Weighted Calderón-Zygmund and Rellich inequalities in L^p

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In 1956, Rellich proved the inequalities

$$\left(\frac{N(N-4)}{4}\right)^2 \int_{\mathbb{R}^N} |x|^{-4}|u|^2 \, dx \leq \int_{\mathbb{R}^N} |\Delta u|^2 \, dx$$

for $N \neq 2$ and for every $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$. These inequalities have been then extended to L^p-norms: in 1996, Okazawa proved the validity of

$$\left(\frac{N}{p}-2\right)^p \left(\frac{N}{p'}\right)^p \int_{\mathbb{R}^N} |x|^{-2p}|u|^p \, dx \leq \int_{\mathbb{R}^N} |\Delta u|^p \, dx$$

for $1 < p < \frac{N}{2}$. Weighted Rellich inequalities have also been studied. In 1998, Davies and Hinz obtained for $N \geq 3$ and for $2 - \frac{N}{p} < \alpha < 2 - \frac{2}{p}$

$$C(N,p,\alpha) \int_{\mathbb{R}^N} |x|^{(\alpha-2)p}|u|^p \, dx \leq \int_{\mathbb{R}^N} |x|^{\alpha p}|\Delta u|^p \, dx \tag{1}$$

with the optimal constants $C(N,p,\alpha) = \left(\frac{N}{p}-2+\alpha\right)^p \left(\frac{N}{p'}-\alpha\right)^p$. Later Mitidieri showed that (1) holds in the wider range $2 - \frac{N}{p} < \alpha < N - \frac{N}{p}$ and with the same constants.

In a recent paper, in 2012, Caldiroli and Musina improved weighted Rellich inequalities for $p = 2$ by giving necessary and sufficient conditions on α for the validity of (1) and finding also the optimal constants $C(N,2,\alpha)$. In particular they proved that (1) is verified for $p = 2$ if and only if $\alpha \neq N/2 + n$, $\alpha \neq -N/2 + 2 - n$ for every $n \in \mathbb{N}_0$. Similar results have been also obtained by Ghoussoub and Moradifam under the restriction $\alpha \geq (4 - N)/2$ and with different methods.

We extend Caldiroli-Musina result to $1 \leq p \leq \infty$, computing also best constants in some cases. We show that (1) holds if and only if $\alpha \neq N/p' + n$, $\alpha \neq -N/p + 2 - n$ for every $n \in \mathbb{N}_0$. Moreover, we use Rellich inequalities to find necessary and sufficient conditions for the validity of weighted Calderón-Zygmund estimates when $1 < p < \infty$

$$\int_{\mathbb{R}^N} |x|^{\alpha p}|D^2 u|^p \, dx \leq C \int_{\mathbb{R}^N} |x|^{\alpha p}|\Delta u|^p \, dx \tag{2}$$

for $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$. We find that (2) holds if and only if $\alpha \neq N/p' + n$ for every $n \in \mathbb{N}_0$ and , $\alpha \neq -N/p + 2 - n$ for every $n \in \mathbb{N}, n \geq 2$.