WEIGHTED-AVERAGING FINITE-ELEMENT METHOD FOR SCALAR WAVE EQUATION IN THE FREQUENCY DOMAIN

DONG-JOO MIN[1], HAI SOO YOO[1], CHANGSOO SHIN[2], HYE-JA HYUN[3] and JUNG HEE SUH[2]

[1] *Korea Ocean Research and Development Institute, Marine Environment and Climate Change Laboratory, Ansan P.O. Box 29, Kyungki 425-600, Korea.*
[2] *Seoul National University, School of Civil, Urban and Geosystem Engineering, San 56-1, Shillim-dong, Kwanak-Gu, Seoul 151-742, Korea.*
[3] *Korea Institute of Geology, Mining, and Materials, P.O. Box 111, Taedok Science Town 305-350, Korea.*

(Received December 4, 2001; revised version accepted February 1, 2002)

ABSTRACT

Min, D.-J, Yoo, H.S., Shin, C., Hyun, H.-J. and Suh, J.H., 2002. Weighted-averaging finite-element method for scalar wave equation in the frequency domain. *Journal of Seismic Exploration*, 11: 197-222.

We develop a new weighted-averaging finite-element method which can be applied to generate synthetic seismograms using scalar wave equation in the frequency domain. Our method introduces three kinds of supplementary element sets in addition to a basic element set which is used in standard finite-element method. By constructing global stiffness and mass matrices for four kinds of element sets and then averaging them with weighting coefficients, we obtain a new global stiffness and mass matrix. With the optimal weighting coefficients minimizing grid dispersion and grid anisotropy, we can reduce the number of grid points required per wavelength to 4 for a 1% upper limit of error. This reduction of the number of grid points, achieved by using the weighted-averaging finite-element method, makes it possible to reduce computer memory to 32.7% of that for the eclectic finite-element method. We confirm the accuracy of our weighted-averaging finite-element method through accuracy analyses for a homogeneous and a horizontal-layer model. By synthetic data example, we reconfirm that our method is more efficient for simulating a geological model than previous finite-element methods.

KEY WORDS: weighted-averaging finite-element method, scalar wave equation, frequency domain, weighting coefficients, synthetic seismograms.

0963-0651/02/$5.00

INTRODUCTION

Seismic forward modeling plays an important role in seismic inversion and migration, which are useful tools for interpretation of seismic data. Efficiency of seismic inversion and migration is strongly dependent on seismic modeling algorithm used in the seismic inversion and migration. The seismic forward modeling can be carried out in either the time-space, time-wavenumber (pseudospectral method), or frequency-space domains by using finite-difference or finite-element methods. Since all of these methods have their own respective advantages, we solve a given problem using the modeling technique suitable for our purpose.

The frequency-domain method, among them, is valuable for simulating a viscoelastic medium with a multiple source. This is accomplished by the advantages of frequency-domain method that the complex impedance matrix independent of source location is constructed and the wave equation which describes the viscosity properties of a medium is very simple. Since the frequency-domain modeling technique does not exchange any information across different frequencies, parallelization in the frequency domain can easily be implemented by distributing frequencies across the computation processors. For these reasons, Pratt and Worthington (1990), Pratt (1990a; 1990b; 1999) and Pratt and Shipp (1999) have applied the frequency-domain modeling to the inversion of crosshole tomography, seismic waveform inversion, and seismic imaging. The frequency-domain modeling, however, has a major drawback that it needs more grid points per wavelength than the time-domain methods. As a result, the frequency-domain modeling has been found to be impractical, even though a number of vector parallel computers have been developed.

In recent years, extensive studies have been done to make the frequency-domain modeling technique practical. Most of the studies were completed by performing dispersion analysis as Alford et al. (1974), Marfurt (1984) and Holberg (1987) did. As a practical frequency-domain modeling technique of 2D scalar wave equation, Jo et al. (1996) and Shin and Sohn (1998) suggested a weighted-averaging finite-difference scheme. Jo et al. (1996) used 9 points around the collocation to solve the 2-D scalar wave equation; Shin and Sohn (1998) used 25 points. They used a few sets of rotated finite-difference operators in addition to a standard operator for discretizing the Laplacian operator. To approximate the mass acceleration term, Jo et al. (1996) and Shin and Sohn (1998) applied an eclectic method which Marfurt (1984) proposed in the finite-element method. The eclectic method is characterized by composing and combining the lumped mass and the consistent mass matrix operators. They applied the eclectic method to finite-difference method by distributing mass at the collocation point into the adjacent 9 or 25 points. The 9-point weighted-averaging scheme proposed by Jo et al. (1996) reduced the number of grid points to 5 per wavelength, keeping errors within 1%; The

25-point weighted-averaging scheme designed by Shin and Sohn (1998) reduced the number of grid points to 2.5 per wavelength within 1% error.

For the elastic wave equations, Štekl and Pratt (1998) and Min et al. (2000) also suggested a weighted-averaging method. Štekl and Pratt (1998) introduced a rotated operators within a standard operator of the elastic wave equations. The method used by Štekl and Pratt (1998) needs approximately 9 grid points per wavelength within 1% errors in the case where weighting coefficients dependent on Poisson's ratio are used. Min et al. (2000) designed the weighted-averaging finite-difference operators using 25 grid points around the collocation point for solving the elastic wave equations like in the method suggested by Shin and Sohn (1998), but they did not introduce any rotated operator. They composed almost all the possible finite-difference operators for the approximation of a spatial second-order derivatives, and then averaged them with weighting coefficients. Min et al. (2000) reduced the number of grid points per wavelength from 33.3 (using the conventional method) to 3.3 (using the weighted-averaging method).

In this paper, we apply these ideas to the finite-element method. We also propose a practical weighted-averaging finite-element method for the scalar wave equation in the frequency domain. The finite-element method, which uses the basis function defined on the element, solves a given problem on the ground of integral principle unlike finite-difference method. The integrals are finally expressed by stiffness matrix, mass matrix and force vector. The weighted-averaging method applied to the finite-element method, therefore, has to be different from that of the finite-difference method. The method which we present in this paper is to use three kinds of supplementary quadrilateral element sets as well as a fundamental quadrilateral element set. We construct global stiffness and mass matrices for four kinds of element sets and then average them with weighting coefficients to yield a new global stiffness and mass matrix. The weighting coefficients are determined to give the ideal, numerical phase and group velocities for the full ranges of propagation angles.

This paper is organized as follows: in the first place, we explain the weighted-averaging finite-element method of scalar wave equation in detail, and present how to obtain the optimal weighting coefficients used in the weighted-averaging finite-element method. Next, we demonstrate the accuracy of the weighted-averaging finite-element method by comparing the numerical solutions computed by the weighted-averaging finite-element method with analytic solutions for a homogeneous model. We also examine whether or not the weighted-averaging finite-element method gives reliable solutions for a horizontal-layer model by comparing the numerical solutions computed by the eclectic method and the weighted-averaging finite-element method with each other. To estimate computational efficiency of the weighted-averaging method, we compare computer memory requirements of the weighted-averaging method for a band-type matrix solver and a nested dissection method with those of the standard and the eclectic finite-element method. Finally, we synthesize the seismogram using the weighted-averaging finite-element method for a syncline model.

FINITE-ELEMENT FORMULAS OF SCALAR WAVE EQUATION

The frequency-domain scalar wave equation can be expressed in a heterogeneous medium with a source as

$$\rho\omega^2\phi + (\partial/\partial x)[k(\partial\phi/\partial x)] + (\partial/\partial z)[k(\partial\phi/\partial z)] = f(x,z,\omega) \quad , \tag{1}$$

where ϕ is the displacement or the pressure field in the frequency domain, k is the bulk modulus, ρ is the density, and f is the source. If we derive the finite-element formula using the Galerkin method which chooses interpolation function for a weighting function as a special case of the weighted residual methods (Marfurt, 1984), we obtain

$$\int\int w_i\{(\partial/\partial x)[k(\partial\phi^e/\partial x)] + (\partial/\partial z)[k(\partial\phi^e/\partial z)] - f + \rho\omega^2\phi^e\}dx\,dz = 0 \quad , \tag{2}$$

where $\phi^e = \Sigma_{j=1}^{r} N_j\phi_j$, N_j is the basis function, w_i is the weighting function (which is written as N_i for the Galerkin method), and r is the number of nodal points. If we integrate equation (2) by parts, we can obtain

$$\omega^2\mathbf{M\Phi} + \mathbf{K\Phi} = \mathbf{f} \quad , \tag{3}$$

with

$$K_{ij} = \int\int [k(\partial N_i/\partial x)(\partial N_j/\partial x) + k(\partial N_i/\partial z)(\partial N_j/\partial z)dx\,dz \quad , \tag{4}$$

$$M_{ij} = \int\int \rho N_i N_j\,dx\,dz \quad , \tag{5}$$

$$f_i = \int\int fN_i\,dx\,dz \quad , \tag{6}$$

and

$$i = 1, 2, \ldots\ldots, r \quad \text{and} \quad j = 1, 2, \ldots\ldots, r \quad , \tag{7}$$

where **K** and **M** are the (r × r) global stiffness and mass matrices, respectively, **Φ** is the (r × 1) pressure field vector, and **f** is the (r × 1) source vector. Equation (3) can be expressed in a simple form, as follows

$$\mathbf{S\Phi} = \mathbf{f} \quad \text{or} \quad \mathbf{\Phi} = \mathbf{S}^{-1}\mathbf{f} \ , \tag{8}$$

with

$$\mathbf{S} = \omega^2\mathbf{M} + \mathbf{K} \ , \tag{9}$$

where $\mathbf{S}$ is the $(r \times r)$ complex impedance matrix. We obtain frequency-domain solutions by decomposing the huge sparse matrix $\mathbf{S}$ with numerical method and then obtain the time-domain solutions by taking the inverse Fourier transform of the frequency-domain solutions.

If we introduce the damping term [for example, resulting from applying sponge boundary condition suggested by Shin (1995)] into the problem, equation (3) can be written as

$$\omega^2\mathbf{M\Phi} + i\omega\mathbf{C\Phi} + \mathbf{K\Phi} = \mathbf{f} \ . \tag{10}$$

A WEIGHTED-AVERAGING FINITE-ELEMENT METHOD

The general type of element set used in finite-element method is either a triangular or a quadrilateral element. We adopted the simplest rectangular element in the weighted-averaging finite-element method. If we apply a standard finite-element method to solve equation (10), a set of rectangular element [whose size is $(\Delta x,\Delta z)$ as shown in Fig. 1a] is used. In the weighted-averaging finite-element method, we use three supplementary sets of rectangular elements [whose sizes are $(2\Delta x,2\Delta z)$, $(2\Delta x,\Delta z)$, and $(\Delta x,2\Delta z)$ as shown in Figs. 1b, 1c and 1d] in addition to the elementary rectangular set used in the standard finite-element method.

If we take the $(\Delta x, \Delta z)$-element set as the criterion of grid sets, all the nodal points are located on the apex of the $(\Delta x, \Delta z)$-element. In the $(2\Delta x,2\Delta z)$-, $(2\Delta x,\Delta z)$- and $(\Delta x,2\Delta z)$-element sets, however, some of nodal points lie on the sides of the element or within the element (e.g., the • points in Figs. 2a, 3a and 4a). The nodal points located on the sides of the element or within the element do not contribute to making up the stiffness and mass matrix at all, which leads to inaccurate solutions. As a possible solution for the problem, we use three additional $(2\Delta x,2\Delta z)$-element sets shown in Figs. 2b, 2c and 2d as well as the fundamental $(2\Delta x,2\Delta z)$-element set shown in Fig. 2a. For the $(2\Delta x,\Delta z)$- and $(\Delta x,2\Delta z)$-element sets, we also add one supplementary element set (Figs. 3b and 4b) to the basic element set (Figs. 3a and 4a), respectively. As a result, we construct 9 global stiffness and mass matrices for the 9 element sets: one $(\Delta x,\Delta z)$-, four $(2\Delta x,2\Delta z)$-, two $(2\Delta x,\Delta z)$- and two $(\Delta x,2\Delta z)$-element sets.

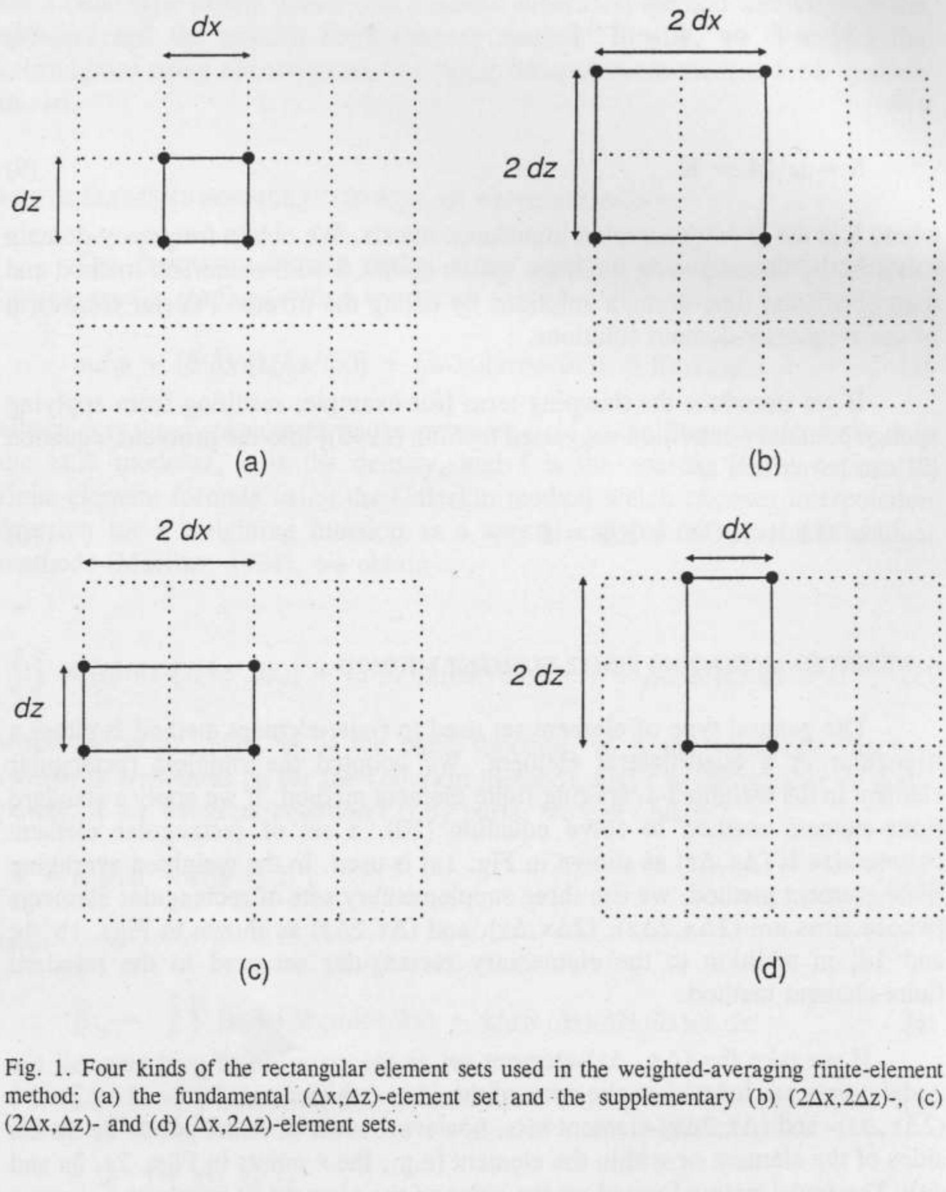


Fig. 1. Four kinds of the rectangular element sets used in the weighted-averaging finite-element method: (a) the fundamental $(\Delta x,\Delta z)$-element set and the supplementary (b) $(2\Delta x,2\Delta z)$-, (c) $(2\Delta x,\Delta z)$- and (d) $(\Delta x,2\Delta z)$-element sets.

Next, we have to combine the 9 global stiffness and mass matrices into a new global stiffness and mass matrix. In the process of combining the stiffness and mass matrices, we need weighting coefficients. By assuming that the same weighting coefficients are given to the same-size element sets and the $(2\Delta x,\Delta z)$- and $(\Delta x,2\Delta z)$-element sets has the same coefficients, we only require 3 weighting coefficients c_1, c_2 and c_3 for combining the 9 stiffness matrices: c_1 is used for the $(\Delta x,\Delta z)$-element set; c_2 for the four $(2\Delta x,2\Delta z)$-element sets; and c_3 for the two $(2\Delta x,\Delta z)$- and two $(\Delta x,2\Delta z)$-element sets.

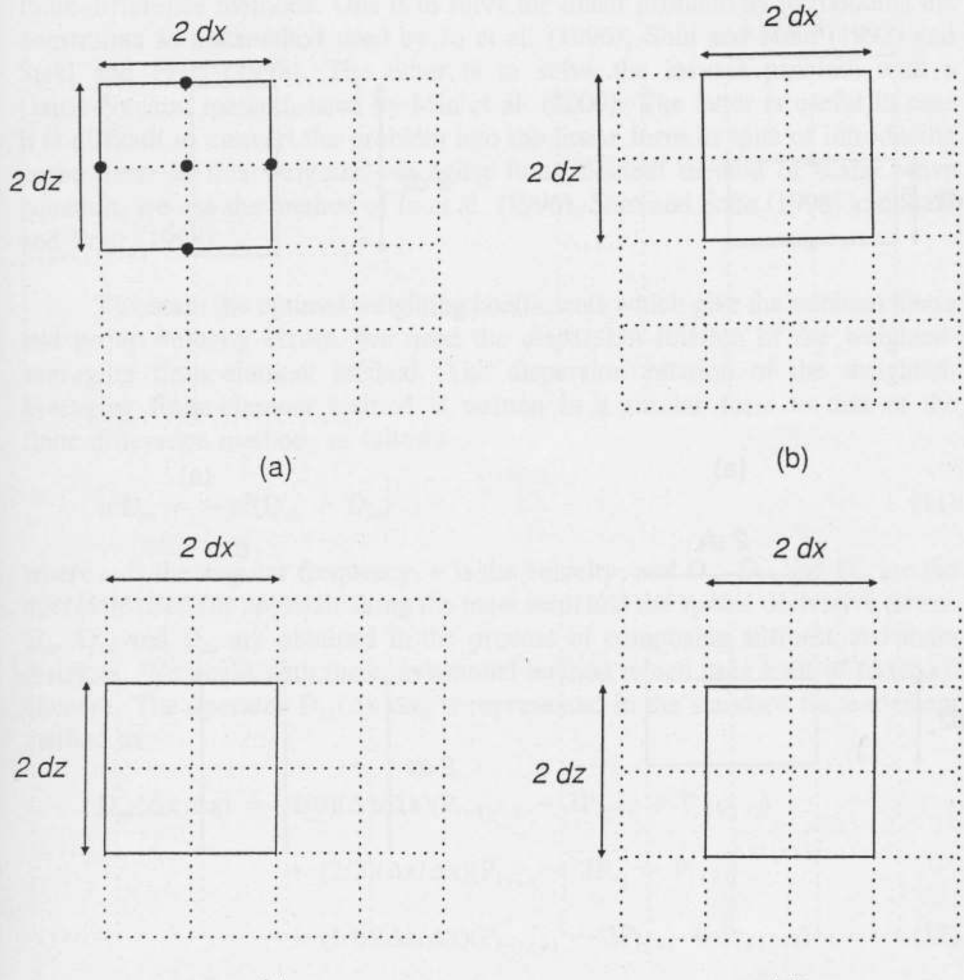


Fig. 2. Four $(2\Delta x, 2\Delta z)$-element sets used for all the nodal points to be included in the construction of the stiffness and mass matrix. The • indicates the nodal points excluded in the construction of the stiffness and mass matrix.

For the mass term, we use the eclectic method to combine the lumped mass with the consistent mass linearly. Since the lumped mass matrices are constructed in the same form for the four kinds of element sets, we only use the lumped mass matrix constructed in the fundamental $(\Delta x, \Delta z)$-element set. As a result, we use 4 weighting coefficients e_1, e_2, e_3 and f: e_1 is used for the consistent mass matrix of the $(\Delta x, \Delta z)$-element set; e_2 for the four $(2\Delta x, 2\Delta z)$-element sets; e_3 for the two $(2\Delta x, \Delta z)$- and two $(\Delta x, 2\Delta z)$-element sets; and f is for the lumped mass matrix of the $(\Delta x, \Delta z)$-element set.

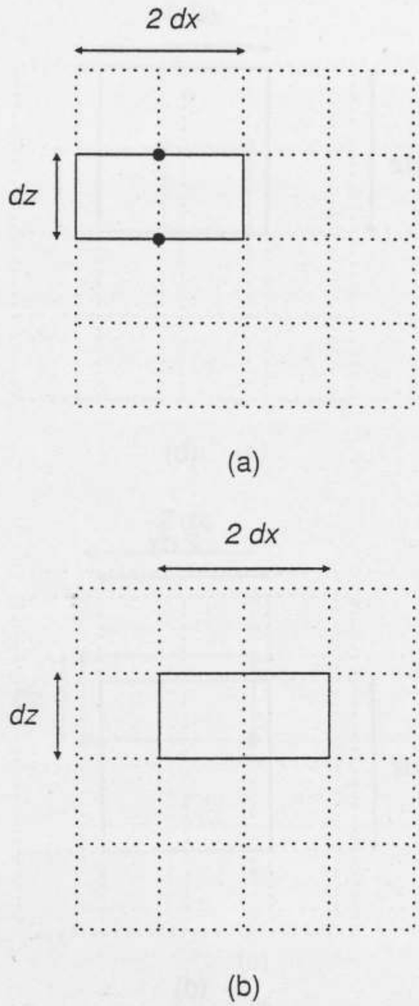


Fig. 3. Two ($2\Delta x, \Delta z$)-element sets used for all the nodal points to be included in the construction of the stiffness and mass matrix. The • indicates the nodal points excluded in the construction of the stiffness and mass matrix.

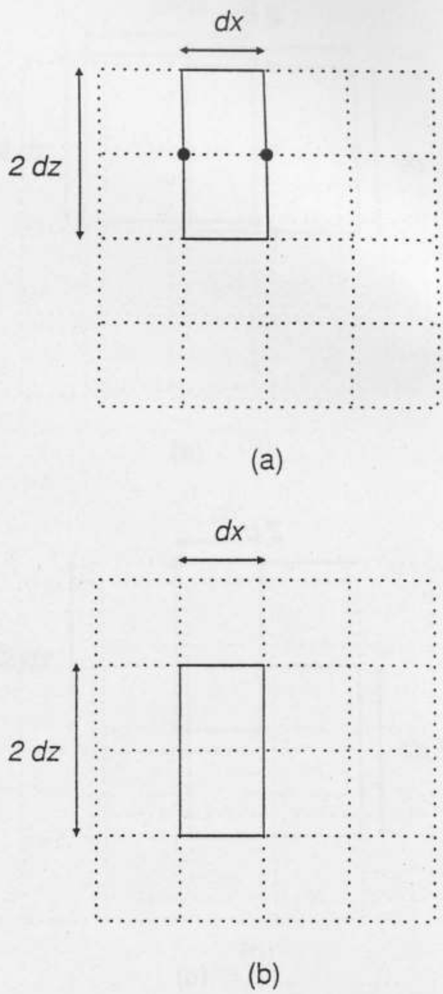


Fig. 4. Two ($\Delta x, 2\Delta z$)-element sets used for all the nodal points not to be included in the construction of the stiffness and mass matrix. The • indicates the nodal points excluded in the construction of the stiffness and mass matrix.

WEIGHTING COEFFICIENTS

The weighted-averaging finite-element method is completed by determining the optimal weighting coefficients. There were two methods for determining the weighting coefficients in the previous weighted-averaging finite-difference methods. One is to solve the linear problem by introducing the constraints as the method used by Jo et al. (1996), Shin and Sohn (1998) and Štekl and Pratt (1998). The other is to solve the inverse problem with a Gauss-Newton method, used by Min et al. (2000). The latter is useful in case it is difficult to convert the problem into the linear form in spite of introducing constraints. In this weighted-averaging finite-element method of scalar wave equation, we use the method of Jo et al. (1996), Shin and Sohn (1998) and Štekl and Pratt (1998).

To obtain the optimal weighting coefficients which give the minimal phase and group velocity errors, we need the dispersion relation of the weighted-averaging finite-element method. The dispersion relation of the weighted-averaging finite-element method is written in a similar form to that of the finite-difference method, as follows

$$\omega^2 D_m = -v^2(D_{xx} + D_{zz}) \ , \tag{11}$$

where ω is the angular frequency, v is the velocity, and D_m, D_{xx} and D_{zz} are the operators used for approximating the mass term and the spatial derivative terms. D_m, D_{xx} and D_{zz} are obtained in the process of composing stiffness and mass matrices. We begin with the conventional method which uses a set of $(\Delta x,\Delta z)$-element. The operator $D_{xx}(\Delta x,\Delta z)$ is represented in the standard finite-element method as

$$\begin{aligned} D_{xx}(\Delta x,\Delta z) \approx\ & (1/6)(\Delta z/\Delta x)(P_{i-1,j-1} - 2P_{i,j-1} + P_{i+1,j-1}) \\ & + (2/3)(\Delta z/\Delta x)(P_{i-1,j} - 2P_{i,j} + P_{i+1,j}) \\ & + (1/6)(\Delta z/\Delta x)(P_{i-1,j+1} - 2P_{i,j+1} + P_{i+1,j+1}) \ , \end{aligned} \tag{12}$$

where $P_{i,j}$ are the pressures or the displacements. $(\Delta z/\Delta x)$ comes from the process of constructing the stiffness matrix. From the viewpoint of the finite-difference technique, the finite-element method is already the form of being weighted-averaged.

To obtain the D_{xx} operator of the weighted-averaging finite-element method, we compose operators for the $(2\Delta x,2\Delta z)$-, $(2\Delta x,\Delta z)$- and $(\Delta x,2\Delta z)$-element sets in addition to the D_{xx} operator for the $(\Delta x,\Delta z)$-element set and then average them with weighting coefficients c_1, c_2 and c_3:

$$\begin{aligned} D_{xx} \approx\ & c_1[D_{xx}(\Delta x,\Delta z)] + c_2[D_{xx}(2\Delta x,2\Delta z)] \\ & + c_3[D_{xx}(2\Delta x,\Delta z)] + c_3[D_{xx}(\Delta x,2\Delta z)] \ , \end{aligned} \tag{13}$$

where

$$D_{xx}(2\Delta x,2\Delta z) \approx (1/6)(2\Delta z/2\Delta x)(P_{i-2,j-2} - 2P_{i,j-2} + P_{i+2,j-2}) + (2/3)(2\Delta z/2\Delta x)(P_{i-2,j} - 2P_{i,j} + P_{i+2,j}) + (1/6)(2\Delta z/2\Delta x)(P_{i-2,j+2} - 2P_{i,j+2} + P_{i+2,j+2}) \; , \quad (14)$$

$$D_{xx}(2\Delta x,\Delta z) \approx (1/6)(\Delta z/2\Delta x)(P_{i-2,j-1} - 2P_{i,j-1} + P_{i+2,j-1}) + (2/3)(\Delta z/2\Delta x)(P_{i-2,j} - 2P_{i,j} + P_{i+2,j}) + (1/6)(\Delta z/2\Delta x)(P_{i-2,j+2} - 2P_{i,j+2} + P_{i+2,j+2}) \; , \quad (15)$$

and

$$D_{xx}(\Delta x,2\Delta z) \approx (1/6)(2\Delta z/\Delta x)(P_{i-1,j-2} - 2P_{i,j-2} + P_{i+1,j-2}) + (2/3)(2\Delta z/\Delta x)(P_{i-1,j} - 2P_{i,j} + P_{i+1,j}) + (1/6)(2\Delta z/\Delta x)(P_{i-1,j+2} - 2P_{i,j+2} + P_{i+1,j+2}) \; . \quad (16)$$

The D_{zz} operator of the weighted-averaging finite-element method is constructed in a similar manner. For the D_{zz} operator, $(\Delta x/\Delta z)$ is substituted for $(\Delta z/\Delta x)$ in the D_{xx} operator. The D_{zz} operator is written as

$$D_{zz} \approx c_1[D_{zz}(\Delta x,\Delta z)] + c_2[D_{zz}(2\Delta x,2\Delta z)] + c_3[D_{zz}(2\Delta x,\Delta z)] + c_3[D_{zz}(\Delta x,2\Delta z)] \; , \quad (17)$$

where

$$D_{zz}(\Delta x,\Delta z) \approx (1/6)(\Delta x/\Delta z)(P_{i-1,j-1} - 2P_{i-1,j} + P_{i-1,j+1}) + (2/3)(\Delta x/\Delta z)(P_{i,j-1} - 2P_{i,j} + P_{i,j+1}) + (1/6)(\Delta x/\Delta z)(P_{i+1,j-1} - 2P_{i+1,j} + P_{i+1,j+1}) \; , \quad (18)$$

$$D_{zz}(2\Delta x,2\Delta z) \approx (1/6)(2\Delta x/2\Delta z)(P_{i-2,j-2} - 2P_{i-2,j} + P_{i-2,j+2}) + (2/3)(2\Delta x/2\Delta z)(P_{i,j-2} - 2P_{i,j} + P_{i,j+2}) + (1/6)(2\Delta x/2\Delta z)(P_{i+2,j-2} - 2P_{i+2,j} + P_{i+2,j+2}) \; , \quad (19)$$

$$D_{zz}(2\Delta x,\Delta z) \approx (1/6)(2\Delta x/\Delta z)(P_{i-2,j-1} - 2P_{i-2,j} + P_{i-2,j+1})$$
$$+ (2/3)(2\Delta x/\Delta z)(P_{i,j-1} - 2P_{i,j} + P_{i,j+1})$$
$$+ (1/6)(2\Delta x/\Delta z)(P_{i+2,j-1} - 2P_{i+2,j} + P_{i+2,j+1}) \ , \qquad (20)$$

and

$$D_{zz}(\Delta x,2\Delta z) \approx (1/6)(\Delta x/2\Delta z)(P_{i-1,j-2} - 2P_{i-1,j} + P_{i-1,j+2})$$
$$+ (2/3)(\Delta x/2\Delta z)(P_{i,j-2} - 2P_{i,j} + P_{i,j+2})$$
$$+ (1/6)(\Delta x/2\Delta z)(P_{i+1,j-2} - 2P_{i+1,j} \ P_{i+1,j+2}) \ . \qquad (21)$$

The D_m operator is obtained from mass matrices. The lumped mass operator approximates the mass term only with the collocation point; the consistent mass operator approximates the mass term with the distributed mass to the adjacent nodal points. Therefore, the lumped mass operator is expressed by $P_{i,j}$ and the consistent mass operator is obtained from the mass matrix formed using the conventional finite-element method. For the $(\Delta x,\Delta z)$-element, the D_m operator by the consistent mass operator is expressed as

$$D_m(\Delta x,\Delta z) = (1/36)\Delta x\Delta z(P_{i-1,j-1} + 4P_{i,j-1} + P_{i+1,j-1})$$
$$+ (1/9)\Delta x\Delta z(P_{i-1,j} + 4P_{i,j} + P_{i+1,j})$$
$$+ (1/36)\Delta x\Delta z(P_{i-1,j+1} + 4P_{i,j+1} + P_{i+1,j+1}) \ . \qquad (22)$$

We can also formulate the D_m operators for the $(2\Delta x,2\Delta z)$-, $(2\Delta x,\Delta z)$- and $(\Delta x,2\Delta z)$-element sets. For the lumped mass, we only formulate an operator for the $(\Delta x,\Delta z)$-element set. If we average the lumped mass and the consistent mass operators with weighting coefficients, we obtain

$$D_m = e_1D_m(\Delta x,\Delta z) + e_2D_m(2\Delta x,2\Delta z) + e_3D_m(2\Delta x,\Delta z)$$
$$+ e_3D_m(\Delta x,2\Delta z) + fP_{i,j}\Delta x\Delta z \ , \qquad (23)$$

where $D_m(2\Delta x,2\Delta z)$, $D_m(2\Delta x,\Delta z)$, and $D_m(\Delta x,2\Delta z)$ are expressed as

$$D_m(2\Delta x,2\Delta z) = (1/36)4\Delta x\Delta z(P_{i-2,j-2} + 4P_{i,j-2} + P_{i+2,j-2})$$
$$+ (1/9)4\Delta x\Delta z(P_{i-2,j} + 4P_{i,j} + P_{i+2,j})$$
$$+ (1/36)4\Delta x\Delta z(P_{i-2,j+2} + 4P_{i,j+2} + P_{i+2,j+2}) \ , \qquad (24)$$

$$D_m(2\Delta x,\Delta z) = (1/36)2\Delta x\Delta z(P_{i-2,j-1} + 4P_{i,j-1} + P_{i+2,j-1}) + (1/9)2\Delta x\Delta z(P_{i-2,j} + 4P_{i,j} + P_{i+2,j}) + (1/36)2\Delta x\Delta z(P_{i-2,j+1} + 4P_{i,j+1} + P_{i+2,j+1}) \quad , \tag{25}$$

and

$$D_m(\Delta x,2\Delta z) = (1/36)2\Delta x\Delta z(P_{i-1,j-2} + 4P_{i,j-2} + P_{i+1,j-2}) + (1/9)2\Delta x\Delta z(P_{i-1,j} + 4P_{i,j} + P_{i+1,j}) + (1/36)2\Delta x\Delta z(P_{i-1,j+2} + 4P_{i,j+2} + P_{i+1,j+2}) \quad . \tag{26}$$

The dispersion relation for the weighted-averaging finite-element method is obtained by substituting equations (13), (17) and (23) into equation (11).

The phase and group velocities can be obtained from the dispersion relation. The phase and group velocities are defined as

$$v_{ph} = \omega/k \quad , \tag{27}$$

and

$$v_{gr} = d\omega/dk \quad , \tag{28}$$

where v_{ph} and v_{gr} are the numerical phase and group velocities, respectively, ω is the angular frequency and k is the wavenumber. Substituting equation (11) into equations (27) and (28) gives

$$v_{ph}/v = [1/(2\pi/G)]\sqrt{[(-D_{xx}-D_{zz})/D_m]} \quad , \tag{29}$$

and

$$v_{gr}/v = [1/(2\pi/G)][1/2D_m^2] \times [\{(D'_{xx} + D'_{zz})/\Delta\}D_m - (D_{xx} + D_{zz})D'_m)/\Delta] \quad , \tag{30}$$

for the special case of $\Delta = \Delta x = \Delta z$, where G is the number of grid points per wavelength and D'_{xx}, D'_{zz} and D'_m are the first derivatives of D_{xx}, D_{zz} and D_m with respect to k. We can obtain the optimal weighting coefficients by making equations (29) and (30) be unity and by introducing the constraints $c_1 + 4c_2 + 4c_3 = 1$ and $e_1 + 4e_2 + 4e_3 + f = 1$ (e.g., Jo et al., 1996; Shin and Sohn, 1998; Štekl and Pratt, 1998). The optimal set of weighting coefficients obtained for the weighted-averaging finite-element method is

$$c_1 = 1.63034868, \qquad c_2 = 0.0663752854, \qquad c_3 = -0.223962456,$$

$$e_1 = 0.168119922, \qquad e_2 = -0.0953879654, \qquad e_3 = 0.15665926,$$

$$f = 0.586794913. \tag{31}$$

Note that the optimal weighting coefficients were determined to be independent of material properties of model.

DISPERSION ANALYSIS

We need to estimate the validity of the weighting coefficients obtained in the previous section through a dispersion relation. The dispersion relation is investigated by computing and displaying the normalized phase and group velocities for different propagation angles with respect to the grid and for different grid intervals. To compare the dispersion relation of the weighted-averaging finite-element method with those of the previous finite-element methods, we calculate the phase and group velocities by using the lumped mass, the consistent mass, the eclectic and the weighted-averaging method. Fig. 5 shows phase and group velocities computed by the lumped mass operator. In Fig. 5 we can see that the numerical phase and group velocities are very dispersive and the numerical waves propagate with smaller velocities than the real velocities for the small G. G indicates the number of grid points per wavelength. Fig. 6 shows phase and group velocities computed by the consistent mass operator. The numerical phase and group velocities are also dispersive and the numerical waves propagate with larger velocities than the real velocities for the small G. To bound the errors within 1% with the lumped and consistent mass operator, we need 33.3 grid points per wavelength. Phase and group velocities calculated by the eclectic and the weighted-averaging method are shown in Figs. 7 and 8. Since the eclectic method is the combination of the lumped and the consistent mass operator, the above effects that the numerical waves propagate with faster or slower velocities than the real velocities disappear in parts. The phase and group velocities are less dispersive. To maintain the errors within 1% by using the eclectic method, the number of grid points required per wavelength is 14. Fig. 8 shows the phase and group velocities obtained by the weighted-averaging finite-element method. The weighted-averaging finite-element method gives the best results. The weighted-averaging method requires 4 grid points per wavelength, achieving errors within 1%. From the results, we know that when we solve the problem that requires large computer memory, it is the most useful to apply the weighted-averaging finite-element method.

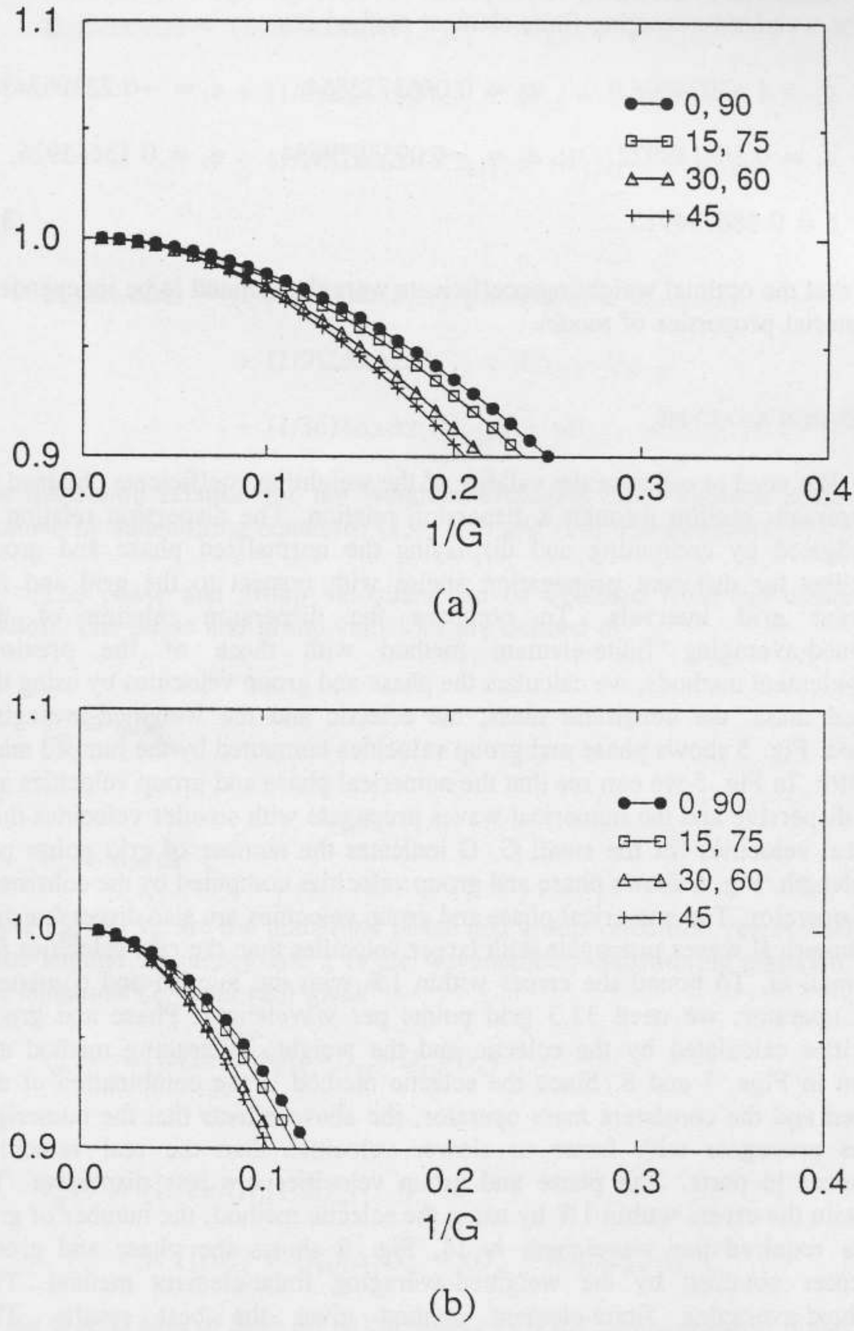
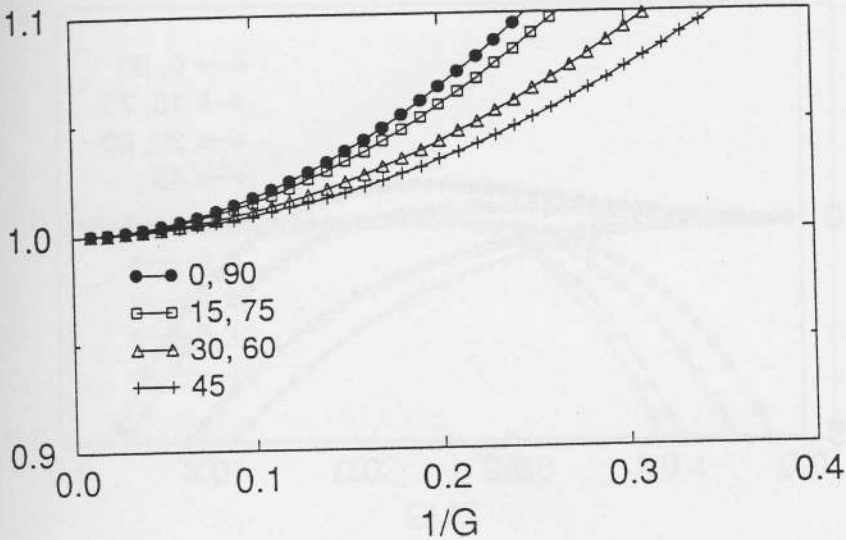
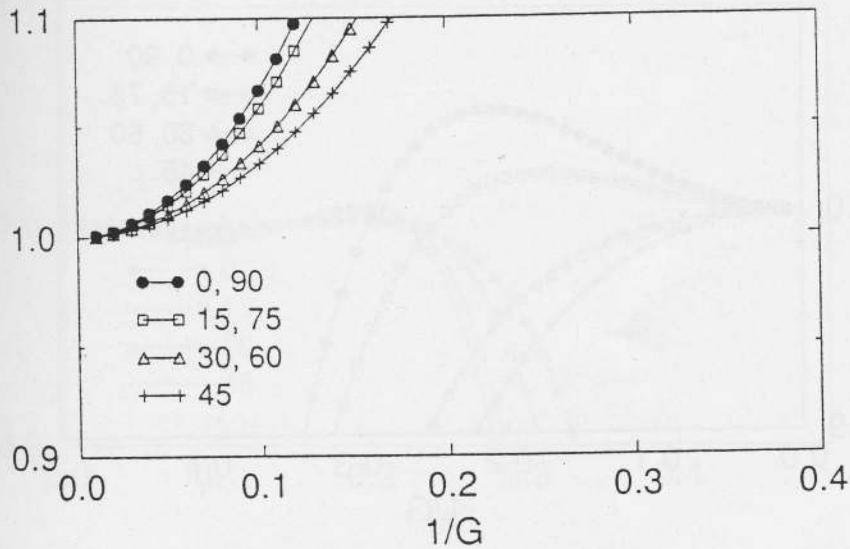


Fig. 5. Normalized (a) phase and (b) group velocities obtained by the lumped mass operator for propagation angles of 0°, 15°, 30°, 45°, 60°, 75°, and 90° with respect to the x-axis. G is the number of grid points per wavelength.

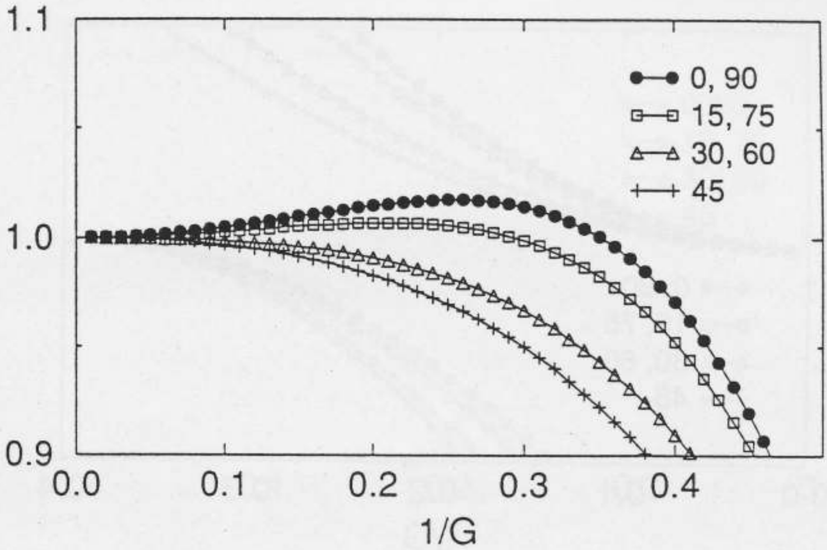


(a)

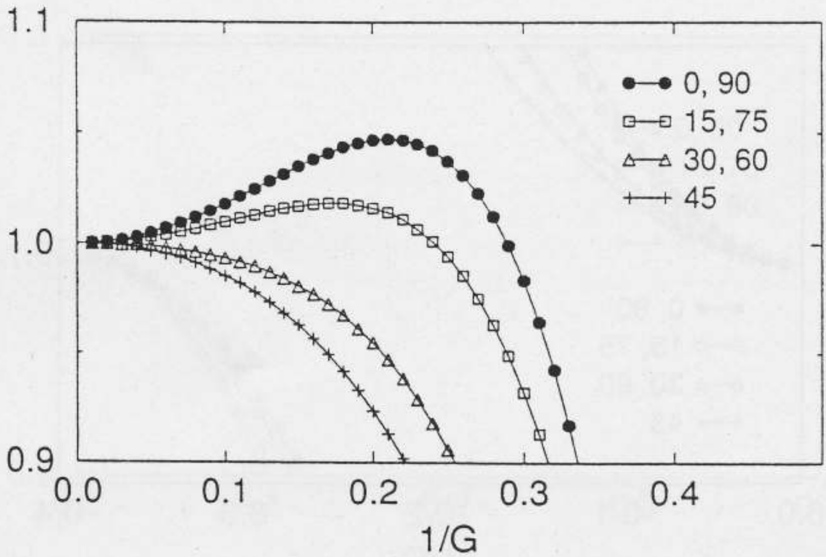


(b)

Fig. 6. Normalized (a) phase and (b) group velocities obtained by the consistent mass operator for propagation angles of 0°, 15°, 30°, 45°, 60°, 75° and 90° with respect to the x-axis. G is the number of grid points per wavelength.

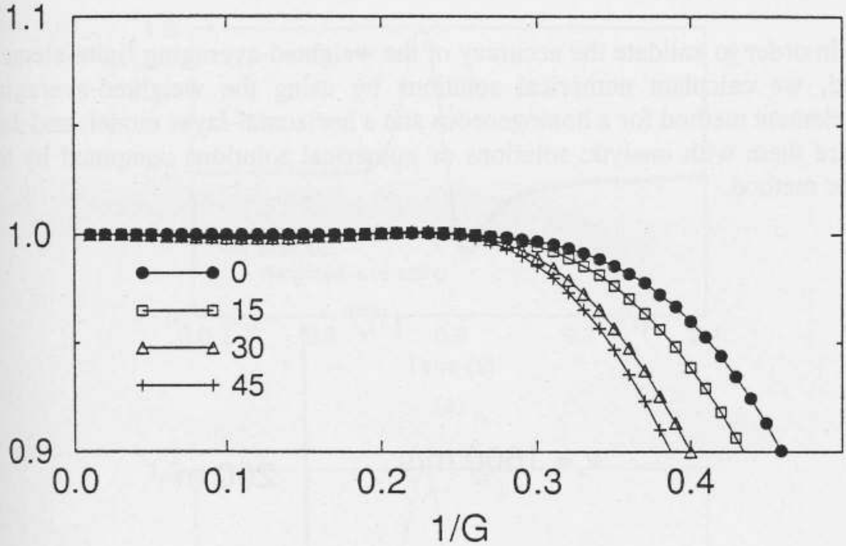


(a)

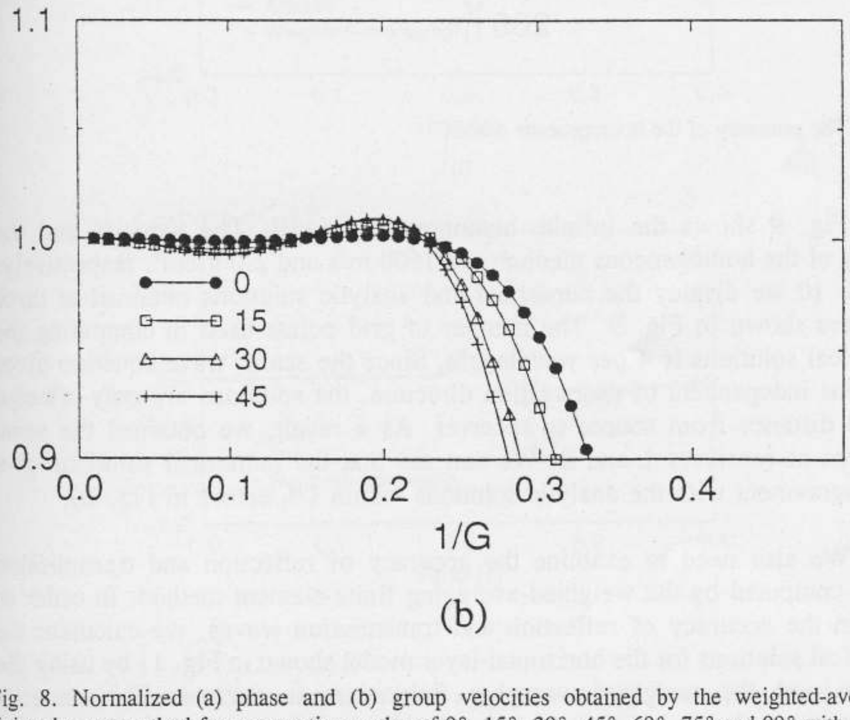


(b)

Fig. 7. Normalized (a) phase and (b) group velocities obtained by the eclectic method for propagation angles of 0°, 15°, 30°, 45°, 60°, 75°, and 90° with respect to the x-axis. G is the number of grid points per wavelength.



(a)



(b)

Fig. 8. Normalized (a) phase and (b) group velocities obtained by the weighted-averaging finite-element method for propagation angles of 0°, 15°, 30°, 45°, 60°, 75° and 90° with respect to the x-axis. G is the number of grid points per wavelength.

ACCURACY ANALYSIS

In order to validate the accuracy of the weighted-averaging finite-element method, we calculate numerical solutions by using the weighted-averaging finite-element method for a homogeneous and a horizontal-layer model, and then compare them with analytic solutions or numerical solutions computed by the eclectic method.

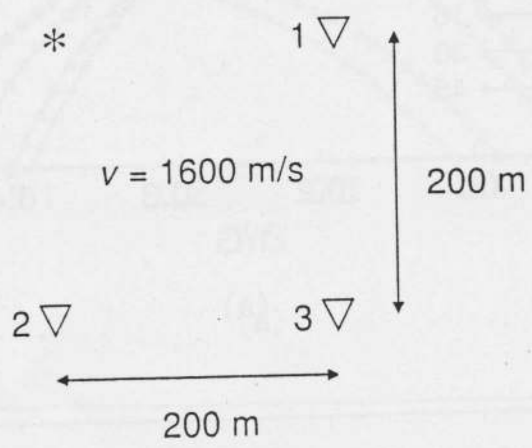


Fig. 9. The geometry of the homogeneous model.

Fig. 9 shows the infinite homogeneous model. The velocity and the density of the homogeneous medium are 1600 m/s and 2.0 g/cm^3, respectively. In Fig. 10 we display the numerical and analytic solutions obtained at three receivers shown in Fig. 9. The number of grid points used in computing the numerical solutions is 4 per wavelength. Since the scalar wave equation gives solutions independent of propagation direction, the solutions are only affected by the distance from source to receiver. As a result, we obtained the same solutions at receivers 1 and 2. We can see that the numerical solutions give good agreement with the analytic solutions within 1% errors in Fig. 10.

We also need to examine the accuracy of reflection and transmission waves computed by the weighted-averaging finite-element method. In order to confirm the accuracy of reflection and transmission waves, we calculate the numerical solutions for the horizontal-layer model shown in Fig. 11 by using the eclectic and the weighted-averaging finite-element method. The material properties are shown in Table 1. A vertical force with the maximum frequency of 30 Hz is applied at the surface. The two receivers are located at the distances

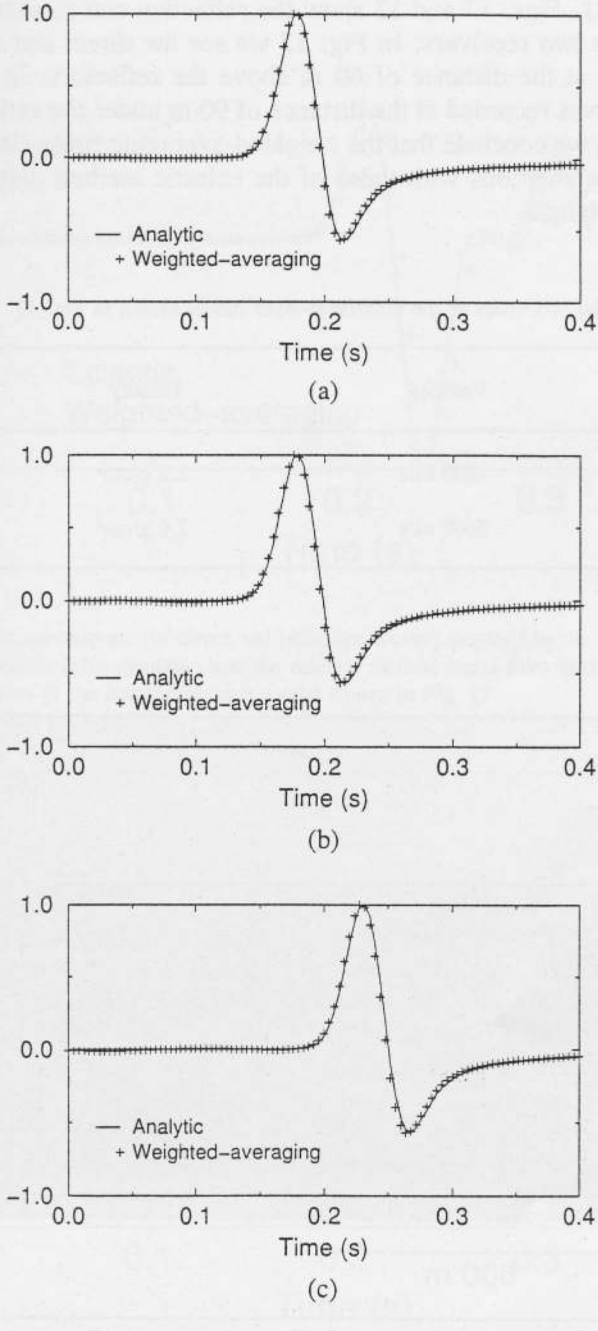


Fig. 10. Numerical solutions computed by the weighted-averaging finite-element method (plus symbols) and analytic solutions (solid line) at (a) receiver 1, (b) receiver 2, and (c) receiver 3 of Fig. 9. The number of grid points per wavelength is 4.

of 60 m above the reflectors and 90 m under the reflectors, respectively, as shown in Fig. 11. Figs. 12 and 13 show the reflection and transmission waves calculated at the two receivers. In Fig. 12 we see the direct and the reflection waves recorded at the distance of 60 m above the reflectors; in Fig. 13, the transmission waves recorded at the distance of 90 m under the reflectors. From Figs. 12 and 13, we conclude that the weighted-averaging finite-element method gives compatible solutions with those of the eclectic method only with 4 grid points per wavelength.

Table 1. The material properties of the horizontal-layer model shown in Fig. 11.

Layer	Velocity	Density
1	1800 m/s	2.2 g/cm^3
2	3600 m/s	2.6 g/cm^3

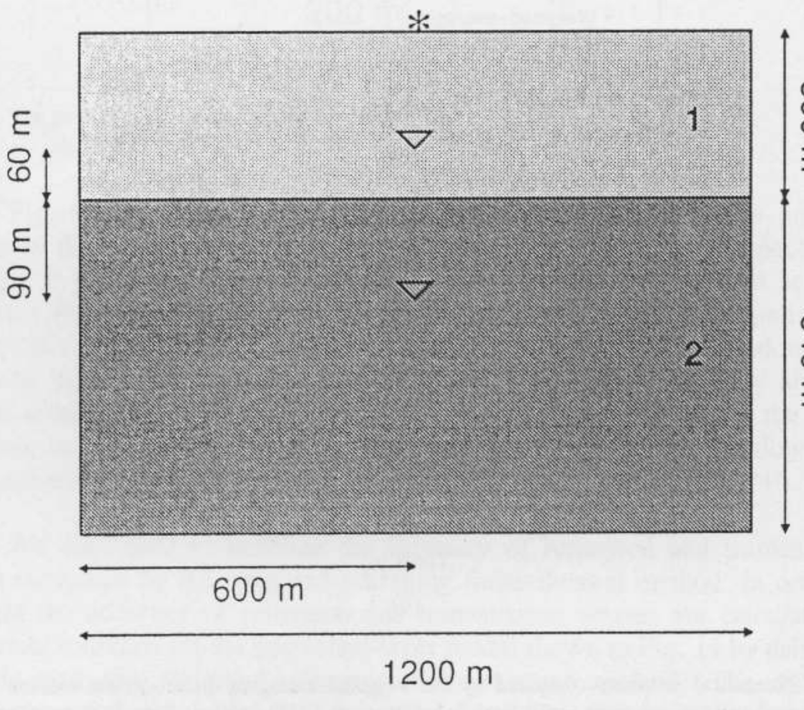


Fig. 11. The geometry of the horizontal-layer model.

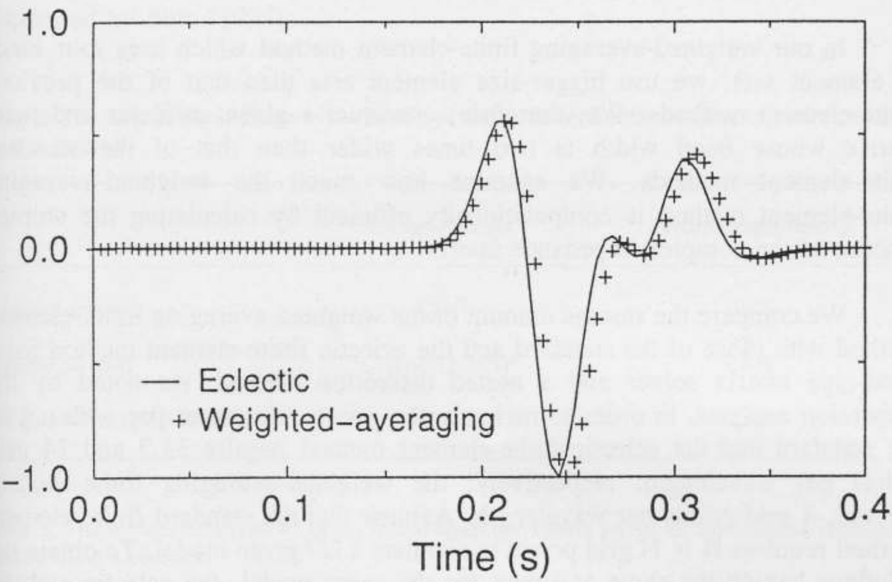


Fig. 12. Synthetic seismogram (of direct and reflection waves) obtained by the weighted-averaging finite-element method (plus symbols) and the eclectic method (solid line) at the distance of 60 m above the reflectors of the horizontal-layer model shown in Fig. 11.

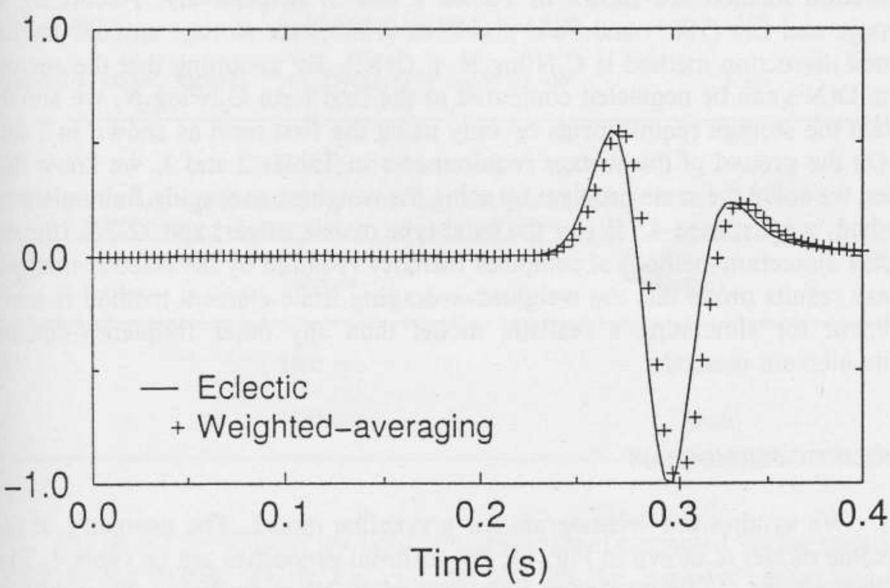


Fig. 13. Synthetic seismogram (of transmission waves) obtained by the weighted-averaging finite-element method (plus symbols) and the eclectic method (solid line) at the distance of 90 m under the reflectors of the horizontal-layer model shown in Fig. 14.

COMPUTATIONAL EFFICIENCY

In our weighted-averaging finite-element method which uses four kinds of element sets, we use bigger-size element sets than that of the previous finite-element methods. We, therefore, construct a global stiffness and mass matrix whose band width is two times wider than that of the standard finite-element methods. We examine how much the weighted-averaging finite-element method is computationally efficient by calculating the storage amount of the complex impedance matrix.

We compare the storage amount of the weighted-averaging finite-element method with those of the standard and the eclectic finite-element method for a band-type matrix solver and a nested dissection method. As noted by the dispersion analysis, in order to maintain the group velocity errors within 1%, the standard and the eclectic finite-element method require 33.3 and 14 grid points per wavelength, respectively; the weighted-averaging finite-element method, 4 grid points per wavelength. Assume that the standard finite-element method requires $N \times N$ grid points to simulate a 2D given model. To obtain the solutions having the same accuracy for the same model, the eclectic and the weighted-averaging finite-element method need $(14/33.3)N \times (14/33.3)N$ and $(4/33.3)N \times (4/33.3)N$ grid points, respectively.

The storage amounts for the band-type matrix solver and the nested dissection method are shown in Tables 2 and 3, respectively. According to George and Liu (1981) and Štekl and Pratt (1998), the storage amount for the nested dissection method is $C_2N^2\log_2N + O(N^2)$. By assuming that the second term $O(N^2)$ can be neglected compared to the first term $C_2N^2\log_2N$, we simply obtain the storage requirements by only using the first term as shown in Table 3. On the ground of the storage requirements in Tables 2 and 3, we know that when we solve the same problem by using the weighted-averaging finite-element method, we just need 4.7% (for the band-type matrix solver) and 32.7% (for the nested dissection method) of computer memory required by the eclectic method. These results prove that the weighted-averaging finite-element method is more efficient for simulating a realistic model than any other frequency-domain finite-element methods.

SYNTHETIC SEISMOGRAM

We synthesized seismogram for a syncline model. The geometry of the syncline model is shown in Fig. 14; the material properties are in Table 4. The vertical source with a maximum frequency of 40 Hz is excited in the middle of surface; The receivers are spread on the surface. The spatial grid spacings are $\Delta x = \Delta z = 15.625$ m and the numbers of spatial grid points for horizontal and vertical directions, $N_x = 154$ and $N_z = 129$. To remove edge reflections resulting from the finite-size model, we applied the sponge boundary condition developed by Shin (1995).

Table 2. The storage requirements of the band-type matrix solver for the complex impedance matrix. C_1 is a constant.

	Storage Amount	Percent Improvement
Standard method	$C_1 N^3$	100%
Eclectic method	$C_1 [(14/33.3)N]^3$	7.4%
Weighted-averaging method	$2C_1 [(14/33.3)N]^3$	0.35%

Table 3. The storage requirements of the nested dissection matrix solver for the complex impedance matrix. C_2 is a constant.

	Storage Amount	Percent Improvement
Standard method	$C_2 N^2 \log_2 N$	100%
Eclectic method	$C_2 [(14/33.3)N]^2 \log_2(14/33.4)N$	17.7
Weighted-averaging method	$4C_2 [(4/33.3)N]^2 \log_2(4/33.4)N$	5.8%

Table 4. The material properties of the syncline model shown in Fig. 14.

Layer	Velocity	Density
1	2500 m/s	2.3 g/cm^3
2	4000 m/s	2.7 g/cm^3

In Fig. 15, we display the synthetic seismogram computed for the syncline model shown in Fig. 14. In Fig. 15, we can see the "bow-tie" generated by synclinal structure. Although we use different-size element sets, no ringing reflections are observed. From this result, we know that the weighted-averaging finite-element method can be applicable to any model.

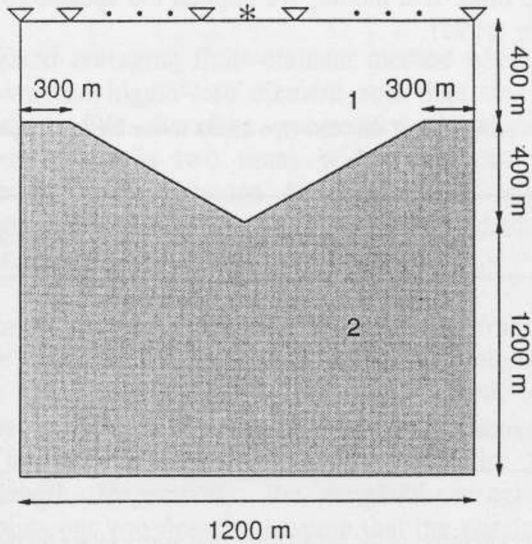


Fig. 14. The geometry of the syncline model.

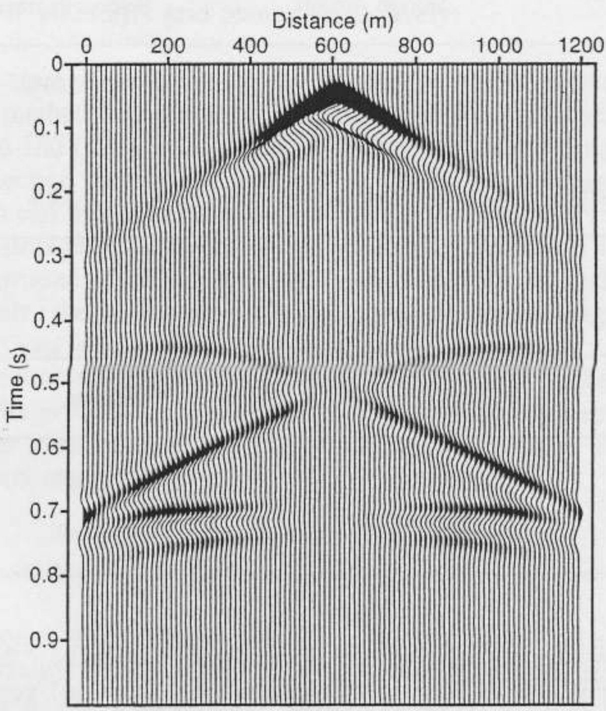


Fig. 15. Synthetic seismogram obtained by the weighted-averaging finite-element method for the syncline model.

CONCLUSIONS

We developed the weighted-averaging finite-element method which uses four kinds of element sets. The method is to construct the stiffness and mass matrices for the four kinds of element sets and then average them with weighting coefficients. The weighting coefficients were determined to give the numerical phase and group velocities which are almost consistent with the real velocities. By using the weighted-averaging finite-element method, we reduced the number of grid points per wavelength from 33.3 (using the standard finite-element operator) and 14 (using the eclectic method) to 4. By substantially reducing the number of grid points per wavelength with the weighted-averaging finite-element method, we achieved a 95.3% reduction (for the band-type solver) and a 67.3% reduction (for the nested dissection method) in the storage amount of the complex impedance matrix. From comparing the numerical solutions obtained by using the weighted-averaging finite-element method for the homogeneous and horizontal-layer models with the analytic solutions or the numerical solutions calculated by the eclectic method, we concluded that the weighted-averaging finite-element method gives accurate solutions for all of the direct, reflection and transmission waves with less grid points per wavelength than those required by the previous finite-element methods. From the synthetic seismogram generated for the syncline model, we also found out that the weighted-averaging finite-element method simulates the geological models with fewer grid points than the previous frequency-domain finite-element methods and is applicable to any model.

ACKNOWLEDGEMENT

This work was financially supported by grant No. R05-2000-00003 from the Basic Research Program of the Korea Science & Engineering Foundation, grant No. PM10300 from Korea Ocean Research & Development Institute, National Laboratory Project of Ministry of Science and Technology, and Brain Korea 21 project of the Ministry of Education.

REFERENCES

Alford, R.M., Kelly, K.R. and Boore, D.M., 1974, Accuracy of finite-difference modeling of the acoustic wave equation. Geophysics, 39: 834-842.

George, A. and Liu, J.W., 1981. Computer Solution of Large Sparse Positive Definite Systems. Prentice-Hall, Inc., New York.

Holberg, O., 1987, Computational aspects of the choice of operator and sampling interval for numerical differentiation in large-scale simulation of wave phenomena. Geophys. Prosp., 35: 629-655.

Jo, C.H., Shin, C. and Suh, J.H., 1996. An optimal 9-point, finite-difference, frequency-space, 2-D scalar wave extrapolator. Geophysics, 61: 529-537.

Marfurt, K.J., 1984. Accuracy of finite-difference and finite-element modeling of the scalar and elastic wave equations. Geophysics, 49: 533-549.

Min, D.-J., Shin, C., Kwon, B.-D. and Chung, S., 2000. Improved frequency-domain elastic modeling using weighted-averaging difference operators. Geophysics, 65: 884-895.

Pratt, R.G., 1990a. Inverse theory applied to multi-source cross-hole tomography. Part II: Elastic wave-equation method. Geophys. Prosp., 38: 287-310.

Pratt, R.G., 1990b. Frequency-domain elastic wave modeling by finite differences: a tool for crosshole seismic imaging. Geophysics, 55: 626-632.

Pratt, R.G., 1999. Seismic waveform inversion in the frequency domain. Part I: Theory and verification in a physical scale model. Geophysics, 64: 888-901.

Pratt, R.G. and Worthington, M.H., 1990. Inverse theory applied to multi-source cross-hole tomography. Part I: Acoustic wave-equation method. Geophys. Prosp., 38: 287-310.

Shin, C., 1995. Sponge boundary condition for frequency-domain modeling. Geophysics, 60: 1870-1874.

Shin, C. and Sohn, H.J., 1998. A frequency-space 2D scalar wave extrapolator using extended 25-point finite-difference operators. Geophysics, 63: 289-296.

Štekl, I. and Pratt, R.G., 1998. Accurate viscoelastic modeling by frequency-domain finite differences using rotated operators. Geophysics, 63: 1779-1794.