Weighted approximation, Mergelyan's theorem and quasi-analytic classes

Guido Zapata*

1. Introduction

It is well known that the classical Hadamard problem on the characterization of quasi-analytic classes of functions admits several equivalent formulations and solutions in terms of holomorphic functions, asymptotic series, divergent series, etc. (See [4] and [7].) Some of the techniques employed to attack this problem proceeded from the weighted approximation theory which originated with S. Bernstein. More precisely, from the very beginning of weighted approximation theory, there was an interdependence with quasi-analytic function theory. For applications of weighted approximation theory to the Hadamard problem, see, for instance, [2], [3], [5] and [6]. On the other hand, the Denjoy—Carleman theorem on quasi-analyticity has been widely used to get results in weighted approximation theory. In fact, almost all these results contain a hypothesis implying that some weight is fundamental in the sense of S. Bernstein by using the above theorem. See, for instance, [7], [9], [10], [11] and [12]. Furthermore, the classical problem which consisted of characterizing the fundamental weights on the real line was completely solved by S. Mergelyan [8]. The key to Mergelyan's solution is a result which is appropriate for approximation, not only for the real line, but also for closed nowhere dense sets in the complex plane.

Our purpose here is twofold. First, we give a simple proof of Mergelyan's theorem (Theorem 1). Then, by using this result we establish that quasi-analyticity is equivalent to the fact that some weight is fundamental (Theorem 2). So, in some sense, we cement the interdependence mentioned above. As an application, we get another solution of Hadamard's problem (Theorem 3). All this is done by using

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techniques of weighted approximation theory such as those employed by Nachbin [10], and no appeal is made to classical results on quasi-analyticity. This approach seems to be interesting in looking for a solution of the generalized quasi-analyticity problem posed by S. Mandelbrojt [7].

2. Preliminaries

Let u be a locally bounded and nonnegative function defined on $\mathbf{R}$ (u is called a *weight* on $\mathbf{R}$). $Cu_\infty(\mathbf{R})$ will denote the vector space of all continuous and complex valued functions f on $\mathbf{R}$ such that uf vanishes at infinity. The topology on $Cu_\infty(\mathbf{R})$ will be given by the semi-norm

$$f \mapsto \|uf\| = \sup\{|uf|(t);\ t\in\mathbf{R}\}.$$

The weight u is called *fundamental* when the set $\mathscr{P}(\mathbf{R})$ of all complex valued polynomials on $\mathbf{R}$ is a dense subset of $Cu_\infty(\mathbf{R})$. We put $\mathscr{P}_u$ for the set of all $P\in\mathscr{P}(\mathbf{R})$ such that uP is bounded and $\|uP\|\leqq 1$ where, for a bounded function g on $\mathbf{R}$, $\|g\|$ stands for $\sup\{|g(t)|,\ t\in\mathbf{R}\}$. Also, we put

$$M_u(z) = \sup\{|P(z)|;\ P\in\mathscr{P}_u\}, \quad \text{for} \quad z\in\mathbf{C},$$

and

$$u^*(t) = \frac{u(t)}{1+|t|}, \quad \text{for} \quad t\in\mathbf{R}.$$

D will denote the set of all $z\in\mathbf{C}$ such that $\operatorname{Im}(z)\neq 0$. For any such z, we put

$$g_z(t) = \frac{1}{t-z}, \quad \text{for} \quad t\in\mathbf{R}.$$

Remark 1. The set $\mathscr{K}(\mathbf{R})$ of all continuous and complex valued functions on $\mathbf{R}$ having compact support is dense in $Cu_\infty(\mathbf{R})$. For weights u and v such that $u\leqq v$, u is fundamental when v is fundamental. Furthermore, writing $C_\infty(\mathbf{R})$ for $C1_\infty(\mathbf{R})$, if the weight u is bounded, $C_\infty(\mathbf{R})$ is a dense subset of $Cu_\infty(\mathbf{R})$, while the inclusion $C_\infty(\mathbf{R})\subset Cu_\infty(\mathbf{R})$ is continuous.

Remark 2. Let $z\in D$ and $A=\mathbf{C}[g_z, \bar{g}_z]$ be the complex algebra generated by the functions g_z and $\bar{g}_z$. Notice that $A\subset C_\infty(\mathbf{R})$. Since A separates points, is non-vanishing at every point, and is self-adjoint, by the Stone—Weierstrass theorem (applied to a compactification of $\mathbf{R}$), it follows that A is dense in $C_\infty(\mathbf{R})$. Whence, by Remark 1, A is dense in $Cu_\infty(\mathbf{R})$ for all u bounded.

Remark 3. Let $a, b\in\mathbf{R}$, $a\neq 0$, and let $\sigma(t)=at+b$, for $t\in\mathbf{R}$. For a complex valued function f on $\mathbf{R}$, let $T(f)=f\circ\sigma$. Then $T(\mathscr{K}(\mathbf{R}))=\mathscr{K}(\mathbf{R})$ and $T(\mathscr{P}(\mathbf{R}))=\mathscr{P}(\mathbf{R})$. Also, the restriction $T|Cu_\infty(\mathbf{R})$ is an isometry of $Cu_\infty(\mathbf{R})$ onto $C(u\circ\sigma)_\infty(\mathbf{R})$. Hence u is fundamental if (and only if) $u\circ\sigma$ is fundamental.

3. Mergelyan's theorem

In [8], Mergelyan has proved the following important result: if u is a weight on $\mathbf{R}$ such that $\mathscr{P}(\mathbf{R})\subset Cu_\infty(\mathbf{R})$, then $\mathscr{P}(\mathbf{R})$ is dense in $Cu_\infty(\mathbf{R})$ if, and only if, $M_{u^*}(z)=+\infty$ for some $z\in D$. The original proof depends on several properties of the function M_{u^*}. Also, it uses the result, due to *N.* Akiezer, that if $H\subset D$ is an infinite set having an accumulation point in D, the set $\{g_z, \bar{g}_z; z\in H\}$ is total in $C_\infty(\mathbf{R})$; the proof of this result is rather complicated [1]. Further, in proving the sufficiency of the condition on $M_{u^*}(z)$ the additional hypothesis on u is used.

We will prove the following more general version of Mergelyan's result.

Theorem 1. *A weight u on* $\mathbf{R}$ *is fundamental if and only if $M_{u^*}(z)=+\infty$ for some $z\in D$.*

Proof. a) *Necessity*: since u is fundamental, $\mathscr{P}(\mathbf{R})$ is a dense subset of $Cu_\infty(\mathbf{R})$. In particular, u is bounded. Let $z\in D$. Given $\varepsilon>0$, since $g_z\in Cu_\infty(\mathbf{R})$, there exists $P\in\mathscr{P}(\mathbf{R})$ such that

(1) $$\|u(g_z-P)\|\leqq\varepsilon.$$

Let $C=\inf\{(1+|t|)|g_z(t)|;\ t\in R\}$ and let $Q=C(1-P/g_z)/\varepsilon$. Then $Q\in\mathscr{P}_{u^*}$ by (1), whence $M_{u^*}(z)\geqq|Q(z)|=C/\varepsilon$. Since C is a positive constant depending only on z, letting $\varepsilon\to 0$ implies $M_{u^*}(z)=+\infty$.

Before proving the sufficiency, we will state two lemmas.

Lemma 1. *Let v be a weight on* $\mathbf{R}$. *If $M_v(z)=+\infty$ for some $z\in D$, then* $\mathscr{P}(\mathbf{R})\subset$ $\subset Cv_\infty(\mathbf{R})$.

Proof. Let $\mathscr{E}$ be the vector space of all $P\in\mathscr{P}(\mathbf{R})$ such that vP is bounded. For any such P, put $\|P\|_v=\|vP\|$. Then $\|\cdot\|_v$ is a semi-norm on $\mathscr{E}$. Assume that $\|\cdot\|_v$ is not a norm. Then there exists $P\in\mathscr{P}(\mathbf{R})$, $P\neq 0$, such that $vP=0$. This implies that v has a finite set as its support whence $\mathscr{E}=\mathscr{P}(\mathbf{R})$. Otherwise, $\|\cdot\|_v$ is a norm. Assume that $\mathscr{E}$ has finite dimension, say m. We have $m\geqq 1$ since $m=0$ would imply $M_v(z)=0$. Notice that $P\in\mathscr{E}$ and $n=\text{degree}(P)$ imply $t^n\in\mathscr{E}$, whence the set $\{1, \dots, t^{m-1}\}$ is a basis for $\mathscr{E}$ and the mapping $\psi_i\colon\mathscr{E}\to\mathbf{C}$ given by $\psi_i(a_0+\dots+a_{m-1}t^{m-1})=a_i$ is continuous for $i=0, \dots, m-1$. Also, the set $\mathscr{P}_v$ is closed and bounded, hence $\mathscr{P}_v$ is compact by our assumption. From the compactness of $\psi_i(\mathscr{P}_v)$, there exists a positive constant C such that for all $P\in\mathscr{P}_v$, $P=a_0+\dots+a_{m-1}t^{m-1}$, we have $|a_i|\leqq C$

for all $i=0, \dots, m-1$. Thus $|P(z)| \leqq C \sum_{i=0}^{m-1} |z|^i$ for all $P \in \mathscr{P}_v$; that is $M_v(z) < +\infty$, contradicting the hypothesis. Then $\mathscr{E}$ has infinite dimension. Since $t^n \in \mathscr{E}$ implies $t^m \in \mathscr{E}$ for all $0 \leqq m \leqq n$ it follows $\mathscr{E} = \mathscr{P}(\mathbf{R})$. To finish the proof, note that vP vanishes at infinity for all $P \in \mathscr{P}(\mathbf{R})$ since $vP(1+t^2)$ is bounded.

Lemma 2. *Let u be a weight on $\mathbf{R}$ such that $\mathscr{P}(\mathbf{R}) \subset Cu_\infty(\mathbf{R})$. If, for some $z \in D$, $g_z \in \overline{\mathscr{P}(\mathbf{R})}$, the closure of $\mathscr{P}(\mathbf{R})$, then $\mathbf{C}[g_z, \bar{g}_z] \subset \overline{\mathscr{P}(\mathbf{R})}$.*

Proof. Let $\mathscr{E} = \overline{\mathscr{P}(\mathbf{R})}$. Since $\mathscr{E}$ is a vector subspace of $Cu_\infty(\mathbf{R})$, it is enough to prove that $g_z^n (\bar{g}_z)^m \in \mathscr{E}$ for all $n, m \in \mathbf{N}$. First, we will prove by induction that

$$g_z^n \mathscr{P}(\mathbf{R}) \subset \mathscr{E} \quad \text{for all} \quad n \in \mathbf{N}. \tag{2}$$

In fact, for $n=0$ this is clear. Assume that it is true for some n and let $P \in \mathscr{P}(\mathbf{R})$. Since $Q = g_z(P - P(z)) \in \mathscr{P}(\mathbf{R})$ and $g_z^{n+1}(P - P(z)) = g_z^n Q$, we have

$$g_z^{n+1}(P - P(z)) \in \mathscr{E}. \tag{3}$$

Note that if g is a continuous and bounded complex valued function on $\mathbf{R}$, then the mapping $f \mapsto gf$ from $Cu_\infty(\mathbf{R})$ into itself is continuous. So

$$g\overline{\mathscr{F}} \subset \overline{g\mathscr{F}} \quad \text{for all} \quad \mathscr{F} \subset Cu_\infty(\mathbf{R}). \tag{4}$$

Since $g_z^{n+1} \in g_z^n \mathscr{E} = g_z^n \overline{\mathscr{P}(\mathbf{R})}$, the above remark and the induction hypothesis imply $g_z^{n+1} \in \mathscr{E}$. Then, from $g_z^{n+1} P = g_z^{n+1}(P - P(z)) + P(z) g_z^{n+1}$ and (3), we get $g_z^{n+1} P \in \mathscr{E}$. Thus (2) is proved. Notice that $h \in \mathscr{E}$ implies $\bar{h} \in \mathscr{E}$ whence, from (2), $(\bar{g}_z)^m = \overline{g_z^m} \in \mathscr{E}$ for all $m \in \mathbf{N}$, and then, from (4), $g_z^n (\bar{g}_z)^m \in g_z^n \overline{\mathscr{P}(\mathbf{R})} \subset \overline{g_z^n \mathscr{P}(\mathbf{R})}$ for all $n, m \in \mathbf{N}$. Using (2) and the fact that $\mathscr{E}$ is closed, we conclude that $g_z^n (\bar{g}_z)^m \in \mathscr{E}$ for all $n, m \in \mathbf{N}$.

Proof of Theorem 1, continued. b) *Sufficiency:* We have $M_{u^*}(z) = +\infty$ for some $z \in D$. From Lemma 1, it follows that $(1+t^2)\mathscr{P}(\mathbf{R}) \subset Cu_\infty^*(\mathbf{R})$ whence $\mathscr{P}(\mathbf{R}) \subset Cu_\infty(\mathbf{R})$. Let $Q \in \mathscr{P}_{u^*}$ be such that $Q(z) \neq 0$, and put

$$P = -\frac{g_z}{Q(z)}(Q - Q(z)).$$

Then $P \in \mathscr{P}(\mathbf{R})$ and $g_z - P = g_z Q / Q(z)$. Hence,

$$u(t)|g_z(t) - P(t)| \leqq |Q(z)|^{-1}(1+|t|)|g_z(t)|u^*(t)|Q(t)| \leqq \text{constant} \cdot |Q(z)|^{-1}$$

for all $t \in \mathbf{R}$. Letting $Q(z) \to \infty$, we have that $\|u(g_z - P)\| \to 0$. So $g_z \in \overline{\mathscr{P}(\mathbf{R})}$. From Lemma 2, it follows that $\mathbf{C}[g_z, \bar{g}_z] \subset \overline{\mathscr{P}(\mathbf{R})}$. Since u is bounded, by Remark 2 we have that $\mathbf{C}[g_z, \bar{g}_z]$ is dense in $Cu_\infty(\mathbf{R})$, and so we conclude that $\mathscr{P}(\mathbf{R})$ is dense in $Cu_\infty(\mathbf{R})$.

Remark 4. From the proof of the necessity of the condition in Theorem 1, it follows that $M_{u^*}(z) = +\infty$ for every $z \in D$ whenever u is fundamental.

4. Fundamental weights and quasi-analytic classes of functions

In the following, M will denote a sequence (M_n), $n\in\mathbf{N}$, of positive real numbers.

$C(M)$ will denote the vector space of all complex valued C^∞ functions f on $\mathbf{R}$ such that there exist positive constants C and c (depending on f) for which

$$(5)\qquad |f^{(n)}(t)| \leqq Cc^n M_n \quad \text{for all} \quad t\in\mathbf{R},\ n\in\mathbf{N}.$$

$C(M)$ is called *quasi-analytic* if $f\in C(M)$ vanishes identically when there exists $s\in\mathbf{R}$ such that $f^{(n)}(s)=0$ for all $n\in\mathbf{N}$.

Remark 5. Let σ be as in Remark 3. Then $C(M)\circ\sigma\subset C(M)$.

We write γ_M for the weight on $\mathbf{R}$ given by

$$\gamma_M(t) = \inf\{M_n|t|^{-n};\ n\in\mathbf{N}\} \quad \text{for all} \quad t\in\mathbf{R}.$$

Remark 6. We have $\mathscr{P}(\mathbf{R})\subset C(\gamma_M)_\infty(\mathbf{R})$. Further, either γ_M is an upper-semi-continuous function with compact support (when $\limsup M_n^{1/n}<+\infty$) or it is a continuous function that never vanishes. In the first case, γ_M is fundamental by the Weierstrass theorem.

Theorem 2. *$C(M)$ is quasi-analytic if and only if γ_M is fundamental.*

Proof. The necessity of the condition follows from the proof of Lemma 2.29 in Nachbin [10] and so we just sketch the proof. In fact, assume that $C(M)$ is quasi-analytic, and let L be a continuous linear form on $C(\gamma_M)_\infty(\mathbf{R})$ that vanishes on $\mathscr{P}(\mathbf{R})$ (see Remark 6). Letting $e_x(t)=e^{ixt}$ for all $t, x\in\mathbf{R}$, define $f\colon \mathbf{R}\to\mathbf{C}$ by $f(x)= =L(e_x)$ for all $x\in\mathbf{R}$. Then $f\in C(M)$ and, from the assumption, $f^{(n)}(0)=0$ for all $n\in\mathbf{N}$; that is, $f\equiv 0$ by quasi-analyticity. Since the vector space generated by $\{e_x;\ x\in\mathbf{R}\}$ is dense in $C(\gamma_M)_\infty(\mathbf{R})$, it follows that L vanishes identically. By the Hahn—Banach theorem, we conclude that $\mathscr{P}(\mathbf{R})$ is dense in $C(\gamma_M)_\infty(\mathbf{R})$.

Now let us prove the sufficiency. Assume that $C(M)$ is not quasi-analytic. From Remark 5, there exists $f\in C(M)$ such that

$$(6)\qquad f^{(n)}(0) = 0 \quad \text{for all} \quad n\in\mathbf{N} \quad \text{and} \quad f|[0, +\infty) \not\equiv 0.$$

Let

$$U = \{z\in\mathbf{C};\ \mathrm{Re}(z) > 0\},\quad V = \{z\in\mathbf{C};\ \mathrm{Re}(z) > 1\}$$

and put

$$F(z) = \int_0^{+\infty} f(t)e^{-zt}\,dt \quad \text{for all} \quad z\in U.$$

Integrating by parts and using induction, we get, in view of (6),

$$(7)\qquad z^n F(z) = \int_0^{+\infty} f^{(n)}(t)e^{-zt}\,dt \quad \text{for all} \quad z\in U,\ n\in\mathbf{N}.$$

Let C, c be as in (5). From this and (7), it follows that

$$(8)\qquad |F(z)| \leqq Cc^n M_n|z|^{-n} \quad \text{for all} \quad z\in\overline{V},\ n\in\mathbf{N}.$$

Since F is a holomorphic function not identically zero, we have

$$F(\alpha) \neq 0 \quad \text{for some} \quad \alpha \in V.$$

Let u be the weight given by

$$u(t) = (1+|t|)|F(1+it)||1+it|^{-1} \quad \text{for all} \quad t \in \mathbf{R},$$

fix $P \in \mathcal{P}_{u^*}$, and let

$$G(z) = P(i-iz)F(z)z^{-1} \quad \text{for all} \quad z \in U.$$

Then G is holomorphic in V and continuous on $\bar{V}$. Letting $t \in \mathbf{R}$, since $|G(1+it)| = |u^*(t)P(t)|$, we have that $|G| \leqq 1$ on ∂V because $P \in \mathcal{P}_{u^*}$. Also, G is bounded on V from (8). Since $\alpha \in V$, it follows from the maximum modulus theorem that $|G(\alpha)| \leqq 1$ whence $|P(i-i\alpha)| \leqq |\alpha| |F(\alpha)|^{-1}$. Notice that, P being arbitrary, $M_{u^*}(i-i\alpha) < +\infty$. Since $i-i\alpha \in D$, Remark 4 implies that u is not fundamental. Furthermore, (8) implies

$$u(t) \leqq \sqrt{2}Cc^n M_n |1+it|^{-n} \leqq \sqrt{2}CM_n \left|\frac{t}{c}\right|^{-n}$$

for all $t \in \mathbf{R}$, $n \in \mathbf{N}$ whence

$$u(t) \leqq \sqrt{2}C\gamma_M\left(\frac{t}{c}\right) \quad \text{for all} \quad t \in \mathbf{R}.$$

From this and Remarks 1 and 3, we conclude that γ_M is not fundamental since u is not fundamental.

Corollary 1. *Let φ be a complex valued C^∞ function on* $\mathbf{R}$ *with compact support and not identically zero. Put $M_n = \|\varphi^{(n)}\|$ for all $n \in \mathbf{N}$. Then γ_M is not fundamental.*

Proof. Since $\varphi \in C(M)$, $C(M)$ is not quasi-analytic and the conclusion follows from Theorem 2.

Remark 7. The above corollary provides a simple counterexample to localizability (see Chapter 31 of Nachbin [10]). Notice also that, in this case, γ_M is a continuous and positive function by Remark 6.

Corollary 2. *Let u be a weight on* $\mathbf{R}$ *such that $\mathcal{P}(\mathbf{R}) \subset Cu_\infty(\mathbf{R})$ and $\mathcal{P}(\mathbf{R})$ is not dense. Let $M_n = \|ut^n\|$, for all $n \in \mathbf{N}$. Then $C(M)$ is not quasi-analytic.*

Proof. Since $u \leqq \gamma_M$, the conclusion follows from Theorem 2 and Remark 1.

Lemma 3. *For M fixed, let $u(t) = (1+|t|)\gamma_M(t)$ for all $t \in \mathbf{R}$. Then u is fundamental if, and only if, γ_M is fundamental.*

Proof. Since $\gamma_M \leqq u$, in view of Remark 1 it is enough to prove that u is fundamental when γ_M is fundamental. In fact, let M' be defined by $M_n' = M_{n+1}$ for all $n \in \mathbf{N}$. For $t \neq 0$, we have

$$\gamma_{M'}(t)|t|^{-1} = \inf\{M_{n+1}|t|^{-(n+1)},\ n \in \mathbf{N}\} \geqq \gamma_M(t),$$

and hence

$$|t|\gamma_M(t) \leqq \gamma_{M'}(t) \quad \text{for all} \quad t \in \mathbf{R}.$$

Also

$$\gamma_M(t) \leqq \gamma_M(0) \leqq \gamma_M(0)\gamma_{M'}(t)/\gamma_{M'}(1)$$

for $|t| \leqq 1$. So, there exists a positive constant C such that $u \leqq C\gamma_{M'}$. Since γ_M is fundamental, it follows from Theorem 2 that $C(M)$ is quasi-analytic. Then $C(M')$ is quasi-analytic whence $\gamma_{M'}$ is fundamental by Theorem 2. So, from Remark 1, we conclude that u is fundamental.

$$\text{Put} \quad T_M(t) = \sup\{|t|^n M_n^{-1};\ n \in \mathbf{N}\} \quad \text{for all} \quad t \in \mathbf{R}.$$

Theorem 3. *$C(M)$ is quasi-analytic if, and only if, there exist a complex number z and a sequence of polynomials (P_n) such that*

$$\operatorname{Im}(z) \neq 0, \quad |P_n| \leqq T_M \quad \textit{for all } n, \textit{ and} \quad P_n(z) \to \infty.$$

Proof. Let u be as in Lemma 3. From this and Theorem 2, we have that $C(M)$ is quasi-analytic if, and only if, u is fundamental. Since $\mathscr{P}_{u^*} = \{P \in \mathscr{P}(\mathbf{R});\ |P| \leqq T_M\}$, Theorem 3 follows from Theorem 1.

Remark 8. From Theorem 2, it follows that to each characterization of fundamental weights, there corresponds a characterization of quasi-analytic classes. See [8] for other applications of this.

Remark 9. Sufficiency of the condition in Theorem 2 could also be obtained by using the Corollary of Theorem 1 in Mergelyan [8] in connection with the classical Denjoy—Carleman theorem.

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Guido Zapata
Instituto de Matemática
Universidade Federal do Rio de Janeiro
Caixa Postal 1835, ZC-00
20.000 — Rio de Janeiro, RJ — Brazil