lambda)^{s-q}(T - \mu)^{-s-1}\, d\mu\,. \tag{6.19}$$

Now we replace $\mathfrak{S}$ by the new contour $\mathfrak{S}_\lambda$ consisting of both sides of our cut along $R_-^{(\lambda)}$, with the direction from $-\infty$ to λ at the lower side:

$$\begin{aligned}(T - \lambda)^{-q} &= c_{s,q} \int_{-\infty}^{0} |\sigma|^{s-q}(T - \lambda - \sigma)^{-s-1}\, d\sigma \\ &= c_{s,q} \int_0^\infty t^{s-q}(T - \lambda + t)^{-s-1}\, dt\,,\end{aligned} \tag{6.20}$$

where

$$c_{s,q} = \frac{(-1)^{s-1} s!\, \sin \pi(s - q)}{\pi(1 - q)\dots(s - q)}\,. \tag{6.21}$$

Here we will need to assume that $|s - q| < 1$; if $q = s$, then we have to replace t^{s-q} in (6.20) and $\sin \pi(s - q)/\pi(s - q)$ in (6.21) by 1. Cf. (Krasnosel'skiĭ et al. 1966, Chapter IV).

We apply formula (6.20) to $T = A_B$ assuming, without loss of generality, that this operator has no eigenvalues in $\mathcal{L}_\theta$:

$$R(\lambda)^q = c_{s,q} \int_0^\infty t^{s-q} R(\lambda - t)^{s+1}\, dt\,. \tag{6.22}$$

Theorem 6.4. *The asymptotic formula* (1.13) *holds in* $\mathcal{L}_\theta$ *for all real* $q > n/2m$ *with* c_q *indicated in* (1.14).

Proof. Assume that q is greater than $n/2m$ and that q is not an even integer. Denote by s the odd integer such that $|s - q| < 1$. Then clearly $s + 1 > n/2m$, and, by our assumption,

$$\operatorname{tr} R(\mu)^{s+1} = c_{s+1}\, (-\mu)^{\frac{n}{2m}-(s+1)} + \rho(\mu)\,, \tag{6.23}$$

where

$$\rho(\mu) = o\left(|\mu|^{\frac{n}{2m}-(s+1)}\right) \quad \text{as } \mu \to \infty \text{ in } \mathcal{L}_\theta\,. \tag{6.24}$$

Using the linearity of the trace, we obtain from (6.22)

$$\operatorname{tr} R(\lambda)^q = c_{s,q} \int_0^\infty t^{s-q} \operatorname{tr} R(\lambda - t)^{s+1}\, dt\,, \tag{6.25}$$

this integral being absolutely convergent. Inserting (6.23) into (6.25), we see that

$$\begin{aligned} \operatorname{tr} R(\lambda)^q \quad &= \quad c_{s,q}\, c_{s+1} \int_0^\infty t^{s-q} (t - \lambda)^{\frac{n}{2m}-(s+1)}\, dt \\ &\quad + \text{const} \int_0^\infty t^{s-q} \rho(t - \lambda)\, dt\,. \end{aligned} \tag{6.26}$$

The first integral is a holomorphic function of λ inside $\mathcal{L}_\theta$. Let at first λ be real (and negative); setting $t = |\lambda|\tau$, we see that this integral is equal to

$$|\lambda|^{\frac{n}{2m}-q} \int_0^\infty \tau^{s-q} (\tau + 1)^{\frac{n}{2m}-(s+1)}\, d\tau\,.$$

Using the holomorphic continuation, we can replace $|\lambda|^{\frac{n}{2m}-q}$ by $(-\lambda)^{\frac{n}{2m}-q}$. In the second integral in (6.26), for any $\varepsilon > 0$ we have[4]

$$|\rho(t - \lambda)| \le \varepsilon\, (t + |\lambda|)^{\frac{n}{2m}-(s+1)}$$

for $\lambda \in \mathcal{L}_\theta$ with sufficiently large $|\lambda|$. Using again the substitution $t = |\lambda|\tau$, we obtain

$$\operatorname{tr} R(\lambda)^q = \hat{c}_q\, (-\lambda)^{\frac{n}{2m}-q} + o\Big(|\lambda|^{\frac{n}{2m}-q}\Big) \quad (\lambda \to \infty \text{ in } \mathcal{L}_\theta)\,,$$

[4] $|t - \lambda| \ge c\,(t + |\lambda|)$ for $t \ge 0$ and $\lambda \in \mathcal{L}_\theta$, where $c =$ const > 0.

where

$$\hat{c}_q = c_{s,q}\, c_{s+1} \int_0^\infty \tau^{s-q} (\tau+1)^{\frac{n}{2m}-(s+1)}\, d\tau\,. \tag{6.27}$$

It can be proved that $\hat{c}_q = c_q$. We omit the corresponding elementary calculations. □

6.3. Now we assume that the boundary problem (1.1)–(1.2) is elliptic with parameter in two closed angles $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ with only one common point, $\lambda = 0$. Let Λ_1 and Λ_2 be two open angles forming the complement to $\mathcal{L}^{(1)} \cup \mathcal{L}^{(2)}$. Without loss of generality we assume that Λ_1 contains $\mathbb{R}_+$ and is symmetric with respect to $\mathbb{R}_+$. All the eigenvalues of A_B, except possibly a finite number of them, lie in Λ_1 and Λ_2.

The set of values of the principal symbol $a_0(x,\xi)$ ($\xi \neq 0$) is also contained in Λ_1 and Λ_2, but in the scalar case, which we consider, it is connected; let us assume that it is contained in Λ_1. Denote by $\operatorname{tr}^{(k)} R(\lambda)^q$ the two parts of the trace of $R(\lambda)^q$ that correspond to the eigenvalues of A_B lying in Λ_k ($k = 1, 2$).

Theorem 6.5. *Let $n/2m < q < 1 + n/2m$. Then formula* (1.13) *remains true for* $\operatorname{tr}^{(1)} R(\lambda)^q$ *with the same c_q. The estimate of the remainder is uniform in any closed angle that has no common point with $\overline{\Lambda}_1$ except* 0.

Proof. Without loss of generality we assume that all the eigenvalues of A_B lie in Λ_1 and Λ_2. Then

$$\operatorname{tr}^{(1)} R(\lambda)^q + \operatorname{tr}^{(2)} R(\lambda)^q = c_q\, (-\lambda)^{\frac{n}{2m}-q} + o\Big(|\lambda|^{\frac{n}{2m}-q}\Big)\,, \tag{6.28}$$

uniformly in $\mathcal{L}^{(1)} \cup \mathcal{L}^{(2)}$. Denote here the remainder by $\rho(\lambda)$. Denote by $\mathfrak{T}$ the boundary of Λ_1 with the negative orientation with respect to Λ_1. We replace λ by μ in (6.28), divide all the terms by $2\pi i(\mu - \lambda)$ and integrate them along $\mathfrak{T}$:

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\mathfrak{T}} \frac{\operatorname{tr}^{(1)} R(\mu)^q}{\mu - \lambda}\, d\mu + \frac{1}{2\pi i} \int_{\mathfrak{T}} \frac{\operatorname{tr}^{(2)} R(\mu)^q}{\mu - \lambda}\, d\mu \\ &\quad = \frac{c_q}{2\pi i} \int_{\mathfrak{T}} \frac{(-\mu)^{\frac{n}{2m}-q}}{\mu - \lambda}\, d\mu + \frac{1}{2\pi i} \int_{\mathfrak{T}} \frac{\rho(\mu)}{\mu - \lambda}\, d\mu\,. \end{aligned} \tag{6.29}$$

Take a point λ to the left of $\mathfrak{T}$. In the first and the third terms we can replace $\mathfrak{T}$ by a closed contour surrounding λ and lying to the left of $\mathfrak{T}$. We see that these terms are equal to $\operatorname{tr}^{(1)} R(\lambda)^q$ and $c_q(-\lambda)^{\frac{n}{2m}-q}$, respectively. In the second term we can replace $\mathfrak{T}$ by a closed contour lying in Λ_1. We see that this term is equal to zero. Thus we obtain

$$\operatorname{tr}^{(1)} R(\lambda)^q = c_q\, (-\lambda)^{\frac{n}{2m}-q} + \rho_1(\lambda)\,,$$

where

$$\rho_1(\lambda) = \frac{1}{2\pi i} \int_{\mathfrak{T}} \frac{\rho(\mu)}{\mu - \lambda}\, d\mu\,,$$

and if $\mathbb{C}\backslash\Lambda_1 = \mathcal{L}_\theta$ and $\theta' > \theta$, we only have to verify that

$$\rho_1(\lambda) = o\Big(|\lambda|^{\frac{n}{2m}-q}\Big) \tag{6.30}$$

uniformly in $\arg\lambda$, $\lambda \in \mathcal{L}_{\theta'}$. Let $\varepsilon > 0$ be fixed; find $R > 0$ such that

$$|\rho(\mu)| \le \varepsilon\, |\mu|^{\frac{n}{2m}-q} \ \text{ for } |\mu| \ge R \text{ on } \mathfrak{T}\,.$$

On the remaining part $\mathfrak{T}_R$ of $\mathfrak{T}$ we have

$$|\rho(\mu)| \le C\, |\mu|^{\frac{n}{2m}-q}$$

with some constant C; in addition

$$|\mu - \lambda|^{-1} \le C'\,(|\mu| + |\lambda|)^{-1}$$

on the whole of $\mathfrak{T}$. Thus

$$|\rho_1(\lambda)| \le \frac{2\,C'\,\varepsilon}{2\pi}\int_R^\infty \frac{t^{\frac{n}{2m}-q}}{t+|\lambda|}\,dt + \frac{2\,C\,C'}{2\pi}\int_0^R \frac{t^{\frac{n}{2m}-q}}{t+|\lambda|}\,dt\,.$$

Using the substitution $t = |\lambda|\tau$, we easily obtain (6.30). □

Remark 6.6. In the matrix case the eigenvalues $\lambda_j(x,\xi)$ of the principal symbol $a_0(x,\xi)$ can lie in Λ_1 and Λ_2. In this case we can obtain the formula

$$\operatorname{tr}^{(1)} R(\lambda)^q = c_q^{(1)}\,(-\lambda)^{\frac{n}{2m}-q} + o\Big(|\lambda|^{\frac{n}{2m}-q}\Big)\,, \tag{6.31}$$

where

$$c_q^{(1)} = \frac{1}{(2\pi)^n}\int_G dx \int_{\mathbb{R}^n} \sum_{\lambda_j(x,\xi)\in\Lambda_1} (\lambda_j(x,\xi)+1)^{-q}\,d\xi\,. \tag{6.32}$$

Cf. (Agranovich and Markus 1989).

6.4. Let us indicate the spectral corollaries of Theorem 6.5. The assumptions are exactly those as above, see the beginnings of Subsections 6.1 and 6.3. Denote by $N_\lambda^{(1)}(t)$ the counting function for the moduli of the eigenvalues of A_B lying in Λ_1.

Theorem 6.7. *Let $d \ne 0$. Then $N_\lambda^{(1)}(t) \asymp t^{\frac{n}{2m}}$.*

Theorem 6.8. *Assume that $a_0(x,\xi) > 0$ and $\Lambda_1 = \mathbb{R}_+$ is an isolated ray without ellipticity with parameter. Then*

$$N_\lambda^{(1)}(t) = d\cdot t^{\frac{n}{2m}} + o(t^{\frac{n}{2m}}) \ \text{ as } \ t \to \infty\,. \tag{6.33}$$

Cf. Theorem 3.4. In the matrix case Theorems 6.7 and 6.8 remain valid with $d^{(1)}$ instead of d, where $c_q^{(1)} = \beta_q d^{(1)}$. Cf. (Agranovich and Markus 1989). Note that if $n/2m \notin \mathbb{N}$, then we can use $q \in \mathbb{N}$, and in this case we can directly apply the Tauberian and Abelian theorems indicated in that paper. If $n/2m \in \mathbb{N}$, we have to use $q \notin \mathbb{N}$; in this case we apply the analogues of those theorems for noninteger q; they are well known or can be checked without difficulties.

7 Boundary Problems with Transmission Conditions

7.1. Here we will use the notation introduced in Section 1.7. Let Γ_k be the common part of the boundaries of the subdomains G_l and $G_{l'}$. We consider the transmission conditions on Γ_k of the form

$$B_{lkj}(x,\mathcal{D})u^{(l)}(x) + B_{l'kj}(x,\mathcal{D})u^{(l')}(x) = h_j(x) \quad (j=1,\dots,2m) \text{ on } \Gamma_k\,. \tag{7.1}$$

Here $B_{lkj}(x,\mathcal{D})$ and $B_{l'kj}(x,\mathcal{D})$ are boundary operators of orders $m_{kj} < 2m$. By $u^{(l)}(x)$ and $u^{(l')}(x)$ we denote the solution in G_l and $G_{l'}$. The boundary problem consists of the equation (1.1) in each G_l, boundary conditions (1.2) on the outer boundary Γ, and the transmission conditions (7.1) on each Γ_k. Actually this is a problem with $N+1$ unknowns $u^{(l)}$, each of which is defined in its own domain G_l. As we mentioned in Section 1.7, in the minimal smoothness assumptions the top order coefficients of $A(x,\mathcal{D})$ are continuous in each G_l up to the boundary; in general they undergo a jump when we cross any Γ_k. As to Γ_k and B_{lkj}, in the minimal smoothness assumptions $\Gamma_k \in C^{2m-1,1}$, and the coefficients in $B_{lkj}(x,\mathcal{D})$ belong to $C^{2m-m_{kj}-1,1}(\Gamma_k)$.

In the definition of the ellipticity with parameter we have to add a condition at any point $x_0 \in \Gamma_k$ for each k. Let the operators A, B_{lkj}, and $B_{l'kj}$ be written in a coordinate system associated with the point $x_0 \in \Gamma_k$ on the boundary of, say, $G_{l'}$. Denote by $a_{l0}(x_0,\xi)$ and $a_{l'0}(x_0,\xi)$ the limit values of the principal symbol of $A(x,\mathcal{D})$ at x_0 from G_l and $G_{l'}$, respectively, and by $b_{lkj0}(x_0,\xi)$ and $b_{l'kj0}(x_0,\xi)$ the principal symbols of B_{lkj} and $B_{l'kj}$ at x_0. Now we formulate the additional condition at x_0.

(3) The problem on the line

$$\begin{gathered}
[a_{l0}(x_0,\xi',\mathcal{D}_n) - \lambda]\, v^{(l)}(t) = 0 \quad (t = x_n < 0)\,,\\
[a_{l'0}(x_0,\xi',\mathcal{D}_n) - \lambda]\, v^{(l')}(t) = 0 \quad (t = x_n > 0)\,,\\
b_{lkj0}(x_0,\xi',\mathcal{D}_n)v^{(l)}(0) + b_{l'kj0}(x_0,\xi',\mathcal{D}_n)v^{(l')}(0) = 0 \quad (j=1,\dots,2m)\,,\\
v^{(l)}(t) \to 0 \quad (t\to -\infty), \qquad v^{(l')}(t) \to 0 \quad (t \to +\infty)
\end{gathered} \tag{7.2}$$

has only the trivial solution if $0 \neq (\xi',\lambda) \in \mathbb{R}^{n-1} \times \mathcal{L}$.

The analogue of the Basic Theorem is obtained under the minimal smoothness conditions using the same tools. We only indicate the a priori estimate for the case $g_j \equiv 0$ and $h_j \equiv 0$:

$$\sum_{l=1}^{N} |\!|\!| u^{(l)} |\!|\!|_{2m,p,G_l} \le C\, \|f\|_{0,p,G}\,. \tag{7.3}$$

In the conditions of weak smoothness we assume the existence of the formally adjoint problem. We do not consider the question when the adjoint problem exists.

In the case of transmission conditions (1.29) we can check Condition (3) (and the absolute ellipticity of these transmission conditions) using the following remark. Let

$v_j(t)$ $(j = 1, \ldots, m)$ be linearly independent solutions of an equation $a_1(\mathcal{D}_n)v(t) = 0$ with constant coefficients, and let $v_j(t) \to 0$ $(t \to -\infty)$. Further, let $w_j(t)$ $(j = 1, \ldots, m)$ be linearly independent solutions of another equation $a_2(\mathcal{D}_n)w(t) = 0$ with constant coefficients, and let $w_j(t) \to 0$ $(t \to +\infty)$. Then $\{v_1(t), \ldots, v_m(t), w_1(t), \ldots, w_m(t)\}$ is a fundamental system of solutions of an equation $a_3(\mathcal{D}_n)v(t) = 0$ with constant coefficients (namely, $a_3 = a_1 a_2$). Because of this their Wronskian is nonzero.

The results of Sections 2–6 remain true almost without alterations in the statements and the proofs.

8 Boundary Problems with Indefinite Weight

8.1. Recalling again the assumptions, definitions, and notation introduced in Subsections 1.7 and 1.8, we shall be concerned in this section with the spectral properties of the boundary problem (1.30), (1.2). We shall call the boundary problem (1.30), (1.2) minimally or weakly smooth according to whether (1.1)–(1.2) is minimally or weakly smooth. *In both cases, $\omega(x)$ is assumed to be continuous in each G_k up to the boundary.* Furthermore, it will always be supposed here that the Γ_k are of class $C^{2m-1,1}$, and, unless otherwise stated, that the problem (1.30), (1.2) is minimally smooth. Note that in the case of weak smoothness we require Γ to be more smooth than Γ_k. The analogue of Definition 1.1 for the problem (1.30), (1.2) is as follows.

Definition 8.1. Let $\mathcal{L}$ be a closed angle in the complex plane with vertex at the origin. The boundary problem (1.30), (1.2) is called *elliptic with parameter in $\mathcal{L}$* if the following two conditions are satisfied:

(1) For $k = 0, \ldots, N$, $a_0(x, \xi) - \lambda\omega(x) \neq 0$ for $(x, \xi) \in \overline{G}_k \times \mathbb{R}^n$ and $\lambda \in \mathcal{L}$ if $|\xi| + |\lambda| \neq 0$, where $\omega(x)$ is defined by continuity on ∂G_k.

(2) Let x_0 be a point of Γ and let $\omega(x_0)$ be defined by continuity. Assume that the boundary problem (1.30), (1.2) is rewritten in a coordinate system associated with x_0. Then for $\xi' \in \mathbb{R}^{n-1}$ and $\lambda \in \mathcal{L}$ the boundary problem on the half-line

$$\begin{gathered} a_0(x_0, \xi', \mathcal{D}_n)v(t) - \lambda\omega(x_0)v(t) = 0 \quad (t = x_n > 0)\,, \\ b_{j0}(x_0, \xi', \mathcal{D}_n)v(t) = 0 \quad (j = 1, \ldots, m) \ \text{ at } \ t = 0\,, \\ v(t) \to 0 \quad (t \to \infty) \end{gathered} \tag{8.1}$$

has only the trivial solution if $|\xi'| + |\lambda| \neq 0$.

Then as a consequence of the results mentioned in Section 7 for the transmission conditions (1.29) we have

Theorem 8.2. *Suppose that the boundary problem* (1.30), (1.2) *is elliptic with parameter in an angle $\mathcal{L}$. Let $1 < p < \infty$. Then there exists a $\lambda_0 = \lambda_0(p) > 0$ such that for $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda_0$ the boundary problem has a unique solution $u \in W_p^{2m}(G)$*

for any $f \in L_p(G)$ and $g_j \in W_p^{2m-m_j-\frac{1}{p}}(\Gamma)$, and the a priori estimate (2.10) *holds, where the constant C does not depend upon f, g_j, and λ.*

8.2. In this subsection we shall always suppose that the hypotheses of Theorem 8.2 are satisfied. Let V_p denote the operator of multiplication by ω in $L_p(G)$. Then recalling the definition of the operator $A_{B,p}$ introduced in Section 3 it follows from Theorem 8.2 that the set of regular values of the pencil $S_p(\lambda) = A_{B,p} - \lambda V_p$ is not empty. Here, as usually, λ is called a regular value of $S_p(\lambda)$ if $S_p(\lambda)$ is invertible. We refer to (Markus 1986) for the relevant terminology concerning pencils. Furthermore, it is easy to show that the set of all regular values of $S_p(\lambda)$ is precisely the resolvent set of $V_p^{-1}A_{B,p}$, and we conclude from the Rellich–Kondrashov theorem that the spectrum of $S_p(\lambda)$ consists of isolated eigenvalues only. Direct calculations also show that at an eigenvalue of $S_p(\lambda)$ (and hence of $V_p^{-1}A_{B,p}$), the corresponding eigenvectors are precisely the eigenfunctions of $V_p^{-1}A_{B,p}$ and the eigenvectors and associated vectors of $S_p(\lambda)$ are precisely the generalized eigenfunctions of $V_p^{-1}A_{B,p}$. Thus we conclude that the eigenvalues of $S_p(\lambda)$ are all of finite multiplicity.

Because $S_p(\lambda)$ and $V_p^{-1}A_{B,p}$ are spectrally equivalent in the sense just described, we will henceforth fix our attention upon $V_p^{-1}A_{B,p}$. We observe from Theorem 8.2 that if $\lambda \in \mathcal{L}$ and $|\lambda| \geq \lambda_0$, then

$$\|\!| (V_p^{-1}A_{B,p} - \lambda I)^{-1} f \|\!|_{2m,p,G} \leq C_p \|f\|_{0,p,G} \tag{8.2}$$

for $f \in L_p(G)$, where the constant C_p does not depend upon f and λ. As in Section 3, the eigenvalues and generalized eigenfunctions of $V_p^{-1}A_{B,p}$ do not depend upon p, and because of this we shall mainly be concerned with the operator $V^{-1}A_B$ acting in $L_2(G)$, where we have written V for V_2 and A_B for $A_{B,2}$. As in Section 3, we have the following four theorems.

Theorem 8.3. *Assume that the boundary problem* (1.30), (1.2) *is elliptic with parameter along some rays $\mathcal{L}^{(j)}$ ($j = 1, \dots, N$) and that the angles between any two adjacent rays are not greater than $2m\pi/n$. Then $V^{-1}A_B$ has an infinite number of eigenvalues and the set of all generalized eigenfunctions of $V^{-1}A_B$ is complete in $L_2(G)$.*

We note that under the hypotheses of Theorem 8.3, the set of all generalized eigenfunctions of $V^{-1}A_B$ is also complete in $L_2\big(G; |\omega(x)|dx\big)$. This follows from the fact that $L_2(G)$ and $L_2\big(G; |\omega(x)|dx\big)$ coincide algebraically and have equivalent norms.

Assuming that the spectrum of $V^{-1}A_B$ is not empty, let $\{\lambda_j\}_{j\geq 1}$ be the set of all eigenvalues of $V^{-1}A_B$ arranged so that

$$|\lambda_1| \leq |\lambda_2| \leq \dots$$

and each eigenvalue is repeated according to its multiplicity. Let $\{u_j\}_{j\geq 1}$ be the system of generalized eigenfunctions of $V^{-1}A_B$ composed of bases in each generalized eigenspace in such a way that u_j belongs to the generalized eigenspace corresponding to the eigenvalue λ_j. Then under the assumptions of Theorem 8.3 the u_j's form an infinite complete minimal system and there exists a system $\{w_j\}_1^\infty$ which is biorthogonal to $\{u_j\}_1^\infty$.

Theorem 8.4. *Let $f \in L_2(G)$. Then under the assumptions of Theorem* 8.3 *the series in* (3.3) *admits the summability to f in both $L_2(G)$ and $L_2\big(G; |\omega(x)|\,dx\big)$ by the Abel–Lidskiĭ method of order $\dfrac{n}{2m} + \varepsilon$ if $\varepsilon > 0$ is sufficiently small.*

Theorem 8.5. *Let the boundary problem* (1.30), (1.2) *be elliptic with parameter along the rays $\mathcal{L}(\theta_1)$ and $\mathcal{L}(\theta_2)$, where $0 < \theta_2 - \theta_1 < \min\{2m\pi/n, 2\pi\}$, and not elliptic with parameter along some ray $\mathcal{L}(\theta_0)$, $\theta_1 < \theta_0 < \theta_2$. Then the angle $\{\lambda : \theta_1 < \arg\lambda < \theta_2\}$ contains infinitely many eigenvalues of $V^{-1}A_B$.*

Theorem 8.6. *Let the boundary problem* (1.30), (1.2) *be elliptic with parameter along all the rays $\mathcal{L}(\theta)$ with $\theta_0 - \varepsilon < \theta < \theta_0$ and $\theta_0 < \theta < \theta_0 + \varepsilon$ for some $\varepsilon > 0$, and not elliptic with parameter along $\mathcal{L}(\theta_0)$. Then any angular neighbourhood of $\mathcal{L}(\theta_0)$ contains infinitely many eigenvalues of $V^{-1}A_B$.*

8.3. Here we indicate the analogues of Theorems 4.1 and 5.1 for the boundary problem (1.30), (1.2). Accordingly, suppose that the problem (1.30), (1.2) is elliptic with parameter in an angle $\mathcal{L}$. Then we know from above that the resolvent set of $V^{-1}A_B$ is not empty. We henceforth let $R_\omega(\lambda)$ denote the resolvent of $V^{-1}A_B$.

Theorem 8.7. *Assume that Condition* (1) *of Definition* 8.1 *holds for the angle $\mathcal{L}_\theta$* (*see* (4.11)) *and that the minimal smoothness assumptions are satisfied for G and $A(x,\mathcal{D})$. Let $2mq > n$ and let q be even. Then for the Dirichlet problem* (1.30), (1.15) *we have*

$$\operatorname{tr} R_\omega(\lambda)^q = c_{q\omega}(-\lambda)^{\frac{n}{2m}-q} + o\big(|\lambda|^{\frac{n}{2m}-q}\big) \quad \text{as} \quad |\lambda| \to \infty, \qquad \lambda \in \mathcal{L}_\theta\,, \tag{8.3}$$

uniformly in λ, where

$$c_{q\omega} = \int_G c_{q\omega}(x)\,dx \quad \text{and} \quad c_{q\omega}(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{d\xi}{\big[\omega(x)^{-1}a_0(x,\xi)+1\big]^q}\,. \tag{8.4}$$

Recall that the definition of $(-\lambda)^{\cdots}$ was indicated in Subsection 1.4. Theorem 8.7 follows from the corresponding variant of Theorem 4.1 for transmission problems.

Theorem 8.8. *Let the boundary problem* (1.30), (1.2) *be weakly smooth and elliptic with parameter in $\mathcal{L}_\theta$. Let $2mq > n$ and let q be even. Then the formula* (8.3) *is true, uniformly in λ, where $c_{q\omega}$ is indicated in* (8.4).

Proof. It is clear that for $\lambda \in \mathcal{L}_\theta$ with $|\lambda|$ sufficiently large, $R_\omega(\lambda)^q$ is a trace class operator in $L_2(G)$. Let us set $R_1(\lambda) = R_\omega(\lambda)^k V^{-1}$ and $R_2(\lambda) = \big(V^{-1}(A_B)^* - \lambda I\big)^{-k}$, where $k = q/2$. Then, using the formulas $(V^{-1}A_B - \lambda I)^{-1} = (A_B - \lambda V)^{-1}V$ and

$\big(V^{-1}(A_B)^* - \overline{\lambda} I\big)^{-1} = \big((A_B)^* - \overline{\lambda} V\big)^{-1} V$, we see that $R_\omega(\lambda)^q$ admits the decomposition $R_\omega(\lambda)^q = R_1(\lambda) R_2(\overline{\lambda})^* V$. Now we argue as we did in Subsection 5.2 but use Remark 5.6. We see that $R_1(\lambda)$ is an integral operator in $L_2(G)$ (it is in fact a Hilbert–Schmidt operator) with a kernel $K_1(x,y,\lambda)$ having the same properties as those asserted for $K(x,y)$ in the statements preceding (5.1) and satisfying

$$\left(\int_G \big|K_1(x,y,\lambda)\big|^2\, dy\right)^{1/2} \le c\,|\lambda|^{\frac{n}{4m}-k}$$

for $x \in \overline{G}$ and $\lambda \in \mathcal{L}_\theta$ with $|\lambda| \ge \lambda_0$, where the constant c does not depend upon x and λ. Since analogous results also hold for the operator $R_2(\overline{\lambda})$, we can now appeal to Remark 5.6 and argue as we did in Subsection 5.2 to show that for $\lambda \in \mathcal{L}_\theta$ with $|\lambda|$ sufficiently large, $R_\omega(\lambda)^q$ is an integral operator with kernel $K_\omega(x,y,\lambda) = K(x,y,\lambda)\omega(y)$, where $K(x,y,\lambda)$ is continuous in $\overline{G}\times\overline{G}$ and

$$\big|K(x,y,\lambda)\big| \le C\,|\lambda|^{\frac{n}{2m}-q}\,; \tag{8.5}$$

here the constant C does not depend upon x, y, and λ.

Next for $0 \le j \le N$, let $x_0 \in F$, where F is a compact subset of G_j, and let $0 < \delta < \operatorname{dist}\,\{x_0, \partial G_j\}$. Then it follows from an obvious modification of the arguments of Subsection 5.3 that for $\lambda \in \mathcal{L}_\theta$ with $|\lambda|$ sufficiently large, $K_\omega(x_0,x_0,\lambda) - c_{q\omega}(x_0)(-\lambda)^{\frac{n}{2m}-q}$ is bounded in modulus by $C\big(\Phi(\delta) + \delta^{-1}|\lambda|^{-\frac{1}{2m}}\big)|\lambda|^{\frac{n}{2m}-q}$, where the constant C does not depend upon λ, x_0, δ, and j, and $\Phi(\delta)\to 0$ as $\delta \to 0$. Hence

$$K_\omega(x,x,\lambda) = c_{q\omega}(x)(-\lambda)^{\frac{n}{2m}-q} + o\left(|\lambda|^{\frac{n}{2m}-q}\right) \quad \text{as } |\lambda| \to \infty \tag{8.6}$$

uniformly in λ and x for $\lambda \in \mathcal{L}_\theta$ and x belonging to any compact subset of G_j. It is a simple matter to deduce from (8.5) and (8.6) that

$$\int_G K_\omega(x,x,\lambda)dx = c_{q\omega}(-\lambda)^{\frac{n}{2m}-q} + o\left(|\lambda|^{\frac{n}{2m}-q}\right) \quad \text{as } |\lambda| \to \infty$$

uniformly in $\mathcal{L}_\theta$, and thus the proof of the theorem is complete. □

Note that

$$c_{q\omega} = b_{\frac{n}{2m},q} d_\omega\,, \tag{8.7}$$

where

$$d_\omega = \frac{1}{(2\pi)^n n}\int_G dx \int_{|\xi|=1} \big[\omega(x)^{-1}a_0(x,\xi)\big]^{-\frac{n}{2m}}\, dS_\xi\,, \tag{8.8}$$

$b_{t,q}$ is defined in the footnote in Section 6, and the power is defined by using a cut along $\overline{\mathbb{R}}_-$.

8.4. In this subsection we are going to derive rough and precise asymptotics for the eigenvalues of $V^{-1}A_B$. Accordingly, we shall henceforth suppose that the boundary problem (1.30), (1.2) is elliptic with parameter in an angle or angles, and that we have there a formula of the form (8.3). Note that the results established below hold under the assumptions similar to those in Theorem 8.7 or Theorem 8.8. Furthermore, in deriving our estimates, we shall suppose from now on that 0 is in the resolvent set of $V^{-1}A_B$ (clearly this involves no loss of generality).

Let $N_\lambda(t)$ and $N_s(t)$ be defined as in (6.2) and (6.3), respectively, where now λ_j are the eigenvalues of $V^{-1}A_B$ and s_j are the s–numbers of $(V^{-1}A_B)^{-1}$. Let

$$\Delta_\omega = \frac{1}{(2\pi)^n n}\int_G dx \int_{|\xi|=1} \left|\omega(x)^{-1}a_0(x,\xi)\right|^{-\frac{n}{2m}} dS_\xi\,. \tag{8.9}$$

Then we have the analogue of (6.4), namely

$$N_s(t) = \Delta_\omega t^{\frac{n}{2m}} + o\left(t^{\frac{n}{2m}}\right) \quad \text{as} \quad t \to \infty\,. \tag{8.10}$$

The proof is similar to that of Proposition 6.1, see Subsection A.7.

Hence it follows from (8.10) and (Agranovich and Markus 1989) that (6.5) and (6.6) hold with Δ replaced by Δ_ω. Furthermore, (6.7) holds if $d_\omega \neq 0$. Thus we obtain

Theorem 8.9. *If $d_\omega \neq 0$, then $N_\lambda(t) \asymp t^{\frac{n}{2m}}$.*

By arguing precisely as in the proof of Theorem 6.4, we can also show that

Theorem 8.10. *The asymptotic formula* (8.3) *holds for all real $q > n/2m$ with $c_{q\omega}$ indicated in* (8.4).

Suppose next that the boundary problem (1.30), (1.2) is elliptic with parameter in two closed angles $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ which intersect only at the origin. Let Λ_1 and Λ_2 denote the two open angles complementary to $\mathcal{L}^{(1)}\cup\mathcal{L}^{(2)}$, and let us suppose that $\overline{\mathbb{R}}_+$ is the bisectrix of Λ_1. Let $G^{(1)}$ denote the union of the G_k for which $\omega(x)^{-1}a_0(x,\xi) \in \Lambda_1$ for $x \in G_k$ and $\xi \in \mathbb{R}^n\backslash\{0\}$, and let $\operatorname{tr}^{(1)} R_\omega(\lambda)^q$ denote the part of the trace of $R_\omega(\lambda)^q$ that corresponds to the eigenvalues of $V^{-1}A_B$ lying in Λ_1. Finally, let

$$d_\omega^{(1)} = \frac{1}{(2\pi)^n n}\int_{G^{(1)}} dx \int_{|\xi|=1} \left[\omega(x)^{-1}a_0(x,\xi)\right]^{-\frac{n}{2m}} dS_\xi\,, \tag{8.11}$$

where the power is defined by using a cut along $\overline{\mathbb{R}}_-$. Following arguments similar to those used in the proof of Theorem 6.5, we obtain

Theorem 8.11. *Let $n/2m < q < 1 + n/2m$. Then*

$$\operatorname{tr}^{(1)} R_\omega(\lambda)^q = b_{\frac{n}{2m},q}\, d_\omega^{(1)}(-\lambda)^{\frac{n}{2m}-q} + o\left(|\lambda|^{\frac{n}{2m}-q}\right) \quad \text{as} \quad |\lambda| \to \infty\,, \tag{8.12}$$

uniformly in any closed angle that has no common points with $\overline{\Lambda}_1$ except 0.

Let us indicate a corollary of Theorem 8.11. Accordingly, let $N_\lambda^{(1)}(t) = \max\{j : |\lambda_j| \leq t,\, \lambda_j \in \Lambda_1\}$.

Theorem 8.12. *If $d_\omega^{(1)} \neq 0$, then $N_\lambda^{(1)}(t) \asymp t^{\frac{n}{2m}}$.*

Proof. The proof follows from the same kind of arguments that were used in establishing Theorem 8.9, but now we make use of (8.12) instead of (8.3). □

Assuming that $\omega(x)$ changes its sign in G, we are now going to use Theorem 8.11 to obtain an analogue of Theorem 6.8. Accordingly, suppose that $a_0(x,\xi) > 0$ for $x \in \overline{G}$ and $\xi \in \mathbb{R}^n\backslash\{0\}$ and that the boundary problem (1.30), (1.2) is elliptic with parameter along every ray emanating from the origin with the exception of the rays $\overline{\mathbb{R}}_\pm$. Then it follows from Theorem 8.6 that for any ε satisfying $0 < \varepsilon < \pi/2$, there are infinitely many eigenvalues of $V^{-1}A_B$ lying in each of the angles $|\arg\lambda| < \varepsilon$ and $|\pi - \arg\lambda| < \varepsilon$, while there are at most a finite number of eigenvalues lying in each of the angles $\varepsilon \le \arg\lambda \le \pi-\varepsilon$ and $-\pi+\varepsilon \le \arg\lambda \le -\varepsilon$. Let $N_\lambda^+(t) = \max\{j : |\lambda_j| \le t, \operatorname{Re}\lambda_j \ge 0\}$, $N_\lambda^-(t) = \max\{j : |\lambda_j| \le t, \operatorname{Re}\lambda_j < 0\}$, so that $N_\lambda(t) = N_\lambda^+(t) + N_\lambda^-(t)$. Lastly, let $\omega^+(x) = \max\{\omega(x), 0\}$, $\omega^-(x) = \max\{-\omega(x), 0\}$.

Theorem 8.13. *Let $a_0(x,\xi) > 0$ for $x \in \overline{G}$ and $\xi \in \mathbb{R}^n\backslash\{0\}$. Let the boundary problem* (1.30), (1.2) *be elliptic with parameter along every ray emanating from the origin with the exception of the rays $\overline{\mathbb{R}}_\pm$. Then*

$$N_\lambda^\pm(t) = \kappa^\pm t^{\frac{n}{2m}} + o\left(t^{\frac{n}{2m}}\right) \quad \textit{as } t \to \infty, \tag{8.13}$$

$$N_\lambda(t) = \kappa t^{\frac{n}{2m}} + o\left(t^{\frac{n}{2m}}\right) \quad \textit{as } t \to \infty, \tag{8.14}$$

where

$$\kappa^\pm = \frac{1}{(2\pi)^n}\int_G (\omega^\pm)^{\frac{n}{2m}}\,dx \int_{a_0(x,\xi)<1} d\xi \tag{8.15}$$

and

$$\kappa = \Delta_\omega = \frac{1}{(2\pi)^n}\int_G |\omega|^{\frac{n}{2m}}\,dx \int_{a_0(x,\xi)<1} d\xi\,. \tag{8.16}$$

Proof. It follows from Theorem 8.11 that

$$\sum_{\operatorname{Re}\lambda_j \ge 0} \frac{1}{(\lambda_j + t)^q} = b_{\frac{n}{2m},q}\kappa^+ t^{\frac{n}{2m}-q} + o\left(t^{\frac{n}{2m}-q}\right) \quad \text{as } t \to \infty,$$

and hence by arguing as in the proof of Theorem 6.3 we obtain

$$\sum_{\operatorname{Re}\lambda_j \ge 0} \frac{1}{\left(|\lambda_j| + t\right)^q} = b_{\frac{n}{2m},q}\kappa^+ t^{\frac{n}{2m}-q} + o\left(t^{\frac{n}{2m}-q}\right) \quad \text{as } t \to \infty.$$

The assertion for $N_\lambda^+(t)$ now follows from the Hardy–Littlewood Tauberian theorem. The assertion for $N_\lambda^-(t)$ can be obtained in the same way if we replace ω in (1.30) by $-\omega$. The assertion for $N_\lambda(t)$ then follows from these results. □

9 Some Generalizations

9.1. In this section we again consider the boundary problem (1.1)–(1.2), where we now make the following smoothness assumptions: 1) Γ is of class $C^{2m-1,1}$; 2) the coefficients $a_\alpha(x)$ are measurable and bounded and the top order coefficients $a_\alpha(x)$ for $|\alpha| = 2m$ are continuous in $\overline{G}$; 3) the coefficients $b_{j\beta}(x)$ belong to the Hölder space $C^{2m-m_j-1,\gamma}(\Gamma)$ with a fixed γ, $0 < \gamma < 1$. Thus we have minimal smoothness for Γ and $A(x,\mathcal{D})$ and a weaker smoothness for $B_j(x,\mathcal{D})$.

From the proof of the Basic Theorem (Theorem 2.1) it can be seen that under these smoothness assumptions the Basic Theorem holds for any *fixed* p with $1 < p < (1-\gamma)^{-1}$. For this we only note that the continuity of $B_j(x,\mathcal{D})$ as an operator from $W_p^{2m}(G)$ to $W_p^{2m-m_j-\frac{1}{p}}(\Gamma)$ follows from the remarks at the end of Subsection A.1 of the Appendix.

Now assume that these smoothness assumptions are fulfilled with $\gamma > 1/2$. Then the generalized eigenfunctions of the operator $A_{B,p}$ belong to the intersection of all spaces $W_p^{2m}(G)$ with $1 < p < (1-\gamma)^{-1}$ and the spectrum of $A_{B,p}$ does not depend upon p for these values of p. Moreover, the results of Section 3 can be extended, where now in Theorem 3.1 the generalized eigenfunctions are complete in $L_p(G)$ for $1 < p < (1-\gamma)^{-1}$. We can also extend Theorem 4.1 and its consequences stated in Section 6, the proofs remaining literally the same. The smoothness assumptions in Section 7 and Section 8 can also be somewhat weakened.

9.2. An essential property of the coefficients of $B_j(x,\mathcal{D})$ is that they must be multipliers in the space $W_p^{2m-m_j-\frac{1}{p}}(\Gamma)$. Here a function is called a multiplier if the operator of multiplication by it is a continuous operator in the corresponding Sobolev space. The space of all multipliers in a given Sobolev space is described in (Maz'ya and Shaposhnikova 1986). With the tools and results from this book, our minimal smoothness assumptions can be somewhat further weakened. In the present paper we preferred to avoid the corresponding complicated notions and used simpler conditions.

Appendix

In this Appendix we prove Theorem V (see Subsection 2.1), the results stated in Subsection 2.3 and Proposition 6.1. In the following, c_1, $c_2, \ldots$ denote constants not depending upon parameters and functions entering in the corresponding inequalities.

A.1. We begin with the proof of Theorem V. Let $b \in C^{s-1,1}(\Gamma)$ with $1 \le s \le 2m$. As Γ is of class $C^{2m-1,1}$, there exists an open covering $\Gamma \subset \bigcup_{j=1}^N U_j$ of Γ and local coordinates

$$\eta^{(j)} : U_j \to \tilde{U}_j \subset \{y \in \mathbb{R}^n : |y_i| < 1 \quad (i = 1, \ldots, n)\}$$

of class $C^{2m-1,1}(U_j)$ with $\eta^{(j)}(U_j \cap \Gamma) = \tilde{U}_j \cap \mathbb{R}^{n-1}$, cf. (Grisvard 1985, Section 1.2.1). Using a C^∞ partition of unity subordinated to this covering, we see that it is sufficient to consider functions on $\mathbb{R}^{n-1}$ with support contained in $\{y' \in \mathbb{R}^{n-1} : |y_i| < 1\ (i = 1, \ldots, n-1)\}$. Note that in local coordinates $b \in C^{s-1,1}(\mathbb{R}^{n-1})$; assume that the derivatives of b of order $s-1$ are Lipschitz continous with Lipschitz constant L.

We fix a $\delta > 0$ and use the equivalence of the norm in $W_p^{s-\frac{1}{p}}(\mathbb{R}^{n-1})$ to the norm

$$\|v\|^{(1)}_{s-\frac{1}{p},p,\mathbb{R}^{n-1}} = \|v\|_{0,p,\mathbb{R}^{n-1}} + \sum_{j=1}^{n-1} \Big[\int_0^\delta t^{-p} \int_{\mathbb{R}^{n-1}} |\Delta_{t,j} \mathcal{D}_j^{s-1} v(y')|^p \, dy'\, dt \Big]^{\frac{1}{p}}, \quad (A.1)$$

where

$$\Delta_{t,j} v(y') = v(y_1, \ldots, y_j + t, \ldots, y_{n-1}) - v(y')$$

(see (Triebel 1978, Theorem 2.5.1)). We have to estimate $\|bv\|^{(1)}_{s-\frac{1}{p},p,\mathbb{R}^{n-1}}$. Set

$$S = \sup\{|\mathcal{D}^{\alpha'} b(x')| : |\alpha'| \le s-1,\ x' \in \mathbb{R}^{n-1}\}. \quad (A.2)$$

Then obviously

$$\|bv\|_{0,p,\mathbb{R}^{n-1}} \le S\, \|v\|_{0,p,\mathbb{R}^{n-1}}. \quad (A.3)$$

To estimate the sum in (A.1), we use the Leibniz rule and see that each term of this sum can be estimated by a linear combination of terms of the form

$$\Big[\int_0^\delta t^{-p} \int\limits_{\mathbb{R}^{n-1}} |\mathcal{D}_j^k b(y_1, \ldots, y_j + t, \ldots, y_{n-1})\, \Delta_{t,j} \mathcal{D}_j^{s-1-k} v(y')|^p \, dy'\, dt \Big]^{\frac{1}{p}} \quad (A.4)$$

and

$$\Big[\int_0^\delta t^{-p} \int\limits_{\mathbb{R}^{n-1}} |\Delta_{t,j} \mathcal{D}_j^k b(y') \cdot \mathcal{D}_j^{s-1-k} v(y')|^p \, dy'\, dt \Big]^{\frac{1}{p}}, \quad (A.5)$$

where k is an integer between 0 and $s-1$. The expression in (A.4) is not greater than

$$S \Big[\int_0^\delta t^{-p} \int\limits_{\mathbb{R}^{n-1}} |\Delta_{t,j} \mathcal{D}_j^{s-1-k} v(y')|^p \, dy'\, dt \Big]^{\frac{1}{p}} \le S\, \|v\|^{(1)}_{s-\frac{1}{p},p,\mathbb{R}^{n-1}}. \quad (A.6)$$

For $0 \le k < s-1$ we apply the Lagrange formula to the real and imaginary part of $\mathcal{D}_j^k b$ and obtain the inequality

$$|\Delta_{t,j} \mathcal{D}_j^k b(y')| \le 2t\, S, \quad (A.7)$$

and hence the expression in (A.5) is not greater than

$$c_1\, S \|v\|_{s-1,p,\mathbb{R}^{n-1}}. \quad (A.8)$$

For $k = s-1$ we use the fact that $\mathcal{D}_j^{s-1}b$ is Lipschitz continuous. We have the inequality $|\Delta_{t,j}\mathcal{D}_j^{s-1}b(y')| \le Lt$, and therefore the expression in (A.5) is not greater than a constant times $L\|v\|_{0,p,\mathbb{R}^{n-1}}$. This completes the proof. □

We remark that from the proof above it is easily seen that the norm of this multiplication operator tends to zero if $S \to 0$, where S is defined as in (A.2) using local coordinates. Indeed, for $0 < \mu < 1/p$ we get from the Lipschitz continuity the inequality

$$|\Delta_{t,j}\mathcal{D}_j^{s-1}b(y')| \le (Lt)^{1-\mu}\,(2S)^{\mu}\,, \tag{A.9}$$

and therefore for $k = s-1$ the expression in (A.5) is not greater than

$$(2S)^{\mu}L^{1-\mu}\Big(\int_0^{\delta} t^{-\mu p}\,dt\Big)^{\frac{1}{p}}\|v\|_{0,p,\mathbb{R}^{n-1}}\,. \tag{A.10}$$

It follows that the norm of the operator of multiplication by b is not greater than $c_2(S + L^{1-\mu}S^{\mu})$ with $c_2 = c_2(\mu)$.

These considerations also show that the assumptions on the function b can be weakened: it is sufficient for b to belong to the Hölder class $C^{s-1,\gamma}(\Gamma)$ with $\gamma > 1-\dfrac{1}{p}$. Indeed, in this case we have the inequality $|\Delta_{t,j}\mathcal{D}_j^{s-1}b(y')| \le \tilde{L}t^{\gamma}$ for some constant $\tilde{L}$. Therefore, (A.9) holds with the right-hand side replaced by $(\tilde{L}t^{\gamma})^{1-\mu}(2S)^{\mu}$, and the corresponding integral in (A.10) converges for $\mu < 1-\gamma^{-1}\Big(1-\dfrac{1}{p}\Big)$. Cf. also (Grisvard 1985, Theorem 1.4.1.1).

A.2. In the following we will use the notations ρ and γu defined at the beginning of Subsection 2.3.

Proof of Proposition 2.2. First we want to remark that parts a) and b) of this proposition hold even in the case when Γ is of class $C^{0,1}$. Part a) is an immediate consequence of (Grisvard 1985, Theorem 1.5.1.10), part b) is formulated in (Grisvard 1985, Theorem 1.4.3.3). To prove c), we first note that $\gamma u \in W_p^{s-\frac{1}{p}}(\Gamma)$ and

$$\|\gamma u\|_{s-\frac{1}{p},p,\Gamma} \le c_3\|u\|_{s,p,G}\,, \tag{A.11}$$

see for instance (Grisvard 1985, Theorem 1.5.1.2). From a) we see that

$$\rho^{s-\frac{1}{p}}\|\gamma u\|_{0,p,\Gamma} \le C_1\Big(\rho^{s-1}\|u\|_{1,p,G} + \rho^{s}\|u\|_{0,p,G}\Big)\,, \tag{A.12}$$

and from part b) with $k = 1$ the desired result follows. □

A.3. Let us formulate a variant of Michlin's multiplier theorem, cf. (Triebel 1978, Section 2.2.4).

Let $p > 1$ and let $w(\xi)$ be a function on $\mathbb{R}^n \backslash \{0\}$ for which the derivatives of order not greater than $\left[\frac{n}{2}\right] + 1$ exist (where $\left[\frac{n}{2}\right]$ denotes the largest integer not greater than $\frac{n}{2}$). Assume that the estimate

$$|\xi|^{|\beta|}\,|\mathcal{D}_\xi^\beta w(\xi)| \le c_4 < \infty \tag{A.13}$$

holds for all $\xi \in \mathbb{R}^n \backslash \{0\}$ and for all β with $|\beta| \le \left[\frac{n}{2}\right] + 1$, where the constant c_4 does not depend upon ξ and β. Then the function w is a Fourier multiplier in $L_p(\mathbb{R}^n)$, i.e. the operator $u \mapsto F^{-1} w F u$ is a continuous operator in $L_p(\mathbb{R}^n)$. Moreover, the norm of this operator is not greater than $c(p)c_4$ with a constant $c(p)$ depending only on n and p.

We will use this theorem replacing n by $n-1$.

Proof of Proposition 2.3. a) First we derive some estimates for the operator corresponding to the Fourier multiplier $\Omega(\xi', x_n, \rho)$. Setting $r = |\xi'|$ and $\Omega(\xi', x_n, \rho) = \tilde{\Omega}(r, x_n, \rho)$, we see that for all multi-indices $\beta' = (\beta_1, \ldots, \beta_{n-1})$ the expression $|\xi'|^{|\beta'|}\,\mathcal{D}_{\xi'}^{\beta'}\Omega(\xi', x_n, \rho)$ is a finite sum of terms of the form

$$w_k(\xi') r^k \Big(\frac{\partial}{\partial r}\Big)^k \tilde{\Omega}(r, x_n, \rho)\,, \tag{A.14}$$

where $0 \le k \le |\beta'|$ and $w_k(\xi')$ is a bounded function in $\mathbb{R}^{n-1} \backslash \{0\}$. As

$$r^k \Big| \Big(\frac{\partial}{\partial r}\Big)^k \tilde{\Omega}(r, x_n, \rho) \Big| = (r x_n)^k \exp(-(r+\rho)x_n) \le k! \exp(-\rho x_n)\,, \tag{A.15}$$

we see that the condition of the multiplier theorem is fulfilled with the constant c_4 in (A.13) replaced by a constant times $\exp(-\rho x_n)$. Due to this theorem, for $u = F'^{-1}\Omega F' v$ the inequality

$$\|u(\cdot, x_n)\|_{0,p,\mathbb{R}^{n-1}} \le c_5\, \exp(-\rho x_n)\, \|v\|_{0,p,\mathbb{R}^{n-1}} \tag{A.16}$$

holds with c_5 independent (also) of x_n.

Similarly, the function

$$|\xi'|^{|\beta'|}\,\mathcal{D}_{\xi'}^{\beta'}\Big(|\xi'|\Omega(\xi', x_n, \rho)\Big)$$

is a finite sum of terms of the form (A.14) with r^k replaced by r^{k+1}. Therefore, we obtain the inequality

$$\Big\| \Big(F'^{-1} |\xi'| \Omega(\xi', x_n, \rho) F' v\Big)(\cdot, x_n) \Big\|_{0,p,\mathbb{R}^{n-1}} \le c_6\, \frac{\exp(-\rho x_n)}{x_n}\, \|v\|_{0,p,\mathbb{R}^{n-1}} \tag{A.17}$$

with c_6 independent (also) of x_n.

b) Using the integration with respect to x_n, we obtain from (A.16) that

$$\rho^s \|u\|_{0,p,\mathbb{R}^n_+} \le c_7\, \rho^{s-\frac{1}{p}}\, \|v\|_{0,p,\mathbb{R}^{n-1}}\,. \tag{A.18}$$

Now fix $\alpha' = (\alpha_1, \ldots, \alpha_{n-1})$ and $l \ge 0$ with $|\alpha'| + l \le s$. We want to estimate $\|F'^{-1}\xi'^{\alpha'}\mathcal{D}_n^l \Omega F' v\|_{0,p,\mathbb{R}^n_+}$ and write

$$\xi'^{\alpha'}\mathcal{D}_n^l\Omega = (-1)^l\, \frac{\xi'^{\alpha'}(|\xi'| + \rho)^l}{|\xi'|^s + \rho^s}\, (|\xi'|^s\Omega + \rho^s\Omega)\,. \tag{A.19}$$

For $\rho \ge \rho_0 > 0$ the quotient on the right-hand side of (A.19) is a Fourier multiplier whose norm can be estimated by a constant not depending upon x_n and ρ. As the term $\rho^s\Omega$ was already considered above, it remains to estimate

$$\left\| F'^{-1}|\xi'|^s\Omega F' v \right\|_{0,p,\mathbb{R}^n_+}\,. \tag{A.20}$$

For this we fix an extension $w \in W_p^s(\mathbb{R}^n_+)$ of v. From the explicit construction of such an extension in (Adams 1975, p. 201) we see that we may assume

$$\|w\|_{s,p,\mathbb{R}^n_+} \le c_8\, \|v\|_{s-\frac{1}{p},p,\mathbb{R}^{n-1}} \quad \text{and} \quad \|w(\cdot, x_n)\|_{0,p,\mathbb{R}^{n-1}} \le c_8\, \|v\|_{0,p,\mathbb{R}^{n-1}} \tag{A.21}$$

with a constant c_8 independent of v and x_n. Following (Volevich 1965), we write

$$F'^{-1}|\xi'|^s\Omega F' v = -\int_0^\infty \frac{\partial}{\partial\tau}\Big(F'^{-1}|\xi'|^s\Omega(\xi', x_n + \tau, \rho)(F'w)(\xi', \tau)\Big)\, d\tau = -u_1 - u_2 \tag{A.22}$$

with

$$\begin{aligned} u_1(x) &= \int_0^\infty F'^{-1}|\xi'|^s\mathcal{D}_n\Omega(\xi', x_n + \tau, \rho)(F'w)(\xi', \tau)\, d\tau\,, \\ u_2(x) &= \int_0^\infty F'^{-1}|\xi'|^s\Omega(\xi', x_n + \tau, \rho)(F'\mathcal{D}_n w)(\xi', \tau)\, d\tau\,. \end{aligned}$$

c) To estimate $\|u_1\|_{0,p,\mathbb{R}^n_+}$, we write $|\xi'|^s\mathcal{D}_n\Omega(\xi', x_n + \tau, \rho)$ in the form

$$-\frac{|\xi'|^{s-1}(|\xi'| + \rho)}{(1 + |\xi'|^2)^{\frac{s}{2}} + \rho^s}\, |\xi'|\, \Omega(\xi', x_n + \tau, \rho)\, \Big[(1 + |\xi'|^2)^{\frac{s}{2}} + \rho^s\Big]\,. \tag{A.23}$$

Again the norm of the quotient in (A.23) as a Fourier multiplier can be estimated by a constant independent of x_n and ρ. If we set

$$a(x', \tau) = F'^{-1}\Big[(1 + |\xi'|)^{\frac{s}{2}} + \rho^s\Big]\, (F'w)(\xi', \tau)\,, \tag{A.24}$$

we can apply (A.17) with v replaced by $a(\cdot, x_n)$ and get

$$\begin{aligned} \|u_1\|_{0,p,\mathbb{R}^n_+} &= \Big(\int_0^\infty \|u_1(\cdot,x_n)\|^p_{0,p,\mathbb{R}^{n-1}}\,dx_n\Big)^{\frac{1}{p}} \\ &\le c_9\Big(\int_0^\infty \Big[\int_0^\infty \frac{\exp(-\rho\tau)}{x_n+\tau}\|a(\cdot,\tau)\|_{0,p,\mathbb{R}^{n-1}}\,d\tau\Big]^p dx_n\Big)^{\frac{1}{p}}. \end{aligned} \tag{A.25}$$

The inner integral in (A.25) is the Hilbert transform $\tilde{\Phi}(x_n)$ of the function

$$\Phi(\tau) = \begin{cases} \exp(-\rho\tau)\|a(\cdot,\tau)\|_{0,p,\mathbb{R}^{n-1}}, & \tau \ge 0, \\ 0, & \tau < 0. \end{cases} \tag{A.26}$$

As the Hilbert transform is continuous in $L_p(\mathbb{R})$ (see, e.g., (Titchmarsh 1948, Chapter V)), we get

$$\|u_1\|_{0,p,\mathbb{R}^n_+} \le c_9\|\tilde{\Phi}\|_{0,p,\mathbb{R}} \le c_{10}\|\Phi\|_{0,p,\mathbb{R}}. \tag{A.27}$$

It remains to estimate

$$\begin{aligned} \|\Phi\|_{0,p,\mathbb{R}} &= \Big(\int_0^\infty \Big[\exp(-\rho\tau)\|a(\cdot,\tau)\|_{0,p,\mathbb{R}^{n-1}}\Big]^p d\tau\Big)^{\frac{1}{p}} \\ &\le \Big(\int_0^\infty \Big[\exp(-\rho\tau)\|F'^{-1}(1+|\xi'|^2)^{\frac{s}{2}}F'w(\cdot,\tau)\|_{0,p,\mathbb{R}^{n-1}}\Big]^p d\tau\Big)^{\frac{1}{p}} \\ &\quad + \Big(\int_0^\infty \Big[\exp(-\rho\tau)\,\rho^s\,\|w(\cdot,\tau)\|_{0,p,\mathbb{R}^{n-1}}\Big]^p d\tau\Big)^{\frac{1}{p}}. \end{aligned} \tag{A.28}$$

Using the inequality $\exp(-\rho\tau) \le 1$ and the first inequality in (A.21), the first integral on the right-hand side of (A.28) can be estimated by

$$\|w\|_{s,p,\mathbb{R}^n_+} \le c_8\,\|v\|_{s-\frac{1}{p},p,\mathbb{R}^{n-1}}. \tag{A.29}$$

Noting that

$$\int_0^\infty \rho\exp(-p\rho\tau)\,d\tau = \frac{1}{p}$$

and using the second inequality in (A.21), we see that the last integral in (A.28) is not greater than

$$c_{11}\,\rho^{s-\frac{1}{p}}\,\|v\|_{0,p,\mathbb{R}^{n-1}}. \tag{A.30}$$

From (A.27)–(A.30) we obtain that

$$\|u_1\|_{0,p,\mathbb{R}^n_+} \le c_{12}\,\|v\|_{s-\frac{1}{p},p,\mathbb{R}^{n-1}}. \tag{A.31}$$

d) To estimate $\|u_2\|_{0,p,\mathbb{R}^n_+}$, we write $|\xi'|^s\Omega(\xi', x_n+\tau,\rho)$ in the form

$$\frac{|\xi'|^{s-1}}{(1+|\xi'|^2)^{\frac{s-1}{2}}}\,|\xi'|\;\Omega(\xi',x_n+\tau,\rho)\left[(1+|\xi'|^2)^{\frac{s-1}{2}}\right] \tag{A.32}$$

and use the same steps as in c). Setting

$$b(x',\tau) = {F'}^{-1}(1+|\xi'|^2)^{\frac{s-1}{2}}F'\mathcal{D}_n w(\xi',\tau)\,, \tag{A.33}$$

we see that

$$\begin{aligned}\|u_2\|_{0,p,\mathbb{R}^n_+} &\le c_{13}\Big(\int_0^\infty\Big[\int_0^\infty \frac{1}{x_n+\tau}\|b(\cdot,\tau)\|_{0,p,\mathbb{R}^{n-1}}\,d\tau\Big]^p dx_n\Big)^{\frac1p}\\ &\le c_{14}\,\|v\|_{s-\frac1p,p,\mathbb{R}^{n-1}}\,.\end{aligned} \tag{A.34}$$

e) From (A.18), (A.31) and (A.34) we see that $|\!|\!|u|\!|\!|_{s,p,\mathbb{R}^n_+}$ is not greater than a constant times $|\!|\!|v|\!|\!|_{s-\frac1p,p,\mathbb{R}^{n-1}}$, which finishes the proof of Proposition 2.3. □

A.4. *Proof of Proposition* 2.4. We obviously have

$$\|A(x,\mathcal{D})u\|_{0,p,G}+\sum_{j=1}^m\|B_j(x,\mathcal{D})u\|_{2m-m_j-\frac1p,p,\Gamma}\le c_{15}\|u\|_{2m,p,G} \tag{A.35}$$

and

$$\begin{aligned}\rho^{2m-m_j-\frac1p}\,\|B_j(x,\mathcal{D})u\|_{0,p,\Gamma} &\le c_{16}\,\rho^{2m-m_j-\frac1p}\sum_{|\beta|\le m_j}\|(\mathcal{D}^\beta u)|_\Gamma\|_{0,p,\Gamma}\\ &\le c_{16}\sum_{|\beta|\le m_j}|\!|\!|(\mathcal{D}^\beta u)|_\Gamma|\!|\!|_{2m-m_j-\frac1p,p,\Gamma}\,.\end{aligned} \tag{A.36}$$

From Proposition 2.2 c) and 2.2 b) we see that

$$|\!|\!|(\mathcal{D}^\beta u)|_\Gamma|\!|\!|_{2m-m_j-\frac1p,p,\Gamma}\le c_{17}\,|\!|\!|\mathcal{D}^\beta u|\!|\!|_{2m-m_j,p,G}\le c_{18}\,|\!|\!|u|\!|\!|_{2m,p,G}\,, \tag{A.37}$$

which proves the inequality (2.15). □

A.5. *Proof of Proposition* 2.5. Let $a(\xi)$ be the symbol of $A(\mathcal{D})$, and fix an α with $|\alpha|\le 2m$. It is easily checked that the estimate

$$|\mathcal{D}_\xi^\beta[\xi^\alpha(a(\xi)-\lambda)^{-1}]|\le c_{19}\,(|\xi|+\rho)^{|\alpha|-|\beta|-2m}\le c_{19}\,\rho^{-2m+|\alpha|}\,|\xi|^{-|\beta|} \tag{A.38}$$

holds, where c_{19} is independent of ξ. If u is a solution of $(A(\mathcal{D})-\lambda)u=f$, we have $Fu=(a(\xi)-\lambda)^{-1}Ff$. From (A.38) and the multiplier theorem, we obtain for $|\alpha|\le 2m$

$$\|\mathcal{D}^\alpha u\|_{0,p,\mathbb{R}^n}\le c_{20}\,\rho^{-2m+|\alpha|}\,\|f\|_{0,p,\mathbb{R}^n}\,. \tag{A.39}$$

Therefore, the a priori estimate (2.16) holds, which also shows the uniqueness of the solution. On the other hand, from (A.39) we see that $u = F^{-1}(a(\xi) - \lambda)^{-1} Ff$ is an element of $W_p^{2m}(\mathbb{R}^n)$ and a solution. $\square$

A.6. *Proof of Proposition* 2.6. a) First we prove the a priori estimate for the case $f = 0$. Let $u \in W_p^{2m}(\mathbb{R}^n_+)$ be a solution of (2.17) with $f = 0$. Then $u = F'^{-1}(\sum_{j=1}^m \Omega_j F' g_j)$, where $\{\Omega_1, \dots, \Omega_m\}$ is the canonical basis of solutions of the equation on the half-line

$$\Big(a(\xi', \mathcal{D}_n) - \lambda\Big) v(x_n) = 0 \quad (x_n \geq 0)\,, \tag{A.40}$$

i.e. the basis of the space of all stable solutions of (A.40) which is determined by the boundary conditions

$$B_j(\xi', \mathcal{D}_n)\Omega_k(\xi', x_n, \lambda) = \delta_{jk} \;\text{ at } x_n = 0\,.$$

The function Ω_j can be written in the form

$$\Omega_j(\xi', x_n, \lambda) = \int_{\mathfrak{S}} e^{i\tau x_n} \tilde{\Omega}_j(\xi', \tau, \lambda)\, d\tau\,. \tag{A.41}$$

Here $\mathfrak{S} = \mathfrak{S}(\xi', \lambda)$ is a smooth contour in the half plane $\operatorname{Im}\tau > 0$ enclosing all zeros of the function $\tau \mapsto a(\xi', \tau) - \lambda$ with positive imaginary part, and the functions $\tilde{\Omega}_j(\xi', \tau, \lambda)$ are homogeneous in $(\xi', \tau, \lambda^{1/2m})$ of degree $-m_j - 1$ in all arguments. Cf. (Agranovich and Vishik 1964, Proposition 3.2) and (Agmon et al. 1959, Section 1).

For $\alpha' = (\alpha_1, \dots, \alpha_{n-1})$ and $l \geq 0$ with $|\alpha'| + l = 2m$, we can estimate

$$\|F'^{-1}\xi'^{\alpha'}\mathcal{D}_n^l \Omega_j F' g_j\|_{0,p,\mathbb{R}^n_+} + \rho^{2m}\|F'^{-1}\Omega_j F' g_j\|_{0,p,\mathbb{R}^n_+} \tag{A.42}$$

analogously to (Volevich 1965) and along the same steps as in the proof of Proposition 2.3. We differentiate in (A.41) under the integral sign and substitute $\tau = \tilde{\tau}(|\xi'|^2 + \rho^2)^{1/2}$. Using spherical coordinates $(r, \eta') \in \mathbb{R}^n$ with respect to (ξ', ρ), i.e. with $r = (|\xi'|^2 + \rho^2)^{1/2}$, we obtain in exactly the same way as in (Volevich 1965)

$$\Big|\mathcal{D}_{\xi'}^{\beta'}(|\xi'|^2 + \rho^2)^{-\frac{2m - m_j}{2}} \xi'^{\alpha'} \mathcal{D}_n^{l+1}\Omega_j(\xi', x_n, \lambda)\Big| \leq \frac{c_{21}}{x_n}\, |\xi'|^{-|\beta'|} \tag{A.43}$$

and

$$\Big|\mathcal{D}_{\xi'}^{\beta'}(|\xi'|^2 + \rho^2)^{-\frac{2m - m_j - 1}{2}} \xi'^{\alpha'} \mathcal{D}_n^{l}\Omega_j(\xi', x_n, \lambda)\Big| \leq \frac{c_{22}}{x_n}\, |\xi'|^{-|\beta'|}\,, \tag{A.44}$$

where the constants c_{21} and c_{22} are independent of ξ' and x_n. Now we fix an extension $h_j \in W_p^{2m-m_j}(\mathbb{R}^n_+)$ of g_j with

$$|\!|\!| h_j |\!|\!|_{2m-m_j,p,\mathbb{R}^n_+} \leq C_4\, |\!|\!| g_j |\!|\!|_{2m-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}}\,, \tag{A.45}$$

cf. Proposition 2.3. Writing ${F'}^{-1}{\xi'}^{\alpha'}\mathcal{D}_n^l\Omega_j F' g_j = -u_{j1} - u_{j2}$ with

$$\begin{aligned} u_{j1}(x) &= \int_0^\infty {F'}^{-1}{\xi'}^{\alpha'}\mathcal{D}_n^{l+1}\Omega_j(\xi', x_n+\tau,\lambda)(F'h_j)(\xi',\tau)\,d\tau\,, \\ u_{j2}(x) &= \int_0^\infty {F'}^{-1}{\xi'}^{\alpha'}\mathcal{D}_n^{l}\Omega_j(\xi', x_n+\tau,\lambda)(F'\mathcal{D}_n h_j)(\xi',\tau)\,d\tau\,, \end{aligned}$$

we get in the same way as in the proof of Proposition 2.3, using (A.43) and (A.44), the estimate

$$\|\!|{F'}^{-1}\Omega_j F' g_j|\!\|_{2m,p,\mathbb{R}^n_+} \le c_{23}\big(\|a\|_{0,p,\mathbb{R}^n_+} + \|b\|_{0,p,\mathbb{R}^n_+}\big) \tag{A.46}$$

where

$$a(x',x_n) = {F'}^{-1}(|\xi'|^2+\rho^2)^{\frac{2m-m_j}{2}}(F'h_j)(\xi',x_n)$$

and

$$b(x',x_n) = {F'}^{-1}(|\xi'|^2+\rho^2)^{\frac{2m-m_j-1}{2}}(F'\mathcal{D}_n h_j)(\xi',x_n)\,.$$

Therefore

$$\begin{aligned} \|\!|u|\!\|_{2m,p,\mathbb{R}^n_+} &\le c_{24}\sum_{j=1}^m \Big(\|\!|h_j|\!\|_{2m-m_j,p,\mathbb{R}^n_+} + \|\!|\mathcal{D}_n h_j|\!\|_{2m-m_j-1,p,\mathbb{R}^n_+}\Big) \\ &\le c_{25}\sum_{j=1}^m \|\!|g_j|\!\|_{2m-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}}\,, \end{aligned} \tag{A.47}$$

where we have used Proposition 2.2 b) and (A.45).

b) To prove the a priori estimate in the general case, we extend f to $\mathbb{R}^n$ by zero outside $\mathbb{R}^n_+$ and set

$$\tilde{u}_0 = F^{-1}(a(\xi)-\lambda)^{-1}Ff \in W_p^{2m}(\mathbb{R}^n)\,. \tag{A.48}$$

Then we can apply part a) to $u_1 = u - u_0$, where u_0 denotes the restriction of $\tilde{u}_0$ on $\mathbb{R}^n_+$. The function u_1 is a solution of (2.17) with $f \equiv 0$ and g_j replaced by $g_j - B_j(\mathcal{D})u_0$. Due to a), we have

$$\|\!|u_1|\!\|_{2m,p,\mathbb{R}^n_+} \le c_{26}\Big(\sum_{j=1}^m \|\!|B_j(\mathcal{D})u_0|\!\|_{2m-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}} + \sum_{j=1}^m \|\!|g_j|\!\|_{2m-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}}\Big)\,. \tag{A.49}$$

From Proposition 2.4 and Proposition 2.5 we see that the first sum in (A.49) is not greater than

$$c_{27}\|\!|u_0|\!\|_{2m,p,\mathbb{R}^n_+} \le c_{28}\|f\|_{0,p,\mathbb{R}^n_+}\,. \tag{A.50}$$

From (A.49) and (A.50) we obtain the a priori estimate (2.18).

c) Clearly the a priori estimate implies the uniqueness. On the other hand, if we define $\tilde{u}_0$ by (A.48), u_0 as the restriction of $\tilde{u}_0$, and u_1 by

$$u_1 = \sum_{j=1}^{m} F'^{-1} \Omega_j F' \Big(g_j - B_j(D) u_0 \Big) , \tag{A.51}$$

then the calculations above show that $u = u_0 + u_1 \in W_p^{2m}(\mathbb{R}^n_+)$ is a solution of (2.17). □

A.7. *Proof of Proposition* 6.1. Arguing as in Subsection 4.3, we can assume without loss of generality that $\Gamma \in C^\infty$. We also can assume that the operator $A(x, \mathcal{D})$ coincides with its principal part. We construct a smooth approximation to $A(x, \mathcal{D})$ as in Subsection 4.4. We also construct approximations $B_j^{(h)}(x, \mathcal{D})$ for $B_j(x, \mathcal{D})$ with C^∞ coefficients on Γ in local representations, such that these coefficients converge uniformly to the corresponding coefficients in $B_j(x, \mathcal{D})$ as $h \to 0$. The operators $A^{(h)}$ and $B_j^{(h)}$ define a smooth boundary problem, and we can assume that it is elliptic with parameter in $\mathcal{L}$ for all $h,\ 0 < h \le h_0$. Thus, a formula of the form (6.4) is true for these smooth boundary problems, and obviously the corresponding quantity $\Delta^{(h)}$ tends to Δ as $h \to 0$.

Now we will use the following result from (Beals 1967, p. 1059). Denote by $L^{(0,2m)}$ the space of bounded operators from $L_2(G)$ to $W_2^{2m}(G)$. Then for $T \in L^{(0,2m)}$ the quantities

$$\alpha(T) = \varliminf_{j\to\infty} s_j(T)\, j^{n/2m} \quad \text{and} \quad \beta(T) = \varlimsup_{j\to\infty} s_j(T)\, j^{n/2m} \tag{A.52}$$

are bounded and uniformly continuous on each bounded set in $L^{(0,2m)}$. We apply this result to $T = R(\lambda)$ and $T = R^{(h)}(\lambda)$, where $R^{(h)}(\lambda)$ is the resolvent of $A^{(h)}_{B^{(h)}}$. Note that the quantities $\alpha(R(\lambda))$, $\alpha(R^{(h)}(\lambda))$, $\beta(R(\lambda))$, and $\beta(R^{(h)}(\lambda))$ do not depend on λ (see (Beals 1967, Theorem 3.2)). It remains to check that for any $\varepsilon > 0$ we have

$$\|R(\lambda) - R^{(h)}(\lambda)\| < \varepsilon$$

for sufficiently small h and $\lambda \in \mathcal{L}$ with sufficiently large modulus. Here and below $\|\cdot\|$ is the norm in $L^{(0,2m)}$.

For this we set

$$\mathcal{A}(\lambda) = (A - \lambda, B_1, \ldots, B_m), \quad \mathcal{A}^{(h)}(\lambda) = (A^{(h)} - \lambda, B_1^{(h)}, \ldots, B_m^{(h)}) \tag{A.53}$$

and denote by $\mathcal{R}(\lambda)$ and $\mathcal{R}^{(h)}(\lambda)$ the corresponding inverse operators. Setting $f_0 = (f, 0, \ldots, 0)'$, we have

$$[R^{(h)}(\lambda) - R(\lambda)]f = [\mathcal{R}^{(h)}(\lambda) - \mathcal{R}(\lambda)]f_0 = \mathcal{R}^{(h)}(\lambda)[\mathcal{A}(\lambda) - \mathcal{A}^{(h)}(\lambda)]\mathcal{R}(\lambda) f_0. \tag{A.54}$$

For $u = \mathcal{R}(\lambda)f_0$ we have

$$|||u|||_{2m,2,G} \leq C_1\|f\|_{0,2,G} \tag{A.55}$$

with a constant C_1 not depending upon λ and f for $\lambda \in \mathcal{L}$ with sufficiently large modulus, according to the Basic Theorem. For

$$F = (\tilde{f}, \tilde{g}_1, \ldots, \tilde{g}_m) = [\mathcal{A}(\lambda) - \mathcal{A}^{(h)}(\lambda)]u$$

and any $\delta > 0$ we have

$$|||F||| = \|\tilde{f}\|_{0,2,G} + \sum |||\tilde{g}_j|||_{2m-m_j-\frac{1}{2},2,\Gamma} \leq (\delta + C_2(\delta)|\lambda|^{-\frac{1}{2m}})|||u|||_{2m,2,G} \tag{A.56}$$

if h is sufficiently small; again here $C_2(\delta)$ does not depend upon λ and u. Finally,

$$\|\mathcal{R}^{(h)}(\lambda)F\|_{2m,2,G} \leq C_3|||F||| \,, \tag{A.57}$$

where C_3 does not depend upon F, h and $\lambda \in \mathcal{L}$ with sufficiently large modulus. Inequalities (A.55)–(A.57) imply the desired estimate for $\|R(\lambda) - R^{(h)}(\lambda)\|$. □

References

Adams, R. A. (1975): *Sobolev Spaces.* Academic Press, New York etc.

Agmon, S. (1962): On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems. *Comm. Pure Appl. Math.* **15**, 119-147.

Agmon, S. (1965a): On kernels, eigenvalues, and eigenfunctions of operators related to elliptic problems. *Comm. Pure Appl. Math.* **18**, 627-663.

Agmon, S. (1965b): *Lectures on Elliptic Boundary Value Problems.* Van Nostrand Math. Studies, Van Nostrand, Princeton etc.

Agmon, S., Douglis, A., Nirenberg, L. (1959): Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. *Comm. Pure Appl. Math.* **22**, 623-727.

Agmon, S., Nirenberg, L. (1963): Properties of solutions of ordinary differential equations in Banach spaces. *Comm. Pure Appl. Math.* **16**, 121-239.

Agranovich, M. S. (1965): Elliptic singular integro-differential operators (Russian). *Uspekhi Mat. Nauk* **20**, No. 5, 3-120. (English transl. in *Russian Math. Surveys* **20**, No. 5, 1-121 (1965).)

Agranovich, M. S. (1977): Spectral properties of diffraction problems (Russian). Supplement to (Vojtovich et al. 1977), 289-416.

Agranovich, M. S. (1987): Some asymptotic formulas for elliptic pseudodifferential operators (Russian). *Funktsional. Anal. i Prilozhen.* **21**, No. 1, 63-65. (English transl. in *Funct. Anal. Appl.* **21**, No. 1 (1987).)

Agranovich, M. S. (1990): Elliptic Operators on Closed Manifolds (Russian). *Itogi Nauki i Tekhniki, Ser. Sovrem. Probl. Mat. Fund. Napravl.* **63**, VINITI, Moscow, 5-129. (English transl. in *Encyclopaedia Math. Sci.* **63**, Springer, Berlin etc., 1-130 (1994).)

Agranovich, M. S. (1992): On moduli of eigenvalues for non-self-adjoint Agmon–Douglis–Nirenberg elliptic boundary problems with a parameter (Russian). *Funktsional. Anal. i Prilozhen.* **26**, No. 2, 51-55. (English transl. in *Funct. Anal. Appl.* **26**, No. 2 (1992).)

Agranovich, M. S., Markus, A. S. (1989): On spectral properties of elliptic pseudo-differential operators far from self-adjoint ones. *Z. Anal. Anwend.* **8**, No. 3, 237-260.

Agranovich, M. S., Vishik, M. I. (1964): Elliptic problems with parameter and parabolic problems of general form (Russian). *Uspekhi Mat. Nauk* **19**, No. 3, 53-161. (English transl. in *Russian Math. Surv.* **19**, No. 3, 53-157 (1964).)

Beals, R. (1967): Classes of compact operators and eigenvalue distributions for elliptic operators. *Amer. J. Math.* **89**, 1056-1072.

Beals, R. (1970): Asymptotic behavior of the Green's function and spectral function of an elliptic operator. *J. Funct. Anal.* **5**, 484-503.

Boĭmatov, K. Kh. and Kostjuchenko, A. G. (1991): Spectral asymptotics of polynomial pencils of differential operators in a bounded domain (Russian). *Funktsional. Anal. i Prilozhen.* **25**, No. 1, 7-20. (English transl. in *Funct. Anal. Appl.* **25**, No. 1, 5-16 (1991).)

Faierman, M. (1988): On the eigenvalues of nonselfadjoint problems involving indefinite weights. *Math. Ann.* **282**, 369-377.

Faierman, M. (1990a): Elliptic problems involving an indefinite weight. *Trans. Amer. Math. Soc.* **320**, 253-279.

Faierman, M. (1990b): Non-selfadjoint elliptic problems involving an indefinite weight. *Comm. Partial Differential Equations* **15**, 939-982.

Faierman, M. (1995a): Eigenvalue asymptotics for a non-selfadjoint elliptic problem involving an indefinite weight. *Rocky Mountain J. Math.* **25**, 179-187.

Faierman, M. (1995b): On the eigenvalue asymptotics for a non-selfadjoint elliptic problem involving an indefinite weight. *Proc. Roy. Soc. Edinburgh*, *Sect.* A, **125**, 859-875.

Gagliardo, E. (1959): Ulteriori proprieta di alcune classi di funzioni in piu variabili. *Ricerche Mat.* **8**, 24–51.

Geymonat, G., Grisvard, P. (1967): Alcuni risultati di teoria spettrale per i problemi ai limiti lineari ellittici. *Rend. Sem. Mat. Univ. Padova* **38**, 127-173.

Gohberg, I. C., Kreĭn, M. G. (1965): *Introduction to the Theory of Linear Nonselfadjoint Operators in a Hilbert Space* (Russian). Nauka, Moscow. (English transl.: Amer. Math. Soc., Providence, R. I. (1969).)

Grisvard, P. (1985): *Elliptic Problems in Nonsmooth Domains.* Pitman, Boston etc.

Grubb, G. (1986): *Functional Calculus of Pseudo-Differential Boundary Problems.* Birkhäuser, Boston etc. (Second edition 1996.)

Hörmander, L. (1958): On the regularity of solutions of boundary problems. *Acta Math.* **99**, 225-264.

Krasnosel'skiĭ, M. A., Zabreĭko, P. P., Pustyl'nik, E. I., Sobolevskiĭ, P. E. (1966): *Integral Operators in Spaces of Summable Functions* (Russian). Nauka, Moscow. (English transl.: Noordhoff, Leyden (1976).)

Lidskiĭ, V. B. (1962): Summability of series in the principal vectors of non-selfadjoint operators (Russian). *Trudy Moskov. Mat. Obshch.* **11**, 3-35. (English transl.: *Amer. Math. Soc. Transl., Ser.* 2, **40**, 193-228 (1964).)

Markus, A. S. (1986): *Introduction to the Spectral Theory of Polynomial Operator Pencils* (Russian). Kishinev, Stiintsa. (English transl.: Amer. Math. Soc., Providence, R. I. (1988).)

Maz'ya, V. G. (1985): *Sobolev Spaces* (Russian). Leningrad University. (English transl.: Springer, Berlin etc. (1985).)

Maz'ya, V. G., Shaposhnikova, T. O. (1986): *Theory of Multipliers in Spaces of Differentiable Functions* (Russian). Leningrad University. (English transl.: Pitman, Boston etc. (1985).)

Mizohata, S. (1965): Sur les propriétés asymptotiques des valeurs propres pour les opérateurs elliptiques. *J. Math. Kyoto Univ.* **4-5**, 399-428.

Nirenberg, L. (1959): On elliptic partial differential equations. *Ann. Scuola Norm. Super. Pisa* **13**, 115–162.

Pattisier, G. (1977): Sur les functions d'un opérateur pseudo-differentiel elliptique. *Publ. Dép. Math.* **14**, No. 3, 21-58.

Roĭtberg, Ya. A. (1991): *Elliptic Boundary Value Problems in the Spaces of Distributions. Preprints* I–IV (Russian). Chernigov State Pedagogical Institute, Chernigov, Ukraine. (English transl.: Kluwer, Dordrecht (1996).)

Roĭtberg, Ya. A., Sheftel', Z. G. (1967): Boundary value problems with a parameter in L_p for systems elliptic in the Douglis–Nirenberg sense (Russian). *Ukrain. Mat. Zh.* **119**, 115-120.

Schechter, M. (1960): A generalization of the problem of transmission. *Ann. Scuola Norm. Sup. Pisa* **14**, 207-236.

Seeley, R. T. (1969): The resolvent of an elliptic boundary problem. *Amer. J. Math.* **91**, No. 4, 889-920.

Stein, E. M. (1970): *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, Princeton, N. J.

Titchmarsh, E. C. (1948): *Introduction to the Theory of Fourier Integrals.* Second edition. Oxford University Press, London.

Triebel, H. (1978): *Interpolation. Function Spaces. Differential Operators.* North-Holland, Amsterdam etc.

Vojtovich, N. N., Katsenelenbaum, B. Z., Sivov, A. N. (1977): *Generalized Method of Eigenoscillations in Diffraction Theory* (Russian). Nauka, Moscow. (Revised edition, in English: Akademie-Verlag, Berlin, in preparation.)

Volevich, L. R. (1965): Solvability of boundary value problems for general elliptic systems (Russian). *Mat. Sb.* **68** (110), No. 3, 373-416. (English transl.: *Amer. Math. Soc. Transl.*, *Ser.* 2, **67**, 182-225 (1968).)

Addresses:

M. S. Agranovich, Moscow State Institute of Electronics and Mathematics (MGIEM), Moscow 109028, Russia.
msa@funcan.msk.su

R. Denk, NWF I – Mathematik, Universität Regensburg, D-93040 Regensburg, Germany.
robert.denk@mathematik.uni-regensburg.de

M. Faierman, Department of Mathematics, University of the Witwatersrand, Johannesburg, WITS 2050, South Africa.
036mef@cosmos.wits.ac.za