Weighted Chebyshev Approximation for the Design of Broadband Beamformers Using Quadratic Programming

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Abstract—**A method to solve a general broadband beamformer design problem is formulated as a quadratic program. As a special case, the minimax near-field design problem of a broadband beamformer is solved as a quadratic programming formulation of the weighted Chebyshev approximation problem. The method can also be applied to the design of multidimensional digital FIR filters with an arbitrarily specified amplitude and phase. For linear phase multidimensional digital FIR filters, the quadratic program becomes a linear program. Examples are given that demonstrate the minimax near-field behavior of the beamformers designed.**

I. Introduction

The near-field design of broadband beamformers [1] is related to the design of multidimensional digital FIR filters with arbitrarily specified amplitude and phase. The minimax design of 1-D and 2-D linear phase digital FIR filters has been extensively studied [2]–[5], and a successful and flexible approach is to use a linear programming technique [4], [6], [7]. The linear programming methods as well as the exchange algorithms [2], [3] assume that the FIR filters have linear phase. For a broadband beamformer, this can be the case if we consider an equispaced linear array and if the array is designed only for the far field.

II. The Minimax Near-Field Design Problem

The transfer function from a spatial point with position vector $\mathbf{r}$ to the nth weight w_n of the broadband beamformer is denoted by $d_n(\mathbf{r}, f)$. Let $G_d(\mathbf{r}, f)$ and $G(\mathbf{r}, f)$ be the specified desired response and the actual response of the broadband beamformer, respectively, defined in space and frequency. The actual response is given by $G(\mathbf{r}, f) = \mathbf{w}^T\mathbf{d}$, where $\mathbf{w} = [w_1 \cdots w_N]^T$ is a vector of real coefficients, and $\mathbf{d} = \mathbf{d}(\mathbf{r}, f) = [d_1(\mathbf{r}, f) \cdots d_N(\mathbf{r}, f)]^T$ is the array response vector.

Define a dense grid of I spatial-frequency points, and let G_{di} and $\mathbf{d}_i$ be the functions $G_d(\mathbf{r}, f)$ and $\mathbf{d}(\mathbf{r}, f)$ evaluated at a particular spatial-frequency point for $i = 1, \ldots, I$. The minimax near-field design problem considered here is to find a coefficient vector $\mathbf{w}$ that solves the weighted Chebyshev approximation problem [2]

$$\min_{\mathbf{w} \in R^N} \max_{i \in \{1,\ldots,I\}} v_i |\mathbf{w}^T\mathbf{d}_i - G_{di}| \tag{1}$$

where the v_i's are positive weights.

III. A Quadratic Programming Formulation of a General Design Problem

Define the real penalty function

$$\varepsilon_i^2(\mathbf{w}) = |\mathbf{w}^T\mathbf{d}_i - G_{di}|^2 \tag{2}$$

for $i = 1, \ldots, I$. Expansion of (2) yields the quadratic form

$$\varepsilon_i^2(\mathbf{w}) = \mathbf{w}^T\mathbf{A}_i\mathbf{w} + \mathbf{a}_i^T\mathbf{w} + a_i \tag{3}$$

where $\mathbf{A}_i = \alpha_i\alpha_i^T + \beta_i\beta_i^T, \alpha_i = \mathrm{Re}\{\mathbf{d}_i\}, \beta_i = \mathrm{Im}\{\mathbf{d}_i\}, \mathbf{a}_i^T = -2 \cdot \mathrm{Re}\{\mathbf{d}_i^H G_{di}\}$ and $a_i = |G_{di}|^2$.

A convex quadratic program associated with a general design problem is

$$\begin{cases} \text{maximize } \delta, \text{ subject to} \\ v_i^2 \cdot \varepsilon_i^2(\mathbf{w}) + h_i \cdot \delta \le c_i \quad \text{for } i = 1, \ldots, I \end{cases} \tag{4}$$

where δ is a real variable, and the v_i's, h_i's, and c_i's are given real constants.

With all v_i's and h_i's chosen as $v_i = 1$ and $h_i \ge 0$, the problem is to find a coefficient vector $\mathbf{w}$ so that a given design specification corresponding to positive upper bounds c_i on $\varepsilon_i^2(\mathbf{w})$ is satisfied. The design problem is feasible if the quadratic program above results in an optimum δ that is nonnegative.

In order to obtain the weighted Chebyshev approximation, proceed as follows. Let all h_i's and c_i's be chosen as $h_i = 1$ and $c_i = c$, where c is an arbitrary constant. For every fixed $\mathbf{w}$, let $\delta(\mathbf{w})$ be the largest δ such that $v_i^2 \cdot \varepsilon_i^2(\mathbf{w}) + \delta \le c$ or equivalently $v_i \cdot |\mathbf{w}^T\mathbf{d}_i - G_{di}| \le \sqrt{c - \delta}$ for $i = 1, \ldots, I$. Then

$$\max_{i \in \{1,\ldots,I\}} v_i \cdot |\mathbf{w}^T\mathbf{d}_i - G_{di}| = \sqrt{c - \delta(\mathbf{w})}. \tag{5}$$

An optimum solution $(\mathbf{w}_o, \delta_o)$ to the quadratic program in (4) gives a solution $\mathbf{w}_o$ to the minimax problem in (1) since

$$\min_{\mathbf{w} \in R^N} \max_{i \in \{1,\ldots,I\}} v_i \cdot |\mathbf{w}^T\mathbf{d}_i - G_{di}| = \min_{\mathbf{w} \in R^N} \sqrt{c - \delta(\mathbf{w})} = \sqrt{c - \delta_o}. \tag{6}$$

Note that the left side of (6) is independent of the constant c and so is therefore a solution $\mathbf{w}_o$ to the quadratic program

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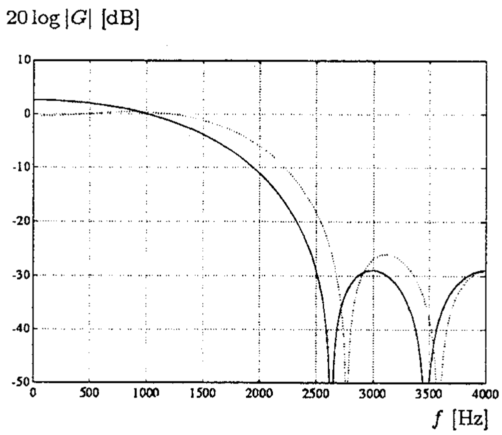

Fig. 1. Frequency response of a one-element beamformer. Solid line: minimax design. Dotted line: least-square design.

above. Note also that the matrix $\mathbf{A}_i = \alpha_i\alpha_i^T + \beta_i\beta_i^T$ is of rank 2 or less and is therefore, in general, not positive definite. Hence, the quadratic form in (3) is, in general, not strictly convex and the solution to (4) not unique.

IV. Examples of Minimax Design

In the two examples below, a sequential quadratic programming (SQP) method [8] was used to give an approximate solution to the quadratic program in (4), which was set up to solve the weighted Chebyshev approximation problem in (1).

In the first example, the quadratic programming design was applied to a one-element beamformer with a seven-tap FIR filter. The passband and stopband frequency intervals were $f = 0 - 1.5$ and $f = 2.5 - 4$ kHz, respectively, and the sampling frequency was 8 kHz. The result is shown in Fig. 1. The solid line shows a weighted Chebyshev approximation over $I = 32$ frequency points and with passband and stopband weights chosen to $v_i = 1$ and $v_i = 10$, respectively. The resulting design was almost identical to that for the Remes exchange algorithm [9]. A least-square design (dotted line) has been included for comparison.

In the second example, the quadratic programming design was applied to an equispaced linear array with five elements and a seven-tap FIR filter behind each element. The element spacing was 5 cm, and the sampling frequency was 8 kHz. The beamformer was specified on an x-axis parallel with, and 1 m in front of, the array. The passband region was defined as

$$(|x| \leq 0.4 \text{ m}, f = 0.5 - 1.5 \text{ kHz.})$$

The stopband regions were defined as

$$(|x| \leq 0.4 \text{ m}, f = 2.5 - 4 \text{ kHz})$$
$$(1.5 \leq |x| \leq 2.5 \text{ m}, f = 0.5 - 1.5 \text{ kHz})$$

and

$$(1.5 \leq |x| \leq 2.5 \text{ m}, f = 2.5 - 4 \text{ kHz.})$$

The result of a weighted Chebyshev approximation over $I = 204$ spatial-frequency points is shown in Fig. 2. The grid spacing used was 0.2 m for the x axis and 250 Hz for the f axis. The weights for the second stopband above were chosen to $v_i = 10$. All other weights were chosen to $v_i = 1$. Table I shows the resulting filter coefficients. These were obtained using the SQP method after 2896 iterations and with termination tolerance for the optimum value δ_0 set to 0.0001. Fig. 3 shows the frequency response in a least-square design.

The examples demonstrate that the weighted Chebyshev approximation can be used for broadband beamformer design in order to emphasize certain frequency-spatial regions. Further, a minimax (or equiripple) behavior is obtained for the 2-D near-field design as well as for the 1-D design.

TABLE I
FIR Coefficients in a Near-Field Chebyshev Design

ω_1–ω_7	ω_8–ω_{14}	ω_{15}–ω_{21}	ω_{22}–ω_{28}	ω_{29}–ω_{35}
0.0826	−0.9488	1.3095	−0.9499	0.0835
0.7666	−2.3487	3.1577	−2.3478	0.7652
1.2260	−3.5051	4.8366	−3.5034	1.2267
1.2235	−3.6373	5.2750	−3.6404	1.2234
1.0821	−3.1415	4.3716	−3.1391	1.0817
0.6128	−1.9969	2.6471	−1.9967	0.6127
0.0290	−0.6620	0.9083	−0.6630	0.0296

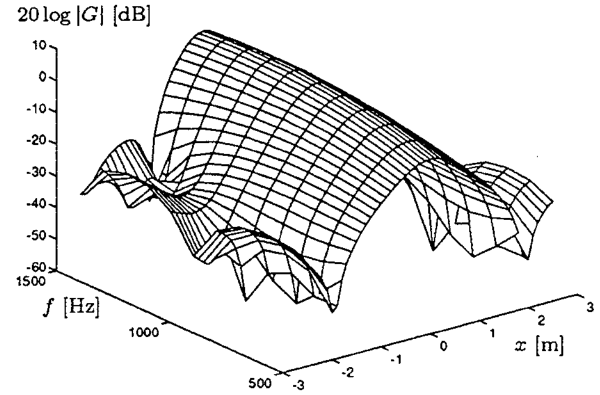

Fig. 2. Frequency response of a five-element beamformer. Weighted Chebyshev approximation.

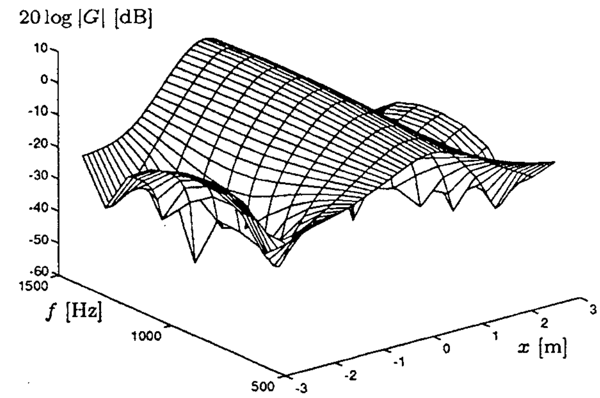

Fig. 3. Frequency response of a five-element beamformer. Least-square approximation.

V. Summary and Conclusion

In this letter, a general broadband beamformer design problem is formulated as a quadratic program. In particular, the minimax near-field design problem of a broadband beamformer is solved as a quadratic programming formulation of the weighted Chebyshev approximation problem.

In the examples given, numerical results were obtained with a standard computer program for optimization using a sequential quadratic programming (SQP) method [8]. The execution time for fairly small size problems was, however, not insignificant. Future work will therefore include investigation of optimization algorithms for this particular quadratic program, aiming at an efficient method to solve the design problem.

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