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WEDGE PRODUCT OF POSITIVE CURRENTS AND BALANCED MANIFOLDS

LUCIA ALESSANDRINI AND GIOVANNI BASSANELLI

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Abstract. We define on a manifold X a wedge product $S \wedge T$ of a closed positive $(1, 1)$-current S, smooth outside a proper analytic subset Y of X, and a positive pluriharmonic (k, k)-current T, when k is less than the codimension of Y. Using this tool, we prove that if M is a compact complex manifold of dimension $n \geq 3$, which is Kähler outside an irreducible curve, then M carries a balanced metric.

1. Introduction. The wedge product of closed positive currents was studied by Bedford and Taylor, who proposed to define $i\partial\bar{\partial}u \wedge T$ as $i\partial\bar{\partial}(uT)$, where T is a closed positive (k, k)-current and u is a plurisubharmonic function (see [8]). Obviously the potential u must satisfy some conditions, to get a well-defined current uT. For instance, if u is not locally bounded, one may require that the unbounded locus of u (intersected with the support of T) is contained in an analytic subset Y of the manifold X, with $k + \dim Y < \dim X$ (see [14], III.4.10; more generally, see Remark 2).

In [7] the second author considered such a product when $i\partial\bar{\partial}u$ is smooth outside a smooth curve and T is a positive pluriharmonic (i.e., $i\partial\bar{\partial}$-closed) $(1, 1)$-current on a three-dimensional manifold. Moreover, in [11] Dinh and Sibony show that in a compact Kähler manifold the wedge product is well-defined when S is a closed positive $(1, 1)$-current with continuous potential and T is a positive pluriharmonic (k, k)-current.

Here we are interested in a "geometrical setting"of the problem, to obtain some information on the existence of particular metrics on the ambient manifold. Hence we improve the results of [7] as follows:

THEOREM 2.2. *Let Y be a proper analytic subset of a complex manifold X. Let S be a positive closed $(1, 1)$-current on X, smooth on $X - Y$, and let T be a positive pluriharmonic (k, k)-current on X, with $k+\dim Y < \dim X$. Then there exists a unique current on X, denoted by $S \wedge T$, with the following property*:

If g is a solution of $S = i\partial\bar{\partial}g$ in an open subset $U \subset X$, and if $\{g_j\}$ is a sequence of smooth plurisubharmonic functions on U, which converge to g in $C^\infty(U - Y)$, then $\lim_j i\partial\bar{\partial}g_j \wedge T = S \wedge T$ in U.

Moreover, when X is compact, $S \wedge T$ belongs to the right (Aeppli) cohomology class (Proposition 2.4).

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Let us make a couple of remarks. We do not need a Kähler ambient, nevertheless this hypothesis would probably simplify the proof (as noted in [7], Remark 1.4). Our main hypothesis, that is, the smoothness of the potential u outside Y, is probably a technical hypothesis, whereas the request on the dimension of Y reflects the classical (i.e., closed) case.

These results allow us to get some interesting geometric information on Kähler or balanced metrics on the ambient (compact) manifold. Recall that a balanced metric on a n-dimensional manifold is a hermitian metric whose Kähler form ω satisfies $d\omega^{n-1} = 0$. In particular, we consider the case of a compact n-dimensional manifold M, $n \geq 3$, which is Kähler outside an irreducible curve C. We proved in [4] the following result: "If M does not carry itself a Kähler metric, then either C is the component of a boundary (i.e., $[C] = \partial\bar{A} + \bar{\partial}A$ for a suitable current A) or C is part of the component of a boundary (i.e., there is a positive current $B \neq 0$ on M such that $\chi_C B = 0$ and $[C] + B = \partial\bar{A} + \bar{\partial}A$)."

In this last case, M carries a balanced metric (see [5] when C is smooth, but the proof also works for an irreducible curve, as noted in Proposition 3.2).

The results proved in Chapter 2 allow us to manage also the other case, inspired by [7]; thus, summing up, we get:

THEOREM 3.4. *If M is a compact complex manifold of dimension $n \geq 3$, and C is an irreducible curve in M such that $M - C$ is Kähler, then either M is Kähler itself, or $[C]$ is the component of a boundary and M is p-Kähler for every $p \geq 2$, or $[C]$ is part of the component of a boundary and $M \in \mathcal{C}$. In all cases, M is balanced.*

Finally, we get also some partial results when M is Kähler outside an analytic subset of dimension bigger than one.

2. Wedge product of positive currents. As regards forms and currents, we shall use the notation of [14]. A (k, k)-current T on a n-dimensional manifold X is a current of bidegree (k, k) or bidimension (p, p), where $p + k = n$; $T \in \mathcal{D}'_{p,p}(X)^+$ means that T is a positive (k, k)-current ($T \geq 0$), while $T \in \mathcal{D}'_{p,p}(X)_{\boldsymbol{R}}$ means that T is a real (k, k)-current; a current T is called pluriharmonic (plurisubharmonic) if $i\partial\bar{\partial}T = 0$ ($i\partial\bar{\partial}T \geq 0$).

When a function u is plurisubharmonic (and thus in $L^1_{\mathrm{loc}}(X)$), then $T := i\partial\bar{\partial}u$ is a closed positive $(1, 1)$-current; the converse holds locally.

Suppose Y is an analytic subset of X. Following [12], we say that $u \in L^1_{\mathrm{loc}}(Y)$ is weakly plurisubharmonic if it is locally essentially bounded from above and $i\partial\bar{\partial}u \in \mathcal{D}'_{n-1,n-1}(Y)^+$.

Let Y be an irreducible analytic subset of dimension d. We denote by $[Y]$ the current of integration on Y, that is, $[Y](\alpha) := \int_{Y_{reg}} \alpha$ for any test form α; $[Y] \in \mathcal{D}'_{d,d}(X)^+$ and it is closed.

A current T on $M - Y$ has locally finite mass across Y if and only if there exists its trivial extension T^0, which is characterized by the condition: $\|T^0\|(Y) = 0$.

Let us recall the following result (see [6] Theorems 1.13 and 4.10):

THEOREM 2.1. *If T is a positive plurisubharmonic (k, k)-current on a manifold X, and Y is a proper analytic subset of X, then $T = (T|_{X-Y})^0 + \chi_Y T$, where $\chi_Y T = 0$ if*

$k + \dim Y < \dim X$, *while if* $k + \dim Y = \dim X$ *and* $\{Y_r\}$ *are the irreducible components of* Y *of maximal dimension*, *then* $\chi_Y T = \sum_r h_r[Y_r]$, *for non negative weakly plurisubharmonic functions* h_r *on* Y_r.

We shall need also the Aeppli cohomology groups of a manifold X, which can be described using forms or currents of the same bidegree:

$$\Lambda^{k,k}_{\boldsymbol{R}}(X) := \frac{\{\varphi \in \mathcal{E}^{k,k}(X)_{\boldsymbol{R}}\,;\, d\varphi = 0\}}{\{i\partial\bar\partial\psi\,;\,\psi \in \mathcal{E}^{k-1,k-1}(X)_{\boldsymbol{R}}\}} \simeq \frac{\{T \in \mathcal{D}'_{p,p}(X)_{\boldsymbol{R}}\,;\, dT = 0\}}{\{i\partial\bar\partial P\,;\, P \in \mathcal{D}'_{p+1,p+1}(X)_{\boldsymbol{R}}\}}\,.$$

$$V^{k,k}_{\boldsymbol{R}}(X) := \frac{\{\varphi \in \mathcal{E}^{k,k}(X)_{\boldsymbol{R}}\,;\, i\partial\bar\partial\varphi = 0\}}{\{\varphi = \partial\bar\eta + \bar\partial\eta\,;\,\eta \in \mathcal{E}^{k,k-1}(X)\}} \simeq \frac{\{T \in \mathcal{D}'_{p,p}(X)_{\boldsymbol{R}}\,;\, i\partial\bar\partial T = 0\}}{\{\partial\bar A + \bar\partial A\,;\, A \in \mathcal{D}'_{p,p+1}(X)\}}\,.$$

In the notation of [15], $\Lambda^{k,k}_{\boldsymbol{R}}(X) = H^{k,k}_{\partial\bar\partial}(X)$; moreover, $\Lambda^{1,1}_{\boldsymbol{R}}(X) \simeq H^1(X, \mathcal{H})$, where $\mathcal{H}$ is the sheaf of germs of real pluriharmonic functions.

A pluriharmonic (k, k)-current T is called the component of a boundary if its class vanishes in $V^{k,k}_{\boldsymbol{R}}(M)$; since these groups are locally trivial, every pluriharmonic current is locally the component of a boundary; when T is positive, we may also suppose that $T = \partial\bar A + \bar\partial A$ where A has L^1_{loc}-coefficients (see (1.15) in [6]).

Now we go to our main technical result, which allows the definition of a wedge product of positive currents, under suitable hypotheses.

THEOREM 2.2. *Let* Y *be a proper analytic subset of a complex manifold* X. *Let* S *be a positive closed* $(1, 1)$-*current on* X, *smooth on* $X - Y$, *and let* T *be a positive pluriharmonic* (k, k)-*current on* X, *with* $k + \dim Y < \dim X$. *Then there exists a* (*unique*) *current on* X, *denoted by* $S \wedge T$, *with the following property*:

If g *is a solution of* $S = i\partial\bar\partial g$ *in an open subset* $U \subset X$, *and if* $\{g_j\}$ *is a sequence of smooth plurisubharmonic functions on* U, *which converge to* g *in* $C^\infty(U - Y)$, *then* $\lim_j i\partial\bar\partial g_j \wedge T = S \wedge T$ *in* U.

REMARK 1. The above result, in the case $\dim X = 3$, $k = 1$ and Y a smooth complex curve, is proved in Proposition 1.1 of [7].

REMARK 2. When T is a closed current, the above definition of $S \wedge T$ coincides with the classical one, i.e., locally $S \wedge T := i\partial\bar\partial(gT)$, where $S = i\partial\bar\partial g$ (see III.3 in [14]). In fact, in our hypotheses ($k + \dim Y < \dim X$), if T is closed, Corollary III.4.10 in [14] applies and thus the classical definition is well-posed. Moreover, taken a decreasing sequence $\{g_j\}$ as in Theorem 2.2 (e.g. regularize g by convolutions), by Proposition III.4.9 in [14] we can apply Theorem III.3.7 ibidem, and so $\lim_j i\partial\bar\partial g_j \wedge T = i\partial\bar\partial(g \wedge T)$. Thus the two definitions coincide by Theorem 2.2.

When T is closed and Y is replaced by a closed set whose Hausdorff measure vanishes in a suitable degree, the above results are improved in Theorem III.4.5 ([14]) and in [9].

PROOF OF THEOREM 2.2. Let $n := \dim X$, $d := \dim Y$, $m := n - d$. When $k = 0$, T is a positive pluriharmonic distribution; in this case, T is smooth, so that $S \wedge T$ is well-defined, and the case $n = 1$ is solved. Moreover, when $m = 1$, then $k = 0$. Thus from now on we suppose $n \geq 2,\ m \geq 2,\ 1 \leq k \leq m-1$.

Step 1. Suppose Y is smooth and $m < n$, and hence $n \geq 3$.

Take an open coordinate pseudoconvex set U in X such that in U, $S = i\partial\bar{\partial}g$ and $T = \bar{\partial}F + \partial\bar{F}$, where F has L^1_{loc}-coefficients and take $\{g_j\}$ as in the Hypothesis (notice that this is always possible); let us control the mass of $i\partial\bar{\partial}g_j \wedge T$ near Y, following [18] (see also [7]).

Let us fix $y \in Y$, and a neighborhood $\Omega = \Delta' \times \Delta'' \subset\subset U$ of y in X, such that: Δ' is a polydisc in $\boldsymbol{C}^d$ with coordinates $(t_1, \ldots, t_d)$, Δ'' is a polydisc in $\boldsymbol{C}^m$ with coordinates $(z_1, \ldots, z_m)$, $\Omega \cap Y = \{(t, z) \in \Delta' \times \Delta''; z = 0\}$ and $y = (0, 0)$ in Ω. Choose a unitary linear coordinates change $w = w(t, z)$ of $\boldsymbol{C}^n$ such that $(w_I(t, z), z) := (w_{i_1}, \ldots, w_{i_d}, z_1, \ldots, z_m)$ are coordinates on $\boldsymbol{C}^n$, for every multi-index $I = (i_1, \ldots, i_d)$, $1 \leq i_1 < \cdots < i_d \leq n$.

Call $W_I(r, s) = \{(t, z) \in \Delta' \times \Delta''; \|w_I(t, z)\| < r, \|z\| < s\}$. When r and s are positive and small (such that $W_I(r, s) \subset\subset \Delta' \times \Delta''$), the set $\{W_I(r, s)\}$ gives a fundamental system of neighborhoods of y.

Fix r and s, and consider $W := \bigcap_I W_I(r, s)$, which is an open neighborhood of y. Since $(i/2)\partial\bar{\partial}\|w\|^2 = (i/2)\partial\bar{\partial}\|t\|^2 + (i/2)\partial\bar{\partial}\|z\|^2$ and $m > k$, there is a constant $c > 0$ such that

$$\left(\frac{i}{2}\partial\bar{\partial}\|w\|^2\right)^{n-k-1} \leq c\sum_I \left(\frac{i}{2}\partial\bar{\partial}\|w_I\|^2\right)^{n-m} \wedge \left(\frac{i}{2}\partial\bar{\partial}\|z\|^2\right)^{m-k-1}.$$

Hence

$$\begin{aligned}\|i\partial\bar{\partial}g_j \wedge T\|(W) &= \int_W i\partial\bar{\partial}g_j \wedge T \wedge \left(\frac{i}{2}\partial\bar{\partial}\|w\|^2\right)^{n-k-1} \\ &\leq \int_W i\partial\bar{\partial}g_j \wedge T \wedge \left(c\sum_I \left(\frac{i}{2}\partial\bar{\partial}\|w_I\|^2\right)^{n-m} \wedge \left(\frac{i}{2}\partial\bar{\partial}\|z\|^2\right)^{m-k-1}\right).\end{aligned}$$

But $i\partial\bar{\partial}g_j \wedge T \geq 0$, so we need only to control, for every I, $\int_{W_I} i\partial\bar{\partial}g_j \wedge T \wedge \theta$, where $\theta := ((i/2)\partial\bar{\partial}\|w_I\|^2)^{n-m} \wedge ((i/2)\partial\bar{\partial}\|z\|^2)^{m-k-1}$.

Since $T = \bar{\partial}F + \partial\bar{F}$, if $\{F_\varepsilon\}$ is a family of forms which regularize F by convolution, we get (see (3.6) in [2])

$$\lim_{\varepsilon \to 0} \int_{W_I} i\partial\bar{\partial}g_j \wedge (\bar{\partial}F_\varepsilon + \partial\bar{F}_\varepsilon) \wedge \theta = \int_{W_I} i\partial\bar{\partial}g_j \wedge T \wedge \theta\,,$$

when we choose r, s such that $W_I = W_I(r, s)$, $\|T\|(bW_I) = 0$; this holds for almost all r, s.

Remark that $bW_I = A_I \cup B_I$, where

$$\begin{aligned} A_I &= \{(t, z) \in \Delta' \times \Delta''\,;\ \|w_I(t, z)\| = r, \|z\| < s\}\,, \\ B_I &= \{(t, z) \in \Delta' \times \Delta''\,;\ \|w_I(t, z)\| < r, \|z\| = s\}\,. \end{aligned}$$

Thus $B_I \cap Y = \emptyset$, and moreover, by dimension reasons, the pull-back of θ to A_I vanishes (we call “good” the sets W_I satisfying these conditions and $\|T\|(bW_I) = 0$).

Hence we get

$$\int_{W_I} i\partial\bar{\partial}g_j \wedge T \wedge \theta = \lim_{\varepsilon\to 0}\int_{W_I} i\partial\bar{\partial}g_j \wedge (\bar{\partial}F_\varepsilon + \partial\bar{F}_\varepsilon)\wedge\theta$$
$$= \lim_{\varepsilon\to 0}\int_{B_I} i\partial\bar{\partial}g_j \wedge (F_\varepsilon + \bar{F}_\varepsilon)\wedge\theta$$
$$= \int_{B_I} i\partial\bar{\partial}g_j \wedge [F+\bar{F}]|_{B_I}\wedge\theta\,,$$

where $[F+\bar{F}]|_{B_I}$ denotes the slice of $F+\bar{F}$ along B_I (see (10.3) in [18]; use a small cylinder M containing B_I, $\pi : M \to \boldsymbol{R}$ given by $\pi(t,z) = \|z\|$ so that B_I becomes a fibre of π).

Since $B_I \subset\subset \Omega - Y$, $i\partial\bar{\partial}g_j$ converges uniformly to $i\partial\bar{\partial}g$ on B_I, so that

$$\lim_j \int_{W_I} i\partial\bar{\partial}g_j \wedge T \wedge\theta = \lim_j \int_{B_I} i\partial\bar{\partial}g_j \wedge [F+\bar{F}]|_{B_I}\wedge\theta$$
$$= \int_{B_I} i\partial\bar{\partial}g \wedge [F+\bar{F}]|_{B_I}\wedge\theta < \infty\,.$$

Suppose now that Y is smooth and $m = n$; we may assume $Y = \{y\}$, and $\Omega \subset\subset U$ be a polydisc in $\boldsymbol{C}^n$ with coordinates $(z_1,\dots,z_n)$, where y is at the origin.

We do not need any change of coordinates, because on $b\Omega \subset\subset U - Y$ the sequence $\{g_j\}$ converges uniformly to g; thus we get the estimate

$$\lim_j \int_{\Omega} i\partial\bar{\partial}g_j \wedge T \wedge\theta = \int_{b\Omega} i\partial\bar{\partial}g \wedge [F+\bar{F}]|_{b\Omega}\wedge\theta < \infty\,,$$

where $\theta = ((i/2)\partial\bar{\partial}\|z\|^2)^{n-k-1})$ (θ does not appear when $k = n-1$).

Summing up, we got that for any $K \subset\subset U$, $\sup_j \|i\partial\bar{\partial}g_j \wedge T\|(K) < \infty$; in fact, on neighborhoods of points outside Y, the hypothesis "$g_j \to g$ in $\mathcal{C}^\infty(U - Y)$" provides the estimate.

Hence we get that there is a subsequence $\{g_{j_\nu}\}$ of $\{g_j\}$ such that $i\partial\bar{\partial}g_{j_\nu} \wedge T$ converges weakly to a $(k+1,k+1)$-current $S \wedge T$ on U, which is positive and pluriharmonic.

Step 2. Let $Y_{\text{sing}} \neq \emptyset$; we may suppose $2 \le m < n$, $1 \le k \le m-1$, so the first significative case is that of a curve Y in a threefold M, and T a $(1,1)$-current.

When $y \in Y_{\text{reg}}$, we can argue exactly as in Step 1, and get the same results on Y_{reg}. Let $y \in Y_{\text{sing}}$; for every irreducible component Y_i of Y. We may argue as in Chapter II (4.8), (4.11), (4.19) of [14], and hence the following Claim holds:

CLAIM 1. *There is a basis $(e_1,\dots,e_n)$ of $\boldsymbol{C}^n$ with coordinates $(t_1,\dots,t_d,z_1,\dots,z_m)$, arbitrarily close to a preassigned basis, such that*:

let $\Delta' = \Delta'(0,r')$ be a polydisc in $\boldsymbol{C}^d$ with coordinates $(t_1,\dots,t_d) = t$,

let $\Delta'' = \Delta''(0,r'')$ be a polydisc in $\boldsymbol{C}^m$ with coordinates $(z_1,\dots,z_m) = z$;

if r'' is small enough and $r' \le r''/C_i$ for some constant $C_i > 0$, then the projection map $\pi : Y_i \cap (\Delta'\times\Delta'') \to \Delta'$, $\pi(t,z) = t$, is a ramified covering, and $Y_i \cap (\Delta'\times\Delta'')$ is contained in the cone $\{\|z\| \le C_i\|t\|\}$ (here y corresponds to the origin $(t,z) = (0,0)$).

Since the above basis is "generic", we may suppose that Claim 1 also holds for the whole Y, so that $\pi : Y \cap (\Delta' \times \Delta'') \to \Delta'$ is a ramified covering, and for some $C > 0$, $Y \cap (\Delta' \times \Delta'')$ is contained in the cone $\{\|z\| \leq C\|t\|\}$.

Moreover, replacing $(e_1, \dots, e_n)$ by $(Ke_1, \dots, Ke_d, e_{d+1}, \dots, e_n)$ for a suitable $K > 0$, we may suppose that C is small ($C << 1$).

A linear algebra argument gives the following result:

CLAIM 2. *There is a new basis* $(v_1, \dots, v_n)$ *of* $\mathbf{C}^n$ *such that*:

(1) *the matrix A of the base change is unitary*,

(2) *for every multi-index* $I = (i_1, \dots, i_d)$, $(v_{i_1}, \dots, v_{i_d}, e_{d+1}, \dots, e_n)$ *is a basis of* $\mathbf{C}^n$,

(3) *for each* j, v_j *is contained in* $\{\|z\| > C\|t\|\}$, *so that no axis of this coordinate system intersects* $Y - (0, 0)$ *in* $\Delta' \times \Delta''$.

Now we can go on as in Step 1: call (w_j) the coordinates with respect to the basis $(v_1, \dots, v_n)$ given in Claim 2. Since the matrix A is unitary, $(i/2)\partial\bar{\partial}\|w\|^2 = (i/2)\partial\bar{\partial}\|t\|^2 + (i/2)\partial\bar{\partial}\|z\|^2$, hence we have only to control $\int_{W_I} i\partial\bar{\partial}g_j \wedge T \wedge \theta$, where $W_I(r, s)$, A_I and B_I are defined as above. The pull-back of θ to A_I vanishes, and for every s there is $r_0 > 0$ such that B_I does not intersect Y when $r < r_0$, and we conclude as in Step 1. (As for the uniqueness, see Corollary 2.3).

Under the hypotheses of Theorem 2.2, the current $S \wedge T$ is obviously well-defined on $X - Y$ and has locally finite mass across Y; we will denote by $(S \wedge T)^0$ its trivial extension to X. As regards this current and the current $S \wedge T$ we get the following

COROLLARY 2.3. *Under the hypotheses of Theorem* 2.2, *the following hold.*

(1) *The current* $S \wedge T$ *is positive and pluriharmonic.*

(2) *If* $k + \dim Y < \dim X - 1$, *then* $S \wedge T = (S \wedge T)^0$.

(3) *If* $k + \dim Y = \dim X - 1$ *and* $\{Y_r\}$ *are the irreducible components of* Y *of maximal dimension*, *then there exist non negative*, *weakly plurisubharmonic functions* h_r *on* Y_r *such that* $S \wedge T = (S \wedge T)^0 + \sum_r h_r[Y_r]$.

PROOF. By Theorem 2.2, $S \wedge T$ is locally the limit of a sequence of positive pluriharmonic currents, thus (1) holds.

Since $g_j \to g$ in $C^\infty(U - Y)$, $S \wedge T$ turns out to be a positive extension of $S|_{X-Y} \wedge T$ in U across Y; thus there is also the trivial extension $(S|_{X-Y} \wedge T)^0 = (S \wedge T)^0$, and on U we have $S \wedge T = (S|_{U-Y} \wedge T)^0 + \chi_{U \cap Y} S \wedge T$.

When $k + 1 < m$, by Theorem 2.1 we get $\chi_{U \cap Y} S \wedge T = 0$, so that $S \wedge T$ is globally defined as $(S \wedge T)^0$, and it does not depend on $\{g_{j_\nu}\}$ nor on the sequence $\{g_j\}$. This proves that $S \wedge T$ is unique.

When $k + 1 = m$, by Theorem 2.1 there is a non-negative weakly plurisubharmonic function h on $U \cap Y$ such that $\chi_{U \cap Y} S \wedge T = h[U \cap Y]$. If Y is smooth, we got in the proof of Theorem 2.2 (Step 1) for $m < n$ that

$$\int_{B_I} i\partial\bar{\partial}g \wedge [F + \bar{F}]|_{B_I} \wedge \theta = \lim_{j_\nu} \int_{W_I} i\partial\bar{\partial}g_{j_\nu} \wedge T \wedge \theta$$
$$= \int_{W_I} S \wedge T \wedge \theta = \int_{W_I} (i\partial\bar{\partial}g \wedge T)^0 \wedge \theta + \int_{W_I \cap Y} h\theta\,.$$

The last term, when we allow y ($\in Y$) and r to vary (but assuring that W_I is "good"), determines the values of h on $\Omega \cap Y$.

Notice that neither $\int_{W_I} (i\partial\bar{\partial}g \wedge T)^0 \wedge \theta$ nor $\int_{B_I} i\partial\bar{\partial}g \wedge [F + \bar{F}]|_{B_I} \wedge \theta$ depend on the subsequence $\{g_{j_\nu}\}$ nor on the sequence $\{g_j\}$; thus $S \wedge T$ and h do not depend on them, but only on g. Moreover, by means of the same formula, h turns out to be globally defined on Y, and also $S \wedge T$ turns out to be globally defined as $S \wedge T = (S|_{X-Y} \wedge T)^0 + h[Y]$, and in U, $S \wedge T = \lim_j i\partial\bar{\partial}g_j \wedge T$.

If $m = n, k = n - 1$, we may assume $Y = \{y_1, \dots, y_r\}$, and that only one of them is contained in U. Hence on U, $S \wedge T = (S|_{X-Y} \wedge T)^0 + c_l[y_l]$ and as above, c_l does not depend on $\{g_{j_\nu}\}$ nor on the sequence $\{g_j\}$, and $S \wedge T$ is globally defined as $S \wedge T = (S|_{X-Y} \wedge T)^0 + \sum_1^k c_l[y_l]$.

When Y is singular, $k + 1 = m < n$, we can argue as before, because $h \in L^1_{\mathrm{loc}}(Y \cap U)$ and their values can be determined on regular points. This completes the proof.

REMARK 3. Looking at $S \wedge T$ as a positive pluriharmonic current on $X - Y$, what can be said about $(S \wedge T)^0$, using extension theorems? Here we show that this approach would give only partial results:

(i) If $k + \dim Y < \dim X - 2$, by Theorem 5.4 in [3] $(S \wedge T)^0$ exists and is pluriharmonic. By Theorem 5 in [10], the same result holds when Y is only a closed set.

Of course, the argument does not say anything about $\lim_j i\partial\bar{\partial}g_j \wedge T$, while we know from Corollary 2.3 that $S \wedge T = (S \wedge T)^0$.

(ii) If $k + \dim Y = \dim X - 2$, by Theorem 5.6 in [3] $(S \wedge T)^0$ exists and is plurisuperharmonic; hence $i\partial\bar{\partial}(S \wedge T)^0$ is a closed negative current supported on Y. By Theorem 6 in [10], the same result holds when Y is only a closed set. By Remark 5.7 in [3], $i\partial\bar{\partial}(S \wedge T)^0 = -\sum_r c_r[Y_r]$ with $c_r \geq 0$, while we know from Corollary 2.3 that $i\partial\bar{\partial}(S \wedge T)^0 = 0$.

(iii) When $k + \dim Y = \dim X - 1$, the results we got here are completely new, because in general we cannot extend currents across subvarieties which have the same dimension as the current. Notice that by Corollary 2.3, we get $i\partial\bar{\partial}(S \wedge T)^0 = -\sum_r i\partial\bar{\partial}h_r \wedge [Y_r]$; are the functions h_r pluriharmonic? (Recall that when T is closed, then $(S \wedge T)^0$ is closed too). The answer is negative, as the following example shows.

EXAMPLE. Let h be a non negative subharmonic function in a neighborhood U of $0 \in \boldsymbol{C}$, and take

$$S := \frac{i}{\pi}\partial\bar{\partial}\log\sqrt{|z_1|^2 + |z_2|^2}\,, \quad T := -\log|z_1|\frac{i}{\pi}\partial\bar{\partial}h(z_3) + h(z_3)[\{z_1 = 0\}]\,.$$

Let $X := \{|z_1| < 1\} \times \boldsymbol{C} \times U$ and $Y := \{z \in X / z_1 = z_2 = 0\}$.

The hypotheses of Theorem 2.2 are satisfied, and in $X-Y$ we get

$$S \wedge T = -\log|z_1| \frac{i}{\pi} \partial\bar\partial \log\sqrt{|z_1|^2+|z_2|^2} \wedge \frac{i}{\pi}\partial\bar\partial h(z_3)$$

but in X it holds

$$S \wedge T = -\log|z_1| \frac{i}{\pi} \partial\bar\partial \log\sqrt{|z_1|^2+|z_2|^2} \wedge \frac{i}{\pi}\partial\bar\partial h(z_3) + h(z_3)[Y]\,.$$

In the next Proposition, we consider also the Aeppli cohomology, in the compact case.

PROPOSITION 2.4. *Under the hypotheses of Theorem* 2.2, *if X is compact, then $S \wedge T$ is "natural" with respect to cohomology in the sense that, if $S = \alpha + i\partial\bar\partial u$ and $T = \psi + \partial\bar A + \bar\partial A$, for suitable smooth forms α and ψ and currents u and A on X, then also $S \wedge T = \alpha \wedge \psi + \partial\bar Q + \bar\partial Q$ for a suitable current Q on X.*

PROOF. Let us consider the following Claim, which is used in [7] and is a variant of the Regularization Theorem of Demailly ([13]).

CLAIM. *Let S be a closed positive* $(1,1)$*-current on X, smooth on $X-Y$. For every $\gamma \in \mathcal{E}^{1,1}(X)_{\boldsymbol{R}}$ such that $S \geq \gamma \geq 0$ on X, for every $\alpha \in \mathcal{E}^{1,1}(X)_{\boldsymbol{R}}$ and for every distribution f, smooth on $X-Y$, such that $S = \alpha + i\partial\bar\partial f$, there exist*:

(1) *a sequence $\{f_j\}$ of smooth functions on X, decreasing to f, such that on $X-Y$ the sequence converges to f in $C^\infty(X-Y)$, and for every compact set $K \subset\subset X-Y$, it holds $f_j = f$ when $j >> 1$.*

(2) *a sequence $\{\lambda_j\}$ of continuous functions on X such that, for any $z \in X$, $\{\lambda_j(z)\}$ decreases to the Lelong number $n(S,z)$. Moreover, for any j, $S_j := \alpha + i\partial\bar\partial f_j \geq \gamma - \lambda_j u$ for a suitable hermitian metric u on X.*

Let us prove that $S \wedge T = \lim_j S_j \wedge T$, where the forms S_j are defined in the Claim.

Let U be a small polydisc in a coordinate set (V,z), where $\alpha = i\partial\bar\partial k$, $k \in C^\infty(U)$. Let $C > 0$ be a constant, and $g_{C,j} := k + f_j + C\|z\|^2 \in C^\infty(U)$, where the sequence $\{f_j\}$ is defined in the Claim; call $g_C := k + f + C\|z\|^2$.

The functions $g_{C,j}$ are plurisubharmonic for every j, if C is sufficently big; moreover, g_C is a plurisubharmonic function on U, smooth on $U-Y$, and the sequence $\{g_{C,j}\}$ converges to g in $C^\infty(U-Y)$, due to the properties of $\{f_j\}$ stated in the Claim.

Apply Theorem 2.2 to U and to the positive closed current $S + Ci\partial\bar\partial\|z\|^2 = i\partial\bar\partial g_C$ to get $(S + Ci\partial\bar\partial\|z\|^2) \wedge T = \lim_j i\partial\bar\partial g_{C,j} \wedge T$.

Since $S \wedge T$ also exists, we get on U

$$S \wedge T + Ci\partial\bar\partial\|z\|^2 \wedge T = \lim_j i\partial\bar\partial g_{C,j} \wedge T = \lim_j S_j \wedge T + Ci\partial\bar\partial\|z\|^2 \wedge T\,.$$

Hence, on U, $S \wedge T = \lim_j S_j \wedge T$, and also on X. Now,

$$\begin{aligned} S \wedge T - \alpha \wedge \psi &= \lim_j S_j \wedge T - \alpha \wedge (T - \bar\partial F - \partial\bar F) \\ &= \lim_j i\partial\bar\partial f_j \wedge T + \bar\partial(\alpha \wedge F) + \partial(\alpha \wedge \bar F)\,. \end{aligned}$$

For every j, $i\partial\bar{\partial} f_j \wedge T$ is the component of a boundary, for it vanishes on every closed test form; since X is compact, also $\lim_j i\partial\bar{\partial} f_j \wedge T$ is the component of a boundary, and this gives the result.

Corollary 2.3 (2) gives a simple characterization of $S \wedge T$, when $k + \dim Y < \dim X - 1$. In the other case, we have:

COROLLARY 2.5. *Under the hypothesis of Theorem* 2.2, *if* $k + \dim Y = \dim X - 1$, *and when moreover* X *is compact*, Y *is irreducible and* $[Y]$ *is not the component of a boundary*, *then the current* $S \wedge T$ *is characterized by the following properties*:

(1) $S \wedge T$ *is positive and pluriharmonic.*

(2) $S \wedge T$ *extends* $S|_{X-Y} \wedge T|_{X-Y}$.

(3) $S \wedge T$ *is "natural" with respect to cohomology.*

PROOF. By Corollary 2.3 we get $S \wedge T = (S \wedge T)^0 + h[Y]$, where h is constant because Y is compact. Since the Aeppli class of $[Y]$ does not vanish, the constant h is determined by the cohomology class of $S \wedge T$ (that is, by Proposition 2.4, by the classes of S and T) and by the class of the pluriharmonic current $(S \wedge T)^0$.

3. Manifolds which are Kähler outside a curve. In this section, we consider the following situation: M is a compact complex manifold of dimension $n > 1$, Y is a proper analytic subset of M, and $M - Y$ carries a Kähler metric h with Kähler form ω ($d\omega = 0$).

If Y is discrete, by a result of Miyaoka [17] M is itself a Kähler manifold, while this is not true in general; for instance, there exist Moishezon non projective manifolds. Nevertheless, we may look for a weaker condition, as one of the following:

(1) M carries a balanced metric, that is, a hermitian metric whose Kähler form ω satisfies $d\omega^{n-1} = 0$.

(2) M has a closed strictly weakly positive (p, p)-form Ω, i.e., M has a p-Kähler form, for an index p, $1 < p < n - 1$ (when $n \geq 3$) (see, e.g., [1]).

(3) M belongs to the Fujiki class $\mathcal{C}$, i.e., M is bimeromorphic to a Kähler manifold.

(Notice that (2) does not imply (1), in general, while (3) implies (1) by Corollary 4.5 in [2]).

REMARK 4. When $1 < p < n-1$, there are several notions of positivity for forms and currents: see, e.g., the Appendix of [1]. A manifold is called p-Kähler if it has a closed strictly weakly positive (p, p)-form (called also a closed transverse (p, p)-form), or equivalently if it has no non-zero strongly positive currents of bidimension (p, p) which are components of boundaries (Theorem 3.2 in [1]). Since strongly positive currents are positive, we can use in this context the results of Chapter 2.

The Hopf surface gives an example of a manifold which is Kähler outside a curve, but not balanced, and this allows to build examples in any dimension assuring that, when Y is a hypersurface, M may be not balanced, nor p-Kähler.

Therefore, the first significative case is that of manifolds of dimension $n \geq 3$, which are Kähler outside a curve C; in this situation we proved in [4] the following

THEOREM 3.1 (Theorem 5.5 in [4]). *Let M be a compact complex manifold of dimension $n \geq 3$, which is Kähler outside an irreducible curve C. Then one and only one of the following cases may occur*:

(1) *M is Kähler.*

(2) *C is the component of a boundary.*

(3) *C is part of the component of a boundary, i.e., there is a positive current $B \neq 0$ on M such that $\chi_C B = 0$ and the class of $S + B$ vanishes in $V_{\boldsymbol{R}}^{n-1,n-1}(M)$.*

This result assures that, if M is not itself a Kähler manifold, then C is either the component of a boundary or a part of it. Let us study the last case:

PROPOSITION 3.2. *Under the hypotheses of Theorem 3.1, if C is part of the component of a boundary, then M is balanced.*

PROOF. The proof of Theorem 2.7 in [5], when C is a smooth curve, also works when C is irreducible. Indeed, only Lemma 2.2 there refers to the smooth case; but Aeppli cohomology is also defined for singular spaces, even if it may loose some properties of the smooth case (see e.g. [15], page 1262). Moreover, working on the normalization of C, we get that also in the irreducible case $\dim \Lambda_{\boldsymbol{R}}^{1,1}(C) = 1$, so that we can conclude as in [5]. Thus, if C is part of the component of a boundary, then $M \in \mathcal{C}$ and so M is balanced.

Let us study now the case when C is the component of a boundary, following the ideas of Theorem II in [7], (where $\dim X = 3$ and C is smooth).

THEOREM 3.3. *Let M be a compact complex manifold of dimension $n \geq 3$, let C be an irreducible curve in M, which is the component of a boundary, and such that $M - C$ carries a Kähler metric. Then M is not Kähler, but is p-Kähler for every $p \geq 2$; in particular, it is balanced.*

PROOF. M is not Kähler, since C is the component of a boundary. Every Kähler form ω on $M - C$ extends to a closed positive $(1, 1)$-current on M (see [16]), called ω too. Fix $p \geq 2$, let $k = n - p$. Let T be a strongly positive (k, k)-current on M, which is the component of a boundary; as we said in Remark 4, we need only to prove that $T = 0$. By Theorem 2.2 and Proposition 2.4 we consider $\omega \wedge T, \dots, \omega^{p-1} \wedge T := \omega \wedge (\omega^{p-2} \wedge T)$, which are components of boundaries.

In particular, by Corollary 2.3, $\omega^{p-1} \wedge T = (\omega^{p-1} \wedge T)^0 + c[C]$, with $c \geq 0$, thus $(\omega^{p-1} \wedge T)^0$ is the component of a boundary. Using the notation of the Claim in the proof of Proposition 2.4, we get

$$0 \leq \int_M \gamma \wedge (\omega^{p-1} \wedge T)^0 \leq \int_M S_j \wedge (\omega^{p-1} \wedge T)^0 + \int_M \lambda_j u \wedge (\omega^{p-1} \wedge T)^0 .$$

The first summand on the right hand side vanishes because S_j is smooth and closed, whereas $\lim_j \int_M \lambda_j u \wedge (\omega^{p-1} \wedge T)^0 = 0$ because $(\omega^{p-1} \wedge T)^0$ vanishes on C, but the Lelong numbers of $S := \omega$ vanish outside C. Thus $\int_M \gamma \wedge (\omega^{p-1} \wedge T)^0 = 0$, which implies $(\omega^{p-1} \wedge T)^0 = 0$.

But ω^{p-1} is strictly positive on $M - C$, so $T|_{M-C} = 0$; since T cannot be supported on C, because $p > 1 = \dim C$ ([1], Theorem 1.1), and hence $T = 0$.

Collecting these results, we get the proof of the following Theorem.

THEOREM 3.4. *If M is a compact complex manifold of dimension $n \geq 3$, and C is an irreducible curve in M, such that $M - C$ is Kähler, then either M is Kähler itself, or $[C]$ is the component of a boundary and M is p-Kähler for every $p \geq 2$, or $[C]$ is part of the component of a boundary and $M \in \mathcal{C}$. In all cases, M is balanced.*

Let us consider briefly also the case $\dim Y > 1$.

PROPOSITION 3.5. *Let M be a compact complex manifold of dimension $n \geq 4$, let Y be a proper analytic subset of M of pure dimension d, $1 < d < n - 1$, such that $M - Y$ carries a Kähler metric.*

(1) *If M is q-Kähler, $q \geq d$, then it is p-Kähler for every $p > q$.*

(2) *If M is $(d-1)$-Kähler and Y is irreducible and is the component of a boundary, then M is not d-Kähler, but it is p-Kähler for every $p > d$.*

(3) *If Y is smooth and Kähler, with $b_2(Y) = 1$, and it is not the component of a boundary, then $M \in \mathcal{C}$.*

In all cases, M is balanced.

PROOF. Take $p < n$ and $k = n - p$. Let as argue exactly as in the proof of Theorem 3.3: let T be a strongly positive (k, k)-current on M, which is the component of a boundary, and consider $\omega \wedge T, \dots, \omega^{p-d} \wedge T$, which are positive components of boundaries.

In the first case, let $p > q$, and let Ω be a q-Kähler form, thus closed and strictly positive; since $(\omega^{p-q} \wedge T)(\Omega) = 0$, the mass of $\omega^{p-q} \wedge T$ vanishes, and in particular $\omega^{p-q} \wedge T|_{M-Y} = 0$. This gives $T|_{M-Y} = 0$, but T cannot be supported on Y, because $p > \dim Y$, and hence $T = 0$.

In the second case, let $p > d$ and let Ω be a $(d-1)$-Kähler form. As in Theorem 3.3 we get $\int_M \gamma \wedge (\Omega \wedge (\omega^{p-d} \wedge T)^0) = 0$, which implies $\Omega \wedge (\omega^{p-d} \wedge T)^0 = 0$ and also $\Omega \wedge T|_{M-Y} = 0$. To get $T|_{M-Y} = 0$, it sufficies to prove that for every $K \subset\subset M - Y$, $\int_K T \wedge \omega^p = 0$, but on K, $T \wedge \omega^p$ is dominated by a multiple of $T \wedge \Omega \wedge \omega^{p-d+1}$.

For the last case, let us follow the proof of Theorem 2.7 in [5]. By the hypothesis, the map $i^* : \Lambda^{1,1}_{\boldsymbol{R}}(M) \to \Lambda^{1,1}_{\boldsymbol{R}}(Y) \simeq H^2(Y, \boldsymbol{R}) \simeq \boldsymbol{R}$ induced by the inclusion $i : Y \to M$ cannot be the zero-map, because Y is not the component of a boundary. Thus, starting from a Kähler form η on Y, by Proposition 3.3 in [15] we get a $(1, 1)$-form γ on M and a smooth function u on M such that $\gamma + i\partial\bar{\partial}u$ is strictly positive in a neighborhood of Y in M. If ω^0 is the trivial extension of a Kähler form on $M - Y$, $K\omega^0 + \gamma + i\partial\bar{\partial}u$ becomes a Kähler current on M (when $K >> 0$), and hence $M \in \mathcal{C}$.

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DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DEGLI STUDI DI PARMA
VIALE USBERTI 53/A
I-43100 PARMA
ITALY

E-mail address: lucia.alessandrini@unipr.it

DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DEGLI STUDI DI PARMA
VIALE USBERTI 53/A
I-43100 PARMA
ITALY

E-mail address: giovanni.bassanelli@unipr.it