Research Article

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Weighted CBMO estimates for commutators of matrix Hausdorff operator on the Heisenberg group

https://doi.org/10.1515/math-2020-0175
received August 7, 2019; accepted April 28, 2020

Abstract: In this article, we study the commutators of Hausdorff operators and establish their boundedness on the weighted Herz spaces in the setting of the Heisenberg group.

Keywords: Hausdorff operator, commutators, Herz space, CBMO function, Heisenberg group, weights

MSC 2010: 42B35, 42B30, 46E30, 22E25

1 Introduction

The matrix Hausdorff operators defined on n-dimensional Euclidean space $\mathbb{R}^n$ was first reported in [1], in the following form:

$$H_{\Phi,A}f(x) = \int_{\mathbb{R}^n} \Phi(y) f(xA(y))\mathrm{d}y. \tag{1.1}$$

Taking into consideration the duality of the Hardy space H^1 and bounded mean oscillation (BMO) space, Lerner and Liflyand in [1] have shown that $H_{\Phi,A}$ is bounded on Hardy spaces. Subsequently, similar boundedness of $H_{\Phi,A}$ was reconsidered in [2] using atomic decomposition of Hardy spaces. The above cited publications are important as their results are the first attempts to study the high-dimensional Hausdorff operators on $H^1(\mathbb{R}^n)$. Recently, Liflyand and Miyachi [3] extended these results on $H^p(\mathbb{R}^n)$ spaces with $0 < p < 1$.

In 2012, Chen et al. [4] modified the form of (1.1) by replacing the kernel function $\Phi(y)$ with $\Phi(y)/|y|^n$:

$$H_{\Phi,A}f(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} f(xA(y))\mathrm{d}y. \tag{1.2}$$

As a subcase, when $A(y) = \operatorname{diag}[1/|y|, 1/|y|, \ldots, 1/|y|]$, they give another definition of the n-dimensional Hausdorff operator:

$$H_{\Phi}f(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} f\left(\frac{x}{|y|}\right)\mathrm{d}y. \tag{1.3}$$

Their results include the boundedness of Hausdorff operators on Hardy spaces, local Hardy spaces, Herz and Herz-type Hardy spaces with a conclusion that these operators have better performance on Herz-type

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Hardy spaces than their performance on Hardy spaces. In the same year, with different co-authors, Chen et al. [5] extended the problem on the boundedness of $H_{\Phi,A}$ to the product of Hardy-type spaces. The boundedness results regarding Hausdorff operators on $H^1(\mathbb{R}^n)$ were improved in [6]. The continuity of (1.2) on Morrey spaces, Hardy-Morrey spaces, Block spaces and rectangularly defined spaces has also been discussed in [7], [8], [9] and [10], respectively. Similarly, some results regarding the boundedness of H_Φ can be found in [11–13].

In the same way, the study of commutators to integral operators is important as it has many applications in the theory of partial differential equations and in characterizing function spaces (see, for instance, [14–16]). An attempt has been made in [17] to discuss the boundedness of commutators of $H_{\Phi,A}$, defined by:

$$H^b_{\Phi,A}f(x) = \int\limits_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n}(b(x) - b(xA(y)))f(xA(y))\mathrm{d}y, \tag{1.4}$$

on function spaces when the symbol function b is either from the Lipschitz space or central BMO space. However, when the matrix $A(y)$ is diagonal, we get the commutators of H_Φ which were studied in [18–20]. For detailed history and other developments regarding Hausdorff operators, we refer the interested readers to the review articles [21,22].

Besides the Euclidean space $\mathbb{R}^n$, the matrix Hausdorff operator can be defined on the p-adic linear space $\mathbb{Q}_p^n$, which is a locally compact commutative group under addition (see, for instance, [23,24]), and on the Heisenberg group $\mathbb{H}^n$ [25–27]. Since, we are mainly concerned with the study of the commutators of Hausdorff operators defined on the Heisenberg group $\mathbb{H}^n$, therefore, it is mandatory to introduce this group briefly and the definition of matrix Hausdorff operators on it first.

With underlying manifold $\mathbb{R}^{2n} \times \mathbb{R}$, the Heisenberg group $\mathbb{H}^n$ is the Lie group under the law of non-commutative multiplication

$$x\cdot y = (x_1, x_2, \ldots, x_{2n+1})\cdot(y_1, y_2, \ldots, y_{2n+1}) = \left(x_1 + y_1,\ldots,x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} + 2\sum_{j=1}^{n}(y_jx_{n+j} - x_jy_{n+j})\right).$$

The above definition suggests that for $x \in \mathbb{H}^n$, we have $x\cdot 0 = x$ and $x\cdot -x = 0$. Therefore, the identity and inverse elements of $\mathbb{H}^n$ are same as that of $\mathbb{R}^{2n+1}$ Euclidean space. The basis for the corresponding Lie algebra is formed by the vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2x_{n+j}\frac{\partial}{\partial x_{2n+1}}, \quad 1 \le j \le n, \quad X_{n+j} = \frac{\partial}{\partial x_{n+j}} - 2x_j\frac{\partial}{\partial x_{2n+1}}, \quad 1 \le j \le n, \quad X_{2n+1} = \frac{\partial}{\partial x_{2n+1}}.$$

The only non-vanishing commutator relations satisfied by these vector fields are

$$[X_j, X_{n+j}] = -4X_{2n+1}, \quad 1 \le j \le n.$$

The dilation, on the Heisenberg group $\mathbb{H}^n$, is defined as

$$\delta_r(x_1, x_2, \ldots, x_{2n}, x_{2n+1}) = (rx_1, rx_2, rx_{2n}, r^2x_{2n+1}), \quad r > 0.$$

Also, the Haar measure on $\mathbb{H}^n$ coincides with the usual Lebesgue measure on $\mathbb{R}^{2n} \times \mathbb{R}^1$. Thus, for any measurable set $E \subset \mathbb{H}^n$, we denote its measure by $|E|$. Moreover, it is easy to see that

$$|\delta_r(E)| = r^Q|E|, \quad d(\delta_r x) = r^Q\mathrm{d}x,$$

where $Q = 2n + 2$ is the so-called homogeneous dimension of $\mathbb{H}^n$.

The Heisenberg group is a homogeneous group with the norm:

$$|x|_h = \left[\left(\sum_{i=1}^{2n} x_i^2\right)^2 + x_{2n+1}^2\right]^{1/4},$$

and the Heisenberg distance d, generated by this norm is given by

$$d(p,q) = d(q^{-1}p, 0) = |q^{-1}p|_h.$$

Note that d satisfies triangular inequality and is left-invariant in the sense that

$$d(r \cdot p, r \cdot q) = d(p,q), \quad \forall\ p,q,r \in \mathbb{H}^n.$$

The ball and sphere on $\mathbb{H}^n$, for $r > 0$ and $x \in \mathbb{H}^n$, can be defined as

$$B(x,r) = \{y \in \mathbb{H}^n : d(x,y) < r\}$$

and

$$S(x,r) = \{y \in \mathbb{H}^n : d(x,y) = r\},$$

respectively. To compute the measure of this ball on $\mathbb{H}^n$, we proceed as follows:

$$|B(x,r)| = |B(0,r)| = \Omega_Q r^Q,$$

where Ω_Q, being a function of n only, is the volume of the unit ball $B(0,1)$. Also, the area of unit sphere $S(0,1)$ on $\mathbb{H}^n$ is $w_Q = Q\Omega_Q$. For further readings on the Heisenberg group, we refer the interested reader to the book by Folland and Stein [28] and previous studies [29–31].

Now, we are in position to define the Hausdorff operator and its commutators on the Heisenberg group $\mathbb{H}^n$. Let Φ be a locally integrable function on $\mathbb{H}^n$. The Hausdorff operators on $\mathbb{H}^n$ are defined by:

$$T_\Phi f(x) = \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} f\left(\delta_{|y|_h^{-1}} x\right) \mathrm{d}y, \quad T_{\Phi,A} f(x) = \int_{\mathbb{H}^n} \frac{\Phi(y)}{|y|_h^Q} f(A(y)x) \mathrm{d}y,$$

where $A(y)$ is a matrix-valued function, and we assume that $\det A(y) \neq 0$ almost everywhere in the support of Φ. Also, we define the commutators $T^b_{\Phi,A}$ of $T_{\Phi,A}$ with locally integrable function b as

$$T^b_{\Phi,A}(f) = bT_{\Phi,A}(f) - T_{\Phi,A}(bf). \tag{1.5}$$

In this article, we will study the boundedness of $T^b_{\Phi,A}$ on the weighted Herz spaces $\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n; w)$, defined in Section 2, with the Heisenberg group as underlying space. Section 2 contains some basic definitions and notations likewise some necessary propositions which will be used in the succeeding sections. Finally, Section 3 is reserved for the main results of this study along with their proofs.

2 Some definitions and notations

In 1972, Muckenhoupt [32] studied the Hardy-Littlewood maximal function on weighted L^p spaces and introduced the theory of A_p weights as a result. The theory was well studied in the later work by García-Cuerva and Rubio de Francia [33]. An extension of this theory, in the settings of the Heisenberg group $\mathbb{H}^n$, was provided in [29] and studied in [30,31]. Any non-negative, locally integrable function w on $\mathbb{H}^n$ can be given the role of a weight. The notation $w(E)$ serves to define weighted measure of $E \subset \mathbb{H}^n$, that is, $w(E) = \int_E w(x)\mathrm{d}x$. Also, if p and p' satisfy $1/p + 1/p' = 1$, then they will be called mutually conjugate indices. Next, let us recall some basic definitions and properties of A_p weights on the Heisenberg group which will be used in the sequel.

Definition 2.1. We say that w belongs to the Muckenhoupt class $A_p(\mathbb{H}^n)$, $1 < p < \infty$, if there exists a $C > 0$ such that for every ball $B \subset \mathbb{H}^n$,

$$\left(\frac{1}{|B|}\int_B w(x)\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B w(x)^{-p'/p}\mathrm{d}x\right)^{p/p'} \leq C.$$

Also, $w \in A_1$ if there exists a constant $C > 0$ such that for every ball $B \subset \mathbb{H}^n$,

$$\left(\frac{1}{|B|}\int_B w(x)\mathrm{d}x\right) \le C \operatorname*{essinf}_{x\in B} w(x).$$

When $p = \infty$, we define $A_\infty = \cup_{1\le p<\infty} A_p$.

According to Proposition 2.2 in [25], we have $A_p(\mathbb{H}^n) \subset A_q(\mathbb{H}^n)$, for $1 \le p < q < \infty$, and if $w \in A_p(\mathbb{H}^n)$, $1 < p < \infty$, then there is an $\varepsilon > 0$ such that $p - \varepsilon > 1$ and $w \in A_{p-\varepsilon}(\mathbb{H}^n)$. Therefore, we may use $q_w := \inf\{q > 1 \colon w \in A_q\}$ to denote the critical index of w.

Definition 2.2. We say that w belongs to the reverse Hölder class $RH_r(\mathbb{H}^n)$, if there exists a fixed constant $C > 0$ and $r > 1$, such that for every ball $B \subset \mathbb{H}^n$,

$$\left(\frac{1}{|B|}\int_B w^r(x)\mathrm{d}x\right)^{1/r} \le \frac{C}{|B|}\int_B w(x)\mathrm{d}x.$$

In [31], it was proved that $w \in A_\infty(\mathbb{H}^n)$ if and only if there exist some $r > 1$ such that $w \in RH_r(\mathbb{H}^n)$. In addition, if $w \in RH_r(\mathbb{H}^n)$, $r > 1$, then for some $\varepsilon > 0$ we have $w \in RH_{r+\varepsilon}(\mathbb{H}^n)$. We therefore use $r_w := \sup\{r > 1 \colon w \in RH_r(\mathbb{H}^n)\}$ to denote the critical index of w for the reverse Hölder condition.

A particular case of Muckenhoupt $A_p(\mathbb{H}^n)$ weights is the power weight function $|x|_h^\alpha$. From Proposition 2.3 in [25], for $x \in \mathbb{H}^n$, we have $|x|_h^\alpha \in A_1(\mathbb{H}^n)$ if and only if $-Q < \alpha \le 0$. Also, for $1 < p < \infty$, $|x|_h \in A_p(\mathbb{H}^n)$, if and only if $-Q < \alpha < Q(p-1)$. In view of these observations, it is easy to see that for $0 < \alpha < \infty$,

$$|x|_h^\alpha \in \bigcap_{\frac{Q+\alpha}{Q}<p<\infty} A_p(\mathbb{H}^n),$$

where $(Q+\alpha)/Q$ is known as the critical index of $|x|_h^\alpha$.

The following two Propositions, proved in [25], concerning $A_p(\mathbb{H}^n)$ weights will be useful in establishing weighted estimates for $T^b_{\Phi,A}$ on Herz-type spaces on $\mathbb{H}^n$.

Proposition 2.3. *Let $w \in A_p \cap RH_r(\mathbb{H}^n)$, $p \ge 1$ and $r > 1$. Then, there exist constants $C_1, C_2 > 0$ such that*

$$C_1\left(\frac{|E|}{|B|}\right)^p \le \frac{w(E)}{w(B)} \le C_2\left(\frac{|E|}{|B|}\right)^{(r-1)/r},$$

for any measurable subset E of a ball B. In general, for any $\lambda > 1$,

$$w(B(x_0, \lambda R) \le \lambda^{Qp} w(B(x_0, R)).$$

Proposition 2.4. *If $w \in A_p(\mathbb{H}^n)$, $1 \le p < \infty$, then for any $f \in L^1_{\mathrm{loc}}(\mathbb{H}^n)$ and any ball $B \subset \mathbb{H}^n$*

$$\frac{1}{|B|}\int_B |f(x)|\mathrm{d}x \le C\left(\frac{1}{w(B)}\int_B |f(x)|^p w(x)\mathrm{d}x\right)^{1/p}.$$

For any measurable set $E \subset \mathbb{H}^n$, the weighted Lebesgue space $L^p(E; w)$ is the space of all functions f satisfying the norm condition

$$\|f\|_{L^p(E;w)} = \left(\int_E |f(x)|^p w(x)\mathrm{d}x\right)^{1/p} < \infty,$$

where $1 \le p < \infty$ and w is a weight function on $\mathbb{H}^n$. When $p = \infty$, we have $L^\infty(\mathbb{H}^n; w) = L^\infty(\mathbb{H}^n)$ and $\|f\|_{L^\infty(\mathbb{H}^n;w)} = \|f\|_{L^\infty(\mathbb{H}^n)}$.

Let $B_k := \{x \in \mathbb{H}^n\colon |x|_h < 2^k\}$, $E_k = B_k/B_{k-1}$ for $k \in \mathbb{Z}$. Then, the homogeneous weighted Herz space in the setting of the Heisenberg group can be defined as follows.

Definition 2.5. [25] Let $\alpha \in \mathbb{R}$, $0 < p, q < \infty$, and w is a weight function on $\mathbb{H}^n$. The homogeneous weighted Herz space $\dot{K}_q^{\alpha,p}(\mathbb{H}^n)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{H}^n; w) := \left\{ f \in L^q_{\mathrm{loc}}(\mathbb{H}^n/\{0\}; w)\colon \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n;w)} < \infty \right\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n;w)} = \left\{ \sum_{k=-\infty}^{\infty} w(B_k)^{\alpha p/Q} \|f\|^p_{L^p(E_k;w)} \right\}^{1/p}.$$

When $w = 1$, we obtain $\dot{K}_q^{\alpha,p}(\mathbb{H}^n)$ introduced in [34]. It is easy to verify that $\dot{K}_p^{\alpha,p}(\mathbb{H}^n) = L^p(\mathbb{H}^n, |\cdot|_h^{\alpha p})$. Hence, Herz space can be considered as an extension of power weighted Lebesgue space. Some relevant papers on Herz-type spaces and Hardy spaces associated with them along with their application include [35–43].

Definition 2.6. [44] Let $1 < q < \infty$ and w be a weight function on $\mathbb{H}^n$. Then, we say a function $f \in L^q_{\mathrm{loc}}(\mathbb{H}^n; w)$ belongs to the weighted central bounded mean oscillation (CBMO) space $C\dot{M}O^q(\mathbb{H}^n; w)$ if

$$\|f\|_{C\dot{M}O^q(\mathbb{H}^n;w)} = \sup_{R>0} \left(\frac{1}{w(B(0,R))} \int_{B(0,R)} |f(x) - f_B|^q w(x) \right)^{1/q} < \infty,$$

where

$$f_B = \frac{1}{|B(0,r)|} \int_{B(0,r)} f(x)\mathrm{d}x. \tag{2.1}$$

For detailed study of CBMO space on $\mathbb{R}^n$, we refer the reader to [45,46].

Recently, weighted boundedness of matrix Hausdorff operators and their commutators defined on different underlying spaces are reported in [44,47–53].

Lemma 2.7. [25] *Suppose that the* $(2n+1) \times (2n+1)$ *matrix* M *is invertible. Then,*

$$\|M\|^{-Q} \le |\det M^{-1}| \le \|M^{-1}\|^Q, \tag{2.2}$$

where

$$\|M\| = \sup_{x \in \mathbb{H}^n, x \neq 0} \frac{|Mx|_h}{|x|_h}. \tag{2.3}$$

Also, when A_p weights are reduced to the power function, we shall use the notation $v(\cdot)$ instead of $w(\cdot)$, that is, $v(\cdot) = |\cdot|_h^\beta$. In that case, an easy computation results in:

$$v(B_k) = \int_{|x|_h \le 2^k} |x|_h^\beta \mathrm{d}x = \omega_Q 2^{k(Q+\beta)}/(\beta + Q). \tag{2.4}$$

Moreover, in the case of boundedness of $T^b_{\Phi,A}$ on the power-weighted Herz space, we shall frequently use the piecewise defined function G:

$$G(M, \delta\beta) = \begin{cases} \|M\|^{\delta\beta} & \text{if } \beta > 0, \\ \|M^{-1}\|^{-\delta\beta} & \text{if } \beta \le 0, \end{cases}$$

where M is any invertible matrix, $\alpha \in \mathbb{R}$ and δ is a positive real number. Then, it is easy to see that G satisfies:

$$G(M, \beta(1/q + 1/p)) = G(M, \beta/q)G(M, \beta/p), \tag{2.5}$$

where $p, q \in \mathbb{Z}^+$.

Proposition 2.8. *Suppose that the $(2n+1) \times (2n+1)$ matrix M is invertible. Let $\beta > -n$, $v(x) = |x|_h^\beta$ and $x \in \mathbb{H}^n$, then*

$$v(Mx) \le \begin{cases} \|M\|^\beta v(x) & \text{if } \beta > 0, \\ \|M^{-1}\|^{-\beta} v(x) & \text{if } \beta \le 0, \end{cases} = G(M, \beta)v(x).$$

From this point forward, the notations $A \preccurlyeq B$ will imply that $A \le CB$, for some $C > 0$. Similarly, for some positive constants C_1 and C_2, if $A \le C_1 B$ and $B \le C_2 A$, then we will write $A \simeq B$. Also, we set $\lambda B(0, R) = B(0, \lambda R)$, for $\lambda > 0$.

3 Main results and their proofs

This section contains the main results of this study and the relevant proofs. Our first result is as follows.

Theorem 3.1. *Let $1 \le p, q, q_1, q_2 \le \infty$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 < 0$. Suppose that $1/s = 1/q_1 + 1/q$ and $\alpha_1/Q + 1/q_1 = \alpha_2/Q + 1/q_2$. In addition, let $w \in A_1$ with the critical index r_w for the reverse Hölder condition and $s > q_2 r_w/(r_w - 1)$.*
(i) *If $1/q_1 + \alpha_1/Q \ge 0$, then for any $1 < \delta < r_w$,*

$$\|T^b_{\Phi,A} f\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)} \le K_1 \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)} \|f\|_{\dot{K}^{\alpha_1,p}_{q_1}(\mathbb{H}^n;w)},$$

where

$$\begin{aligned} K_1 &= \int\limits_{\|A(y)\|<1} \frac{|\Phi(y)|}{|y|_h^Q} (1 + |\det A^{-1}(y)|^{1/q} \|A(y)\|^{Q/q}) |\det A^{-1}(y)|^{1/q_1} \|A(y)\|^{-\alpha_1} \log \frac{2}{\|A(y)\|} dy \\ &+ \int\limits_{\|A(y)\|\ge 1} \frac{|\Phi(y)|}{|y|_h^Q} (1 + |\det A^{-1}(y)|^{1/q} \|A(y)\|^{Q/q}) |\det A^{-1}(y)|^{1/q_1} \|A(y)\|^{Q/q_1 - (\alpha_1 + Q/q_1)(\delta - 1)/\delta} \log 2\|A(y)\| dy; \end{aligned}$$

(ii) *If $\alpha_1/Q + 1/q_1 < 0$, then for any $1 < \delta < r_w$*

$$\|T^b_{\Phi,A}\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)} \le K_2 \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)} \|f\|_{\dot{K}^{\alpha_1,p}_{q_1}(\mathbb{H}^n;w)},$$

where

$$\begin{aligned} K_2 &= \int\limits_{\|A(y)\|\ge 1} \frac{|\Phi(y)|}{|y|_h^Q} (1 + |\det A^{-1}(y)|^{1/q} \|A(y)\|^{Q/q}) |\det A^{-1}(y)|^{1/q_1} \|A(y)\|^{-\alpha_1} \log 2\|A(y)\| dy \\ &+ \int\limits_{\|A(y)\|<1} \frac{|\Phi(y)|}{|y|_h^Q} (1 + |\det A^{-1}(y)|^{1/q} \|A(y)\|^{Q/q}) |\det A^{-1}(y)|^{1/q_1} \|A(y)\|^{Q/q_1 - (\alpha_1 + Q/q_1)(\delta - 1)/\delta} \log \frac{2}{\|A(y)\|} dy. \end{aligned}$$

When general weights are reduced to power weights, then the next theorem is as follows.

Theorem 3.2. *Let* $1 \le p < \infty$, $1 < q, q_1, q_2 < \infty$ *and* $\beta > -n$. *If* $1/q_2 = 1/q + 1/q_1$ *and* $1/q + \alpha_2/Q = \alpha_1/Q$, *then we have*

$$\|T^b_{\Phi,A}\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n;v)} \lesssim K_3\|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)}\|f\|_{\dot{K}^{\alpha_1,p}_{q_1}(\mathbb{H}^n;v)},$$

where K_3 *is*

$$K_3 = \begin{cases} \int\limits_{\mathbb{H}^n} \Theta(y)(1 + \log_2(\|A^{-1}(y)\|\|A(y)\|))\mathrm{d}y, & \text{if } \alpha_1 = 0, \\ \int\limits_{\mathbb{H}^n} \Theta(y)G(A^{-1}(y), \alpha_1(Q+\beta)/Q)\mathrm{d}y, & \text{if } \alpha_1 \ne 0, \end{cases}$$

and

$$\Theta(y) = \frac{|\Phi(y)|}{|y|_h^Q}|\det A^{-1}(y)|^{1/q_1}\left(\log\frac{2}{\|A(y)\|}\chi_{\{\|A(y)\|<1\}} + \log 2\|A(y)\|\chi_{\{\|A(y)\|\ge 1\}}\right)$$
$$\times G(A^{-1}(y), \beta/q_1)(1 + |\det A^{-1}(y)|^{1/q}G(A^{-1}(y), \beta/q)\|A(y)\|^{(Q+\beta)/q}).$$

3.1 Proof of Theorem 3.1

Here, we have to show that

$$\left\{\sum_{k=-\infty}^{\infty} w(B_k)^{\alpha_2 p/Q}\|T^b_{\Phi,A}f\|^p_{L^{q_2}(E_k,w)}\right\}^{1/p} \lesssim \|f\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)}.$$

By the Minkowski inequality and necessary splitting, an upper bound for the inner norm $\|T^b_{\Phi,A}f\|^p_{L^{q_2}(E_k,w)}$ can be obtained as:

$$\begin{aligned}
\|(T^b_{\Phi,A}f)\|_{L^{q_2}(E_k;w)} &= \left\|\left(\int\limits_{\mathbb{H}^n}\frac{\Phi(y)}{|y|_h^Q}(b(x) - b(A(y)x))f(A(y)x)\mathrm{d}y\right)\right\|_{L^{q_2}(E_k;w)} \\
&\le \int\limits_{\mathbb{H}^n}\frac{\Phi(y)}{|y|_h^Q}\|(b(x) - b(A(y)x))f(A(y)x)\|_{L^{q_2}(E_k;w)}\mathrm{d}y \\
&\le \int\limits_{\mathbb{H}^n}\frac{\Phi(y)}{|y|_h^Q}\|(b(x) - b_{B_k})f(A(y)x)\|_{L^{q_2}(E_k;w)}\mathrm{d}y \\
&\quad + \int\limits_{\mathbb{H}^n}\frac{\Phi(y)}{|y|_h^Q}\|(b(A(y)x) - b_{\|A(y)\|B_k})f(A(y)x)\|_{L^{q_2}(E_k;w)}\mathrm{d}y \\
&\quad + \int\limits_{\mathbb{H}^n}\frac{\Phi(y)}{|y|_h^Q}\|(b_{B_k} - b_{\|A(y)\|B_k})f(A(y)x)\|_{L^{q_2}(E_k;w)}\mathrm{d}y = I_1 + I_2 + I_3.
\end{aligned} \tag{3.1}$$

While targeting I_1, we first compute $\|(b(x) - b(A(y)x))f(A(y)x)\|_{L^{q_2}(E_k;w)}$. The condition $s > q_2 r_w/(r_w - 1)$ implies that there exist $1 < r < r_w$ such that $s = q_2 r'$. Therefore, by the Hölder inequality and the reverse Hölder condition, we have

$$\begin{aligned}
\|(b(\cdot) - b_{B_k})f(A(y)\cdot)\|_{L^{q_2}(E_k;w)} &= \left(\int\limits_{E_k}\left|(b(x) - b_{B_k})f(A(y)x)\right|^s\mathrm{d}x\right)^{1/s}\left(\int\limits_{E_k}w(x)^r\mathrm{d}x\right)^{1/rq_2} \\
&\lesssim |B_k|^{-1/s}w(B_k)^{1/q_2}\|(b(\cdot) - b_{B_k})f(A(y)\cdot)\|_{L^s(E_k)}.
\end{aligned} \tag{3.2}$$

Next, using the condition $1/s = 1/q + 1/q_1$, we can have

$$\|\big(b(\cdot) - b_{B_k}\big)f(A(y)\cdot)\|_{L^s(E_k)} \leq \|b(\cdot) - b_{B_k}\|_{L^q(B_k)}\|f(A(y)\cdot)\|_{L^{q_1}(B_k)}. \tag{3.3}$$

In second factor, on the right side of inequality (3.3), a change of variables along with Proposition 2.4 yields

$$\begin{aligned}\|f(A(y)\cdot)\|_{L^{q_1}(B_k)} &= |\det A^{-1}(y)|^{1/q_1}\left(\int\limits_{A(y)B_k}|f(x)|^{q_1}\mathrm{d}x\right)^{1/q_1}\\ &\lesssim |\det A^{-1}(y)|^{1/q_1}\ |B(0,2^k\|A(y)\|)|^{1/q_1}\\ &\quad\times\left(\frac{1}{w(B(0,2^k\|A(y)\|))}\int\limits_{B(0,2^k\|A(y)\|)}|f(x)|^{q_1}w(x)\mathrm{d}x\right)^{1/q_1}\\ &\lesssim (|\det A^{-1}(y)|\|A(y)\|^Q|B_k|)^{1/q_1}\,w(\|A(y)\|B_k)^{-1/q_1}\|f\|_{L^{q_1}(\|A(y)\|B_k;w)}.\end{aligned} \tag{3.4}$$

Similarly, the other factor on the right hand of inequality (3.3), in view of Proposition 2.4, gives

$$\|b(\cdot) - b_{B_k}\|_{L^q(B_k)} \lesssim |B_k|^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)}. \tag{3.5}$$

Inequalities (3.2)–(3.5) together yield

$$\|\big(b(\cdot) - b_{B_k}\big)f(A(y)\cdot)\|_{L^{q_2}(E_k;w)} \lesssim \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)}\|f\|_{L^{q_1}(\|A(y)\|B_k;w)}(|\det A^{-1}(y)|\|A(y)\|^Q)^{1/q_1}\frac{w(B_k)^{1/q_2}}{w(\|A(y)\|B_k)^{1/q_1}}.$$

Hence, we obtain the following estimate for I_1:

$$I_1 \lesssim \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)}\int\limits_{\mathbb{H}^n}\frac{|\Phi(y)|}{|y|_h^Q}(|\det A^{-1}(y)|\|A(y)\|^Q)^{1/q_1}\frac{w(B_k)^{1/q_2}}{w(\|A(y)\|B_k)^{1/q_1}}\|f\|_{L^{q_1}(\|A(y)\|B_k;w)}\mathrm{d}y.$$

Next, we fix to estimate I_2, which is given by

$$I_2 = \int\limits_{\mathbb{H}^n}\frac{|\Phi(y)|}{|y|_h^Q}\|\big(b\big(A\big(y\big)\cdot\big) - b_{\|A(y)\|B_k}\big)f\big(A\big(y\big)\cdot\big)\|_{L^{q_2}(E_k;w)}\mathrm{d}y.$$

Since $s = q_2r'$, therefore, we infer from (3.2) that

$$\|\big(b(A(y)\cdot) - b_{\|A(y)\|B_k}\big)f(A(y)\cdot)\|_{L^{q_2}(E_k;w)} \lesssim |B_k|^{-1/s}w(B_k)^{1/q_2}\|\big(b(A(y)\cdot) - b_{\|A(y)\|B_k}\big)f(A(y)\cdot)\|_{L^s(E_k)}. \tag{3.6}$$

Applying the change of variables formula, Proposition 2.4 and Hölder's inequality, we have

$$\begin{aligned}&\|\big(b(A(y).) - b_{\|A(y)\|B_k}\big)f(A(y).)\|_{L^s(E_k)}\\ &= |\det A^{-1}(y)|^{1/s}\left(\int\limits_{A(y)B_k}|\big(b(x) - b_{\|A(y)\|B_k}\big)f(x)|^s\mathrm{d}x\right)^{1/s}\\ &\lesssim |\det A^{-1}(y)|^{1/s}\ |\|A(y)\|B_k|^{1/s}\left(\frac{1}{w(\|A(y)\|B_k)}\int\limits_{\|A(y)\|B_k}|\big(b(x) - b_{\|A(y)\|B_k}\big)f(x)|^s w(x)\mathrm{d}x\right)^{1/s}\\ &\lesssim |\det A^{-1}(y)|^{1/s}\ |B_k|^{1/s}\|A(y)\|^{Q1/s}w(\|A(y)\|B_k)^{-1/s}\\ &\quad\times\left(\int\limits_{\|A(y)\|B_k}|b(x) - b_{\|A(y)\|B_k}|^q w(x)\mathrm{d}x\right)^{1/q}\left(\int\limits_{\|A(y)\|B_k}|f(x)|^{q_1}w(x)\mathrm{d}x\right)^{1/q_1}\\ &\lesssim |\det A^{-1}(y)|^{1/s}\ |B_k|^{1/s}\|A(y)\|^{Q1/s}w(\|A(y)\|B_k)^{-1/q_1}\|f\|_{L^{q_1}(\|A(y)\|B_k;w)}\|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)}.\end{aligned} \tag{3.7}$$

By virtue of (3.6) and (3.7), the expression for I_2 assumes the following form:

$$I_2 \preccurlyeq \|b\|_{C\dot{M}O^q(\mathbb{H}^n,w)} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} (|\det A^{-1}(y)|\|A(y)\|^Q)^{1/s} \frac{w(B_k)^{1/q_2}}{w(\|A(y)\|B_k)^{1/q_1}} \|f\|_{L^{q_1}(\|A(y)\|B_k;w)} \mathrm{d}y.$$

Now, the estimation of I_3, given by

$$I_3 = \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \|f(A(y)\cdot)\|_{L^{q_2}(E_k)} |b_{B_k} - b_{\|A(y)\|B_k}| \mathrm{d}y,$$

requires the bounds for $\|f(A(y)\cdot)\|_{L^{q_2}(E_k)}$ and $|b_{B_k} - b_{\|A(y)\|B_k}|$. First, we consider $\|f(A(y)\cdot)\|_{L^{q_2}(E_k,w)}$. In view of the condition $s = q_2 r'$, we use the Hölder inequality and the reverse Hölder condition to obtain

$$\begin{aligned} \|f(A(y)\cdot)\|_{L^{q_2}(E_k,w)} &\le \left(\int_{B_k} |f(A(y)x)|^{q_2} w(x)\mathrm{d}x \right)^{1/q_2} \le \left(\int_{B_k} |f(A(y)x)|^s \mathrm{d}x \right)^{1/s} \left(\int_{B_k} w(x)^r \mathrm{d}x \right)^{1/rq_2} \\ &\preccurlyeq |B_k|^{-1/s} w(B_k)^{1/q_2} \|f(A(y)\cdot)\|_{L^s(B_k)}. \end{aligned} \tag{3.8}$$

Furthermore, the condition $1/s = 1/q + 1/q_1$ and inequality (3.4) help us to write

$$\begin{aligned} \|f(A(y)\cdot)\|_{L^s(B_k)} &= |B_k|^{1/q} \|f(A(y)\cdot)\|_{L^{q_1}(B_k)} \\ &\preccurlyeq |B_k|^{1/s} (|\det A^{-1}(y)|\|A(y)\|^Q)^{1/q_1} w(\|A(y)\|B_k)^{-1/q_1} \|f\|_{L^{q_1}(\|A(y)\|B_k;w)}. \end{aligned} \tag{3.9}$$

We combine inequalities (3.8) and (3.9) to substitute the result in the expression for I_3, which now becomes

$$I_3 \preccurlyeq \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} (|\det A^{-1}(y)|\|A(y)\|^Q)^{1/q_1} \frac{w(B(0,2^k))^{1/q_2}}{w(\|A(y)\|(B_k))^{1/q_1}} \|f\|_{L^{q_1}(w(\|A(y)\|B_k),w)} |b_{B_k} - b_{\|A(y)\|B_k}| \mathrm{d}y.$$

Now, it turns to bound $|b_{B_k} - b_{\|A(y)\|B_k}|$. For this purpose, we split the integral as follows:

$$I_3 \preccurlyeq \int_{\|A(y)\|<1} |b_{B_k} - b_{\|A(y)\|B_k}| \Psi(y)\mathrm{d}y + \int_{\|A(y)\|\ge 1} |b_{B_k} - b_{\|A(y)\|B_k}| \Psi(y)\mathrm{d}y = I_{31} + I_{32},$$

where, for the convenience's sake, we used the following notation:

$$\Psi(y) = \frac{|\Phi(y)|}{|y|_h^Q} (|\det A^{-1}(y)|\|A(y)\|^Q)^{1/q_1} \frac{w(B(0,2^k))^{1/q_2}}{w(\|A(y)\|(B_k))^{1/q_1}} \|f\|_{L^{q_1}(\|A(y)\|B_k;w)}.$$

Further decomposition of integral for I_{31} results in:

$$I_{31} = \sum_{j=0}^{\infty} \int_{2^{-j-1}\le\|A(y)\|<2^{-j}} \Psi(y) \left\{ \sum_{i=1}^{j} |b_{2^{-i}B_k} - b_{2^{-i+1}B_k}| + |b_{2^{-j}B_k} - b_{\|A(y)\|B_k}| \right\} \mathrm{d}y.$$

The first term inside the curly brackets can be approximated using Proposition 2.4, that is,

$$\begin{aligned} |b_{2^{-i}B_k} - b_{2^{-i+1}B_k}| &\le \frac{1}{|2^{-i}B_k|} \int_{2^{-i}B_k} |b(y) - b_{2^{-i+1}B_k}|\mathrm{d}y \le \frac{1}{w(2^{-i}B_k)} \int_{2^{-i}B_k} |b(y) - b_{2^{-i+1}B_k}| w(y)\mathrm{d}y \\ &\le \frac{1}{w(2^{-i}B_k)} \left(\int_{2^{-i+1}B_k} |b(y) - b_{2^{-i+1}B_k}|^q w(y)\mathrm{d}y \right)^{\frac{1}{q}} \left(\int_{2^{-i+1}B_k} w(y)\mathrm{d}y \right)^{1/q'} \\ &\le \frac{w(2^{-i+1}B_k)}{w(2^{-i}B_k)} \left(\frac{1}{w(2^{-i+1}B_k)} \int_{2^{-i+1}B_k} |b(y) - b_{2^{-i+1}B_k}|^q w(y)\mathrm{d}y \right)^{\frac{1}{q}} \preccurlyeq \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)}. \end{aligned}$$

Similarly, for the second term inside the curly brackets in the expression of I_{31}, we have

$$|b_{2^{-j}B_k} - b_{\|A(y)\|B_k}| \lesssim \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)}.$$

Therefore, we finish the estimation of I_{31} by writing

$$I_{31} \lesssim \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)} \sum_{j=0}^{\infty} \int\limits_{2^{-j-1}\le\|A(y)\|<2^{-j}} \Psi(y)(j+1)\,dy \lesssim \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)} \int\limits_{\|A(y)\|<1} \Psi(y)\log\frac{2}{\|A(y)\|}dy.$$

In a similar fashion, the integral I_{32} gives us

$$\begin{aligned} I_{32} &= \int\limits_{\|A(y)\|\ge1} \Psi(y)|b_{B_k} - b_{\|A(y)\|B_k}|dy \\ &= \sum_{j=0}^{\infty} \int\limits_{2^j\le\|A(y)\|<2^{j+1}} \Psi(y)\left\{\sum_{i=1}^{j}|b_{2^iB_k} - b_{2^{i+1}B_k}| + |b_{2^{j+1}B_k} - b_{\|A(y)\|B_k}|\right\}dy \\ &\lesssim \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)} \int\limits_{\|A(y)\|\ge1} \Psi(y)\log 2\|A(y)\|dy. \end{aligned}$$

A combination of expressions for I_1, I_2, I_{31} and I_{32} gives

$$\begin{aligned} \|T^b_{\Phi,A}f\|_{L^{q_2}(E_k;w)} \lesssim \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)} &\int\limits_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q}(|\det A^{-1}(y)|\|A(y)\|^Q)^{1/q_1} \\ &\times(1+|\det A^{-1}(y)|^{1/q}\ \|A(y)\|^{Q/q})\frac{w(B(0,2^k))^{1/q_2}}{w(\|A(y)\|B_k))^{1/q_1}}\|f\|_{L^{q_1}(\|A(y)\|B_k;w)} \\ &\times\max\left\{\log\frac{2}{\|A(y)\|},\ \log(2\|A(y)\|)\right\}dy. \end{aligned}$$

Keeping in view the definition of the Herz space, factors containing the index k in the expression of $\Psi(y)$ are important. Therefore, to proceed further and to avoid repetition of unimportant factors relative to the Herz space, we have to modify and rename the expression for Ψ. Hence, in the remaining of this paper we shall use the following notation:

$$\tilde{\Psi}(y) = \frac{|\Phi(y)|}{|y|_h^Q}(|\det A^{-1}(y)|\|A(y)\|^Q)^{1/q_1}(1+|\det A^{-1}(y)|^{1/q}\ \|A(y)\|^{Q/q})\max\left\{\log\frac{2}{\|A(y)\|},\ \log 2\|A(y)\|\right\}.$$

Then,

$$\|T^b_{\Phi,A}f\|_{L^{q_2}(E_k;w)} \le \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)} \int\limits_{\mathbb{H}^n} \tilde{\Psi}(y)\frac{w(B_k)^{1/q_2}}{w(\|A(y)\|B_k)^{1/q_1}}\|f\|_{L^{q_1}(\|A(y)\|B_k;w)}dy.$$

Finally, we take into consideration the definition of Herz space and employ the Minkowski inequality to have

$$\begin{aligned} \|T^b_{\Phi,A}f\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)} &= \left\{\sum_{k=-\infty}^{\infty} w(B_k)^{\frac{\alpha_2 p}{Q}}\|T^b_{\Phi,A}f\|^p_{L^{q_2}(E_k;w)}\right\}^{1/p} \\ &\lesssim \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)} \int\limits_{\mathbb{H}^n} \tilde{\Psi}(y)\left\{\left(\sum_{k=-\infty}^{\infty}\frac{w(B_k)^{\alpha_2/Q+1/q_2}}{w(\|A(y)\|B_k)^{1/q_1}}\|f\|_{L^{q_1}(\|A(y)\|B_k;w)}\right)^p\right\}^{1/p}dy. \end{aligned} \tag{3.10}$$

Comparing inequality (3.10) with inequality (3.9) in [25], we found that the term inside the curly brackets is same in both these inequalities, the only difference lies in the integrands outside the curly brackets along with a constant multiple $\|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)}$ outside the integral. Therefore, inequality (3.10) can be written as:

$$\|T^b_{\Phi,A}f\|_{\dot K^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)} \lesssim \|b\|_{C\dot MO^q(\mathbb{H}^n;w)} \sum_{j=-\infty}^{\infty} \int\limits_{2^{j-1}<\|A(y)\|\le 2^j} \tilde\Psi(y)\left\{\sum_{k=-\infty}^{\infty}\left[\left(\frac{w(B_k)}{w(B_{k+j})}\right)^{\alpha_1/Q+1/q_1} \times \sum_{l=-\infty}^{j}\left(\frac{w(B_{k+j})}{w(B_{k+l})}\right)^{\alpha_1/Q} w(B_{k+l})^{\alpha_1/Q}\|f\|_{L^{q_1}(E_{k+l};w)}\right]^p\right\}^{1/p} dy, \tag{3.11}$$

where the condition $\alpha_1/Q + 1/q_1 = \alpha_2/Q + 1/q_2$ is utilized in obtaining the last inequality.

Under the stated condition that $\alpha_1 < 0$ and $l \le j$, we use Proposition 2.3 to have

$$\left(\frac{w(B_{k+j})}{w(B_{k+l})}\right)^{\alpha_1/Q} \lesssim \left(\frac{|B_{k+j}|}{|B_{k+l}|}\right)^{\alpha_1(\delta-1)/(Q\delta)} = 2^{(j-l)\alpha_1(\delta-1)/\delta}, \tag{3.12}$$

for any $1 < \delta < r_w$.

In view of Proposition 2.3, if $\alpha_1/Q + 1/q_1 \ge 0$, then

$$\left(\frac{w(B_k)}{w(B_{k+j})}\right)^{\alpha_1/Q+1/q_1} \lesssim \begin{cases} 2^{-jQ(\alpha_1/Q+1/q_1)}, & \text{if } j \le 0,\\ 2^{-jQ(\alpha_1/Q+1/q_1)(\delta-1)/\delta}, & \text{if } j > 0,\end{cases} \tag{3.13}$$

and if $\alpha_1/Q + 1/q_1 < 0$, then

$$\left(\frac{w(B_k)}{w(B_{k+j})}\right)^{\alpha_1/Q+1/q_1} \lesssim \begin{cases} 2^{jQ(\alpha_1/Q+1/q_1)(\delta-1)/\delta}, & \text{if } j \le 0,\\ 2^{jQ(\alpha_1/Q+1/q_1)}, & \text{if } j > 0,\end{cases} \tag{3.14}$$

for any $1 < \delta < r_w$.

Thus, for $\alpha_1/Q + 1/q_1 \ge 0$, from inequalities (3.11)–(3.13), for any $1 < \delta < r_w$, we have

$$\begin{aligned}\|T^b_{\Phi,A}f\|_{\dot K^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)} \lesssim{}& \|b\|_{C\dot MO^q(\mathbb{H}^n;w)} \sum_{j=-\infty}^{0} \int\limits_{2^{j-1}<\|A(y)\|\le 2^j} \tilde\Psi(y)\|A(y)\|^{-\alpha_1-Q/q_1}\\ &\times \sum_{l=-\infty}^{j} 2^{\alpha_1(j-l)(\delta-1)/\delta}\left\{\sum_{k=-\infty}^{\infty} w(B_{k+l})^{\alpha_1 p/Q}\|f\|^p_{L^{q_1}(E_{k+l};w)}\right\}^{1/p} dy\\ &+ \|b\|_{C\dot MO^q(\mathbb{H}^n;w)} \sum_{j=1}^{\infty} \int\limits_{2^{j-1}<\|A(y)\|\le 2^j} \tilde\Psi(y)\|A(y)\|^{(\alpha_1+Q/q_1)(\delta-1)/\delta}\\ &\times \sum_{l=-\infty}^{j} 2^{\alpha_1(j-l)(\delta-1)/\delta}\left\{\sum_{k=-\infty}^{\infty} w(B_{k+l})^{\alpha_1 p/Q}\|f\|^p_{L^{q_1}(E_{k+l};w)}\right\}^{1/p} dy.\end{aligned}$$

Replacing $\tilde\Psi(y)$ with its value in the above inequality, we get

$$\begin{aligned}\|T^b_{\Phi,A}f\|_{\dot K^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)} \lesssim{}& \|b\|_{C\dot MO^q(\mathbb{H}^n;w)}\|f\|_{\dot K^{\alpha_1,p}_{q_1}(\mathbb{H}^n;w)}\\ &\times\left\{\int\limits_{\|A(y)\|<1} \frac{|\Phi(y)|}{|y|_h^Q}(1+|\det A^{-1}(y)|^{1/q}\|A(y)\|^{Q/q})\,|\det A^{-1}(y)|^{1/q_1}\|A(y)\|^{-\alpha_1}\log\frac{2}{\|A(y)\|}dy\right.\\ &+\int\limits_{\|A(y)\|\ge 1}\frac{|\Phi(y)|}{|y|_h^Q}(1+|\det A^{-1}(y)|^{1/q}\|A(y)\|^{Q/q})\\ &\left.\times|\det A^{-1}(y)|^{1/q_1}\|A(y)\|^{Q/q_1-(\alpha_1+Q/q_1)(\delta-1)/\delta}\log(2\|A(y)\|)\,dy\right\}.\end{aligned}$$

This completes the proof of Theorem 3.1(i).

Similarly, when $\alpha_1/Q + 1/q_1 < 0$, by using inequalities (3.11), (3.12) and (3.14), the second part of Theorem 3.1 can be proved easily. Hence, we complete the proof of Theorem 3.1.

3.2 Proof of Theorem 3.2

Following the proof of Theorem 3.1, we write:

$$\|T^b_{\Phi,A}\|_{L^{q_2}(E_k;v)} \le J_1 + J_2 + J_3,$$

where J_1, J_2 and J_3 are similar to I_1, I_2 and I_3 in the previous theorem with $w(\cdot)$ replaced by $v(\cdot) = |\cdot|_h^{\alpha}$. Then, by using the Hölder inequality and change of variables, we obtain

$$\begin{aligned} J_1 &\le \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q}\left(\int_{E_k} |b(x) - b_{B_k}|^q v(x)\mathrm{d}x\right)^{1/q}\left(\int_{E_k} |f(A(y)x)|^{q_1} v(x)\mathrm{d}x\right)^{1/q_1}\mathrm{d}y \\ &\le v(B_k)^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} |\det A^{-1}(y)|^{1/q_1}\left(\int_{A(y)E_k} |f(z)|^{q_1} v(A^{-1}(y)z)\mathrm{d}z\right)^{1/q_1}\mathrm{d}y. \end{aligned}$$

Using Proposition 2.8, we get

$$\begin{aligned} J_1 &\le v(B_k)^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q}|\det A^{-1}(y)|^{1/q_1}\left(\int_{A(y)E_k} |f(x)|^{q_1} G(A^{-1}(y),\beta/q_1)v(x)\mathrm{d}x\right)^{1/q_1}\mathrm{d}y \\ &\le v(B_k)^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q}|\det A^{-1}(y)|^{1/q_1} G(A^{-1}(y),\beta/q_1)\|f\|_{L^{q_1}(\|A(y)\|E_k;v)}\mathrm{d}y. \end{aligned}$$

Next, the expression for J_2 can be written as:

$$J_2 = \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q}\|\left(b\left(A\left(y\right)\cdot\right) - b_{\|A(y)\|B_k}\right)f\left(A\left(y\right)\cdot\right)\|_{L^{q_2}(E_k;v)}\mathrm{d}y. \tag{3.15}$$

Changing variables and using the condition $q_2/q + q_2/q_1 = 1$, we get

$$\begin{aligned} &\|\left(b(A(y)x) - b_{\|A(y)\|B_k}\right)(f(A(y).))\|_{L^{q_2}(E_k;v)} \\ &= \left(\int_{E_k}\left|\left(b\left(A\left(y\right)x\right) - b_{\|A(y)\|B_k}\right)f\left(A\left(y\right)x\right)\right|^{q_2} v(x)\mathrm{d}x\right)^{1/q_2} \\ &= |\det A^{-1}(y)|^{1/q_2}\ G(A^{-1}(y),\beta/q_2)\left(\int_{A(y)E_k}\left|\left(b\left(x\right) - b_{\|A(y)\|B_k}\right)f\left(x\right)\right|^{q_2} v(x)\mathrm{d}x\right)^{1/q_2} \qquad (3.16) \\ &\le |\det A^{-1}(y)|^{1/q_2}\ G(A^{-1}(y),\beta/q_2)\left(\int_{\|A(y)\|B_k} |b(x) - b_{\|A(y)\|B_k}|^q v(x)\mathrm{d}x\right)^{1/q}\left(\int_{A(y)E_k} |f(x)|^{q_1} v(x)\mathrm{d}x\right)^{1/q_1} \\ &= |\det A^{-1}(y)|^{1/q_2}\ G(A^{-1}(y),\beta/q_2)v(\|A(y)\|B_k)^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)}\|f\|_{L^{q_1}(A(y)E_k;v)}. \end{aligned}$$

It is easy to see that $v(\|A(y)\|B_k) = \|A(y)\|^{Q+\beta}v(B_k)$. Using properties (2.5) and (3.16), inequality (3.15) becomes:

$$J_2 = \nu(B_k)^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{H}^n;\nu)} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} |\det A^{-1}(y)|^{1/q_2}\, G(A^{-1}(y),\beta/q)\, G(A^{-1}(y),\beta/q_1)\, \|A(y)\|^{(Q+\beta)/q} \|f\|_{L^{q_1}(A(y)E_k;\nu)}\, dy.$$

It remains to estimate J_3. A change of variables following the Hölder inequality and Proposition 2.8 gives us

$$\begin{aligned} J_3 &= \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \|(b_{B_k} - b_{\|A(y)\|B_k}) f(A(y)\cdot)\|_{L^{q_2}(E_k;\nu)}\, dy \\ &= \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \|f(A(y)x)\|_{L^{q_2}(E_k;\nu)} |b_{B_k} - b_{\|A(y)\|B_k}|\, dy \\ &\le \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} |\det A^{-1}(y)|^{1/q_2} G(A^{-1}(y),\beta/q_2)\, \nu(\|A(y)\|B_k)^{1/q} \|f\|_{L^{q_1}(A(y)E_k;\nu)} |b_{B_k} - b_{\|A(y)\|B_k}|\, dy. \end{aligned}$$

Next, if $\|A(y)\| < 1$, then there exists an integer $j \ge 0$, such that

$$2^{-j-1} \le \|A(y)\| < 2^{-j}.$$

Therefore,

$$|b_{B_k} - b_{\|A(y)\|B_k}| \le \sum_{i=1}^{j} |b_{2^{-i}B_k} - b_{2^{-i+1}B_k}| + |b_{2^{-j}B_k} - b_{A(y)B_k}| \le \|b\|_{C\dot{M}O^q(\mathbb{H}^n;\nu)} \log \frac{2}{\|A(y)\|}.$$

Similarly, for $\|A(y)\| \ge 1$, we have

$$|b_{B_k} - b_{\|A(y)\|B_k}| \le \|b\|_{C\dot{M}O^q(\mathbb{H}^n;\nu)} \log 2\|A(y)\|.$$

Hence,

$$\begin{aligned} J_3 \preccurlyeq\ & \nu(B_k)^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{H}^n;\nu)} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} |\det A^{-1}(y)|^{1/q_1}\, G(A^{-1}(y),\beta/q_2) \\ & \times G(A^{-1}(y),\beta/q)\|A(y)\|^{(Q+\beta)/q} \left(\log \frac{2}{\|A(y)\|}\chi_{\{\|A(y)\|<1\}} + \log 2\|A(y)\|\chi_{\{\|A(y)\|\ge 1\}}\right) \|f\|_{L^{q_1}(A(y)E_k;\nu)}\, dy. \end{aligned}$$

Thus, combining J_1, J_2 and J_3, we get

$$\begin{aligned} \|T^b_{\Phi,A}\|_{L^{q_2}(E_k;\nu)} \preccurlyeq\ & \nu(B_k)^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{H}^n;\nu)} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} |\det A^{-1}(y)|^{1/q_1} \\ & \times G(A^{-1}(y),\beta/q_1)(1 + |\det A^{-1}(y)|^{1/q} G(A^{-1}(y),\beta/q)\|A(y)\|^{(Q+\beta)/q}) \\ & \times \left(\log \frac{2}{\|A(y)\|}\chi_{\{\|A(y)\|<1\}} + \log 2\|A(y)\|\chi_{\{\|A(y)\|\ge 1\}}\right) \|f\|_{L^{q_1}(A(y)E_k;\nu)}\, dy. \end{aligned} \tag{3.17}$$

For the approximation of $\|f(\cdot)\|_{L^q(A(y)C_k)}$, we consider the method used in [25]. Hence, the definition of E_k and (2.2) imply that

$$A(y)E_k \subset \{x\colon \|A^{-1}(y)\|^{-1}2^{k-1} < |x|_h < \|A(y)\|2^k\}.$$

Now, there exists an integer l such that for any $y \in \operatorname{supp}(\Phi)$, we have

$$2^l < \|A^{-1}(y)\|^{-1} < 2^{l+1}. \tag{3.18}$$

Finally, the inequality $\|A^{-1}(y)\|^{-1} \le \|A(y)\|$ implies that there exists a non-negative integer m satisfying:

$$2^{l+m} < \|A(y)\| < 2^{l+m+1}. \tag{3.19}$$

We infer from (3.18) and (3.19) that:

$$\log_2(\|A(y)\|\|A^{-1}(y)\|/2) < m < \log_2(2\|A(y)\|\|A^{-1}(y)\|).$$

Therefore,

$$A(y)E_k \subset \{x\colon 2^{l+k-1} < |x|_h < 2^{k+l+m+1}\}.$$

Hence,

$$\|f\|_{L^{q_1}(A(y)E_k;v)} \le \sum_{j=l}^{l+m+1} \|f\|_{L^{q_1}(E_{k+j};v)}. \tag{3.20}$$

Incorporating inequality (3.20) into inequality (3.17), we obtain

$$\|T^b_{\Phi,A}\|_{L^{q_2}(E_k;v)} \le v(B_k)^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)} \int_{\mathbb{H}^n} \Theta(y) \sum_{j=l}^{l+m+1} \|f\|_{L^{q_1}(E_{k+j};v)}\,\mathrm{d}y, \tag{3.21}$$

where

$$\begin{aligned}\Theta(y) = {}& \frac{|\Phi(y)|}{|y|_h^Q}|\det A^{-1}(y)|^{1/q_1}\left(\log\frac{2}{\|A(y)\|}\chi_{\{\|A(y)\|<1\}} + \log 2\|A(y)\|\chi_{\{\|A(y)\|\ge 1\}}\right)\\ & \times G(A^{-1}(y),\beta/q_1)(1+|\det A^{-1}(y)|^{1/q}G(A^{-1}(y),\beta/q)\|A(y)\|^{(Q+\beta)/q}).\end{aligned}$$

Using the Minkowski inequality and the condition $1/q + \alpha_2/Q = \alpha_1/Q$ yields

$$\begin{aligned}\|T^b_{\Phi,A}\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n;v)} &\lesssim \|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)}\left\{\sum_{k=-\infty}^{\infty}\left(v(B_k)^{1/q+\alpha_2/Q}\int_{\mathbb{H}^n}\Theta(y)\sum_{j=l}^{l+m+1}\|f\|_{L^{q_1}(E_{k+j};v)}\,\mathrm{d}y\right)^p\right\}^{1/p}\\ &\lesssim \|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)}\int_{\mathbb{H}^n}\Theta(y)\sum_{j=l}^{l+m+1}v(B_{-j})^{\alpha_1/Q}\left\{\sum_{k=-\infty}^{\infty}\left(v(B_{k+j})^{\alpha_1/Q}\|f\|_{L^{q_1}(E_{k+j};v)}\right)^p\right\}^{1/p}\mathrm{d}y\\ &\lesssim \|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)}\|f\|_{\dot{K}^{\alpha_1,p}_{q_1}(\mathbb{H}^n;v)}\int_{\mathbb{H}^n}\Theta(y)\sum_{j=l}^{l+m+1}v(B_{-j})^{\alpha_1/Q}\mathrm{d}y.\end{aligned}$$

It is easy to see that

$$\sum_{j=l}^{l+m+1} v(B_{-j})^{\alpha_1/Q} \simeq \sum_{j=l}^{l+m+1} 2^{-j\alpha_1(Q+\beta)/Q}.$$

Next, for $\alpha_1 = 0$,

$$\sum_{j=l}^{l+m+1} 2^{-j\alpha_1(Q+\beta)/Q} = m+2 \lesssim 1 + \log_2(\|A^{-1}(y)\|\|A(y)\|),$$

and for $\alpha_1 \ne 0$,

$$\sum_{j=l}^{l+m+1} 2^{-j\alpha_1(Q+\beta)/Q} \simeq 2^{-l\alpha_1(Q+\beta)/Q} \lesssim \begin{cases}\|A^{-1}(y)\|^{\alpha_1(Q+\beta)/Q}, & \text{if } \alpha_1>0,\\ \|A(y)\|^{-\alpha_1(Q+\beta)/Q}, & \text{if } \alpha_1<0,\end{cases} = G(A^{-1}(y),\alpha_1(Q+\beta)/Q).$$

Therefore,

$$\begin{aligned}\|T^b_{\Phi,A}\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n;v)} &\lesssim \|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)}\|f\|_{\dot{K}^{\alpha_1,p}_{q_1}(\mathbb{H}^n;v)}\begin{cases}\displaystyle\int_{\mathbb{H}^n}\Theta(y)(1+\log_2(\|A^{-1}(y)\|\|A(y)\|))\mathrm{d}y, & \text{if } \alpha_1=0,\\ \displaystyle\int_{\mathbb{H}^n}\Theta(y)G(A^{-1}(y),\alpha_1(Q+\beta)/Q)\mathrm{d}y, & \text{if } \alpha_1\ne 0,\end{cases}\\ &= K_3\|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)}\|f\|_{\dot{K}^{\alpha_1,p}_{q_1}(\mathbb{H}^n;v)}.\end{aligned}$$

Thus, we complete the proof of Theorem 3.2.

4 Conclusion

In this article, the authors obtained the boundedness of the commutator of the matrix Hausdorff operator on the homogeneous weighted Herz space in the settings of the Heisenberg group. As an application, the authors also investigated the particular case of Muckenhoupt A_p weights, namely, the power weights. Some potential directions for the future works include the boundedness of the same operator on homogeneous weighted Herz-Morrey spaces and weighted Herz-type Hardy spaces defined on homogeneous groups.

Acknowledgments: The authors would like to thank the referees for careful reading of the paper and valuable suggestions. Amjad Hussain is supported by Higher Education Commission (HEC) of Pakistan through the National Research Program for Universities (NRPU) Project No: 7098/Federal/NRPU/R&D/HEC/2017 and the Quaid-I-Azam University Research Fund (URF) Project.

Conflicts of interests: The authors declare that they have no competing interest.

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