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WEIERSTRASS GAP SEQUENCES ON CURVES ON TORIC SURFACES

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Abstract

In this paper, we consider a nonsingular curve C on a nonsingular compact toric surface S and intersection points of C and T-invariant divisors on S. We provide a sufficient condition for a positive integer to be a gap value of C at such points. Under a suitable assumption, it becomes the necessary and sufficient condition. We determine several Weierstrass gap sequences at infinitely near points of a point on a plane curve by using this method.

1. Introduction

First we define Weierstrass gap sequences and review several previous results for them. Let C be a complete nonsingular irreducible algebraic curve of genus g defined over an algebraically closed field of characteristic 0. For a point P on C, a positive integer j is called a *gap value* at P if

$$h^0(C, jP) = h^0(C, (j-1)P).$$

The set of all gap values is called the *Weierstrass gap sequence* (or, simply, *gap sequence*) of C at P. By the Riemann-Roch theorem, its cardinality is equal to g. The following classical result is a powerful tool in the study of gap sequences.

Theorem 1.1 (Weierstrass gap theorem). *Let C be a complete nonsingular irreducible algebraic curve of genus $g \geq 1$, and P a point on C. Then any gap value at P is less than $2g$.*

For example, in the case of a hyperelliptic curve, there are two types of gap sequences:

Theorem 1.2. *Let P a point on a hyperelliptic curve C and $\Phi_{|K_C|} : C \to \mathbf{P}^1$ the holomorphic map associated to $|K_C|$.*

(i) *If P is a ramification point of $\Phi_{|K_C|}$, then the gap sequence of C at P is the set of odd numbers $\{1, 3, \ldots, 2g-1\}$.*

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(ii) *If* $\Phi_{|K_C|}$ *is unramified at* P, *then the gap sequence of* C *at* P *is* $\{1, 2, \dots, g\}$.

For trigonal curves, Coppens has computed gap sequences at their ramification points.

THEOREM 1.3 ([1, 2]). *Let* C *be a trigonal curve and* $\varphi : C \to \mathbf{P}^1$ *the trigonal morphism. A point* P *on* C *is called a total* (*resp. an ordinary*) *ramification point if the ramification index of* φ *at* P *is three* (*resp. two*).

(i) *The gap sequence at a total ramification point of* φ *is one of the following two types*:

$$\{1, 2, 4, \dots, 3n-2, 3n-1, 3n+1, 3n+4, \dots, 3(g-n-1)+1\},$$
$$\{1, 2, 4, \dots, 3n-2, 3n-1, 3n+2, 3n+5, \dots, 3(g-n-1)+2\}.$$

(ii) *The gap sequence at an ordinary ramification point of* φ *is one of the following two types*:

$$\{1, 2, 3, \dots, 2n-1, 2n, 2n+1, 2n+3, \dots, 2g-2n-1\},$$
$$\{1, 2, 3, \dots, 2n-1, 2n, 2n+2, 2n+4, \dots, 2g-2n\}.$$

Kato and Horiuchi [4] established a criterion for deciding the kinds of ramification points and their gap sequences. Besides, Kim studied unramified points and completed the classification of the gap sequences in the trigonal case.

THEOREM 1.4 ([5]). *Let* C *and* φ *be as in Theorem* 1.3, *and denote by* g *the genus of* C. *Assume that* $g \geq 5$, *and define* $j_0 = \max\{j \in \mathbf{N} \mid jP \text{ is special}\}$. *If* φ *is unramified at a point* P *on* C, *then the gap sequence of* C *at* P *is of the form* $\{1, 2, \dots, g\}$ *or*

$$\{1, 2, \dots, n-1, n+j_0-g+1, n+j_0-g+2, \dots, j_0+1\}$$

for some integer n *with* $[(j_0+1)/2]+1 \leq n \leq g$, *where* $[x]$ *is the so-called Gauss' symbol*, *that is*, *the greatest integer not greater than* x.

Actually, the notion of gap sequence was extended to singular points by Lax and Widland [6]. In [3], some methods were given by Gatto to compute gap sequences at singular points on a plane curve. They allowed to determine gap sequences at ordinary nodes on quartic curves or at cusps on quintic curves. Notari [7] has developed a technique to compute the gap sequence at a given point on a plane curve, either it is smooth or singular. Note that a projective plane is a typical example of a toric surface.

In general, however, it is not so easy to determine it in its entirety at a given point. In this paper, we restrict ourselves to a curve C on a toric surface S and consider intersection points of C and T-invariant divisors on S. Theorem 1.5 provide a sufficient condition for a positive integer to be a gap value of C at such points. Moreover, as we will see in Corollary 1.6, it becomes the necessary and sufficient condition under the suitable condition. Namely, in such cases, we can detect all the gap values (i.e. the gap sequence). In Section 4, we will apply this technique to three examples. Concretely, we will consider singular plane curves

$$x^6y^3 + x^3y + y - 1 = 0,$$

$$x^5 + x^2y + xy^6 + y^6 = 0,$$

$$x^p + y^q + x^ry^s = 0 \quad (p \geq q \geq 1, r + s \geq 1)$$

and the resolutions of their singularities. In these cases, we can determine the gap sequences at the infinitely near points of singularities.

1.1. Summary of the toric theory and the main theorem

Let S be a nonsingular compact toric surface. The surface S contains an algebraic torus T as a nonempty Zariski open set. The torus action of T on itself naturally extends to S. A prime divisor on X is called a *T-invariant divisor* if it is invariant with respect to the torus action. We denote them by $D_1, \dots, D_d$. Since $\bigcup_{i=1}^d D_i$ is a simple chain of nonsingular rational curves, we can assume the following properties:

$$D_i.D_j = \begin{cases} 1 & (j = i-1, i+1), \\ 0 & (\text{otherwise}), \end{cases}$$

where we formally set $D_0 = D_d$ and $D_{d+1} = D_1$.

The intersections of two adjacent divisors are called *T-fixed points*.

For a compact toric surface S, there is the associated *fan* Δ_S, which is the division of $\mathbf{R}^2$ consisting of half-lines σ_i starting from the origin called *cones* $(i = 1, \dots d)$. Each cone σ_i corresponds to a T-invariant divisor D_i. We denote by (x_i, y_i) the primitive elements of σ_i, i.e., they are the lattice points on the cones σ_i closest to the origin. There are essentially two ways to take the fan associated to S, which depend on whether the value of $x_iy_{i-1} - y_ix_{i-1}$ is one or minus one. In this paper, we adopt the former, that is, assume the equality $x_iy_{i-1} - y_ix_{i-1} = 1$ for any integer $1 \leq i \leq d$. This means that the cones $\sigma_1, \dots, \sigma_d$ are arranged clockwise (Fig. 1). The Picard group of S is generated (not necessarily freely) by the classes of $D_1, \dots, D_d$. Hence, for a divisor D on S, we can write its linear equivalence class as the sum of $D_1, \dots, D_d$ with integral coefficients. For example, the canonical divisor K_S of S is

$$K_S \sim -\sum_{i=1}^d D_i,$$

where the symbol "$\sim$" means linear equivalence.

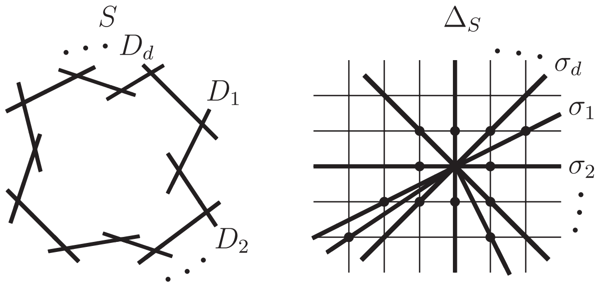

FIGURE 1

For a divisor $D \sim \sum_{i=1}^{d} n_i D_i$ on S, the *lattice polytope* $\square_D \subset \mathbf{R}^2$ is defined as

$$\square_D = \{(z, w) \in \mathbf{R}^2 \mid x_i z + y_i w \le n_i \text{ for } 1 \le i \le d\}.$$

Though $\square_D$ can change according to how we describe the linear equivalence class of D, those differences induce only parallel translations of $\square_D$. Hence the shape of the lattice polytope is determined uniquely. For integers n and i with $1 \le i \le d$, we define the line $l_i(n) \subset \mathbf{R}^2$ by

$$l_i(n) = \{(z, w) \in \mathbf{R}^2 \mid x_i z + y_i w = n\}.$$

Then $\square_D$ is the domain surrounded by the lines $l_i(n_i)$ $(i = 1, \dots, d)$.

Now, our main result can be stated as follows:

THEOREM 1.5. *Let S be a nonsingular compact toric surface defined by a fan composed by d cones, and $C \sim \sum_{i=1}^{d} p_i D_i$ a nonsingular irreducible nef curve on S. Assume that C does not pass through any T-fixed point on S. For positive integers j and i_0 with $1 \le i_0 \le d$, if the line $l_{i_0}(p_{i_0} - j)$ has more than or equal to $C.D_{i_0}$ lattice points in the interior of $\square_C$, then j is a gap value of C at the intersection points of C and D_{i_0}.*

Here we remark that it is not an essential assumption that C does not pass through any T-fixed point. Indeed, if there are T-fixed points lying on C, then by a succession of blowing-ups with those points as centers, we can obtain an embedding of C in a toric surface which satisfies the assumptions of Theorem 1.5.

As declared in Abstract, under a suitable condition, Theorem 1.5 gives the necessary and sufficient condition for j to be a gap value at the intersection points of C and D_{i_0}. Concretely, the following Corollary holds.

COROLLARY 1.6. *Let S, C and i_0 be as in Theorem 1.5. Assume that $C.D_{i_0} = 1$ and the line $l_{i_0}(p_{i_0} - j)$ has at most one lattice point in the interior of $\square_C$ for any integer j. Then j is a gap value of C at $P = C \cap D_{i_0}$ if and only if $l_{i_0}(p_{i_0} - j)$ has a lattice point in the interior of $\square_C$.*

Indeed, under these assumptions, the gap values at P detected by Theorem 1.5 are in one-to-one correspondence with the set of lattice points contained in the interior of $\square_C$. Since $\square_C$ has g lattice points in its interior (cf. Corollary 2.2), this means that all the gap values at P are completely found by Theorem 1.5.

2. Fundamentals of toric surfaces

In this section, we collect several fundamental properties of toric surfaces. For many of them, we refer the readers to [8] without further mention.

Let S be a nonsingular compact toric surface. As in Section 1, for an integer $1 \le i \le d$, we denote by σ_i the cone corresponding to the T-invariant divisor D_i and by (x_i, y_i) the primitive element of σ_i. The labeling of the T-invariant divisors and the correspondence of the surface S and the fan Δ_S follow the ways in Section 1. Let D be a divisor on S. The dimension of the global section space of D can be read off the lattice points contained in $\square_D$:

THEOREM 2.1. *The equation* $h^0(S, D) = \#(\square_D \cap \mathbf{Z}^2)$ *holds.*

COROLLARY 2.2. *The following hold*:
(i) *If D is a nonzero effective divisor, then* $h^0(S, -D) = 0$.
(ii) *For a nonsingular irreducible curve C on S, its genus is equal to the number of lattice points in the interior of* $\square_C$.

In the case where the complete linear system $|D|$ is base point free, we have the following two results.

THEOREM 2.3. *If $|D|$ is base point free, then $h^i(S, D) = 0$ for any positive integer i.*

THEOREM 2.4. *The following are equivalent*:
(i) *$|D|$ is base point free.*
(ii) *$D.D_i \ge 0$ for any T-invariant divisor D_i $(i = 1, \dots, d)$.*

The self-intersection numbers of T-invariant divisors are computed by the following formula.

THEOREM 2.5. *For any integer $0 \le i \le d$ and $1 \le j \le d$,*

$$x_i D_i^2 = -x_{i-1} - x_{i+1},$$
$$y_i D_i^2 = -y_{i-1} - y_{i+1}.$$

PROPOSITION 2.6. *Let (z, w), (z_1, w_1) and (z_2, w_2) be lattice points such that $z_1 w_2 - w_1 z_2 \neq 0$. Then there is a unique pair of real numbers (α, β) such that*

$$(z, w) = \alpha(z_1, w_1) + \beta(z_2, w_2).$$

In particular, if $z_1 w_2 - w_1 z_2 = \pm 1$, *then* α *and* β *are integers.*

PROPOSITION 2.7. *Let* i *and* j *be integers with* $0 \leq i \leq d$ *and* $1 \leq j \leq d$. *Let* α *and* β *be integers such that*

$$(x_j, y_j) = \alpha(x_i, y_i) + \beta(x_{i+1}, y_{i+1}).$$

Then at least one of α *and* β *is non-positive. Furthermore, if either* α *or* β *is equal to zero, then the other is equal to one or minus one.*

LEMMA 2.8. *Let* i *and* j *be distinct integers with* $0 \leq i \leq d$ *and* $1 \leq j \leq d + 1$. *If* $x_i y_j - y_i x_j \leq 0$, *then*

$$x_{i+1} y_j - y_{i+1} x_j \leq 0,$$
$$x_i y_{j-1} - y_i x_{j-1} \leq 0.$$

The equalities hold if and only if $j = i + 1$.

Proof. We will show only the first inequality. One can similarly verify the second one. By Proposition 2.6, we can write

$$(x_j, y_j) = \alpha(x_i, y_i) + \beta(x_{i+1}, y_{i+1})$$

with integers α and β. Then we have

$$0 \leq x_j y_i - y_j x_i = \beta(x_{i+1} y_i - y_{i+1} x_i) = \beta.$$

Recall that $j \neq i$. In the case of $\beta = 0$, by Proposition 2.7, we have $(x_j, y_j) = -(x_i, y_i)$. Hence

$$x_{i+1} y_j - y_{i+1} x_j = -x_{i+1} y_i + y_{i+1} x_i = -1.$$

In the case of $\beta \geq 1$, we have $\alpha \leq 0$ by Proposition 2.7. Hence

$$x_{i+1} y_j - y_{i+1} x_j = \alpha(x_{i+1} y_i - y_{i+1} x_i) = \alpha \leq 0.$$

If $x_{i+1} y_j - y_{i+1} x_j = 0$, then we have $\alpha = 0$. Hence, by Proposition 2.7, we have $\beta = 1$, which means $j = i + 1$. □

3. Proof of the main theorem

We keep the notation in Section 2. By renumbering of T-invariant divisors, we can assume $i_0 = 1$ in Theorem 1.5. We thus consider the case where $i_0 = 1$ henceforth.

In fact, the Picard group of S is freely generated by the classes of T-invariant divisors except two adjacent divisors (e.g. $D_2, \ldots, D_{d-1}$). Hence, for a curve C on S, we can take the linear equivalence class of C as

$$C \sim \sum_{i=1}^{d} p_i D_i \quad (p_i \in \mathbf{Z}, p_1 = p_d = 0) \tag{1}$$

without loss of generality. We thus assume $p_1 = p_d = 0$ henceforth. We denote by $\mathrm{Int}\,\square_C$ the interior of $\square_C$, that is,

$$\mathrm{Int}\,\square_C = \{(z, w) \in \mathbf{R}^2 \mid x_i z + y_i w < p_i \text{ for } 1 \leq i \leq d\}.$$

3.1. Key lemma

The aim of this subsection is to show Lemma 3.9 which is the key to proving Theorem 1.5. We first see the relation between the coefficients of the linear equivalence class of C and the primitive elements of the cones.

LEMMA 3.1. *Assume that $|C|$ is base point free. Then, for any integer $2 \leq k \leq d-1$,*

$$p_k = \sum_{i=1}^{k-1} (x_k y_i - y_k x_i) C.D_i \geq 0.$$

Proof. Recall Theorem 2.5 and that $p_1 = p_d = 0$. An easy computation shows the equality

$$\begin{aligned}
&\sum_{i=1}^{k-1} (x_k y_i - y_k x_i) C.D_i \\
&\quad = x_k \sum_{i=1}^{k-1} y_i (p_{i-1} + p_i D_i^2 + p_{i+1}) - y_k \sum_{i=1}^{k-1} x_i (p_{i-1} + p_i D_i^2 + p_{i+1}) \\
&\quad = x_k \sum_{i=1}^{k-1} (y_i p_{i-1} - (y_{i-1} + y_{i+1}) p_i + y_i p_{i+1}) \\
&\qquad - y_k \sum_{i=1}^{k-1} (x_i p_{i-1} - (x_{i-1} + x_{i+1}) p_i + x_i p_{i+1}) \\
&\quad = x_k (y_1 p_d - y_d p_1 - y_k p_{k-1} + y_{k-1} p_k) - y_k (x_1 p_d - x_d p_1 - x_k p_{k-1} + x_{k-1} p_k) \\
&\quad = (x_k y_{k-1} - y_k x_{k-1}) p_k = p_k.
\end{aligned}$$

Next we shall show that p_k is non-negative. Note that Theorem 2.4 implies that $C.D_i \geq 0$ for any integer $1 \leq i \leq d$. If $x_k y_1 - y_k x_1 \geq 0$, then by Lemma 2.8, we have $x_k y_i - y_k x_i \geq 1$ for any integer $2 \leq i \leq k-1$. This means that $p_k \geq 0$.

Assume that $x_k y_1 - y_k x_1 \le -1$. An easy computation gives the equation

$$\sum_{i=1}^{d} x_i C.D_i = \sum_{i=1}^{d} y_i C.D_i = 0.$$

Namely, we have

$$p_k = -x_k \sum_{i=k+1}^{d} y_i C.D_i + y_k \sum_{i=k+1}^{d} x_i C.D_i = \sum_{i=k+1}^{d} (x_i y_k - y_i x_k) C.D_i. \tag{2}$$

On the other hand, Lemma 2.8 implies that $x_k y_i - y_k x_i \le -1$ for any integer $k+1 \le i \le d$. Hence the inequality $p_k \ge 0$ follows from (2). □

In the remaining part of this subsection, let C be a nonsingular irreducible nef curve of genus g on S, and assume $C.D_1 \ge 1$. Since C is nef, $|C|$ is base point free by Theorem 2.4. Let j be a positive integer such that $l_1(-j) \cap \operatorname{Int} \square_C \cap \mathbf{Z}^2 \neq \emptyset$, and we denote by (z_0, w_0) the lattice point in $l_1(-j) \cap \operatorname{Int} \square_C$ closest to the line $l_d(p_d)$. All the remaining lemmas in this subsection are closely related to the notion of lattice polytope. Hence, for a better understanding, we will argue together with the following example.

Example 3.2. Let S be a toric surface defined by the fan in Fig. 1, and

$$C_0 \sim 2D_2 + 6D_3 + 10D_4 + 5D_5 + 7D_6 + 16D_7 + 10D_8 + 4D_9 + 3D_{10}$$

a nonsingular irreducible nef curve on S. Then the lattice polytope $\square_{C_0}$ is drawn as in Fig. 2.

We next define a certain effective divisor I, which plays an central role in the proof of Theorem 1.5.

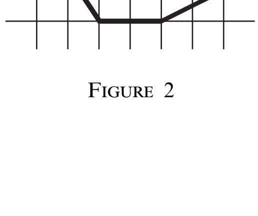

FIGURE 2

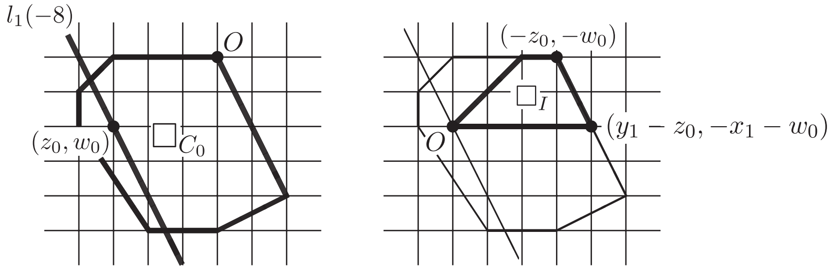

FIGURE 3

DEFINITION 3.3. We define

$$a = \min\{i \geq 2 \mid x_i(z_0 - y_1) + y_i(w_0 + x_1) \geq 0\},$$

$$b = \max\{i \leq d \mid x_i z_0 + y_i w_0 \geq 0\},$$

$$q_i = \begin{cases} x_i(y_1 - z_0) - y_i(x_1 + w_0) & (1 \leq i \leq a - 1), \\ -x_i z_0 - y_i w_0 & (b + 1 \leq i \leq d), \\ 0 & (\text{otherwise}), \end{cases}$$

$$I = \sum_{i=1}^{d} q_i D_i.$$

Note that $b \leq d - 1$. Indeed, by the definition of (z_0, w_0), the inequality $x_d z_0 - y_d w_0 \leq p_d - 1 = -1$ holds. For instance, in the case of Example 3.2, for an integer $j = 8$, we have $a = 5$, $b = 10$ and

$$I = 8D_1 + 4D_2 + 4D_3 + 4D_4 + 2D_{11} + 5D_{12}.$$

The line $l_1(-8)$ and $\square_I$ are as in Fig. 3. Note that the origin has changed.

LEMMA 3.4. *For any integer $b + 1 \leq k \leq d$, the inequality*

$$x_k y_1 - y_k x_1 \leq -1$$

holds. Moreover, if $a \geq 3$, then $x_m y_1 - y_m x_1 \geq 1$ for any integer $2 \leq m \leq a - 1$.

Proof. Since $x_1 z_0 + y_1 w_0 = -j \neq 0$, we can write

$$(x_b, y_b) = \alpha_1(x_1, y_1) + \beta_1(w_0, -z_0),$$

$$(x_{b+1}, y_{b+1}) = \alpha_2(x_1, y_1) + \beta_2(w_0, -z_0)$$

with some real numbers. By the definition of b, we have

$$x_b z_0 + y_b w_0 = \alpha_1(x_1 z_0 + y_1 w_0) = -j\alpha_1 \geq 0,$$

$$x_{b+1} z_0 + y_{b+1} w_0 = \alpha_2(x_1 z_0 + y_1 w_0) = -j\alpha_2 < 0.$$

Hence we have $\alpha_1 \le 0$ and $\alpha_2 > 0$. Now, we suppose that $x_{b+1}y_1 - y_{b+1}x_1 \ge 0$. Then Lemma 2.8 implies that $x_b y_1 - y_b x_1 \ge 0$. Hence we have

$$x_b y_1 - y_b x_1 = \beta_1(x_1 z_0 + y_1 w_0) = -j\beta_1 \ge 0,$$

$$x_{b+1}y_1 - y_{b+1}x_1 = \beta_2(x_1 z_0 + y_1 w_0) = -j\beta_2 \ge 0,$$

which imply $\beta_1 \le 0$ and $\beta_2 \le 0$. Then, by computing, we have

$$x_b y_{b+1} - y_{b+} x_{b+1} = j(\alpha_1\beta_2 - \beta_1\alpha_2) \ge 0.$$

This contradicts the fact that $x_{b+1}y_b - y_{b+1}x_b = 1$. We thus obtain that $x_{b+1}y_1 - y_{b+1}x_1 \le -1$. Then by Lemma 2.8, $x_k y_1 - y_k x_1 \ge 1$ for any integer $b+1 \le k \le d$. Similarly, one can show the second inequality by considering the descriptions of (x_{a-1}, y_{a-1}) and (x_a, y_a) as the sum of (x_1, y_1) and $(x_1 + w_0, y_1 - z_0)$ with real coefficients. □

Note that the inequality $a \le b+1$ follows immediately from Lemma 3.4. Indeed, if $a \ge b+2$, then we have

$$x_{a-1}y_1 - y_{a-1}x_1 \le -1 \tag{3}$$

by Lemma 3.4. However, this contradicts the second statement in Lemma 3.4 in the case where $a \ge 3$. It goes without saying that (3) is a contradiction in the case where $a = 2$ also.

Lemma 3.5. *The complete linear system $|I|$ is base point free.*

Proof. By Theorem 2.4, it is sufficient to verify $I.D_i \ge 0$ for each integer $1 \le i \le d$. Recall Theorem 2.5. Then we have

$$\begin{aligned} I.D_1 &= q_d + q_1 D_1^2 + d_2 \\ &= -x_d z_0 - y_d w_0 - x_1 z_0 D_1^2 - y_1 w_0 D_1^2 + x_2 y_1 - y_2 x_1 - x_2 z_0 - y_2 w_0 = 1. \end{aligned}$$

For integers $2 \le k_1 \le a-2$,

$$I.D_{k_1} = (x_{k_1-1} + x_{k_1}D_{k_1}^2 + x_{k_1+1})(y_1 - z_0) - (y_{k_1-1} + y_{k_1}D_{k_1}^2 + y_{k_1+1})(x_1 + w_0) = 0.$$

For integers $b+2 \le k_2 \le d$,

$$I.D_{k_2} = -(x_{k_2-1} + x_{k_2}D_{k_2}^2 + x_{k_2+1})z_0 - (y_{k_2-1} + y_{k_2}D_{k_2}^2 + y_{k_2+1})w_0 = 0.$$

Moreover, it is obvious that $I.D_{k_3} = 0$ for any integer $a+1 \le k_3 \le b-1$.

Let us check the remaining divisors D_{a-1}, D_a, D_b and D_{b+1}. Recall Lemma 3.4. Then we have

$$I.D_{a-1} = \begin{cases} x_a(z_0 - y_1) + y_a(w_0 + x_1) \ge 0 & (a \le b), \\ -x_{b+1}y_1 + y_{b+1}x_1 \ge 1 & (a = b+1), \end{cases}$$

$$I.D_a = \begin{cases} -x_{a-1}(z_0 - y_1) - y_{a-1}(w_0 + x_1) \geq 1 & (a \leq b-1), \\ -x_{a-1}(z_0 - y_1) - y_{a-1}(w_0 + x_1) - x_{b+1}z_0 - y_{b+1}w_0 \geq 2 & (a = b), \\ x_{a-1}y_1 - y_{a-1}x_1 \geq 1 & (a = b+1). \end{cases}$$

Similarly, we have

$$I.D_b = \begin{cases} -x_{b+1}z_0 - y_{b+1}w_0 \geq 1 & (a \leq b-1), \\ -x_{a-1}(z_0 - y_1) - y_{a-1}(w_0 + x_1) - x_{b+1}z_0 - y_{b+1}w_0 \geq 2 & (a = b), \\ -x_{b+1}y_1 + y_{b+1}x_1 \geq 1 & (a = b+1), \end{cases}$$

$$I.D_{b+1} = \begin{cases} x_b z_0 + y_b w_0 \geq 0 & (a \leq b), \\ x_{a-1}y_1 - y_{a-1}x_1 \geq 1 & (a = b+1). \end{cases} \qquad \square$$

Very roughly speaking, Theorem 1.5 is verified by comparing the cohomology dimension $h^0(C, I|_C)$ with $h^0(C, (I - D_1)|_C)$. In fact, however, it is not enough for the proof to deal with only I. We need to introduce the following auxiliary divisor X and consider the divisor obtained by subtracting it from I. We define

$$X = \sum_{i=2}^{a-1} D_i + \sum_{i=b+1}^{d} D_i,$$

$$L_i(n) = \{(z, w) \in \mathbf{Z}^2 \mid x_i z + y_i w \leq n\}$$

for integers n and i with $1 \leq i \leq d$.

Lemma 3.6. *The vanishing $h^1(S, I - X) = 0$ holds.*

Proof. Consider the cohomology long exact sequence

$$\begin{aligned} 0 &\to H^0(S, I - X) \to H^0(S, I) \to H^0(X, I|_X) \\ &\to H^1(S, I - X) \to H^1(S, I) \to H^1(X, I|_X) \to H^2(S, I - X) \to \cdots. \end{aligned}$$

Lemma 3.5 and Theorem 2.3 imply that $h^1(S, I) = 0$. Besides, $h^2(S, I - X) = h^0(S, K_S + X - I) = 0$ holds by Serre duality and Corollary 2.2. Hence Riemann-Roch theorem yields the equality

$$h^0(X, I|_X) = \deg I|_X + 1 - \frac{1}{2}X.(K_S + X) - 1 = I.X - \frac{1}{2}X.(K_S + X).$$

We thus have

$$h^1(S, I - X) = h^0(S, I - X) - h^0(S, I) + I.X - \frac{1}{2}X.(K_S + X). \tag{4}$$

Since $I.D_i = 0$ for any integer i with $2 \leq i \leq a-2$ or $b+2 \leq i \leq d$, we have

$$I.X = \begin{cases} I.D_{a-1} + I.D_{b+1} & (a \geq 3), \\ I.D_{b+1} & (a = 2). \end{cases} \tag{5}$$

Moreover, by computing, we have

$$X.(K_S + X) = \begin{cases} -4 & (3 \leq a \leq b), \\ -2 & (\text{otherwise}). \end{cases} \tag{6}$$

In order to compute the value of $h^0(S,I) - h^0(S,I-X)$, we first verify the following inclusions:

$$\begin{aligned} L_1(q_1) \cap L_{a-1}(q_{a-1}) &\subset \bigcap_{i=2}^{a-1} L_i(q_i) \quad \text{if } a \geq 3, \\ L_1(q_1) \cap L_{b+1}(q_{b+1}) &\subset \bigcap_{i=b+1}^{d} L_i(q_i). \end{aligned} \tag{7}$$

Assume $a \geq 3$ and let (z_1, w_1) be a lattice point contained in $L_1(q_1) \cap L_{a-1}(q_{a-1})$. We write

$$(z_1, w_1) = (y_1 - z_0, -x_1 - w_0) + \alpha_1(y_1, -x_1) + \beta_1(y_{a-1}, -x_{a-1})$$

with real numbers α_1 and β_1. Then the inequalities

$$\begin{gathered} x_1 z_1 + y_1 w_1 = q_1 + \beta_1(x_1 y_{a-1} - y_1 x_{a-1}) \leq q_1, \\ x_{a-1} z_1 + y_{a-1} w_1 = q_{a-1} + \alpha_1(x_{a-1} y_1 - y_{a-1} x_1) \leq q_{a-1} \end{gathered}$$

implies $\alpha_1 \leq 0$ and $\beta_1 \geq 0$, respectively. Let k_1 be an integer with $2 \leq k_1 \leq a-1$. Then Lemma 3.4 and Lemma 2.8 imply that $x_{k_1} y_1 - y_{k_1} x_1 \geq 1$ and $x_{a-1} y_{k_1} - y_{a-1} x_{k_1} \geq 0$. We thus have

$$x_{k_1} z_1 + y_{k_1} w_1 = q_{k_1} + \alpha_1(x_{k_1} y_1 - y_{k_1} x_1) + \beta_1(x_{k_1} y_{a-1} - y_{k_1} x_{a-1}) \leq q_{k_1}.$$

Hence we obtain the first inclusion of (7). Similarly, for a point (z_2, w_2) contained in $L_1(q_1) \cap L_{b+1}(q_{b+1})$, we write

$$(z_2, w_2) = (-z_0, -w_0) + \alpha_2(y_1, -x_1) + \beta_2(y_{b+1}, -x_{b+1})$$

and can show $\alpha_2 \geq 0$, $\beta_2 \leq 0$ and the second inclusion of (7).

The same argument can be adapted to show

$$\begin{aligned} L_1(q_1) \cap L_{a-1}(q_{a-1} - 1) &\subset \bigcap_{i=2}^{a-1} L_i(q_i - 1) \quad \text{if } a \geq 3, \\ L_1(q_1) \cap L_{b+1}(q_{b+1} - 1) &\subset \bigcap_{i=b+1}^{d} L_i(q_i - 1). \end{aligned} \tag{8}$$

Recall the notation $l_i(n)$ defined in Section 1. Then by (7) and (8), if $a \geq 3$, we have

$$
\begin{aligned}
&h^0(S,I) - h^0(S,I-X)\\
&\quad = \#\left(\bigcap_{i=1}^{d} L_i(q_i)\right) - \#\left(L_1(q_1) \cap \bigcap_{i=2}^{a-1} L_i(q_i-1) \cap \bigcap_{i=a}^{b} L_i(q_i) \cap \bigcap_{i=b+1}^{d} L_i(q_i-1)\right)\\
&\quad = \#\left(L_1(q_1) \cap \bigcap_{i=a-1}^{b+1} L_i(q_i)\right)\\
&\qquad - \#\left(L_1(q_1) \cap L_{a-1}(q_{a-1}-1) \cap \bigcap_{i=a}^{b} L_i(q_i) \cap L_{b+1}(q_{b+1}-1)\right)\\
&\quad = \#\left(L_1(q_1) \cap \bigcap_{i=a-1}^{b+1} L_i(q_i) \backslash (L_{a-1}(q_{a-1}-1) \cap L_{b+1}(q_{b+1}-1))\right)\\
&\quad = \#\left(\left(L_1(q_1) \cap \bigcap_{i=a-1}^{b+1} L_i(q_i) \backslash L_{a-1}(q_{a-1}-1)\right)\right.\\
&\qquad \left.\cup \left(L_1(q_1) \cap \bigcap_{i=a-1}^{b+1} L_i(q_i) \backslash L_{b+1}(q_{b+1}-1)\right)\right)\\
&\quad = \#\left(\left(L_1(q_1) \cap l_{a-1}(q_{a-1}) \cap \bigcap_{i=a}^{b+1} L_i(q_i)\right)\right.\\
&\qquad \left.\cup \left(L_1(q_1) \cap \bigcap_{i=a-1}^{b} L_i(q_i) \cap l_{b+1}(q_{b+1})\right)\right).
\end{aligned}
$$

Similarly, if $a = 2$, one can obtain

$$h^0(S,I) - h^0(S,I-X) = \#\left(\bigcap_{i=1}^{b} L_i(q_i) \cap l_{b+1}(q_{b+1})\right).$$

We define

$$M = L_1(q_1) \cap l_{a-1}(q_{a-1}) \cap \bigcap_{i=a}^{b+1} L_i(q_i),$$

$$N = L_1(q_1) \cap \bigcap_{i=a-1}^{b} L_i(q_i) \cap l_{b+1}(q_{b+1}).$$

Then we have

$$h^0(S,I) - h^0(S,I-X) = \begin{cases} \#(M \cup N) = \#M + \#N - \#(M \cap N) & (a \geq 3), \\ \#N & (a = 2). \end{cases}$$

Here let us see the case of Example 3.2. As we saw after Definition 3.3, in this example, we have $a = 5$ and $b = 10$ for $j = 8$. Hence M and N are the sets

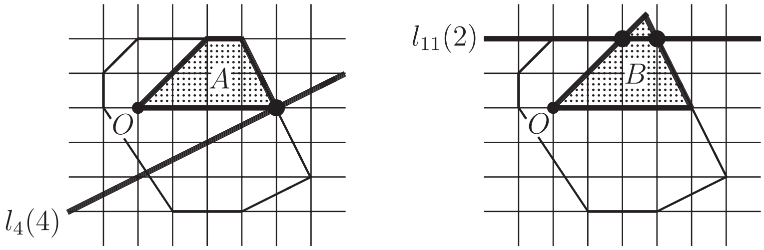

FIGURE 4

of lattice points contained in $l_4(4) \cap A$ and $l_{11}(2) \cap B$, respectively (see Fig. 4), where $A = L_1(8) \cap \bigcap_{i=5}^{11} L_i(q_i)$ and $B = L_1(8) \cap \bigcap_{i=4}^{10} L_i(q_i)$.

We shall examine $\#M$. Let (u, v) be a lattice point contained in M. Since both (u, v) and $(y_1 - z_0, -x_1 - w_0)$ are contained in $l_{a-1}(q_{a-1})$, we can write

$$(u, v) = (y_1 - z_0, -x_1 - w_0) + \gamma(y_{a-1}, -x_{a-1})$$

with some integer γ. We obtain $\gamma \geq 0$ by Lemma 3.4 and the inequality

$$x_1 u + y_1 v = q_1 + \gamma(x_1 y_{a-1} - y_1 x_{a-1}) \leq q_1.$$

Since (u, v) is contained in $L_a(q_a)$, we have

$$\begin{aligned} q_a \geq x_a u + y_a v &= x_a(y_1 - z_0) - y_a(x_1 + w_0) + \gamma \\ &= \begin{cases} -I.D_{a-1} + \gamma & (a \leq b), \\ -I.D_{a-1} + q_a + \gamma & (a = b + 1). \end{cases} \end{aligned}$$

Recall that $q_a = 0$ in the case where $a \leq b$. We thus have $\gamma \leq I.D_{a-1}$.

Conversely, we shall show that for any integer $0 \leq \gamma' \leq I.D_{a-1}$, the lattice point

$$(u', v') = (y_1 - z_0, -x_1 - w_0) + \gamma'(y_{a-1}, -x_{a-1})$$

is contained in M. Since (u', v') is clearly contained in $L_1(q_1) \cap l_{a-1}(q_{a-1})$, it is sufficient to verify that (u', v') is contained in $\bigcap_{i=a}^{b+1} L_i(q_i)$. We remark the equality

$$\begin{aligned} &I.D_{a-1}(x_{a-1}, y_{a-1}) \\ &\quad = (q_{a-2} + q_{a-1} D_{a-1}^2 + q_a)(x_{a-1}, y_{a-1}) \\ &\quad = (-x_a(y_1 - z_0) + y_a(x_1 + w_0) + q_a)(x_{a-1}, y_{a-1}) \\ &\quad = (-w_0 - x_1, z_0 - y_1) + (x_{a-1}(z_0 - y_1) \\ &\qquad + y_{a-1}(w_0 + x_1))(x_a, y_a) + q_a(x_{a-1}, y_{a-1}). \end{aligned} \tag{9}$$

We first show that (u', v') is contained in $L_{b+1}(q_{b+1})$.

(i) If $x_{b+1}y_{a-1} - y_{b+1}x_{a-1} \le 0$, then $x_{b+1}y_1 - y_{b+1}x_1 \le 0$ by Lemma 2.8. We thus have

$$\begin{aligned} x_{b+1}u' + y_{b+1}v' &= x_{b+1}y_1 - y_{b+1}x_1 - x_{b+1}z_0 - y_{b+1}w_0 + \gamma'(x_{b+1}y_{a-1} - y_{b+1}x_{a-1}) \\ &\le -x_{b+1}z_0 - y_{b+1}w_0 = q_{b+1}. \end{aligned}$$

(ii) If $x_{b+1}y_{a-1} - y_{b+1}x_{a-1} \ge 1$, then $x_{b+1}y_a - y_{b+1}x_a \ge 0$ by Lemma 2.8. Moreover, by the equation (9), we have

$$\begin{aligned} &I.D_{a-1}(x_{b+1}y_{a-1} - y_{b+1}x_{a-1}) \\ &\quad = x_{b+1}(z_0 - y_1) + y_{b+1}(w_0 + x_1) \\ &\qquad + (x_{a-1}(z_0 - y_1) + y_{a-1}(w_0 + x_1))(x_{b+1}y_a - y_{b+1}x_a) \\ &\qquad + q_a(x_{b+1}y_{a-1} - y_{b+1}x_{a-1}) \\ &\quad \le x_{b+1}(z_0 - y_1) + y_{b+1}(w_0 + x_1) + q_a(x_{b+1}y_{a-1} - y_{b+1}x_{a-1}). \end{aligned}$$

Hence we have

$$\begin{aligned} x_{b+1}u' + y_{b+1}v' &= x_{b+1}(y_1 - z_0) + y_{b+1}(-x_1 - w_0) + \gamma'(x_{b+1}y_{a-1} - y_{b+1}x_{a-1}) \\ &\le x_{b+1}(y_1 - z_0) + y_{b+1}(-x_1 - w_0) + I.D_{a-1}(x_{b+1}y_{a-1} - y_{b+1}x_{a-1}) \\ &\le q_a(x_{b+1}y_{a-1} - y_{b+1}x_{a-1}). \end{aligned}$$

If $a \le b$, then $q_a = 0$ and we have $x_{b+1}u' + y_{b+1}v' \le 0 < q_{b+1}$. If $a = b+1$, then we have

$$q_a(x_{b+1}y_{a-1} - y_{b+1}x_{a-1}) = q_{b+1}(x_{b+1}y_b - y_{b+1}x_b) = q_{b+1}.$$

Hence we can conclude that (u', v') is contained in $L_{b+1}(q_{b+1})$.

If $a = b+1$, then the above argument is enough to show that (u', v') is contained in M. On the other hand, in the case where $a \le b$, we have to check that (u', v') is contained in $\bigcap_{i=a}^{b} L_i(q_i)$ also. Let m be an integer with $a \le m \le b$. Note that $q_m = 0$ in this case.

(i) If $x_m y_{a-1} - y_m x_{a-1} \ge 0$, then we have $x_m y_a - y_m x_a \ge 0$ by Lemma 2.8. Then by the equation (9), we have

$$\begin{aligned} x_m u' + y_m v' &= (\gamma' - I.D_{a-1})(x_m y_{a-1} - y_m x_{a-1}) \\ &\qquad + (x_{a-1}(z_0 - y_1) + y_{a-1}(w_0 + x_1))(x_m y_a - y_m y_a) \\ &\le 0 = q_m. \end{aligned}$$

(ii) If $x_m y_{a-1} - y_m x_{a-1} \le -1$, then Lemma 2.8 yields the inequalities $x_m y_1 - y_m x_1 \le -1$, $x_b y_1 - y_b x_1 \le -1$ and $x_m y_b - y_m x_b \le 0$. Thus we can write

$$(x_m, y_m) = \delta(x_1, y_1) + \varepsilon(x_b, y_b)$$

with real numbers $\delta \le 0$ and $\varepsilon > 0$. Recall that (z_0, w_0) lies on $l_1(-j)$. Then we have

$$x_m z_0 + y_m w_0 = \delta(x_1 z_0 + y_1 w_0) + \varepsilon(x_b z_0 + y_b w_0) \geq 0,$$

$$x_m u' + y_m v' = x_m y_1 - y_m x_1 - x_m z_0 - y_m w_0 + \gamma'(x_m y_{a-1} - y_m x_{a-1}) < 0 = q_m.$$

Hence we have that (u', v') is contained in $\bigcap_{i=a}^{b} L_i(q_i)$.

In sum, we can conclude that

$$M = \{(y_1 - z_0, -x_1 - w_0) + \gamma(y_{a-1}, -x_{a-1}) \,|\, 0 \leq \gamma \leq I.D_{a-1}\}.$$

A similar argument can be adapted to show that

$$N = \{(-z_0, -w_0) - \zeta(y_{b+1}, -x_{b+1}) \,|\, 0 \leq \zeta \leq I.D_{b+1}\}.$$

Next we examine $M \cap N$ under the assumption that $a \geq 3$. By the definition of M and N, the intersection $M \cap N$ is included in $l_{a-1}(q_{a-1}) \cap l_{b+1}(q_{b+1})$.

(i) In the case where $x_{a-1}y_{b+1} - y_{a-1}x_{b+1} = 0$, we have $(x_{b+1}, y_{b+1}) = -(x_{a-1}, y_{a-1})$. Let (u_1, v_1) be a lattice point on $l_{a-1}(q_{a-1})$. Then, by Lemma 3.4, we have

$$\begin{aligned} x_{b+1}u_1 + y_{b+1}v_1 &= -x_{a-1}u_1 - y_{a-1}v_1 = -q_{a-1} = x_1 y_{a-1} - y_1 x_{a-1} + x_{a-1}z_0 + y_{a-1}w_0 \\ &\leq x_{a-1}z_0 + y_{a-1}w_0 - 1 = -x_{b+1}z_0 - y_{b+1}w_0 - 1 = q_{b+1} - 1. \end{aligned}$$

Hence (u_1, v_1) does not lie on $l_{b+1}(q_{b+1})$. This means $M \cap N = \emptyset$.

Assume $x_{a-1}y_{b+1} - y_{a-1}x_{b+1} \neq 0$. In this case, the intersection $l_{a-1}(q_{a-1}) \cap l_{b+1}(q_{b+1})$ clearly consists of only one lattice point. We denote it by (u_0, v_0).

(ii) Consider the case where $x_{a-1}y_{b+1} - y_{a-1}x_{b+1} \geq 1$. Since both (u_0, v_0) and $(-z_0, -w_0)$ lie on $l_{b+1}(q_{b+1})$, one can write

$$(u_0, v_0) = (-z_0, -w_0) + \eta(y_{b+1}, -x_{b+1})$$

with some integer η. Then the inequality

$$\begin{aligned} &-x_{a-1}z_0 - y_{a-1}w_0 + \eta(x_{a-1}y_{b+1} - y_{a-1}x_{b+1}) \\ &\quad = x_{a-1}u_0 + y_{a-1}v_0 = q_{a-1} = x_{a-1}(y_1 - z_0) - y_{a-1}(x_1 + w_0) \\ &\quad \geq -x_{a-1}z_0 - y_{a-1}w_0 + 1 \end{aligned}$$

implies $\eta \geq 1$. Hence we have

$$x_1 u_0 + y_1 v_0 = q_1 + \eta(x_1 y_{b+1} - y_1 x_{b+1}) \geq q_1 + 1.$$

This means that (u_0, v_0) is not contained in $L_1(q_1)$, that is, $M \cap N = \emptyset$.

(iii) Consider the case where $x_{a-1}y_{b+1} - y_{a-1}x_{b+1} \leq -1$. We write

$$(u_0, v_0) = \theta(y_{a-1}, -x_{a-1}) + \iota(y_{b+1}, -x_{b+1})$$

with real numbers θ and ι. Since (u_0, v_0) is contained in $l_{a-1}(q_{a-1}) \cap l_{b+1}(q_{b+1})$, we have $\theta > 0$ and $\iota < 0$.

(iii)-(a) If $a \leq b$, then $q_b = 0$. Since Lemma 2.8 implies that $x_{a-1}y_b - y_{a-1}x_b \leq -1$, we have

$$x_b u_0 + y_b v_0 = \theta(x_b y_{a-1} - y_b x_{a-1}) - \iota \geq \theta - \iota > 0 = q_b.$$

This means that (u_0, v_0) is not contained in $L_b(q_b)$, that is, $M \cap N = \emptyset$.

(iii)-(b) If $a = b+1$, then $M \cap N = L_1(q_1) \cap l_{a-1}(q_{a-1}) \cap l_{b+1}(q_{b+1})$. Since $q_1 = -x_1 z_0 - y_1 w_0 = j \geq 1$, we have

$$x_1 u_0 + y_1 v_0 = \theta(x_1 y_{a-1} - y_1 x_{a-1}) + \iota(x_1 y_{b+1} - y_1 x_{b+1}) \leq -\theta + \iota < 0 \leq q_1 - 1.$$

Hence, in this case, (u_0, v_0) is contained in $L_1(q_1)$ and we have $M \cap N = \{(u_0, v_0)\}$.

Here we note that $a \leq b$ in the case of (i) and (ii). Indeed, if $a = b+1$, then $x_{a-1} y_{b+1} - y_{a-1} x_{b+1} = -1$. Therefore, we can conclude that

$$\#(M \cap N) = \begin{cases} 0 & (3 \leq a \leq b), \\ 1 & (3 \leq a = b+1). \end{cases}$$

In sum, we have

$$h^0(S, I) - h^0(S, I - X) = \begin{cases} I.D_{a-1} + I.D_{b+1} + 2 & (3 \leq a \leq b), \\ I.D_{a-1} + I.D_{b+1} + 1 & (3 \leq a = b+1), \\ I.D_{b+1} + 1 & (a = 2). \end{cases} \tag{10}$$

Therefore, combining (4), (5), (6) and (10), we can obtain $h^1(S, I - X) = 0$. $\square$

In order to compute the difference between the dimensions of global section spaces of $(I - X)|_C$ and $(I - X - D_1)|_C$, we examine their cohomologies of higher order in Lemma 3.8 below.

Lemma 3.7. *If $\#(l_1(-j) \cap \operatorname{Int} \square_C \cap \mathbf{Z}^2) \geq C.D_1$, then $a \geq 3$.*

Proof. We put $c = C.D_1$. Let (z, w) be a lattice point contained in $l_1(-j) \cap \operatorname{Int} \square_C$. Then we can write

$$(z, w) = (z_0, w_0) + \alpha(y_1, -x_1)$$

with some integer α. Since (z_0, w_0) is the lattice point in $l_1(-j) \cap \operatorname{Int} \square_C$ closest to $l_d(0)$, we have $\alpha \geq 0$. Hence, by assumption, the point $(z_0, w_0) + (c-1)(y_1, -x_1)$ have to be contained in $\operatorname{Int} \square_C$. We thus have

$$x_2(z_0 + (c-1)y_1) + y_2(w_0 - (c-1)x_1) = x_2(z_0 - y_1) + y_2(w_0 + x_1) + c < p_2 = c,$$

where the last equality follows from Lemma 3.1. Hence we have $x_2(z_0 - y_1) + y_2(w_0 + x_1) < 0$, which means $a \geq 3$. $\square$

Lemma 3.8. *If $\#(l_1(-j) \cap \operatorname{Int} \square_C \cap \mathbf{Z}^2) \geq C.D_1$, then*

$$h^0(S, K_S + C - I + X + D_1) = h^0(S, K_S + C - I + X) + C.D_1.$$

Proof. We put $c = C.D_1$. Recall that $p_1 = 0$ and $q_1 = j$. Then by Theorem 2.1, we have

$$h^0(S, K_S + C - I + X + D_1) - h^0(S, K_S + C - I + X)$$

$$= \#\left(L_1(-j) \bigcap_{i=2}^{a-1} L_i(p_i - q_i) \cap \bigcap_{i=a}^{b} L_i(p_i - 1) \cap \bigcap_{i=b+1}^{d} L_i(p_i - q_i)\right)$$

$$- \#\left(L_1(-j-1) \bigcap_{i=2}^{a-1} L_i(p_i - q_i) \cap \bigcap_{i=a}^{b} L_i(p_i - 1) \cap \bigcap_{i=b+1}^{d} L_i(p_i - q_i)\right)$$

$$= \#\left(l_1(-j) \bigcap_{i=2}^{a-1} L_i(p_i - q_i) \cap \bigcap_{i=a}^{b} L_i(p_i - 1) \cap \bigcap_{i=b+1}^{d} L_i(p_i - q_i)\right).$$

We define

$$K = l_1(-j) \bigcap_{i=2}^{a-1} L_i(p_i - q_i) \cap \bigcap_{i=a}^{b} L_i(p_i - 1) \cap \bigcap_{i=b+1}^{d} L_i(p_i - q_i).$$

Then our purpose is to show that $\#K = c$. Let (u, v) be a lattice point contained in K. Since (z_0, w_0) and (u, v) lie on $l_1(-q_1)$, we can write

$$(u, v) = (z_0, w_0) + \alpha(y_1, -x_1)$$

with some integer α. Since $p_d = 0$, (u, v) is contained in $L_d(-q_d)$. Hence we have

$$x_d u + y_d v = -q_d + \alpha(x_d y_1 - y_d x_1) \le -q_d,$$

which implies $\alpha \ge 0$. On the other hand, since $a \ge 3$ by Lemma 3.7, (u, v) is contained in $L_2(p_2 - q_2)$. Hence we have

$$x_2 u + y_2 v = x_2 z_0 + y_2 w_0 + \alpha \le p_2 - q_2 = c + x_2 z_0 + y_2 w_0 - 1,$$

that is, $\alpha \le c - 1$.

Conversely, let us verify that, for an integer α' with $0 \le \alpha' \le c - 1$, the point

$$(u', v') = (z_0, w_0) + \alpha'(y_1, -x_1)$$

is contained in K. Let k_1 be an integer with $2 \le k_1 \le a - 1$. By Lemma 3.4 and Lemma 2.8, we have $x_{k_1} y_m - y_{k_1} x_m \ge 1$ for integers $1 \le m \le k_1 - 1$. Hence we have $p_{k_1} \ge (x_{k_1} y_1 - y_{k_1} x_1)c$ by Lemma 3.1 and

$$\begin{aligned} x_{k_1} u' + y_{k_1} v' &= x_{k_1}(z_0 - y_1) + y_{k_1}(w_0 + x_1) + (\alpha' + 1)(x_{k_1} y_1 - y_{k_1} x_1) \\ &\le -q_{k_1} + c(x_{k_1} y_1 - y_{k_1} x_1) \le p_{k_1} - q_{k_1}. \end{aligned}$$

For integers $b + 1 \le k_2 \le d$, we have

$$x_{k_2} u' + y_{k_2} v' = x_{k_2} z_0 + y_{k_2} w_0 + \alpha'(x_{k_2} y_1 - y_{k_2} x_1) \le -q_{k_2} \le p_{k_2} - q_{k_2}.$$

Finally, we shall check that (u', v') is contained in $\bigcap_{i=a}^{b} L_i(p_i - 1)$. Since (z_0, w_0) is the lattice point in $l_1(-j) \cap \operatorname{Int} \square_C$ closest to $l_d(0)$, we have that $(z_0, w_0) + \beta(y_1, -x_1)$ is not contained in $\operatorname{Int} \square_C$ if $\beta \leq -1$. On the other hand, by the assumption of the lemma, $l_1(-j)$ has at least c lattice points in $\operatorname{Int} \square_C$. We thus have that (u', v') is contained in $\bigcap_{i=a}^{b} L_i(p_i - 1)$ for integers $0 \leq \alpha' \leq c - 1$. In sum, we can conclude that (u', v') is contained in K for integers $0 \leq \alpha' \leq c - 1$. It follows that $\#K = c$. $\square$

By using Lemma 3.6 and 3.8 in cohomology long exact sequences, we obtain the following equality:

Lemma 3.9. *If* $\#(l_1(-j) \cap \operatorname{Int} \square_C \cap \mathbf{Z}^2) \geq C.D_1$, *then*

$$h^0(C, (I - X)|_C) = h^0(C, (I - X - D_1)|_C).$$

Proof. It is sufficient to verify the inequality $h^0(C, (I - X)|_C) \leq h^0(C, (I - X - D_1)|_C)$. By Lemma 3.6, we have the cohomology long exact sequence

$$0 \to H^1(C, (I - X)|_C) \to H^2(S, I - X - C) \to H^2(S, I - X) \to \cdots.$$

By Serre duality and Corollary 2.2, we have

$$h^2(S, I - X - C) = h^0(S, K_S + C - I + X),$$

$$h^2(S, I - X) = h^0\left(S, -I - D_1 - \sum_{i=a}^{b} D_i\right) = 0.$$

Hence, by Riemann-Roch theorem, we have

$$\begin{aligned} h^0(C, (I - X)|_C) &= h^1(C, (I - X)|_C) + \deg(I - X)|_C + 1 - g \\ &= h^0(S, K_S + C - I + X) + (I - X).C + 1 - g. \end{aligned}$$

On the other hand, the cohomology long exact sequence

$$\begin{aligned} \cdots \to H^1(C, (I - X - D_1)|_C) &\to H^2(S, I - X - D_1 - C) \\ &\to H^2(S, I - X - D_1) \to \cdots \end{aligned}$$

and the vanishings $h^2(S, I - X - D_1) = h^0(S, -I - \sum_{i=a}^{b} D_i) = 0$ lead the inequality

$$h^1(C, (I - X - D_1)|_C) \geq h^0(S, K_S + C - I + X + D_1).$$

Hence, by Riemann-Roch theorem and Lemma 3.8, we have

$$\begin{aligned} h^0(C, (I - X - D_1)|_C) &\geq h^0(S, K_S + C - I + X + D_1) + (I - X - D_1).C + 1 - g \\ &= h^0(C, (I - X)|_C). \end{aligned}$$

$\square$

3.2. Proof of the main theorem

We are now in a position to prove the main theorem.

Proof of Theorem 1.5. As mentioned at the beginning of the Section 3, we can assume $i_0 = 1$. We first consider the case where $g = 0$. In this case, the gap sequence at P is empty. Indeed, the equation $h^0(C, jP) = j - 1$ holds for any positive integer j. On the other hand, by Corollary 2.2, there are no lattice points in the interior of $\square_C$. Hence the statement is obviously true.

We assume that $g \geq 1$ and put $D_1|_C = \{P_1, \dots, P_c\}$. Lemma 3.9 implies that

$$h^0(C, (I - X)|_C) = h^0(C, (I - X)|_C - P_1).$$

Namely, P_1 is the base point of $|(I - X)|_C|$. Note that $q_1 = j$. We define

$$I' = I - jD_1 - X = \sum_{i=2}^{a-1}(q_i - 1)D_i + \sum_{i=a}^{b} q_i D_i + \sum_{i=b+1}^{d}(q_i - 1)D_i.$$

It is clear that I' is effective by Definition 3.3. Besides, since P_1 lies on neither D_2 nor D_d by assumption, $I'|_C$ does not contain P_1. Therefore, P_1 is also the base point of

$$|(I - X)|_C - I'|_C - jP_2 - \cdots - jP_c| = |jP_1|,$$

that is, $h^0(C, jP_1) = h^0(C, (j - 1)P_1)$. A similar argument goes through for the points $P_2, \dots, P_c$. $\square$

4. Examples

In this section, we shall apply Corollary 1.6 to concrete examples in practice. Our aim is to compute the gap sequences at the infinitely near points of a (possibly singular) point on a plane curve. Let Q be a point on plane curve C', and consider the resolution of singularities of C' by a succession of blowing-ups. Then, for some cases, we can determine the gap sequences of the nonsingular model of C' at the infinitely near points of Q by Corollary 1.6.

For a toric surface, a composite of a finite succession of blowing-ups with T-fixed points as centers is called a *toric morphism*. Recall that $\mathbf{P}^2$ is a toric surface. Let $\mathbf{P}^2(X_0 : X_1 : X_2)$ be the projective plane. We denote $x = X_1/X_0$, $y = X_2/X_0$ the local coordinates on the affine open subset $U_0 = \{(X_0 : X_1 : X_2) \in \mathbf{P}^2 \mid X_0 \neq 0\}$.

Example 4.1. Let C' be an irreducible plane curve defined by the local equation

$$x^6y^3 + x^3y + y - 1 = 0.$$

One can obtain a toric morphism $\varphi : S \to \mathbf{P}^2$ such that S is a nonsingular compact toric surface and the proper transform $C := \varphi_*^{-1}(C')$ is a nonsingular nef

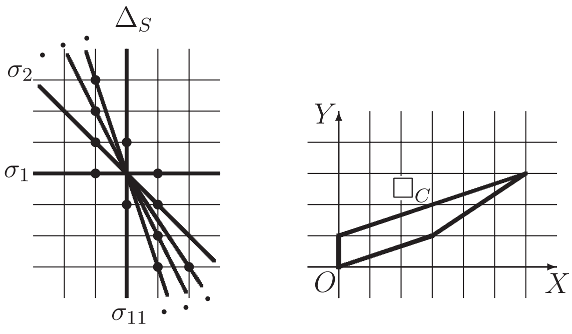

FIGURE 5

curve of genus 3 on S. The fan Δ_S defining the surface S is as in Fig. 5. If we place $\square_C$ as in Fig. 5, then the linear equivalence class of C is written as

$$C \sim D_2 + 2D_3 + 3D_4 + 3D_5 + 6D_6 + 3D_7 + 3D_8 + D_9.$$

Consider the point $Q = (0,1)$ on $C' \cap U_0$. The point Q has only one infinitely near point P on C, which is in fact the intersection point $C \cap D_1$. The cone σ_1 corresponding to D_1 has the primitive element $(-1,0)$. Hence, by Corollary 1.6, the gap sequence of C at P is

$$\{j \in \mathbf{N} \mid \text{the line } X = j \text{ has lattice points in Int } \square_C\} = \{1,2,4\}.$$

Example 4.2. Let C' be an irreducible plane curve defined by the local equation

$$x^5 + x^2y + xy^6 + y^6 = 0,$$

and $\varphi : S \to \mathbf{P}^2$ a toric morphism such that $C := \varphi_*^{-1}(C')$ is a nonsingular nef curve of genus 8 on S. The fan Δ_S is as in Fig. 6. If we place $\square_C$ as in Fig. 6, then the linear equivalence class of C is written as

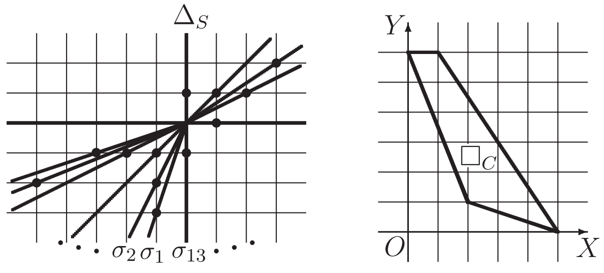

FIGURE 6

$$C \sim -5D_1 - 4D_2 - 3D_3 - 5D_4 - 12D_5 - 6D_6 \\ + 6D_8 + 7D_9 + 15D_{10} + 10D_{11} + 5D_{12}.$$

Consider the origin $Q = (0,0)$ on $C' \cap U_0$. Then the infinitely near points of Q on C are $P_1 = C \cap D_1$ and $P_2 = C \cap D_5$. The primitive elements of σ_1 and σ_5 are $(-1,-3)$ and $(-5,-2)$, respectively.

It is obvious that the lines $X + 3Y = k$ and $5X + 2Y = l$ have at most one lattice point in the interior of $\square_C$ for any integer k and l. Hence, by Corollary 1.6, the gap sequences of C at P_1 and P_2 are

$$\{j \in \mathbf{N} \mid \text{the line } X + 3Y = j + 5 \text{ has a lattice point in } \operatorname{Int} \square_C\} \\ = \{1,2,3,4,6,8,9,11\},$$

$$\{j \in \mathbf{N} \mid \text{the line } 5X + 2Y = j + 12 \text{ has a lattice point in } \operatorname{Int} \square_C\} \\ = \{1,2,3,4,5,6,7,9\},$$

respectively.

Before proceeding to the next example, we define the following function.

Definition 4.3. For a positive integer m and a non-negative integer n, we define a function f as

$$f(m,n) = \begin{cases} \gcd(m,n) & (n \geq 1), \\ m & (n = 0). \end{cases}$$

Example 4.4. Let C' be an irreducible plane curve defined by the local equation of the form

$$x^p + y^q + x^r y^s = 0,$$

where $p \geq q \geq 1$ and $r + s \geq 1$. One can obtain a toric morphism $\varphi : S \to \mathbf{P}^2$ such that $C := \varphi_*^{-1}(C')$ is nonsingular and nef. We write the linear equivalence class of C as $C \sim \sum_{i=1}^d p_i D_i$. The genus of C can be computed by the formula

$$g = \begin{cases} \dfrac{1}{2}(|pq - rq - sp| - f(p, p-q) & (pq - rq - sp \neq 0), \\ \qquad - f(p-r, s) - f(q-s, r)) + 1 & \\ 0 & (pq - rq - sp = 0). \end{cases}$$

Besides, in this case, the lattice polytope $\square_C$ becomes a triangle and we can place it such that its vertices are $(p,0)$, $(0,q)$ and (r,s). Then, by Corollary 1.6, we can compute the gap sequence of C at the infinitely near points of the origin $Q = (0,0)$ in the following cases:

(i) $pq - rq - sp = 0$,
(ii) $pq - rq - sp < 0$ and $f(p, p-q) = 1$,
(iii) $pq - rq - sp > 0$ and $f(p-r, s) = f(q-s, r) = 1$.

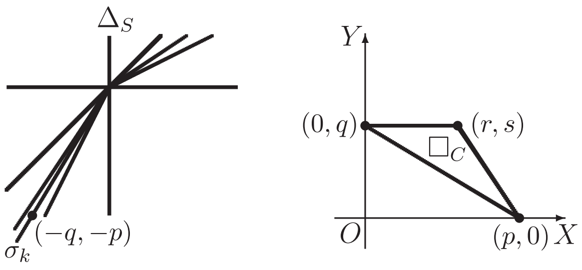

FIGURE 7

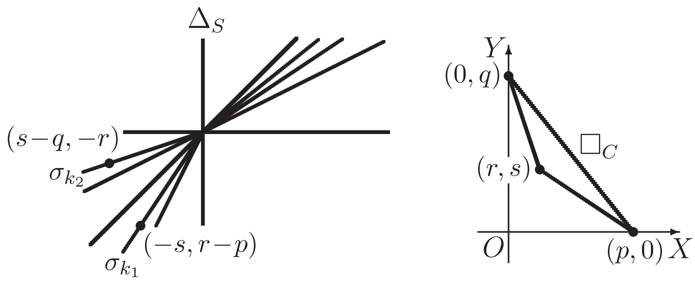

FIGURE 8

The case (i) does not require Corollary 1.6. Since $g = 0$, the gap sequence is empty at every point on C.

In the case (ii), the fan Δ_S is as in Fig. 7. The point Q has one infinitely near point P on C, which is the intersection point $C \cap D_k$. The primitive element of σ_k is $(-q,-p)$ and $p_k = -pq$. Hence, by Corollary 1.6, the gap sequence of C at P is

$$\{j \in \mathbf{N} \mid \text{the line } qX + pY = pq + j \text{ has a lattice point in Int } \square_C\}.$$

In the case (iii), the fan Δ_S and the lattice polytope $\square_C$ are as in Fig. 8. The infinitely near points of Q on C are $P_1 = C \cap D_{k_1}$ and $P_2 = C \cap D_{k_2}$. The primitive elements of σ_{k_1} and σ_{k_2} are $(-s, r-p)$ and $(s-q, -r)$, respectively. Moreover, $p_{k_1} = -sp$ and $p_{k_2} = -rq$ hold. Hence, by Corollary 1.6, the gap sequences of C at P_1 and P_2 are

$$\{j \in \mathbf{N} \mid \text{the line } sX + (p-r)Y = sp + j \text{ has a lattice point in Int } \square_C\},$$

$$\{j \in \mathbf{N} \mid \text{the line } (q-s)X + rY = rq + j \text{ has a lattice point in Int } \square_C\},$$

respectively.

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