WEALTH DISTRIBUTIONS IN ASSET EXCHANGE MODELS

P. L. KRAPIVSKY AND S. REDNER*

How do individuals accumulate wealth as they interact economically? We outline the consequences of a simple microscopic model in which repeated pairwise exchanges of assets between individuals build the wealth distribution of a population. This distribution is determined for generic exchange rules — transactions that involve a fixed amount or a fixed fraction of individual wealth, as well as random or greedy exchanges. In greedy multiplicative exchange, a continuously evolving power law wealth distribution arises, a feature that qualitatively mimics empirical observations.

PACS No.: 02.50.Ga, 05.70.Ln, 05.40.+j

Perspective

The economy is a complex interacting system that responds to a multitude of influences and extends over a wide range of monetary scales. As experience with financial crises continues to demonstrate, understanding how an economy develops and how it is influenced by externalities remains poorly understood. Basic questions about what causes financial crises and how to deal with them continue to be hotly debated, with little sign that a fundamental understanding is emerging[1–3].

What can statistical physics contribute to this discussion? Not much, if the goal is to predict the economy next year. However, statistical physics possesses powerful theoretical tools that have proven useful in describing specific financial phenomena, such as the Black-Scholes options pricing formula[4]. There are many parallels between statistical physics and economic phenomena, and physics-based modeling has helped facilitate conceptual developments in finance and economics[5–7].

In classic economic theories, humans, or companies, are considered as rational actors that respond deterministically to external conditions. More recently, stochastic tools have been applied to the economy, particularly to financial modeling. The stochastic approaches that are conventionally employed are Brownian motion and its generalizations. In physics, a similar approach was followed to describe non-deterministic systems, where the interaction between a particle and its environment was mimicked by noise, while interactions between microscopic entities (such as Brownian particles) were ignored. This development (associated with physicists like Einstein, Langevin, and Stratonovich, and mathematicians like Kolmogorov, Feller, and Itô) led to increasingly sophisticated stochastic processes[8–11], a research thread that is still active.

Over the last 40 years a new approach that combines the stochastic behavior of elemental entities as well as their mutual interactions has emerged[12, 13]. While a few-particle interacting system is hopelessly complicated and beyond the reach of analytical techniques, a dramatic simplification arises for a many-particle system because we can often make *statistical* predictions about its fate. That a

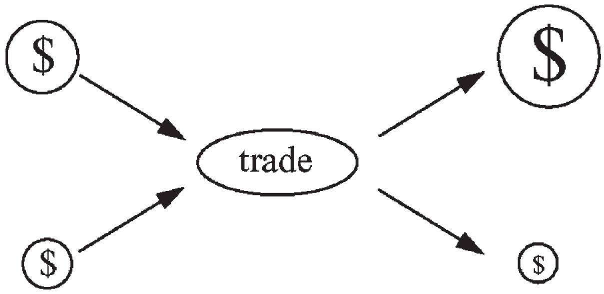

Fig.. 1. Illustration of the asset exchange model.

* Department of Physics, Boston University, Boston, MA 02215, USA.

macroscopic interacting system is simpler than its few-element counterpart has several appellations — the law of large numbers, ergodicity, etc. — and it justifies the utility of statistical physics and probability theory in attempts to understand economic processes.

In this short review, we present an interacting many-agent *asset exchange* model that can be quantitatively analyzed using statistical physics tools. An agent could be a single person or a self-contained economic entity, such as a company. In this model, the interaction between two agents results in a redistribution of their assets. We regard an asset as any economic attribute — cash, goods, or other materials — that contributes to overall individual wealth.

The macroeconomy is viewed as the result of a large number of asset exchanges between randomly-selected pairs of agents. Through these exchanges a global wealth distribution develops, and we want to understand how generic features of this distribution depend on the nature of the exchanges. The notion that the wealth distribution is driven by two-person exchanges appears to have been first considered in the economics literature by Angle[14, 15]. In the physics community, this approach was introduced by Ispolatov *et al.*[16], and related perspectives on this subject include Refs.[17–21]. A comprehensive review of this research topic is given in[22] and an engaging non-technical exposition appears in[23].

Additive Exchange

As a preliminary, we first study additive exchange processes, in which a fixed amount of asset is exchanged between two agents, independent of their wealth before a trade occurs. At the outset, we have to determine how to treat agents whose wealth reaches zero as a result of many unfavorable trades. We treat such penurious agents as economically "dead", so that they no longer participate in the evolution of the wealth. Mathematically, this rule corresponds to imposing an absorbing boundary condition on the density of agents of zero wealth. An alternative is to impose a reflecting boundary condition at zero wealth, so that all agents continue to economically interact, even if they have no wealth[17]. In this latter case, the wealth follows the Boltzmann distribution of equilibrium statistical mechanics, with an effective temperature equal to the average amount of wealth per agent.

Under the condition that bankrupt agents are eliminated from further economic activity, we determine the consequences of: (i) *fair* transactions, where either agent is equally likely to profit in an interaction, and (ii) *greedy* transactions, in which the richer agent profits in an interaction. For simplicity, each agent is assumed to possess an integer-valued amount of assets and that one unit of asset is transferred between traders in each interaction.

Fair Transactions : In a fair exchange, the wealth of two agents evolves as $(j, k) \to (j \pm 1, k \mp 1)$; the direction of the exchange is independent of their starting wealth. The wealth distribution evolves by selecting two agents at random who exchange one unit of wealth and repeating this elemental step *ad infinitum*. We assume that all agents are equally likely to interact with any other agent (corresponding to the mean-field limit in statistical physics).

In this limit, the evolution of the wealth distribution is described by a *master equation* that accounts for the changes in wealth in each microscopic interaction between agents. Let $c_k(t)$ be the density of agents with wealth k. In random additive exchange, the master equation is

$$\frac{dc_k}{dt} = N\left[c_{k+1} + c_{k-1} - 2c_k\right], \qquad (1)$$

where $N(t) \equiv \sum_{k\geq 1} c_k(t)$ is the density of economically viable agents. The first two terms on the right-hand side account for the gain in c_k due to the transactions $(j, k + 1) \to (j + 1, k)$ and $(j, k - 1) \to (j - 1, k)$, respectively, while the last term accounts for the loss in c_k due to the transactions $(j, k) \to (j \pm 1, k \mp 1)$. Since these transactions require the presence of an agent of wealth k or $k \pm 1$ and an agent of arbitrary wealth, all terms on the right-hand side involve N times a concentration. The density of agents with a single unit of wealth evolves by $dc_1/dt = N(c_2 - 2c_1)$; this equation may also be written in the same form as Eq. (1) by imposing the absorbing boundary condition $c_0(T) = 0$.

Introducing the time-like variable, $T = \int_0^t dt'\, N(t')$, we reduce Eq. (1) to the discrete diffusion equation

$$\frac{dc_k}{dT} = c_{k+1} + c_{k-1} - 2c_k\ , \qquad (2)$$

which may be solved for any initial condition[11,13]. When all agents start with unit wealth, $c_k(0) = \delta_{k,1}$, we may account for the absorbing boundary condition by augmenting the initial condition with an "image" contribution due to agents with initial wealth – 1; that is, $c_k(0) = \delta_{k,1} - \delta_{k,-1}$. The solution to Eq. (2) subject to these initial conditions is[13]

$$c_k(T) = e^{-2T}\left[I_{k-1}(2T) - I_{k+1}(2T)\right], \qquad (3)$$

where I_n is the modified Bessel function of order n. Correspondingly, the total density of active agents $N(T)$ is

$$N(T) = e^{-2T}\left[I_0(2T) - I_1(2T)\right]. \tag{4}$$

In the limit $T \to \infty$, the asymptotic behaviors of Eqs. (3) and (4) are :

$$c_k \simeq \frac{k}{\sqrt{4\pi T^3}} e^{-k^2/4T}, \quad N \simeq (\pi T)^{-1/2}. \tag{5}$$

These asymptotics apply, up to an overall factor, to all initial conditions that decay sufficiently rapidly with k. An important feature of (5) is the emergence of *scaling*: the distribution $c_k(T)$ depends on the scaled wealth, $k/\sqrt{T}$, rather than separately on the variables k and T. Similar scaling behavior arises in numerous interacting particle systems[13]. Normally, scaling is *postulated* and then verified analytically or numerically. For asset exchange, we deduce the validity of scaling from the exact solution. We now express the asymptotic solution (5) in terms of the physical time t by using $t(T) = \int_0^T dT'/N(T') \simeq \frac{2}{3}\sqrt{\pi T^3}$ to eliminate T and give

$$c_k \simeq \frac{k}{3t}\exp\left[-\left(\frac{\pi}{144}\right)^{1/3}\frac{k^2}{t^{2/3}}\right], \quad N \simeq \left(\frac{2}{3\pi t}\right)^{1/3} \tag{6}$$

The number of viable agents decreases as $t^{-1/3}$ and their typical wealth grows as $t^{1/3}$. While this model is not realistic, it illustrates the efficacy of a statistical physics perspective in solving an interacting many-body system.

Instead of removing bankrupt agents, let us provide each of them with 'welfare' of a single unit of asset. In this case, the economically viable population density is always $N = 1$ and the master equation for the wealth distribution simplifies to

$$\frac{dc_k}{dt} = c_{k+1} + c_{k-1} - 2c_k \qquad k \geq 2,$$

$$\frac{dc_1}{dt} = c_2 - c_1, \qquad\qquad k = 1, \tag{7}$$

We can extend the first of these equations to all k and also subsume the equation for c_1 by choosing the initial condition $c_{1-k}(0) = c_k(0)$, with $c_1(0) = c_0(0) = 1$, and $c_k(0) = 0$ for $k \neq 0;$ 1. The solution to (7) subject to this initial condition is

$$c_k = e^{-2t}\left[I_{k-1}(2t) + I_k(2t)\right]. \tag{8}$$

Because of this injection of assets to destitute agents, the total wealth density of the population, $M = \sum_{k\geq 1} kc_k$, grows with time as

$$M = e^{-2t}\sum_{k\geq 1} k\left[I_{k-1}(2t) + I_k(2t)\right] \simeq 2\sqrt{t/\pi},$$

as $t \to \infty$. In this toy model, the rate of welfare expenditure to keep everyone solvent decreases with time!

Greedy Transactions : In greedy exchange, the richer agent is exploitative and always takes one unit of wealth from the poorer agent in each interaction, as represented by $(j, k) \to (j+1, k-1)$ for $j \geq k$. The densities $c_k(t)$ now evolve according to

$$\frac{dc_k}{dt} = c_{k-1}\sum_{j=1}^{k-1} c_j + c_{k+1}\sum_{j=k+1}^{\infty} c_j - c_k\left(N + c_k\right). \tag{9}$$

The first term on the right accounts for the gain in c_k due to an agent with wealth $k - 1$ taking one wealth unit from a poorer trading partner. Similarly, the second term accounts for an agent with wealth $k + 1$ losing one unit of wealth to a richer trading partner. The last term accounts for the loss of c_k when an agent of wealth k trades with anyone; the extra factor of c_k accounts for the loss of both agents of wealth k when two such agents interact.

While this set of non-linear equations appears intractable by exact methods, they are readily amenable to a scaling analysis[13]. We first re-write Eq. (9) as

$$\frac{dc_k}{dt} = -c_k(c_k + c_{k+1}) + N(c_{k-1} - c_k) + \left(c_{k+1} - c_{k-1}\right)\sum_{j=k}^{\infty} c_j, \tag{10}$$

and make the *scaling ansatz* $c(t) \simeq k_*^{-2} C\left(k/k_*\right)$, where $k_*(t)$ is the typical wealth of each agent. That is, the wealth distribution at different times is invariant when wealth is measured in units of the time-dependent typical wealth. The prefactor k_*^{-2} ensures that the total wealth of the population, $\sum_k kc_k(t)$, is conserved, while the condition $\sum_k kc_k(t) = N(t)$ gives $k_*(t) \sim 1/N(t)$. Substituting now the scaling form $c_k(t) \simeq N^2 C(x)$, with $x = kN$, in Eq. (10) and taking the continuum limit gives

$$C(0)\left[2C + xC'\right] = 2C^2 + C'\left[1 - 2\int_x^{\infty} dy\, C(y)\right], \tag{11}$$

where $C' = dC/dx$. The scaling function must satisfy

$$\int_0^{\infty} dx\, C(x) = 1, \text{ and } \int_0^{\infty} dx\, x\, C(x) = 1, \tag{12}$$

that follow from $N = \Sigma\, c_k(t) \simeq N \int dx\, C(x)$ and setting the (conserved) wealth density to one, $\sum_k kc_k(t) = 1$.

Equation (11) is soluble by elementary techniques[16], and the asymptotic wealth is simply the step function

$$c_k(t) = \begin{cases} 1/(2t), & k < 2\sqrt{t}, \\ 0, & k \geq 2\sqrt{t}, \end{cases} \tag{13}$$

while the density of active agents decays as $N(t) = t^{-1/2}$. In greedy exchange, the number of viable agents decays faster than in random exchange and the population is slightly wealthier, with the average wealth growing as $t^{1/2}$ rather than as $t^{1/3}$.

Multiplicative Exchange

While additive exchange provides instructive warmup examples, multiplicative exchanges, where a fixed fraction of the current wealth of one of the agents is traded, are economically more realistic. For example, investment returns are generally quoted as percentages rather than absolute amounts. A trade now has the form $(x, y) \to (x - \alpha x,\, y + \alpha x)$, with $0 < \alpha < 1$ the fraction of the loser's assets that are gained by the winner. By multiplicative exchanges agents can never go bankrupt, but they can become arbitrarily poor.

Fair Transactions : If an agent gains or loses with equal probabilities in a transaction, the wealth distribution evolves as

$$\frac{\partial c(x)}{\partial t} = \frac{1}{2} \iint dy\, dz\, c(y)\, (z) \times \left[-\delta(x - z) - \delta(x - y) \right.$$

$$\left. + \delta\left(y(1-\alpha) - x \right) + \delta(z + \alpha y - x) \right]. \tag{14}$$

The delta functions cleanly indicate the origin of the various terms in this equation. For example, the first two terms on the right account for the loss of agents of wealth x due to trades with any other agents. The next two terms account, respectively, for the gain in $c(x)$ due to the exchanges $\left(\frac{x}{1-\alpha}, y \right) \to \left(x,\, y + \frac{\alpha x}{1-\alpha} \right)$ and $\left(y, x - \alpha y \right) \to \left(y(1-\alpha), x \right)$. Integrating over the delta functions, the master equation becomes

$$\frac{\partial c(x)}{\partial t} = -c(x) + \frac{1}{2(1-\alpha)}\, c\left(\frac{x}{1-\alpha} \right) + \frac{1}{2\alpha} \int_0^x dy\, c(y)\, c\left(\frac{x-y}{\alpha} \right), \tag{15}$$

where we set the (conserved) total density to one.

Equation (15) is daunting, and it is simpler to study the evolution of the moments, $M_n(t) \equiv \int_0^\infty dx\, x^n c(x,t)$, that quantify the wealth of a typical agent. It is straightforward to verify that the first two moments, the population M_0 and the wealth density M_1, are conserved; we choose $M_0 = 1$ and $M_1 = M$ without loss of generality. More interesting behavior arises for the second moment equation

$$\frac{dM_2(t)}{dt} = -\alpha(1-\alpha)\, M_2(t) + \alpha M^2,$$

whose solution is

$$M_2(t) = \frac{M^2}{1-\alpha} + \left[M_2(0) - \frac{M^2}{1-\alpha} \right] e^{-\alpha(1-\alpha)t}. \tag{16}$$

All moments beyond the second also converge to non-zero steady-state values. This steady state arises because the wealth of a rich agent substantially diminishes in a losing multiplicative exchange, but its wealth increases only slightly in a winning exchange. Conversely, a poor agent suffers a slight loss in a losing exchange but can gain substantially in a winning exchange. These two countering outcomes tends to move all agents toward a middle class.

Greedy Transactions : When only the richer agent gains in an exchange, the master equation is now

$$\frac{\partial c(x)}{\partial t} = -c(x) + \frac{1}{1-\alpha}\, c\left(\frac{x}{1-\alpha} \right) \int_{x/(1-\alpha)}^\infty dy\, c(y)$$

$$+ \frac{1}{\alpha} \int_{x/(1+\alpha)}^\infty dy\, c(y)\, c\left(\frac{x-y}{\alpha} \right). \tag{17}$$

Numerically, we find that the resulting wealth distribution is a power law (Fig. 2), with most of the population impoverished. Pervasive impoverishment arises because greedy exchange causes the poor to become poorer and the rich to become richer, but wealth conservation forces there to be many more poor than rich agents. In the longtime limit, a small fraction of the population possesses most of the wealth.

An exact formal solution to Eq. (17) is[16]

$$e(x,t) = \frac{A}{xt}, \quad \text{with} \quad A = -\frac{1}{\ln(1-\alpha)}. \tag{18}$$

This distribution is pathological, however, because positive the moments $M_n(t)$ of this distribution are divergent. Thus Eq. (18) can only apply within an intermediate scaling regime $x_1(t) < x < x_2(t)$, a restriction that leads to finiteness of all the moments. To determine this scaling region, we use Eq. (18) to compute the moments and obtain:

$$M_0(t) \sim \int_{x_1}^{x_2} dx\, c(x,t) \sim \frac{A \ln\left(x_2/x_1\right)}{t},$$

$$M_1(t) \sim \int_{x_1}^{x_2} dx\, x\, c(x,t) \sim \frac{Ax_2}{t}. \qquad (19)$$

Since $M_0 = 1$ and M_1 are constants, we infer that $x_1(t) \sim e^{-t/A} = (1-\alpha)^t$ and $x_2(t) \propto t$. These cutoffs correspond to the wealth of the poorest and richest agent, respectively, in the population. It is only within these ranges that the wealth distribution is a power law, as shown in Fig. 2.

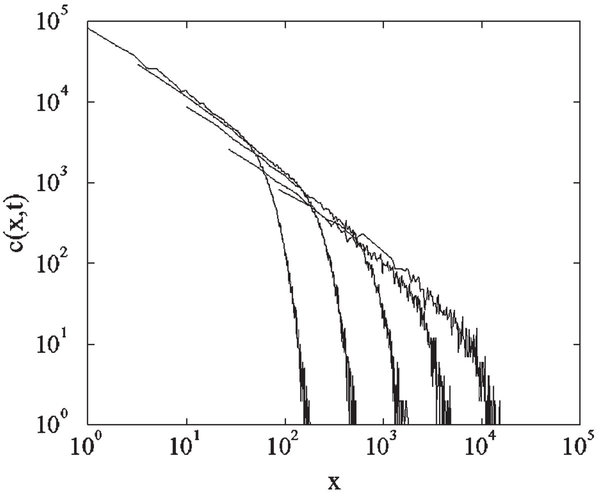

Fig. 2. The power-law wealth distribution $c(x)$ of greedy multiplicative exchange with α = 0:5 on a double logarithmic scale for times t = 1:5n, with n = 7, 10, 13, and 16.

Discussion

Asset exchange represents a parsimonious mechanism for the gain and loss of individual wealth in an economically active population. In spite of the obvious shortcomings of considering only this single factor among the myriad of influences on individual wealth, asset exchange models lead to a rich array of wealth distributions. For additive asset exchange, the wealth distribution can be explicitly derived for a variety of microscopic exchange rules. For greedy multiplicative exchange, where the richer agent always gains in an interaction, a scaling-based approach indicates that the wealth distribution has an evolving power law form, $c(x,t) \propto 1/(xt)$.

Power-law distributions occur in the high-wealth tail of the wealth distribution in various economies, with the associated exponent in the range of 1.6–2.2 (see Refs.[21, 24]). As alluded to in the introduction, a variety of stochastic models, where agent undergoes an independent stochastic process, have also been invoked to argue for this power law[25–29]. In contrast, greedy multiplicative exchange is based on a combination of stochasticity and microscopic interactions between agents. There are many directions in which asset exchange models have been extended to make them more realistic; recent work along these directions can be found in Ref.[30–36]. Specific examples of such additional elements include the incorporation of the saving of assets[19, 21, 37], speculative trading[38], and other forms of wealth redistribution. The notion of exchange of assets has also been applied to construct a migration model for the distribution of city sizes[39]. It should prove interesting to examine the role of such redistribution mechanisms in the ideologically-free setting of statistical physics modeling. The underlying assumption of conserved assets in an exchange neglects the possibility of wealth growth because of the exploitation of a natural resource, technological developments, or by both agents benefiting in exchanges. These are issues that appear ripe for further development.

We thank Slava Ispolatov for his initial collaboration on this project. □

References

1. *The Economy as an Evolving Complex System*, edited by P. W. Anderson *et al.* (Addison-Wesley, Redwood City, CA), (1988).
2. *The Theory of Income and Wealth Distribution*, edited by Y. S. Brenner et al., (St. Martin's Press, New York), (1988).
3. S. A. Kauffman, *The Origin of Order: Self-Organization and Selection in Evolution* (Oxford University Press, New York), (1993).
4. F. Black and M. Scholes, *J. Political Economy* **81** 637 (1973).
5. J.-P. Bouchaud and M. Potters, *Theory of Financial Risk and Derivative Pricing* (Cambridge University Press, Cambridge), (2000).
6. R. N. Mantegna and H. E. Stanley, *An Introduction to Econophysics* (Cambridge University Press, Cambridge), (2000).
7. J. D. Farmer, D. E. Smith, and M. Shubik, *Physics Today* **58**, 37 (2005).
8. S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943). [Reprinted in *Selected Papers on Noise and Stochastic Processes*, ed. N. Wax (Dover, New York), (1954)].
9. K. Itó and H. P. McKean, *Diffusion Processes and Their Sample Paths* (Springer-Verlag, Berlin), (1965).
10. B. Duplantier, in: *Einstein, 1905-2005*, eds. Th. Damour, O. Darrigol, B. Duplantier and V. Rivasseau (Birkhäuser Verlag, Basel), (2006); arXiv:0705.1951.
11. W. Feller, *An Introduction to Probability Theory* (Wiley, New York), (1971).
12. T. M. Liggett, *Stochastic Interacting Systems: Contact, Voter, and Exclusion Processes* (Springer, New York), (1999).

13. P. L. Krapvisky, S. Redner, and E. Ben-Naim, *A Kinetic View of Statistical Physics* (Cambridge University Press, Cambridge), (2010).
14. J. Angle, *J. Math. Sociol.* **18**, 27 (1993).
15. J. Angle, *Physica A* **367**, 388 (2006).
16. S. Ispolatov, P. L. Krapivsky, and S. Redner, *Eur. Phys. J. B* **2**, 267 (1998)
17. A. A. Drǎgulescu and V. M. Yakovenko, *Eur. Phys. J. B* **17**, 723 (2000)
18. J.-P. Bouchaud and M. Mézard, *Physica A* **282**, 536 (2000).
19. A. Chakraborti and B. K. Chakrabarti, *Eur. Phys. J. B* **17**, 167 (2000).
20. F. Slanina, *Phys. Rev. E* **69** 046102 (2004).
21. A. Chatterjee and B. K. Chakrabarti, *Eur. Phys. J. B* **60** 135 (2007).
22. V. M. Yakovenko and J. B. Rosser, *Rev. Mod. Phys.* **81**, 1703 (2009).
23. B. Hayes, *American Scientist* **90**, 400 (2002).
24. M. Levy and S. Solomon, *Physica A* **242**, 90 (1997).
25. D. G. Champenowne, *Economic Journal* **63**, 318 (1953).
26. B. B. Mandelbrot, *Intl. Economic Review* **1**, 79 (1960).
27. M. Levy and S. Solomon, *Int. J. Mod. Phys. C* **7**, 595 (1996).
28. D. Sornette and R. Cont, J. Phys. I France **7**, 431 (1997); D. Sornette, *Physica A* **250**, 295 (1998).
29. A.-H. Sato, H. Takayasu and Y. Sawada, *Phys. Rev. E* **61**, 1081 (2000).
30. N. Ding N, N. Xi N, and Y. G. Wang, *Physica A* **367**, 415 (2006).
31. Z. Q. Lin, J. H. Ke, and G. X.Ye, *Phys. Rev. E* **74**, 046113 (2006).
32. C. F. Moukarzel, S. Goncalves, and J. R. Iglesias, *Eur. Phys. J. Special Topics* **143**, 75 (2007).
33. A. Chatterjee, *Eur. Phys. J. B* **67**, 593 (2009).
34. M. A. Saif and P. M. Gade, *Physica A* **388**, 697 (2009).
35. M. Patriarca, E. Heinsalu, and A. Chakraborti, *Eur. Phys. J. B* **73**, 145 (2010).
36. A. Sokolov, A. Melatos, and T. Kieu, *Physica A* **389**, 2782 (2010).
37. A. Chatterjee, B. K. Chakrabarti, and S. S. Manna, *Physica A* **335**, 155 (2004).
38. S. Cordier, L. Pareschi, and G. Toscani, *J. Stat. Phys.* **120**, 253 (2005).
39. F. Leyvraz and S. Redner, *Phys. Rev. Lett.* **88**, 068301 (2002).