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WEIBULL RENEWAL PROCESSES

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Abstract. We study the class of renewal processes with Weibull lifetime distribution from the point of view of the general theory of point processes. We investigate whether a Weibull renewal process can be expressed as a Cox process. It is shown that a Weibull renewal process is a Cox process if and only if $0 < \alpha \leq 1$, where α denotes the shape parameter of the Weibull distribution. The Cox character of the process is analyzed. It is shown that the directing measure of the process is continuous and singular.

Key words and phrases: Weibull, renewal process, point process, Cox process.

1. Introduction

The most important point process is the Poisson process, where events occur with a constant intensity. Next are the class of renewal processes and the class of Cox processes. The *Cox* or *doubly stochastic Poisson* processes are natural generalizations of the Poisson process. A Cox process can be considered as a Poisson process with a stochastic intensity (the intensity itself is a stochastic process), and it is called the *directing measure* of the process. For properties of such processes see Grandell (1976) and Karr (1991). Although much is known on the properties of Cox processes, there are not so many examples of known point processes which are Cox processes. For theoretical as well as practical purposes it is of interest to have examples of tractable point processes which are Cox processes. The renewal processes are also natural generalizations of the Poisson process, where the renewal character is preserved.

If we want to have a point process which similarly to the Poisson process has the renewal character but contrary to it has a stochastic intensity, we may think of the intersection of the classes of Cox processes and renewal processes. It is attractive to use such processes instead of the Poisson process in applications where the intensity of the process is random. A classical characterization of this class of processes is due to Kingman (1964). A new characterization is given in Yannaros (1989). However, these characterizations are in general not easy to check. For theoretical and practical purposes, it is, therefore, useful to have examples of such processes, particularly, examples which are simple enough as to be possible to calculate explicitly the interesting functionals of the process.

The distribution function of a Weibull distribution on $[0, \infty)$ with parameters $\alpha > 0$ (shape or form parameter) and $\beta > 0$ (scale parameter) is given by

$$F(t) = 1 - \exp(-(t/\beta)^\alpha) = 1 - \exp(-\lambda t^\alpha), \qquad t \geq 0,$$

where $\lambda = 1/\beta^\alpha$. The family can be generalized by introducing a third location parameter, but we do not consider this case here.

The Weibull distribution is the most widely used lifetime distribution. It is used as a model for lifetimes of many types of manufactured items, such as vacuum tubes, ball bearings and electrical insulation. It is also used in biomedical applications, for example, as the distribution for times to occurrence (diagnosis) of tumors in human populations or in laboratory animals (see Pike (1966) and Peto and Lee (1973)). Further, the Weibull distribution is a suitable model for extreme value data, such as the strength of certain materials. Recently, empirical studies have shown that the Weibull distribution is superior to the classical stable distributions, inclusive the normal distribution, for fitting empirical economic data (see Mittnik and Rachev (1990)) and the references therein).

Apart from the simplicity and the ease with which the parameter estimates and the estimated distribution function can be constructed, the wide applicability of the Weibull distribution depends on several attractive features which it possesses, and in the following we mention some of the most important.

(i) A main reason for its popularity is that it has a great variety of shapes. This makes it extremely flexible in fitting data, and it has been found to provide a good description of many kinds of data.

(ii) As $t \to \infty$, the hazard rate of the distribution increases to infinity if $\alpha > 1$, decreases to zero if $\alpha < 1$ and is constant for $\alpha = 1$. Among all usual lifetime distributions no other distribution has this property. This property makes the Weibull distribution suitable in applications where a *decreasing hazard tendency* distribution or an *increasing hazard tendency* distribution is required.

(iii) It contains the exponential distribution as a special case, when $\alpha = 1$. Further, it has the following strong relation to the exponential distribution: if X has a Weibull distribution, then X^α is exponentially distributed. So, by appropriate changing of the scale we get the exponential distribution from the Weibull and vice versa.

(iv) The Weibull distribution is one of the three possible types of minimum stable distributions. Let $X, X_1, X_2, \ldots,$ be iid random variables with common distribution function F. If for each n there exist constants $a_n > 0$ and b_n such that

$$a_n \min_n \{X_1, \ldots, X_n\} + b_n \stackrel{d}{=} X,$$

then the distribution of X is called *minimum stable*. It can be shown that there exist three types of such distributions. For a discussion of the derivation of these distributions and of their properties see Gumbel (1958) or Leadbetter *et al.* (1983). Necessary and sufficient conditions for convergence of minima to each of the three types of distributions are due to Gnedenko (1943).

If the X_i have a Weibull distribution with shape parameter α, it is easily shown that

$$n^{1/\alpha} \min_n \{X_1, \ldots, X_n\} \stackrel{d}{=} X.$$

This stability property is an expression of a deeper limiting property. If $F(0) = 0$ and $F(t)$ behaves as ct^α for some positive c and α, as $t \to 0$ from above, then $n^{1/\alpha} \min_n\{X_1, \dots, X_n\}$ converges in distribution to the Weibull distribution with shape parameter α. This characteristic property makes the Weibull distribution a natural model for extreme value data.

If the lifetime distribution is Weibull, the process of the successive times of renewals constitutes a renewal process which we call a *Weibull renewal process*. So, if the lifetime of a certain type of items has a Weibull distribution and each item is replaced at the time of failure by an item of the same type, the replacements follow a Weibull renewal process. For example, consider a unit (system) consisting of a large number of components (parts) having the same lifetime distribution. If the unit fails with the first component failure, i.e., its lifetime is the minimum of the lifetimes of the components, then the Weibull distribution being stable for the minimum scheme is a suitable model for the lifetime of the unit. If any time the unit fails it is replaced by a unit of the same type, then the Weibull renewal process is an appropriate model for the successive failure times.

We will study the Weibull renewal process from the point of view of the general theory of point processes. In Section 2, it is shown that the Weibull renewal process is a Cox process if and only if $0 < \alpha \leq 1$. The Cox character of the process is analyzed and its directing measure is described.

The results have implications on the properties of the Weibull distribution and its use as a lifetime distribution in the applications. In the case $0 < \alpha \leq 1$, the Weibull renewal process gives us an example of a simple point process which is useful in the applications when we want to go beyond the Poisson process by considering a point process with the renewal character, and which has a random intensity. It is also useful in theoretical contexts as an example of a nice and simple point process which is a Cox process, and which is simple enough as to calculate the interesting functionals of the process.

2. Weibull renewal processes

In this section we investigate whether the Weibull renewal process can be a Cox process. For that we need two preliminary results, which have their own interest.

The first of these results is a variation of a result of O. Thorin given in Grandell (1991) (it is referred as a personal communication), which says that a stationary renewal process with interarrival distribution function (d.f.) F on $[0, \infty)$ satisfying

$$F(t) = 1 - \int_0^\infty e^{-tx} dV(x), \tag{2.1}$$

where V is a proper d.f. satisfying $V(0) = 0$, is a Cox process. The result appears as Theorem 38, and is proved under the requirement that $\int_0^\infty (1/x) dV(x) < \infty$. It turns out that this condition is equivalent to the finiteness of the mean μ of F. In fact, we get from (2.1)

$$\mu = \int_0^\infty (1/x) dV(x).$$

By Bernstein's Theorem (see Feller (1971), p. 439), a d.f. F satisfies (2.1), if and only if its tail is completely monotone (for the definition of this notion see Feller (1971)), or if it has a completely monotone density. Denoting by c the total mass of V we obtain $c = 1 - F(0)$. Thus, $F(0) = 0$ if and only if V is proper. If V is defective, then F has an atom at zero with mass $F(\{0\}) = 1 - c$. Renewal processes having such interarrival distributions are natural models in applications with multiple occurrences.

For our purposes, we need Thorin's result for ordinary renewal processes, which is more delicate than the stationary case. For the sake of completeness, we will prove it in the general case, where the measure V can be defective, and the mean of F infinite. The result is of its own interest since it proves that a wide class of ordinary renewal processes are Cox processes.

LEMMA 2.1. *An ordinary renewal process with interarrival d.f. F satisfying (2.1) is a Cox process.*

PROOF. As in Grandell (1991) we will use a technique based on Stieltjes transforms, and we give the definition of this notion first. A function $f : (0, \infty) \to [0, \infty)$ is called a *Stieltjes transform*, if

$$f(s) = a + \int_0^\infty \frac{1}{s+x} dA(x), \qquad s > 0,$$

for some constant $a \geq 0$, and some positive measure A on $[0, \infty)$. The pair (a, A) is uniquely determined by f, and we have $a = \lim_{s\to\infty} f(s)$. For properties of Stieltjes transforms see Berg and Forst (1975).

Let $\hat{f}(s) = \int_0^\infty e^{-ts} dF(t)$ be the Laplace transform of F. By (2.1),

$$\hat{f}(s) = (1-c) + \int_{0+}^\infty e^{-ts} dF(t) = (1-c) + \int_{0+}^\infty \int_0^\infty x e^{-t(s+x)} dt dV(x).$$

It follows that

$$\hat{f}(s) = (1-c) + \int_{0+}^\infty \frac{x}{s+x} dV(x). \tag{2.2}$$

Define the measure A by $dA(x) = x dV(x)$. Then we have

$$\hat{f}(s) = (1-c) + \int_0^\infty \frac{1}{s+x} dA(x). \tag{2.3}$$

Thus, $\hat{f}(s)$ is a Stieltjes transform. By Propositions 14.11 and 14.6 in Berg and Forst (1975), $\hat{f}$ satisfies $\hat{f}(s) = 1/\phi(s)$, where ϕ is a non-zero function with a completely monotone derivative. Since $\hat{f}(0) = 1$, we have $\phi(0) = 1$. We get

$$\hat{f}(s) = \frac{1}{1+\psi(s)}, \tag{2.4}$$

where $\psi(s) = \phi(s) - 1$. The function ψ has a completely monotone derivative and satisfies $\psi(0) = 0$. This implies that the function $\hat{g}(s) = e^{-\psi(s)}$ is the Laplace transform of an infinitely divisible d.f. (see Feller (1971), Theorem 1, p. 450). It follows that

$$\hat{f}(s) = \frac{1}{1 - \log \hat{g}(s)}, \tag{2.5}$$

and the result follows from Lemma A.1 in the Appendix. □

The second result says that a point process which is both Cox and renewal is overdispersed compared with the Poisson process.

LEMMA 2.2. *The coefficient of variation of the interarrival distribution of any Cox and renewal process is at least one.*

PROOF. By Lemma A.1 in the Appendix, the Laplace transform of the interarrival d.f. F of a Cox and renewal process must satisfy (2.4), where ψ has a completely monotone derivative, that is, $(-1)^n \psi^{(n)}(s) \leq 0$, $n \geq 1$. Differentiating both sides of (2.4) and using the relations between moments and derivatives of the Laplace transform, we get

$$CV = \lim_{s \to 0} \sqrt{1 - \frac{\psi''(s)}{\psi'(s)}},$$

where the coefficient of variation CV is defined by

$$CV = \frac{\text{standard deviation}}{\text{mean}}.$$

Since $\psi''(s) \leq 0$ and $\psi'(s) \geq 0$, we must have $CV \geq 1$. □

The following result gives the definitive answer to the question whether the Weibull renewal process can be expressed as a Cox process.

THEOREM 2.1. *The Weibull renewal process is a Cox process if and only if* $0 < \alpha \leq 1$, *where* α *is the shape parameter of the distribution.*

PROOF. According to Lemma 2.1, it is sufficient to check whether the Weibull distribution satisfies (2.1).

Let $F(t) = 1 - \exp(-\lambda t^\alpha)$, be the distribution function of a Weibull distribution. By easy calculations, the function λt^α has a completely monotone derivative when $\alpha \in (0, 1]$. It follows from Feller ((1971), p. 450, Theorem 1) that $e^{-\lambda t^\alpha}$ is the Laplace transform of some infinitely divisible d.f. $V_{\alpha,\lambda}$. We realize that

$$F(t) = 1 - \int_0^\infty e^{-tx} dV_{\alpha,\lambda},$$

where $V_{\alpha,\lambda}$ is the d.f. of the one-sided stable distribution of characteristic exponent α (see Feller (1971), p. 448). Thus, the Weibull distribution satisfies (2.1) when $0 < \alpha \leq 1$. This proves the sufficiency of the condition $0 < \alpha \leq 1$ for the Weibull renewal process to be a Cox process.

We shall now prove the necessity of the condition. It can be shown that the coefficient of variation of the Weibull distribution satisfies $CV < 1$ when $\alpha > 1$. By Lemma 2.2, in the case $\alpha > 1$ the Weibull distribution cannot be interarrival distribution of a Cox and renewal process. This completes the proof. □

We will describe the Weibull renewal process as a Cox process. In the case $\alpha = 1$ the Weibull renewal process is a Poisson process, so we consider the case $0 < \alpha < 1$.

Since the Weibull d.f. F satisfies (2.1), it follows from the proof of Lemma 2.1 that the Laplace transform $\hat{f}(s)$ satisfies (2.4), where ψ has a completely monotone derivative and satisfies $\psi(0) = 0$. Lemma A.2 in the Appendix implies that

$$\hat{f}(s) = \frac{1}{1 + bs + \int_0^\infty (1 - e^{-sx})dB(x)}, \tag{2.6}$$

where $b \geq 0$, and B is a positive measure on $(0, \infty)$. Combining (2.3) with $c = 1$, and (2.6) we obtain

$$\int_0^\infty \frac{sx}{s+x} dV(x) = \frac{1}{\frac{1}{s} + b + \int_0^\infty \frac{1 - e^{-sx}}{s} dB(x)}. \tag{2.7}$$

Taking limits, as $s \to \infty$, we have

$$\int_0^\infty x dV(x) = \frac{1}{b}. \tag{2.8}$$

In the case $0 < \alpha < 1$, the mean of the one-sided stable distribution $V_{\alpha,\lambda}$ is infinite. Hence, we must have $b = 0$. We also have $F(0) = 0$. Since $b = 0$ and $F(0) = \lim_{s\to\infty} \hat{f}(s)$, it follows from (2.6) that $\int_0^\infty dB(x) = \infty$.

If $b = 0$ and $\int_0^\infty dB(x) = \infty$, then almost all realizations of the directing measure of the corresponding Cox and renewal process are continuous and singular with respect to Lebesgue measure (see Grandell (1976), p. 39). It follows that in the case $0 < \alpha < 1$, the realizations of the directing measure of the Weibull renewal process, being a Cox process, are continuous and singular with respect to Lebesgue measure. It means that we have continuous distributions without densities, and the whole increase takes place in a set of Lebesgue measure zero, i.e., the intervals where the distribution functions are constant add up to length one. As Feller ((1971), p. 141) with right pointed out "singular distributions play an important role" but "this situation is obscured by the cliché that "in practice" singular distributions do not occur". We have here examples where continuous and singular distributions appear in a natural way as distributions for the stochastic intensities of simple Cox and renewal processes.

We have shown that the Weibull renewal process can be expressed as a Cox process with a directing measure which is continuous and singular, and not of the two-state type. We have here a similar behaviour as in the case of gamma renewal processes (see Yannaros (1988)). It is interesting and on the same time surprising that such simple renewal processes are in fact Cox processes.

In the case $\alpha < 1$, the Weibull renewal process being a Cox process can be used as a generalization of the Poisson process, and it is a suitable model in applications where irregular or overdispersed processes appear. For example, it can be used for modeling returns on financial assets in an irregular market, where explosive bubbles are expected to occur. It may also serve as a useful example in theoretical contexts.

Mittnik and Rachev (1990) showed that, compared with many other alternatives, the Weibull distribution represents the best fit to the empirical distribution of stock returns for various types of data. Assuming that the stock return distribution is Weibull, they estimated the parameters of the distribution using various samples from the Standard and Poor 500 index data. In most periods the estimated shape parameter exceeded unity, but for the year 1986 the shape parameter was 0.876, indicating that increasing negative movements were possible. As they noted, 1986 was the year before a stock market crash. This may reflect the fact that in the case $\alpha < 1$, the Weibull renewal process is a suitable model for irregular and overdispersed processes.

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Appendix

In the paper we used the following characterization of processes which are Cox and renewal and which is due to Kingman (1964). For the proof see also Grandell (1976).

LEMMA A.1. *An ordinary renewal process with interarrival d.f. F is a Cox process if and only if the Laplace transform $\hat{f}$ of F satisfies*

$$\hat{f}(s) = \frac{1}{1 - \log \hat{g}(s)}, \tag{A.1}$$

where $\hat{g}(s)$ is the Laplace transform of an infinitely divisible d.f. G.

A nonnegative function $\psi(s)$, $s > 0$, has a completely monotone derivative if it has derivatives of all orders and $(-1)^n \psi^{(n)}(s) \le 0$, $n \ge 1$. Such functions have the following unique representation (for a proof see Berg and Forst (1975), Theorem 9.8).

LEMMA A.2. *A function $\psi : (0,\infty) \to [0,\infty)$ has a completely monotone derivative if and only if*

$$\psi(s) = a + bs + \int_0^\infty (1 - e^{-sx}) dB(x), \tag{A.2}$$

where a, b, are nonnegative constants, and B a positive measure on $(0,\infty)$ verifying

$$\int_0^\infty \frac{x}{1+x} dB(x) < \infty. \tag{A.3}$$

We have,

$$a = \lim_{s\to 0} \psi(s) \quad \text{and} \quad b = \lim_{s\to\infty} (\psi(s)/s).$$

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