Serdica Math. J. **34** (2008), 771–782

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

WEIERSTRASS POINTS WITH FIRST NON-GAP FOUR ON A DOUBLE COVERING OF A HYPERELLIPTIC CURVE II

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Communicated by V. Kanev

Abstract. A 4-semigroup means a numerical semigroup whose minimum positive integer is 4. In [7] we showed that a 4-semigroup with some conditions is the Weierstrass semigroup of a ramification point on a double covering of a hyperelliptic curve. In this paper we prove that the above statement holds for every 4-semigroup.

1. Introduction. Let $\mathbb{N}_0$ be the additive semigroup of non-negative integers. A subsemigroup H of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0 \backslash H$ of H in $\mathbb{N}_0$ is finite. The cardinality of the set $\mathbb{N}_0 \backslash H$ is said to be the *genus* of H, which is denoted by $g(H)$. Let H be an m-semigroup, i.e., a numerical semigroup whose minimum positive integer is m. Then we denote its standard basis by $S(H)$. Namely, $S(H) = \{m, s_1, \ldots, s_{m-1}\}$ where $s_i = \min\{h \in H \mid h \equiv i \ (m)\}$ for $i = 1, 2, \ldots, m-1$.

2000 *Mathematics Subject Classification:* Primary 14H55; Secondary 14H30, 14J26.

Key words: Weierstrass semigroup of a point, double covering of a hyperelliptic curve, 4-semigroup.

Let C be a complete non-singular irreducible curve over an algebraically closed field k of characteristic 0, which is called a *curve* in this paper. We denote by $k(C)$ the field of rational functions on C. For any point P of C we set

$$H(P) = \{n \in \mathbb{N}_0 | \text{ there exists } f \in k(C) \text{ such that } (f)_\infty = nP\},$$

which is called the *Weierstrass semigroup* of the point P. It is known that $H(P)$ is a numerical semigroup of genus g if the genus of the curve C is g. A $2n$-semigroup H is said to be *of double covering type* if there exists a double covering $\pi : C \longrightarrow E$ of a curve with a ramification point $\tilde{P}$ such that $H = H(\tilde{P})$. Let H be a 4-semigroup with $S(H) = \{4, s_1, s_2, s_3\}$. If $g(H) \geqq 3s_2$ there is a cyclic covering $\pi : C \longrightarrow \mathbb{P}^1$ of degree 4 with a total ramification point $\tilde{P}$ satisfying $H(\tilde{P}) = H$ ([4]). Hence, to prove that every 4-semigroup is of double covering type it suffices to show the following:

Main Theorem. *Let H be a 4-semigroup. We set $r = s_2 = \min\{h \in H | h \equiv 2 \bmod 4\}$. If $g(H) \leqq 3r - 1$, then H is of double covering type.*

The authors expected to have proved Main Theorem in the paper [6], but they found an error in its proof. To correct its proof using the method in [6] we needed some more assumptions ([7]). Consequently, to prove Main Theorem with no condition we must've developed another method, whose main tools are the blow ups and blow downs of curves at points. Throughout this paper we use the following notation:

Notation. For a positive integer r, an even integer t with $2 \leqq t \leqq 2r$ and an odd integer s with $1 \leqq s \leqq t-1$ we denote by $H_{r,t,s}$ the 4-semigroup with $S(H) = \{4, 2r+s, 2r+2t-s, 4r+2\}$. For a 4-semigroup with $g(H) \leqq 3r-1$ where $r = \min\{h \in H | h \equiv 2 \bmod 4\}$ there exist t, s within the above range such that $H = H_{r,t,s}$ by Proposition 2.7 in [6].

For any two divisors D_1 and D_2 on a curve $D_1 \sim D_2$ means that D_1 and D_2 are linearly equivalent. For a divisor D on a curve we denote by $|D|$ the complete linear system.

2. The case where $t \geqq r+1$. We note that the material of the first two sections of the paper [6] is unaffected of the wrong Lemma 3.1 and will be used in the present paper.

First we consider the case $t = r+1$.

Theorem 1. *The 4-semigroup $H_{r,r+1,s}$ is of double covering type.*

Proof. In this case we have

$$r+\frac{s+1}{2}=t-1+\frac{s+1}{2}=t+\frac{s-1}{2}\equiv\frac{s-1}{2} \bmod \frac{t}{2}.$$

By Main Theorem C in [7] $H_{r,r+1,s}$ is of double covering type. □

The following is a key lemma to prove Main Theorem in the case where $t \geqq r+2$.

Lemma 2. *Let α be a non-negative integer with $t+2\alpha \leqq 2r$. If $H_{r,t,s}$ is of double covering type, so is $H_{r,t+2\alpha,s+2\alpha}$.*

Proof. By the assumption there exists a double covering $\pi : \tilde{C} \longrightarrow C$ of a curve with a ramification point $\tilde{P}$ satisfying $H(\tilde{P}) = H_{r,t,s}$. We set $P = \pi(\tilde{P})$. Then C is a hyperelliptic curve of genus r and P is a Weierstrass point. Moreover, π has t ramification points $P, P_1, \dots, P_{t-1}$ because of $g(H_{r,t,s}) = 2r-1+\frac{t}{2}$. By Theorem 2.6 in [6] there is a divisor D on C satisfying

$$D \sim \frac{2r+s+1}{2}P - Q_1 - \cdots - Q_{\frac{s+1-t}{2}+r}$$

for some points $Q_1, \dots, Q_{\frac{s+1-t}{2}+r}$ different from P with $h^0(Q_1+\cdots+Q_{\frac{s+1-t}{2}+r}) = 1$ such that $2D \sim P + P_1 + \cdots + P_{t-1}$. Hence we have

$$2(D+\alpha P) = 2D + \alpha \cdot 2P \sim P + P_1 + \cdots + P_{t-1} + P'_1 + \cdots + P'_{2\alpha}$$

where $P, P_1, \dots, P_{t-1}, P'_1, \dots, P'_{2\alpha}$ are distinct points. We set $D' = D + \alpha P$. Then

$$-D' + \left(r + \frac{s+2\alpha+1}{2}\right)P = -D + \left(r+\frac{s+1}{2}\right)P \sim Q_1 + \cdots + Q_{\frac{s+1-t}{2}+r}.$$

By [6] we can construct a double covering $\pi' : \tilde{C}' \longrightarrow C$ of a curve with the ramification point $\tilde{P}'$ over P with $H(\tilde{P}') = H_{r,t+2\alpha,s+2\alpha}$. □

We need two more propositions to use Lemma 2 in the proof of the case $t \geqq r+2$.

Proposition 3. *For any s with $s \geqq 3$ the 4-semigroup $H_{r,r+2,s}$ is of double covering type.*

Proof. We consider the case $t = r+2$. In this case we have

$$r+\frac{s+1}{2}=t-2+\frac{s+1}{2}=t+\frac{s-3}{2}\equiv\frac{s-3}{2} \bmod \frac{t}{2}.$$

Since $s \geqq 3$, we get $0 \leqq \frac{s-3}{2} \leqq \frac{s-1}{2}$. By Main Theorem C in [7] $H_{r,r+2,s}$ is of double covering type. $\Box$

Proposition 4. *For any t with $t \geqq r+2$ the 4-semigroup $H_{r.t.1}$ is of double covering type.*

Proof. Let C be a hyperelliptic curve of genus r with a Weierstrass point P. We set $u = 2r+2-t$. Let $Q_1, \cdots, Q_{\frac{u}{2}}$ be distinct ordinary points of C such that $h^0(\mathcal{O}(Q_1 + \cdots + Q_{\frac{u}{2}})) = 1$. Consider the divisor

$$N = (r+1)P - Q_1 - \ldots - Q_{\frac{u}{2}}.$$

Then we get

$$2N - P \sim (r-u)g_2^1 + 2\iota Q_1 + \cdots + 2\iota Q_{\frac{u}{2}} + P$$

where ι is the hyperelliptic involution on C. Since $t \geqq r+2$, we obtain $r - u = t - r - 2 \geqq 0$. Moreover, any two points among $\iota Q_1, \ldots, \iota Q_{\frac{u}{2}}, P$ are not conjugate under the involution ι and $\iota Q_1, \ldots, \iota Q_{\frac{u}{2}}$ are ordinary points. We note that $\max\{r+1-(u+1), 0\} = r-u$. Hence, by Lemma 3 in [3] the complete linear system $|2N - P|$ is base-point free. Therefore, we get $2N - P \sim P_1 + \cdots + P_{t-1}$ where $P_1, \ldots, P_{t-1}$ are distinct points different from P. We set $\mathcal{L} = \mathcal{O}_C(-N)$. Then we get $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_C(-P - P_1 - \cdots - P_{t-1})$. By Theorem 2.6 in [6] $H_{r,t,1}$ is of double covering type. $\Box$

In the case where $t \geqq r+2$ we can show that $H_{r,t,s}$ is of double covering type using Lemma 2 and Propositions 3, 4.

Theorem 5. *Assume that $t \geqq r+2$. Then the 4-semigroup $H_{r,t,s}$ is of double covering type.*

Proof. First, we consider the case where r is even. Then we have either $t - r - s \geqq 1$ or $t - r - s \leqq -1$. Assume that $t - r - s \geqq 1$. We set $\alpha = \frac{s-1}{2}$. Hence, $t - 2\alpha \geqq r+2$. Consider the 4-semigroup $H_{r,t-2\alpha,1}$, which is of double covering type from Proposition 4 and so is $H_{t,r,s}$ by Lemma 2. Assume that $t - r - s \leqq -1$. We set $\alpha = \frac{t-r-2}{2} \geqq 0$. Then $s - 2\alpha \geqq 3$. Consider the 4-semigroup $H_{r,t-2\alpha,s-2\alpha}$, which is of double covering type by Proposition 3 and so is $H_{r,t,s}$ by Lemma 2.

Second, let r be odd. We have either $t - r - s \geqq 2$ or $t - r - s \leqq 0$. Assume that $t - r - s \geqq 2$. Then the same way as in the case where r is even with $t - r - s \geqq 1$ works well. Let $t - r - s \leqq 0$. We set $\alpha = \frac{t-r-1}{2}$. Then we get

$s-2\alpha \geqq 1$. Consider the 4-semigroup $H_{r,t-2\alpha,s-2\alpha}$, which is of double covering type by Theorem 1 and so is $H_{r,t,s}$ by Lemma 2. □

3. The case where $t \leqq r$. In this section we prove Main Theorem in the case of $t \leqq r$ using a method different from the cases of $t = r+1$ and $t \geqq r+2$.

Proposition 6. *Let $t \leqq r$. Then there exists a hyperelliptic curve C of genus r with $t-s+1$ distinct points $P_1, \dots, P_{t-s+1}$ satisfying $|2P_1| = \cdots = |2P_{t-s+1}| = g_2^1$ such that*

$$P_1 + \cdots + P_{t-s+1} + (r-t+s)g_2^1 \sim 2(Q_1 + \cdots + Q_{r+\frac{s+1-t}{2}})$$

where $Q_1, \dots, Q_{r+\frac{s+1-t}{2}}$ are points of C which are not conjugate each other under the hyperelliptic involution and which are distinct from $P_1, \dots, P_{t-s+1}$.

Proof. Let us consider the rational ruled surface

$$\rho : S = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-(r-t+s))) \longrightarrow \mathbb{P}^1.$$

We set $e = r-t+s$. We note that $e \geqq s \geqq 1$, because $t \leqq r$. Let E_0 be the minimal section. In this case we have $\rho_* \mathcal{O}(E_0) \cong \mathcal{O} \oplus \mathcal{O}(-e)$. Hence,

$$\rho_* \mathcal{O}(2E_0) \cong S^2(\mathcal{O} \oplus \mathcal{O}(-e)) \cong \mathcal{O} \oplus \mathcal{O}(-e) \oplus \mathcal{O}(-2e).$$

For divisors D_1 and D_2 on S, (D_1, D_2) means the intersection number for D_1 and D_2, and (D_1, D_1) is denoted by (D_1^2). Let F be a fiber of ρ. Then we have $(E_0^2) = -e < 0$, $(E_0.F) = 1$ and $(F^2) = 0$. Consider a curve $C \in |2E_0 + (2r-t+s+1)F|$. If C is an irreducible non-singular curve, then it is a hyperelliptic curve, because of $(C.\rho^{-1}(\text{a point})) = (C.F) = 2$. Let $K_S = -2E_0 + (-2 + (-r+t-s))F$ be a canonical divisor on S. Since we have $((K_S + C).C) = 2r-2$, C is of genus r.

By V 2.18 in [1] we may take $H_0 \in |E_0 + eF|$ as an irreducible non-singular curve. Then $E_0 \cap H_0 = \emptyset$, because of $(E_0.H_0) = 0$. Let $P_1, \dots, P_{t-s+1} \in E_0$ be distinct points. For any $1 \leqq i \leqq t-s+1$ the fiber $\rho^{-1}(\rho(P_i))$ is denoted by F_i. We may take points $Q_1, \dots, Q_{r+\frac{s+1-t}{2}} \in H_0$ such that $\rho(Q_1), \dots, \rho(Q_{r+\frac{s+1-t}{2}})$, $\rho(P_1), \dots, \rho(P_{t-s+1})$ are distinct. Let $\sigma_1 : T_1 \longrightarrow S$ be the blowing-up of S at the points $P_1, \dots, P_{t-s+1}, Q_1, \dots, Q_{r+\frac{s+1-t}{2}}$. Let $e_1 = \sigma_1^{-1}(P_1), \dots, e_{t-s+1} = \sigma_1^{-1}(P_{t-s+1}), \varepsilon_1 = \sigma_1^{-1}(Q_1), \dots, \varepsilon_{\frac{s+1-t}{2}} = \sigma_1^{-1}(Q_{r+\frac{s+1-t}{2}})$ be the exceptional divisors. Let f_i be the proper transform of F_i for $1 \leqq i \leqq t-s+1$. E_0' and H_0' denote the proper transforms of E_0 and H_0 respectively. Let P_i^* be the point at which e_i and f_i intersect and Q_j^* the point at which H_0 and ε_j intersect. Let $\sigma_2 : T \longrightarrow T_1$ be the blowing-up of T_1 at the points $P_1^*, \ldots, P_{t-s+1}^*, Q_1^*, \ldots, Q_{r+\frac{s+1-t}{2}}^*$. Let $e_i^* = \sigma_2^{-1}(P_i^*)$ and $\varepsilon_j^* = \sigma_2^{-1}(Q_j^*)$ be the exceptional divisors. Let $\tilde{f}_i$ be the proper transform of f_i. $\tilde{E}_0$ and $\tilde{H}_0$ denote the proper transforms of E_0' and H_0' respectively. Let $\tilde{e}_i$ and $\tilde{\varepsilon}_j$ be the proper transforms of e_i and ε_j respectively. We reset $e_i = \tilde{e}_i + e_i^*$ and $\varepsilon_j = \tilde{\varepsilon}_j + \varepsilon_j^*$ on T. The composition $\sigma_1 \circ \sigma_2$ is denoted by σ. See Figure 1 : Blowing-up in the next page for the above notations.

Consider the divisor

$$\mathcal{L} = \sigma^*(2E_0 + (2r - t + s + 1)F) - e_1 - e_1^* - \cdots - e_{t-s+1} - e_{t-s+1}^*$$
$$-\varepsilon_1 - \varepsilon_1^* - \cdots - \varepsilon_{r+\frac{s+1-t}{2}} - \varepsilon_{r+\frac{s+1-t}{2}}^*$$

on T and the divisor

$$\mathcal{L}_0 = \sigma_1^*(2E_0 + (2r - t + s + 1)F) - e_1 - \cdots - e_{t-s+1} - \varepsilon_1 - \cdots - \varepsilon_{r+\frac{s+1-t}{2}}$$

on T_1. Now we have $(\mathcal{L}_0.e_i) = 1$, $(\mathcal{L}_0.f_i) = 1$ and $(\mathcal{L}_0.\varepsilon_j) = 1$. Moreover, we get

$$(\mathcal{L}_0.H_0') = (\mathcal{L}_0.\sigma_1^*(H_0) - \varepsilon_1 - \cdots \varepsilon_{r+\frac{s+1-t}{2}}) = r + \frac{s+1-t}{2}.$$

We note that

$$\mathcal{L} = \sigma_2^*\mathcal{L}_0 - e_1^* - \cdots - e_{t-s+1}^* - \varepsilon_1^* - \cdots - \varepsilon_{r+\frac{s+1-t}{2}}^*.$$

Hence, we get $(\mathcal{L}.e_i^*) = 1$, $(\mathcal{L}.\varepsilon_j^*) = 1$ and

$$(\mathcal{L}.\tilde{H}_0) = (\mathcal{L}.\sigma_2^*H_0' - \varepsilon_1^* - \cdots - \varepsilon_{r+\frac{s+1-t}{2}}^*) = 0.$$

Moreover, we have $(\mathcal{L}^2) = 2(r + \dfrac{s+1-t}{2}) > 0$.

From now on we want to prove that the complete linear system $|\mathcal{L}|$ is base-point free. Since we have

$$2E_0 + (2r - t + s + 1)F \sim E_0 + (r - t + s)F + H_0 + (t - s + 1)F,$$
$$\sigma^*(E_0 + (r - t + s)F) - \varepsilon_1 - \varepsilon_1^* - \cdots - \varepsilon_{r+\frac{s+1-t}{2}} - \varepsilon_{r+\frac{s+1-t}{2}}^* \sim \tilde{H}_0$$

and $\sigma^*F - e_j - e_j^* \sim \tilde{f}_j$, we get

$$\mathcal{L} \sim \sigma^*(H_0) + \tilde{f}_1 + \cdots + \tilde{f}_{t-s+1} + \tilde{H}_0.$$

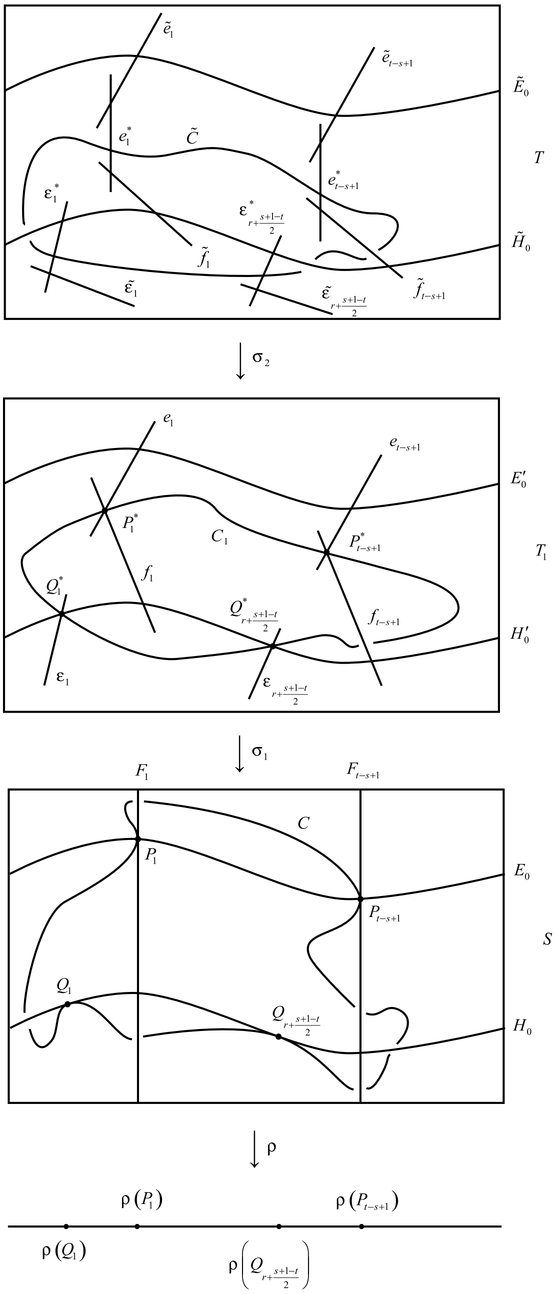

Fig. 1. Blowing-up

Since H_0 is base-point free (V 2.17 in [1]), we obtain

$$B_S|\mathcal{L}| \subseteqq \tilde{f}_1 \cup \cdots \cup \tilde{f}_{t-s+1} \cup \tilde{H}_0,$$

where for any divisor D we denote by $B_S|D|$ the base locus of the complete linear system $|D|$. Hence, it suffices to show that $B_S|\mathcal{L}| \cap \tilde{H}_0 = \emptyset$ and $B_S|\mathcal{L}| \cap \tilde{f}_i = \emptyset$ for any $1 \leqq i \leqq t-s+1$.

First, we show that $h^0(\mathcal{L}) \geqq r-t+s+3$. We set

$$\mathcal{M} = \sigma_1^*(2E_0 + (2r-t+s+1)F) - e_1 - \cdots - e_{t-s+1} - \varepsilon_1 - \cdots - \varepsilon_{r+\frac{s+1-t}{2}}.$$

Then there is an exact sequence

$$0 \longrightarrow \sigma_2^*\mathcal{M} - e_1^* \longrightarrow \sigma_2^*\mathcal{M} \longrightarrow \mathcal{O}_{e_1^*}(\sigma_2^*\mathcal{M}) \longrightarrow 0.$$

Here, we have $\mathcal{O}_{e_1^*}(\sigma_2^*\mathcal{M}) \cong \mathcal{O}_{\mathbb{P}^1}$, because $e_1^* \cong \mathbb{P}^1$ and $(e_1^*.\sigma_2^*\mathcal{M}) = 0$. Hence, we get $h^0(\sigma_2^*\mathcal{M} - e_1^*) \geqq h^0(\sigma_2^*\mathcal{M}) - 1$. Using the similar way to the above repeatedly and $h^0(\sigma^*\mathcal{M}) = h^0(\mathcal{M})$ we get

$$\begin{aligned} h^0(\mathcal{L}) &= h^0\left(\sigma_2^*\mathcal{M} - e_1^* - \cdots - e_{t-s+1}^* - \varepsilon_1^* - \cdots - \varepsilon^*_{r+\frac{s+1-t}{2}}\right) \\ &\geqq h^0(\mathcal{M}) - (t-s+1) - \left(r + \frac{s+1-t}{2}\right) \\ &\geqq h^0(\sigma^*(2E_0 + (2r-t+s+1)F)) - 2(t-s+1) - 2\left(r + \frac{s+1-t}{2}\right). \end{aligned}$$

Now we have

$$\begin{aligned} h^0(\sigma^*(2E_0 + (2r-t+s+1)F)) &= h^0(\rho_*(2E_0 + (2r-t+s+1)F)) \\ &= h^0(\rho_*\mathcal{O}(2E_0) \otimes \mathcal{O}_{\mathbb{P}^1}(2r-t+s+1)) \\ &= h^0(S^2(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-(r-t+s))) \otimes \mathcal{O}_{\mathbb{P}^1}(2r-t+s+1)) \\ &= h^0(\mathcal{O}_{\mathbb{P}^1}(2r-t+s+1) \oplus \mathcal{O}_{\mathbb{P}^1}(r+1) \oplus \mathcal{O}_{\mathbb{P}^1}(t-s+1)) = 3r+6. \end{aligned}$$

Hence, we obtain $h^0(\mathcal{L}) \geqq r-t+s+3$.

Second, we prove that $h^0(\mathcal{L}) = r-t+s+3$ and $B_S|\mathcal{L}| \cap \tilde{H}_0 = \emptyset$. We have

$$\begin{aligned} (\mathcal{L}.\tilde{H}_0) = \Big(&\sigma^*(2E_0+(2r-t+s+1)F) - e_1 - e_1^* - \cdots - e_{t-s+1} - e_{t-s+1}^* - \varepsilon_1 - \varepsilon_1^* - \cdots \\ &- \varepsilon_{r+\frac{s+1-t}{2}} - \varepsilon^*_{r+\frac{s+1-t}{2}} \;.\; \sigma^*(E_0+(r-t+s)F) - \varepsilon_1 - \varepsilon_1^* - \cdots - \varepsilon_{r+\frac{s+1-t}{2}} - \varepsilon^*_{r+\frac{s+1-t}{2}}\Big) = 0. \end{aligned}$$

Hence, we have an exact sequence

$$0 \longrightarrow \mathcal{L}(-\tilde{H}_0) \longrightarrow \mathcal{L} \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0.$$

We want to show that the sequence

$$0 \longrightarrow H^0(\mathcal{L}(-\tilde{H}_0)) \longrightarrow H^0(\mathcal{L}) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}) \longrightarrow 0$$

is exact. We note that

$$\mathcal{L}(-\tilde{H}_0) \sim \sigma^*(H_0) + \tilde{f}_1 + \cdots + \tilde{f}_{t-s+1}.$$

For any i we have $(f_i^2) = -1$, which implies that $(\tilde{f}_i^2) = -2$ because of $f_i = \tilde{f}_i + e_i^*$. Since $(\tilde{f}_i.\sigma^* H_0 + \tilde{f}_i) = -1$, we have an exact sequence

$$0 \longrightarrow \sigma^* H_0 \longrightarrow \sigma^* H_0 + \tilde{f}_1 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow 0,$$

which implies that $h^0(\sigma^* H_0) = h^0(\sigma^* H_0 + \tilde{f}_1)$. We get

$$h^0(\sigma^* H_0) = h^0(\sigma^* H_0 + \tilde{f}_1) = \cdots = h^0(\sigma^* H_0 + \tilde{f}_1 + \cdots + \tilde{f}_{t-s+1}) = h^0(\mathcal{L}(-\tilde{H}_0))$$

by the similar way to the above. Since we have

$$h^0(\mathcal{L}(-\tilde{H}_0)) = h^0(H_0) = h^0(\rho_* \mathcal{O}_S(E_0) \otimes \mathcal{O}_{\mathbb{P}^1}(r-t+s)) = r-t+s+2,$$

we must have $h^0(\mathcal{L}) = r-t+s+3$. Hence, we get an exact sequence

$$0 \longrightarrow H^0(\mathcal{L}(-\tilde{H}_0)) \longrightarrow H^0(\mathcal{L}) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}) \longrightarrow 0.$$

Since $H^0(\mathcal{L}|_{\tilde{H}_0}) \cong H^0(\mathcal{O}_{\mathbb{P}^1}) = k$, we get $B_S|\mathcal{L}| \cap \tilde{H}_0 = \emptyset$.

Third, we show that $B_S|\mathcal{L}| \cap \tilde{f}_i = \emptyset$ for all i. Since we have

$$(\tilde{f}_i.\mathcal{L}) = (F.2E_0 + (2r-t+s+1)F) + (e_i^2) + ((e_i^*)^2) = 0,$$

we get an exact sequence

$$0 \longrightarrow \mathcal{L}(-\tilde{f}_i) \longrightarrow \mathcal{L} \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0.$$

We have

$$\mathcal{L}(-\tilde{f}_{t-s+1}) = \sigma^*(2E_0 + (2r-t+s)F) - \sum_{i=1}^{t-s}(e_i + e_i^*) - \sum_{j=1}^{r+\frac{s+1-t}{2}} (\varepsilon_j + \varepsilon_j^*)$$

$$\sim \sigma^*(H_0 + (t-s)F) - \sum_{i=1}^{t-s}(e_i + e_i^*) + \tilde{H}_0.$$

Consider an exact sequence

$$0 \longrightarrow \mathcal{O}\left(\sigma^*(H_0 + (t-s)F) - \sum_{i=1}^{t-s}(e_i + e_i^*)\right) \longrightarrow$$

$$\mathcal{O}\left(\sigma^*(H_0 + (t-s)F) - \sum_{i=1}^{t-s}(e_i + e_i^*) + \tilde{H}_0\right) \longrightarrow$$

$$\mathcal{O}_{\tilde{H}_0}\left(\sigma^*(H_0 + (t-s)F) - \sum_{i=1}^{t-s}(e_i + e_i^*) + \tilde{H}_0\right) \longrightarrow 0.$$

We have

$$\left(\sigma^*(H_0 + (t-s)F) - \sum_{i=1}^{t-s}(e_i + e_i^*) + \tilde{H}_0.\tilde{H}_0\right)$$

$$= \left(\sigma^*(H_0 + (t-s)F) + \sigma^*H_0 - \sum_{j=1}^{r+\frac{s+1-t}{2}}(\varepsilon_j + \varepsilon_j^*).\sigma^*H_0 - \sum_{j=1}^{r+\frac{s+1-t}{2}}(\varepsilon_j + \varepsilon_j^*)\right)$$

$$= (2H_0 + (t-s)F.H_0) - 2\left(r + \frac{s+1-t}{2}\right) = -1.$$

Hence, we get

$$h^0(\mathcal{L}(-\tilde{f}_{t-s+1})) = h^0\left(\sigma^*(H_0 + (t-s)F) - \sum_{i=1}^{t-s}(e_i + e_i^*)\right)$$

$$= h^0(\sigma^*H_0 + \tilde{f}_1 + \cdots + \tilde{f}_{t-s}) = h^0(\mathcal{L}(-\tilde{H}_0)) = r - t + s + 2,$$

because of $(\sigma^*H_0 + \tilde{f}_1 + \cdots + \tilde{f}_{t-s} + \tilde{f}_{t-s+1}.\tilde{f}_{t-s+1}) = -1$. Consider an exact sequence

$$0 \longrightarrow \mathcal{L}(-\tilde{f}_{t-s+1}) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}|_{\tilde{f}_{t-s+1}} \longrightarrow 0.$$

Since $(\mathcal{L}.\tilde{f}_{t-s+1}) = 0$, we have $\mathcal{L}|_{\tilde{f}_{t-s+1}} \cong \mathcal{O}_{\mathbb{P}^1}$. Hence, in the similar way to the second case we get $B_S|\mathcal{L}| \cap \tilde{f}_{t-s+1} = \emptyset$. Similarly we have $B_S|\mathcal{L}| \cap \tilde{f}_i = \emptyset$ for all i.

Since $(\mathcal{L}^2) > 0$ and $|\mathcal{L}|$ is base-point free, we may take an irreducible smooth curve $\tilde{C} \in |\mathcal{L}|$ by Zariski's Theorem (See Theorem 7.19 in [2]) and Bertini's Theorem. We set $C = \sigma(\tilde{C})$ and $C_2 = \sigma_2(C)$. See Fig. 1. for the above notations. Then C is an irreducible non-singular curve, hence it is a hyperelliptic curve of genus r. We have $E_0|_C = P_1 + \cdots + P_{t-s+1}$, because of $(E_0.C) = t - s + 1$. Moreover, we have $(r - t + s)F|_C \sim (r - t + s)g_2^1$. Since we have $(H_0.C) = 2(r + \dfrac{s+1-t}{2})$, we get $H_0|_C = 2(Q_1 + \cdots + Q_{r+\frac{s+1-t}{2}})$. In view of $H_0 \in |E_0 + (r - t + s)F|$ we obtain

$$P_1 + \cdots + P_{t-s+1} + (r - t + s)g_2^1 \sim 2\left(Q_1 + \cdots + Q_{r+\frac{s+1-t}{2}}\right).$$

Since we have $(H_0.F) = 1$, the points $Q_1, \ldots, Q_{r+\frac{s+1-t}{2}}$ are not conjugate each other. □

Using Proposition 6 we get our desired result in the case where $t \leqq r$.

Theorem 7. *For $t \leqq r$ the 4-semigroup $H_{r,t,s}$ is of double covering type.*

Proof. We use the notation as in Proposition 6. We set

$$\mathcal{L} = \mathcal{O}_C\left(Q_1 + \cdots + Q_{r+\frac{s+1-t}{2}} - \left(r + \frac{s+1}{2}\right)P_1\right).$$

Then we have

$$\mathcal{L}^{\otimes 2} \cong \mathcal{O}_C\left(P_1 + \cdots + P_{t-s+1} - (t - s + 1)g_2^1 - \frac{s-1}{2}g_2^1\right)$$

$$\cong \mathcal{O}_C\left(-P_1 - \cdots - P_{t-s+1} - P_1' - P_1'' - \cdots - P'_{\frac{s-1}{2}} - P''_{\frac{s-1}{2}}\right) \subset \mathcal{O}_C$$

where $P_1, \ldots, P_{t-s+1}, P_1', P_1'', \ldots, P'_{\frac{s-1}{2}}, P''_{\frac{s-1}{2}}$ are distinct points. Let $\pi : \tilde{C} = \mathrm{Spec}(\mathcal{O}_C \oplus \mathcal{L}) \longrightarrow C$ be the canonical morphism. Let $\tilde{P}_1$ be the point of $\tilde{C}$ such that $\pi(\tilde{P}_1) = P_1$. Then by Theorem 2.6 in [6] we have $H(\tilde{P}_1) = H_{t,r,s}$. □

By Theorems 1, 5 and 7 we get Main Theorem. Finally we note that the similar proof to the case $t \leqq r$ also works well in the case where $r + 1 \leqq t \leqq 2r$. But if we use this proof in the case where $r + 1 \leqq t \leqq 2r$, the length of the paper would become long. So we did not adopt the proof in the case where $r + 1 \leqq t \leqq 2r$.

REFERENCES

[1] R. Hartshorne. Algebraic Geometry. Graduate Texts in Mathematics, vol **52**, New York–Heidelberg–Berlin, Springer-Verlag, 1977.

[2] S. Iitaka. Algebraic Geometry. An introduction to birational geometry of algebraic varieties. Graduate Texts in Mathematics, vol. **76**, New York–Heidelberg–Berlin: Springer-Verlag, 1982.

[3] T. Kato, A. Ohbuchi. Very ampleness of multiple of tetragonal linear systems. *Comm. Algebra* **21** (1993), 4587–4597.

[4] J. Komeda. On Weierstrass points whose first non-gaps are four. *J. Reine Angew. Math.* **341** (1983), 68–86.

[5] J. Komeda. Weierstrass semigroups whose minimum positive integres are even. *Arch. Math.* **89** (2007), 52–59.

[6] J. Komeda, A. Ohbuchi. Weierstrass Points with First Non-gap Four on a Double Covering of a Hyperelliptic Curve. *Serdica Math. J.* **30** (2004), 43–54.

[7] J. Komeda, A. Ohbuchi. Corrigendum for Weierstrass Points with First Non-gap Four on a Double Covering of a Hyperelliptic Curve. *Serdica Math. J.* **32** (2006), 375–378.

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Received April 22, 2008