

WEIGHTED COMPOSITION OPERATORS FROM $F(p, q, s)$ INTO LOGARITHMIC BLOCH SPACE

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ABSTRACT. We characterize the boundedness and compactness of the weighted composition operator uC_φ from the general function space $F(p, q, s)$ into the logarithmic Bloch space β_L on the unit disk. Some necessary and sufficient conditions are given for which uC_φ is a bounded or a compact operator from $F(p, q, s)$, $F_0(p, q, s)$ into β_L , β_L^0 respectively.

1. Introduction

Let $D = \{z : |z| < 1\}$ be the open unit disk in the complex plane \mathbb{C} , and $H(D)$ denote the set of all analytic functions on D . For $f \in H(D)$, let

$$\|f\|_{\beta_L} = \sup\{(1 - |z|^2) \ln\left(\frac{2}{1 - |z|}\right) |f'(z)| : z \in D\}.$$

As in [6], the logarithmic Bloch space β_L consists of all $f \in H(D)$ satisfying $\|f\|_{\beta_L} < +\infty$ and the little logarithmic Bloch space β_L^0 consists of all $f \in H(D)$ satisfying $\lim_{|z| \rightarrow 1^-} (1 - |z|^2) \ln\left(\frac{2}{1 - |z|}\right) |f'(z)| = 0$. The β_L is a Banach space under the norm

$$\|f\|_L = |f(0)| + \|f\|_{\beta_L}.$$

In [6], the author proved that β_L^0 is a closed subspace and coincides with the closure of polynomials under the norm. Yoneda [8] studied the composition operator in the β_L space and the β_L^0 space.

We write β_α ($\alpha > 0$) for the space of analytic function f for which $\|f\|_{\beta_\alpha} = |f(0)| + \sup_{z \in D} (1 - |z|^2)^\alpha |f'(z)| < \infty$ and β_α^0 ($\alpha > 0$) for $f \in \beta_\alpha$ such that $\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^\alpha |f'(z)| = 0$. The functions in β_α and β_α^0 will be referred to as α -Bloch functions, little α -Bloch functions respectively. It is easily proved that for $0 < \alpha < 1$, $\beta_\alpha \subsetneq \beta_L \subsetneq \beta_1$. For more information about the β_α (see [12]).

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Let dm denote the Lebesgue area measure in D . Let $0 < p, s < \infty$ and $-2 < q < \infty$, then a function $f \in H(D)$ is said to belong to the general function space $F(p, q, s)$ (see [11]) if

$$\|f\|_{F(p,q,s)} = |f(0)| + \left\{ \sup_{a \in D} \int_D |f'(z)|^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s dm(z) \right\}^{\frac{1}{p}} < \infty$$

and the little general function space $F_0(p, q, s)$ if $f \in F(p, q, s)$ and

$$\lim_{|\alpha| \rightarrow 1^-} \int_D |f'(z)|^p (1 - |z|^2)^q (1 - |\varphi_\alpha(z)|^2)^s dm(z) = 0.$$

Here φ_a is a conformal automorphism defined by $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ for $a \in D$. We can get many function spaces if we take some specific parameters of p, q, s . For example (see [11]), $F(p, q, s) = \beta_{\frac{q+s}{p}}$ and $F_0(p, q, s) = \beta_{\frac{q+s}{p}}^0$ for $s > 1$; $F(p, q, s) \subset \beta_{\frac{q+s}{p}}$ and $F_0(p, q, s) \subset \beta_{\frac{q+s}{p}}^0$ for $0 < s \leq 1$; $F(2, 0, s) = Q_s$ and $F_0(2, 0, s) = Q_{s,0}$; $F(2, 0, 1) = BMOA$ and $F_0(2, 0, 1) = VMOA$. For more information about the $BMOA$ and $VMOA$ (see [1]). If $q + s \leq -1$, then $F(p, q, s)$ is the space of constant functions.

An analytic self-map $\varphi : D \rightarrow D$ induces the composition operator C_φ on $H(D)$, defined by $C_\varphi(f) = f(\varphi(z))$ for f analytic on D . It is a well known consequence of Littlewood's subordination principle that the composition operator C_φ is bounded on the classical Hardy and Bergman spaces (see, for example, [2, 13]).

Recall that a linear operator is said to be bounded if the image of a bounded set is a bounded set, while a linear operator is compact if it takes bounded sets to sets with compact closure. It is interesting to provide a function theoretic characterization of when φ induces a bounded or compact composition operator on various spaces. The book [2] contains plenty of information on this topic.

Let u be a fixed analytic function on the open unit disk. Define a linear operator uC_φ on the space of analytic functions on D , called a weighted composition operator, by $uC_\varphi f = u \cdot (f \circ \varphi)$, where f is an analytic function on D . We can regard this operator as a generalization of a multiplication operator and a composition operator.

In [4, 5], Ohno, Stroethoff and Zhao have characterized the boundedness and compactness of weighted composition operators between α -Bloch spaces, between H^∞ and the Bloch space β_1 , and from the little $F_0(p, q, s)$ into the little Bloch space β_1^0 . The boundedness and compactness of weighted composition operators from $F(p, q, s)$ into the Bloch space β_α were investigated in [9].

In this paper we study the weighted composition operators from the general function spaces $F(p, q, s)$ into the logarithmic Bloch space β_L and the little $F_0(p, q, s)$ into the little logarithmic Bloch space β_L^0 . Throughout this paper, constants are denoted by C , they are positive and may differ from one occurrence to the other.

2. Boundedness and compactness of $uC_\varphi : F(p, q, s) \rightarrow \beta_L$

In this section we characterize the boundedness and compactness of the weighted composition operator $uC_\varphi : F(p, q, s) \rightarrow \beta_L$. For this purpose we need some lemmas. The following lemma can be found in [11].

Lemma 2.1. *Suppose that $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$ and $f \in F(p, q, s)$. Then there is a positive constant C such that $\|f\|_{\beta_\alpha} \leq C\|f\|_{F(p, q, s)}$, moreover, if $f \in F_0(p, q, s)$, then $f \in \beta_\alpha^0$, where $\alpha = \frac{q+2}{p}$.*

Lemma 2.2 ([7]). *Suppose $t > 0$ and $f \in H(D)$. Then $\sup_{z \in D} (1 - |z|)^t |f(z)| < +\infty$ if and only if $\sup_{z \in D} (1 - |z|)^{t+1} |f'(z)| < +\infty$.*

Lemma 2.3. *Let $0 < p, s < \infty$, $-2 < q < \infty$ and $q + s > -1$. The operator $uC_\varphi : F(p, q, s) \rightarrow \beta_L$ is compact if and only if for any bounded sequence $\{f_n\}$ in $F(p, q, s)$ which converges to zero uniformly on compact subsets of D , we have $\|uC_\varphi f_n\|_L \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. Using [6, Lemma 2.1] and Montel’s Theorem, one may prove the lemma. The details are omitted here. □

Lemma 2.4 ([7]). *Let $\alpha > 0$ and $f \in \beta_\alpha$. Then*

- (1) $|f(z)| \leq C\|f\|_{\beta_\alpha}$, where $\alpha < 1$;
- (2) $|f(z)| \leq C \log\left(\frac{2}{1-|z|^2}\right)\|f\|_{\beta_\alpha}$, where $\alpha = 1$;
- (3) $|f(z)| \leq \frac{C}{(\alpha-1)(1-|z|)^{\alpha-1}}\|f\|_{\beta_\alpha}$, where $\alpha > 1$.

Theorem 2.1. *Let $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $u \in H(D)$ and φ be an analytic self-map of D .*

(i) *If $2 + q > p$, then uC_φ is a bounded operator from $F(p, q, s)$ to β_L if and only if*

$$(1) \quad \sup_{z \in D} \frac{(1 - |z|^2) \log \frac{2}{1-|z|}}{(1 - |\varphi(z)|^2)^\alpha} |u(z)\varphi'(z)| < +\infty$$

and

$$(2) \quad \sup_{z \in D} \frac{(1 - |z|^2) \log \frac{2}{1-|z|}}{(1 - |\varphi(z)|^2)^{\alpha-1}} |u'(z)| < +\infty,$$

where $\alpha = \frac{q+2}{p}$.

(ii) *If $2 + q < p$, then uC_φ is a bounded operator from $F(p, q, s)$ to β_L if and only if $u \in \beta_L$ and (1) holds.*

(iii) *If $2 + q = p$, $s > 1$, then uC_φ is a bounded operator from $F(p, q, s)$ to β_L if and only if (1) holds and*

$$(3) \quad \sup_{z \in D} |u'(z)| (1 - |z|^2) \log\left(\frac{2}{1-|z|}\right) \log\left(\frac{2}{1-|\varphi(z)|^2}\right) < \infty.$$

(iv) *Let $2 + q = p$, $0 < s \leq 1$. Then uC_φ is a bounded operator from $F(p, q, s)$ to β_L provided that (1) and (3) hold.*

Conversely, if uC_φ is a bounded operator from $F(p, q, s)$ to β_L , then $u \in \beta_L$ and (1) holds.

Proof. Suppose that uC_φ is bounded from $F(p, q, s)$ to β_L . Then we can easily obtain $u \in \beta_L$ by taking $f(z) = 1$. Let $\alpha = \frac{q+2}{p}$. Fixed $w \in D$, we consider the function

$$(4) \quad f_w(z) = \frac{(z - \varphi(w))(1 - |\varphi(w)|^2)}{(1 - \overline{\varphi(w)}z)^{\alpha+1}}.$$

Then $\|f_w\|_{F(p,q,s)} \leq C$ by [9], where C is not depended on w . Since that $f_w(\varphi(w)) = 0$ and $f'_w(\varphi(w)) = \frac{1}{(1-|\varphi(w)|^2)^\alpha}$, it follows that

$$\begin{aligned} & \frac{(1 - |w|^2) \log \frac{2}{1-|w|}}{(1 - |\varphi(w)|^2)^\alpha} |u(w)\varphi'(w)| \\ &= (1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) |u(w)f'_w(\varphi(w))\varphi'(w)| \\ &\leq \|uC_\varphi f_w\|_L \leq \|uC_\varphi\| \|f_w\|_{F(p,q,s)} \leq C \|uC_\varphi\| < +\infty, \end{aligned}$$

which showing that (1) is necessary for all case.

(i) Let $\alpha = \frac{q+2}{p} > 1$. Fix $w \in D$, we take again the test function

$$(5) \quad g_w(z) = \frac{(\alpha + 1)(1 - |\varphi(w)|^2)}{(1 - \overline{\varphi(w)}z)^\alpha} - \frac{\alpha(1 - |\varphi(w)|^2)^2}{(1 - \overline{\varphi(w)}z)^{\alpha+1}}.$$

Then we can easily prove that $\|g_w\|_{F(p,q,s)} \leq C$ by using the same methods in [9], where C is not depended on w . Since $g'_w(\varphi(w)) = 0$ and $g_w(\varphi(w)) = \frac{1}{(1-|\varphi(w)|^2)^{\alpha-1}}$, it follows that

$$\begin{aligned} & (1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) |u'(w)g_w(\varphi(w))| \\ &= (1 - |w|^2) \log \frac{2}{1 - |w|} |(uC_\varphi g_w)'(w)| \\ &\leq \|uC_\varphi g_w\|_L \leq \|uC_\varphi\| \|g_w\|_{F(p,q,s)} \leq C \|uC_\varphi\| < +\infty. \end{aligned}$$

So,

$$\sup_{w \in D} \frac{(1 - |w|^2) \log \frac{2}{1-|w|}}{(1 - |\varphi(w)|^2)^{\alpha-1}} |u'(w)| < +\infty.$$

This proves that (2) is also necessary.

Conversely, suppose that u and φ satisfy the condition in (i). For arbitrary $f \in F(p, q, s)$, by Lemma 2.1 and 2.2 we have

$$\sup_{z \in D} (1 - |z|^2)^\alpha |f'(z)| \leq C \|f\|_{F(p,q,s)}$$

and $\sup_{z \in D} (1 - |z|^2)^{\alpha-1} |f(z)| \leq C \|f\|_{F(p,q,s)}$. It follows that

$$\begin{aligned} & \|uC_\varphi f\|_{\beta_L} \\ & \leq \sup_{z \in D} (1 - |z|^2) \log\left(\frac{2}{1 - |z|}\right) |u'(z)f(\varphi(z))| \\ & \quad + \sup_{z \in D} (1 - |z|^2) \log\left(\frac{2}{1 - |z|}\right) |u(z)||f'(\varphi(z))\varphi'(z)| \\ & = \sup_{z \in D} (1 - |\varphi(z)|^2)^{\alpha-1} |f(\varphi(z))| \frac{(1 - |z|^2) \log \frac{2}{1 - |z|}}{(1 - |\varphi(z)|^2)^{\alpha-1}} |u'(z)| \\ & \quad + \sup_{z \in D} (1 - |\varphi(z)|^2)^\alpha |f'(\varphi(z))| \frac{(1 - |z|^2) \log \frac{2}{1 - |z|}}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)u(z)| \\ & \leq C \|f\|_{F(p,q,s)} \end{aligned}$$

and by Lemma 2.4,

$$|u(0)f(\varphi(0))| \leq \frac{C|u(0)|}{(\alpha - 1)(1 - |\varphi(0)|)^{\alpha-1}} \|f\|_{\beta_\alpha} \leq C \|f\|_{F(p,q,s)}.$$

Hence uC_φ is bounded from $F(p, q, s)$ to β_L .

Next we will prove (ii). Let $\alpha = \frac{q+2}{p} < 1$. Suppose $u \in \beta_L$ and that (1) holds. Assume that $f \in F(p, q, s)$. Then we have $\sup_{z \in D} |f(z)| \leq C \|f\|_{F(p,q,s)}$ by Lemma 2.1 and 2.4. It follows that

$$\begin{aligned} & \|uC_\varphi f\|_{\beta_L} \\ & \leq \sup_{z \in D} (1 - |z|^2) \log\left(\frac{2}{1 - |z|}\right) |u'(z)f(\varphi(z))| \\ & \quad + \sup_{z \in D} (1 - |z|^2) \log\left(\frac{2}{1 - |z|}\right) |u(z)||f'(\varphi(z))\varphi'(z)| \\ & \leq C \|u\|_{\beta_L} \|f\|_{F(p,q,s)} + C \|f\|_{F(p,q,s)} \sup_{z \in D} \frac{(1 - |z|^2) \log \frac{2}{1 - |z|}}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)u(z)| \\ & \leq C \|f\|_{F(p,q,s)} \end{aligned}$$

and $|u(0)f(\varphi(0))| \leq C|u(0)| \|f\|_{F(p,q,s)}$. Hence uC_φ is bounded from $F(p, q, s)$ to β_L .

Conversely, we have proved that (1) holds and $u \in \beta_L$ above.

Finally let $\alpha = \frac{q+2}{p} = 1$ and $s > 1$, we set

$$(6) \quad h_w(z) = 2 \log \frac{2}{1 - \varphi(w)z} - \frac{1}{\log \frac{2}{1 - |\varphi(w)|^2}} \left(\log \frac{2}{1 - \varphi(w)z} \right)^2$$

for $z, w \in D$. Then

$$h'_w(z) = \frac{2\overline{\varphi(w)}}{1 - \overline{\varphi(w)}z} - 2 \left(\log \frac{2}{1 - \overline{\varphi(w)}z} \right) \frac{\overline{\varphi(w)}}{(1 - \overline{\varphi(w)}z)} \frac{1}{\log \frac{2}{1 - |\overline{\varphi(w)}|^2}}.$$

By a direct calculation we get $\sup_{w \in D} \|h_w\|_{\beta_1} \leq C < +\infty$. Noting $s > 1$, thus we have $h_w \in F(p, q, s)$ and $\sup_{w \in D} \|h_w\|_{F(p, q, s)} \leq C < +\infty$ by introduction. Since $h'_w(\varphi(w)) = 0$ and $h_w(\varphi(w)) = \log \frac{2}{1-|\varphi(w)|^2}$, it follows that

$$\begin{aligned} & \sup_{w \in D} |u'(w)|(1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) \log\left(\frac{2}{1 - |\varphi(w)|^2}\right) \\ &= \sup_{w \in D} (1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) |u'(w)h_w(\varphi(w))| \\ &\leq \|uC_\varphi h_w\|_L \leq \|uC_\varphi\| \|h_w\|_{F(p, q, s)} \leq C \|uC_\varphi\| < +\infty, \end{aligned}$$

which showing that (3) is necessary.

Conversely, suppose (1) and (3) hold. Assume that $f \in F(p, q, s)$. Then $f \in \beta_1$ by Lemma 2.1. From Lemma 2.4, it follows that

$$\begin{aligned} & \|uC_\varphi f\|_{\beta_L} \\ &\leq \sup_{z \in D} (1 - |z|^2) \log\left(\frac{2}{1 - |z|}\right) |u'(z)f(\varphi(z))| \\ &\quad + \sup_{z \in D} (1 - |z|^2) \log\left(\frac{2}{1 - |z|}\right) |u(z)||f'(\varphi(z))\varphi'(z)| \\ &\leq C \|f\|_{F(p, q, s)} \sup_{z \in D} |u'(z)|(1 - |z|^2) \log\left(\frac{2}{1 - |z|}\right) \log\left(\frac{2}{1 - |\varphi(z)|^2}\right) \\ &\quad + \|f\|_{F(p, q, s)} \sup_{z \in D} \frac{(1 - |z|^2) \log \frac{2}{1 - |z|}}{(1 - |\varphi(z)|^2)^{\frac{q+s}{p}}} |\varphi'(z)u(z)| \leq C \|f\|_{F(p, q, s)} \end{aligned}$$

and

$$|u(0)f(\varphi(0))| \leq C|u(0)| \log\left(\frac{2}{1 - |\varphi(0)|^2}\right) \|f\|_{\beta_1} \leq C \|f\|_{F(p, q, s)}.$$

Hence uC_φ is bounded from $F(p, q, s)$ to β_L .

(iv) The results have been proved in case (ii) and (iii). □

Corollary 2.1. *Let $0 < p, s < \infty, -2 < q < \infty, q + s > -1$ and φ be an analytic self-map of D . Then C_φ is a bounded operator from $F(p, q, s)$ into β_L if and only if*

$$\sup_{z \in D} \frac{(1 - |z|^2) \log \frac{2}{1 - |z|}}{(1 - |\varphi(z)|^2)^{\frac{q+s}{p}}} |\varphi'(z)| < +\infty.$$

In the formulation of the corollary, we will use the notation $M(X, Y)$ to denote the set of all multipliers of X into Y : $M(X, Y) = \{u : fu \in Y \text{ for all } f \in X\}$. By M_u we denote the operator of multiplication by u : $M_u f = uf, f \in X$.

Corollary 2.2. *Let $0 < p, s < \infty, -2 < q < \infty, q + s > -1$.*

- (1) *If $2 + q \geq p$, then $M(F(p, q, s), \beta_L) = \{0\}$.*
- (2) *If $2 + q < p$, then $M(F(p, q, s), \beta_L) = \beta_L$.*

Proof. If $2 + q < p$, then M_u is bounded from $F(p, q, s)$ to β_L if and only if $u \in \beta_L$ and

$$(7) \quad \sup_{z \in D} (1 - |z|^2)^{1 - \frac{q+2}{p}} \log \frac{2}{1 - |z|} |u(z)| < \infty$$

by Theorem 2.1. However, if $u \in \beta_L$, by [6, Lemma 2.1], we get

$$|u(z)| \leq (2 + \log(\log \frac{2}{1 - |z|})) \|u\|_L.$$

Hence (7) holds, since $\frac{q+2}{p} < 1$. \square

Theorem 2.2. *Let $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $u \in H(D)$ and φ be an analytic self-map of D .*

(i) *If $2 + q > p$, then uC_φ is a compact operator from $F(p, q, s)$ to β_L if and only if $u, u\varphi \in \beta_L$,*

$$(8) \quad \lim_{|\varphi(z)| \rightarrow 1^-} \frac{(1 - |z|^2) \log \frac{2}{1 - |z|}}{(1 - |\varphi(z)|^2)^\alpha} |u(z)\varphi'(z)| = 0$$

and

$$(9) \quad \lim_{|\varphi(z)| \rightarrow 1^-} \frac{(1 - |z|^2) \log \frac{2}{1 - |z|}}{(1 - |\varphi(z)|^2)^{\alpha-1}} |u'(z)| = 0,$$

where $\alpha = \frac{q+2}{p}$.

(ii) *If $2 + q < p$, then uC_φ is a compact operator from $F(p, q, s)$ to β_L if and only if $u, u\varphi \in \beta_L$ and (8) holds.*

(iii) *If $2 + q = p$, $s > 1$, then uC_φ is a compact operator from $F(p, q, s)$ to β_L if and only if $u, u\varphi \in \beta_L$, (8) holds and*

$$(10) \quad \lim_{|\varphi(z)| \rightarrow 1^-} |u'(z)| (1 - |z|^2) \log\left(\frac{2}{1 - |z|}\right) \log\left(\frac{2}{1 - |\varphi(z)|^2}\right) = 0.$$

Proof. Suppose that uC_φ is compact from $F(p, q, s)$ to β_L . It is easily obtained that $u, u\varphi \in \beta_L$ by taking $f(z) = 1$ and $f(z) = z$ respectively. Let $\alpha = \frac{q+2}{p}$ and $\{z_n\}$ be a sequence in D such that $|\varphi(z_n)| \rightarrow 1^-$ as $n \rightarrow \infty$. We take the test functions

$$f_n(z) = \frac{(z - \varphi(z_n))(1 - |\varphi(z_n)|^2)}{(1 - \overline{\varphi(z_n)}z)^{\alpha+1}}.$$

Then $f_n \in F(p, q, s)$ and $\sup_n \|f_n\|_{F(p, q, s)} \leq C < \infty$ by using the same methods in [9]. Then $\{f_n\}$ is a bounded sequence in $F(p, q, s)$ which converges to 0 uniformly on compact subsets of D . Note that $f_n(\varphi(z_n)) = 0$ and $f'_n(\varphi(z_n)) =$

$\frac{1}{(1-|\varphi(z_n)|^2)^\alpha}$, it follows that

$$\begin{aligned} \|uC_\varphi f_n\|_L &\geq \|uC_\varphi f_n\|_{\beta_L} \\ &\geq (1 - |z_n|^2) \log\left(\frac{2}{1 - |z_n|}\right) |u'(z_n)f_n(\varphi(z_n)) + u(z_n)f'_n(\varphi(z_n))\varphi'(z_n)| \\ &= \frac{(1 - |z_n|^2) \log\frac{2}{1 - |z_n|}}{(1 - |\varphi(z_n)|^2)^\alpha} |u(z_n)\varphi'(z_n)|. \end{aligned}$$

Then, by Lemma 2.3, (8) holds for all case.

(i) **Case $\alpha = \frac{q+2}{p} > 1$.** We consider the functions g_n defined by

$$(11) \quad g_n(z) = \frac{(\alpha + 1)(1 - |\varphi(z_n)|^2)}{(1 - \overline{\varphi(z_n)}z)^\alpha} - \frac{\alpha(1 - |\varphi(z_n)|^2)^2}{(1 - \overline{\varphi(z_n)}z)^{\alpha+1}}.$$

Then $g_n \in F(p, q, s)$ and $\sup_n \|g_n\|_{F(p,q,s)} \leq C < \infty$, so $\{g_n\}$ is a bounded sequence in $F(p, q, s)$ which converges to 0 uniformly on compact subsets of D . Since $g'_n(\varphi(z_n)) = 0$ and $g_n(\varphi(z_n)) = \frac{1}{(1-|\varphi(z_n)|^2)^{\alpha-1}}$, it follows that

$$\begin{aligned} \|uC_\varphi g_n\|_L &\geq \|uC_\varphi g_n\|_{\beta_L} \\ &\geq (1 - |z_n|^2) \log\left(\frac{2}{1 - |z_n|}\right) |u'(z_n)g_n(\varphi(z_n)) + u(z_n)g'_n(\varphi(z_n))\varphi'(z_n)| \\ &= (1 - |z_n|^2) \log\left(\frac{2}{1 - |z_n|}\right) |u'(z_n)| \frac{1}{(1 - |\varphi(z_n)|^2)^{\alpha-1}}. \end{aligned}$$

Then (9) holds by Lemma 2.3.

(ii) **Case $\alpha = \frac{q+2}{p} < 1$.** The necessity in condition (ii) has been proved above.

(iii) **Case $\alpha = \frac{q+2}{p} = 1$ and $s > 1$.** We take the other test functions

$$h_n(z) = \frac{3}{a_n} \left(\log \frac{2}{1 - \overline{\varphi(z_n)}z}\right)^2 - \frac{2}{a_n^2} \left(\log \frac{2}{1 - \overline{\varphi(z_n)}z}\right)^3,$$

where $a_n = \log \frac{2}{1-|\varphi(z_n)|^2}$. Clearly $h_n(z) \rightarrow 0$ uniformly on compact subsets of D . By a direct calculation we get $\sup_n \|h_n\|_{\beta_1} \leq C < +\infty$. Noting $s > 1$, thus we have $h_n \in F(p, q, s)$ and $\sup_n \|h_n\|_{F(p,q,s)} < \infty$ by introduction. Then $\{h_n\}$ is a bounded sequence in $F(p, q, s)$ which converges to 0 uniformly on compact subsets of D . Note that $h'_n(\varphi(z_n)) \equiv 0$ and $h_n(\varphi(z_n)) = a_n$, it follows that

$$\begin{aligned} \|uC_\varphi h_n\|_L &\geq \|uC_\varphi h_n\|_{\beta_L} \\ &\geq (1 - |z_n|^2) \log\left(\frac{2}{1 - |z_n|}\right) |u'(z_n)h_n(\varphi(z_n)) + u(z_n)h'_n(\varphi(z_n))\varphi'(z_n)| \\ &= (1 - |z_n|^2) \log\left(\frac{2}{1 - |z_n|}\right) \log \frac{2}{1 - |\varphi(z_n)|^2} |u'(z_n)|. \end{aligned}$$

Then (10) holds by Lemma 2.3.

Now we prove the sufficient conditions.

Case $\alpha = \frac{2+q}{p} > 1$. Assume that (8) and (9) hold and $u, u\varphi \in \beta_L$. Then it is not difficult to obtain that (1), (2) and

$$(12) \quad \sup_{z \in D} (1 - |z|^2) \log\left(\frac{2}{1 - |z|}\right) |u(z)\varphi'(z)| < \infty$$

hold. Let $\{f_n\}$ be a bounded sequence in $F(p, q, s)$ which converges to 0 uniformly on compact subsets of D . Let $M = \sup_n \|f_n\|_{F(p, q, s)} < +\infty$. We only prove $\lim_{n \rightarrow \infty} \|uC_\varphi(f_n)\|_L = 0$ by Lemma 2.3. This amounts to showing that both

$$\sup_{w \in D} (1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) |u(w)f'_n(\varphi(w))\varphi'(w)| \rightarrow 0$$

and

$$\sup_{w \in D} (1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) |u'(w)f_n(\varphi(w))| \rightarrow 0$$

as $n \rightarrow \infty$.

If $|\varphi(w)| \leq r < 1$, by (12), then

$$(1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) |u(w)f'_n(\varphi(w))\varphi'(w)| \leq C \max_{|z| \leq r} |f'_n(z)|.$$

If $|\varphi(w)| > r$, then by Lemma 2.1 we get

$$\begin{aligned} & (1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) |u(w)f'_n(\varphi(w))\varphi'(w)| \\ & \leq CM \frac{(1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right)}{(1 - |\varphi(w)|^2)^\alpha} |\varphi'(w)u(w)|. \end{aligned}$$

Thus

$$\begin{aligned} & \sup_{w \in D} (1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) |u(w)f'_n(\varphi(w))\varphi'(w)| \\ & \leq C \max_{|w| \leq r} |f'_n(w)| + CM \sup_{|\varphi(w)| > r} \frac{(1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right)}{(1 - |\varphi(w)|^2)^\alpha} |\varphi'(w)u(w)|. \end{aligned}$$

First letting n tend to infinity and subsequently r increase to 1, one obtains that

$$\sup_{w \in D} (1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) |u(w)f'_n(\varphi(w))\varphi'(w)| \rightarrow 0$$

as $n \rightarrow \infty$. The other statement is proved similarly.

If $|\varphi(w)| \leq r < 1$, by $u \in \beta_L$, then

$$(1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) |u'(w)f_n(\varphi(w))| \leq \|u\|_L \max_{|z| \leq r} |f_n(z)|.$$

If $|\varphi(w)| > r$, by Lemma 2.1 and 2.2, then

$$(1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) |u'(w)f_n(\varphi(w))| \leq CM \frac{(1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right)}{(1 - |\varphi(w)|^2)^{\alpha-1}} |u'(w)|.$$

Thus

$$\begin{aligned} & \sup_{w \in D} (1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) |u'(w) f_n(\varphi(w))| \\ & \leq \|u\|_L \max_{|w| \leq r} |f_n(w)| + CM \sup_{|\varphi(w)| > r} \frac{(1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right)}{(1 - |\varphi(w)|^2)^{\alpha - 1}} |u'(w)|, \end{aligned}$$

which also implies that

$$\sup_{w \in D} (1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) |u'(w) f_n(\varphi(w))| \rightarrow 0$$

as $n \rightarrow \infty$.

Case $\alpha = \frac{2+q}{p} < 1$. Similarly, let $\{f_n\}$ be a bounded sequence in $F(p, q, s)$ which converges to 0 uniformly on compact subsets of D . By Lemma 2.1 and [10, Lemma 3.2] we have $\sup_{z \in D} |f_n(z)| \rightarrow 0$ as $n \rightarrow \infty$. Then

$$\sup_{w \in D} (1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) |u'(w) f_n(\varphi(w))| \leq \|u\|_L \sup_{z \in D} |f_n(z)| \rightarrow 0$$

as $n \rightarrow \infty$. As the same proof as in case $\alpha = \frac{2+q}{p} > 1$, we get

$$\sup_{w \in D} (1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) |u(w) f'_n(\varphi(w)) \varphi'(w)| \rightarrow 0.$$

Hence uC_φ is a compact operator from $F(p, q, s)$ to β_L .

Case $\alpha = \frac{2+q}{p} = 1$ and $s > 1$. Let $\{f_n\}$ be a bounded sequence in $F(p, q, s)$ which converges to 0 uniformly on compact subsets of D . Let $M = \sup_n \|f_n\|_{F(p,q,s)} < +\infty$. By Lemma 2.1 and 2.4, we get

$$(13) \quad \sup_n |f_n(z)| \leq CM \log \frac{2}{1 - |z|^2}.$$

If $|\varphi(w)| \leq r < 1$, by $u \in \beta_L$, then

$$(1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) |u'(w) f_n(\varphi(w))| \leq \|u\|_L \max_{|z| \leq r} |f_n(z)|.$$

If $|\varphi(w)| > r$, by (13), we have

$$\begin{aligned} & (1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) |u'(w) f_n(\varphi(w))| \\ & \leq CM (1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) \log\left(\frac{2}{1 - |\varphi(w)|^2}\right) |u'(w)|. \end{aligned}$$

Thus

$$\begin{aligned} & \sup_{w \in D} (1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) |u'(w) f_n(\varphi(w))| \\ & \leq \|u\|_L \max_{|w| \leq r} |f_n(w)| + CM (1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) \log\left(\frac{2}{1 - |\varphi(w)|^2}\right) |u'(w)|, \end{aligned}$$

which also implies that

$$\sup_{w \in D} (1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) |u'(w) f_n(\varphi(w))| \rightarrow 0$$

as $n \rightarrow \infty$. We also similarly prove that

$$\sup_{w \in D} (1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) |u(w) f'_n(\varphi(w)) \varphi'(w)| \rightarrow 0$$

as $n \rightarrow \infty$. Hence $\lim_{n \rightarrow \infty} \|uC_\varphi(f_n)\|_L = 0$. This completes the proof of Theorem 2.2. \square

3. Boundedness and compactness of $uC_\varphi : F_0(p, q, s) \rightarrow \beta_L^0$

In this section we characterize the boundedness and compactness of the weighted composition operator $uC_\varphi : F_0(p, q, s) \rightarrow \beta_L^0$. For this purpose we need another lemmas.

Lemma 3.1. *Let $0 < s < \infty$, $1 \leq p < \infty$, $-2 < q < \infty$ and $q + s > -1$. For $0 < t < 1$, $z \in D$, let $f_t(z) = f(tz)$. If $f \in F(p, q, s)$, then $f_t \in F(p, q, s)$ and $\|f_t\|_{F(p, q, s)} \leq \|f\|_{F(p, q, s)}$.*

Proof. For $f \in F(p, q, s)$ and $0 < t < 1$, by the Poisson formula, we have

$$f_t(z) = \int_0^{2\pi} f(ze^{i\theta}) \frac{1 - t^2}{|e^{i\theta} - t|^2} \frac{d\theta}{2\pi}, \quad z \in D.$$

Then by Hölder's inequality we get,

$$\begin{aligned} & \int_D |f'_t(z)|^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s dm(z) \\ & \leq \int_D \int_0^{2\pi} |f'(e^{i\theta}z)|^p \frac{1 - t^2}{|e^{i\theta} - t|^2} \frac{d\theta}{2\pi} (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s dm(z) \\ & = \int_0^{2\pi} \int_D |f'(e^{i\theta}z)|^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s dm(z) \frac{1 - t^2}{|e^{i\theta} - t|^2} \frac{d\theta}{2\pi} \\ & \leq \|f\|_{p, q, s}^p \int_0^{2\pi} \frac{1 - t^2}{|e^{i\theta} - t|^2} \frac{d\theta}{2\pi} = \|f\|_{p, q, s}^p, \end{aligned}$$

where $\|f\|_{p, q, s}^p = \sup_{a \in D} \int_D |f'(z)|^p (1 - |z|^2)^q (1 - |\varphi_a(z)|^2)^s dm(z)$. That is,

$$\|f_t\|_{F(p, q, s)} \leq \|f\|_{F(p, q, s)}.$$

\square

Lemma 3.2. *Let u be an analytic function on the unit disk D and φ an analytic self-map of D . Let $0 < s < \infty$, $1 \leq p < \infty$, $-2 < q < \infty$ and $q + s > -1$. If uC_φ is a bounded operator from $F_0(p, q, s)$ into β_L^0 , then uC_φ is bounded from $F(p, q, s)$ to β_L .*

Proof. Suppose uC_φ is bounded from $F_0(p, q, s)$ into β_L^0 and for $f \in F(p, q, s)$, we have $f_t \in F_0(p, q, s)$ for all $0 < t < 1$. Then, according to Lemma 3.1,

$$\|uC_\varphi(f_t)\|_{F(p,q,s)} \leq \|uC_\varphi\| \|f_t\|_{F(p,q,s)} \leq \|uC_\varphi\| \|f\|_{F(p,q,s)} < +\infty.$$

Hence $\|uC_\varphi(f)\|_{F(p,q,s)} \leq \|uC_\varphi\| \|f\|_{F(p,q,s)} < +\infty$, which shows uC_φ is bounded from $F(p, q, s)$ to β_L . \square

Theorem 3.1. *Let u be an analytic function on the unit disc D and φ an analytic self-map of D . Let $0 < p, s < \infty$, $-2 < q < \infty$ and $q + s > -1$. Then $uC_\varphi : F_0(p, q, s) \rightarrow \beta_L^0$ is a bounded operator provided that $u \in \beta_L^0$, $uC_\varphi : F(p, q, s) \rightarrow \beta_L$ is bounded and*

$$(14) \quad \lim_{|z| \rightarrow 1^-} (1 - |z|^2) \log \frac{2}{1 - |z|} |u(z)\varphi'(z)| = 0.$$

Conversely, if $uC_\varphi : F_0(p, q, s) \rightarrow \beta_L^0$ is a bounded operator, then $u \in \beta_L^0$ and (14) holds; moreover, if $1 \leq p < \infty$, then $uC_\varphi : F(p, q, s) \rightarrow \beta_L$ is bounded.

Proof. Assume that $u \in \beta_L^0$ and $\lim_{|z| \rightarrow 1^-} (1 - |z|^2) \log \frac{2}{1 - |z|} |u(z)\varphi'(z)| = 0$. Then, for each polynomial $p(z)$, we have that

$$\begin{aligned} & (1 - |z|^2) \log \frac{2}{1 - |z|} |(uC_\varphi p)'(z)| \\ & \leq (1 - |z|^2) \log \frac{2}{1 - |z|} |u'(z)||p(\varphi(z))| + (1 - |z|^2) \log \frac{2}{1 - |z|} |u(z)\varphi'(z)p'(\varphi(z))|, \end{aligned}$$

from which it follows that $uC_\varphi p \in \beta_L^0$. Since the set of all polynomials is dense in $F_0(p, q, s)$, we have that for every $f \in F_0(p, q, s)$ there is a sequence of polynomials $\{p_n\}$ such that $\|f - p_n\|_{F(p,q,s)} \rightarrow 0$ as $n \rightarrow \infty$. Note that $uC_\varphi : F(p, q, s) \rightarrow \beta_L$ is bounded, then

$$\|uC_\varphi f - uC_\varphi p_n\|_L \leq \|uC_\varphi\| \|f - p_n\|_{F(p,q,s)} \rightarrow 0$$

as $n \rightarrow \infty$. Since β_L^0 is closed in subset of β_L , we obtain

$$uC_\varphi(F_0(p, q, s)) \subset \beta_L^0.$$

So $uC_\varphi : F_0(p, q, s) \rightarrow \beta_L^0$ is bounded.

Conversely, assume that $uC_\varphi : F_0(p, q, s) \rightarrow \beta_L^0$ is bounded. Taking the functions $f(z) = 1$ and $f(z) = z$ respectively, we obtain that $u \in \beta_L^0$ and $\lim_{|z| \rightarrow 1^-} (1 - |z|^2) \log \frac{2}{1 - |z|} |u(z)\varphi'(z)| = 0$. If $p \geq 1$, then $uC_\varphi : F(p, q, s) \rightarrow \beta_L$ is bounded by Lemma 3.2. \square

Lemma 3.3. *Let $U \subset \beta_L^0$. Then U is compact if and only if it is closed, bounded and satisfies*

$$\lim_{|z| \rightarrow 1^-} \sup_{f \in U} (1 - |z|^2) \log \left(\frac{2}{1 - |z|} \right) |f'(z)| = 0.$$

The proof is similar to [3, Lemma 1], so we omitted it.

Theorem 3.2. *Let $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $u \in H(D)$ and φ be an analytic self-map of D .*

(i) *If $2 + q > p$, then uC_φ is a compact operator from $F_0(p, q, s)$ to β_L^0 if and only if*

$$(15) \quad \lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2) \log \frac{2}{1 - |z|}}{(1 - |\varphi(z)|^2)^\alpha} |u(z)\varphi'(z)| = 0$$

and

$$\lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2) \log \frac{2}{1 - |z|}}{(1 - |\varphi(z)|^2)^{\alpha-1}} |u'(z)| = 0,$$

where $\alpha = \frac{q+2}{p}$.

(ii) *If $2 + q < p$, then uC_φ is a compact operator from $F_0(p, q, s)$ to β_L^0 if and only if $u \in \beta_L^0$ and (15) holds.*

(iii) *If $2 + q = p$, $s > 1$, then uC_φ is a compact operator from $F_0(p, q, s)$ to β_L^0 if and only if (15) holds and*

$$\lim_{|z| \rightarrow 1^-} |u'(z)| (1 - |z|^2) \log \left(\frac{2}{1 - |z|} \right) \log \left(\frac{2}{1 - |\varphi(z)|^2} \right) = 0.$$

Proof. First we consider the case that $\alpha = \frac{q+2}{p} > 1$. Suppose that u and φ satisfy the condition in (i). By Theorem 2.1 and Theorem 3.1, we know that uC_φ is bounded from $F_0(p, q, s)$ to β_L^0 . Suppose that $f \in F_0(p, q, s)$ is such that $\|f\|_{F(p, q, s)} \leq 1$. By Lemma 2.1, we know that $f \in \beta_\alpha^0$ and $\|f\|_{\beta_\alpha} \leq C\|f\|_{F(p, q, s)}$. Then from Lemma 2.2 it follows that

$$\begin{aligned} & (1 - |z|^2) \log \frac{2}{1 - |z|} |(uC_\varphi f)'(z)| \\ & \leq C \frac{(1 - |z|^2) \log \frac{2}{1 - |z|}}{(1 - |\varphi(z)|^2)^{\alpha-1}} |u'(z)| + C \frac{(1 - |z|^2) \log \frac{2}{1 - |z|}}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)u(z)|, \end{aligned}$$

thus

$$\begin{aligned} & \sup\{(1 - |z|^2) \log \frac{2}{1 - |z|} |(uC_\varphi f)'(z)| : f \in F_0(p, q, s), \|f\|_{F(p, q, s)} \leq 1\} \\ & \leq C \frac{(1 - |z|^2) \log \frac{2}{1 - |z|}}{(1 - |\varphi(z)|^2)^{\alpha-1}} |u'(z)| + C \frac{(1 - |z|^2) \log \frac{2}{1 - |z|}}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)u(z)|, \end{aligned}$$

and it follows that

$$\lim_{|z| \rightarrow 1^-} \sup\{(1 - |z|^2) \log \frac{2}{1 - |z|} |(uC_\varphi f)'(z)| : f \in F_0(p, q, s), \|f\|_{F(p, q, s)} \leq 1\} = 0,$$

so that uC_φ is compact from β_α^0 to β_L^0 by Lemma 3.3.

Conversely, suppose that uC_φ is compact from $F_0(p, q, s)$ to β_L^0 . By Lemma 3.3, we have

$$\lim_{|z| \rightarrow 1^-} \sup\{(1 - |z|^2) \log \frac{2}{1 - |z|} |(uC_\varphi f)'(z)| : f \in F_0(p, q, s), \|f\|_{F(p, q, s)} \leq M\} = 0$$

for some $M > 0$. Note that the proof of Theorem 2.1 and the fact that the functions given in (4) are in $F_0(p, q, s)$ and have norms bounded uniformly in w , we get

$$\lim_{|w| \rightarrow 1^-} \frac{(1 - |w|^2) \log \frac{2}{1 - |w|}}{(1 - |\varphi(w)|^2)^\alpha} |u(w)\varphi'(w)| = 0.$$

Similarly, note that the functions given in (5) are in $F_0(p, q, s)$ and have norms bounded uniformly in w , we get

$$\lim_{|w| \rightarrow 1^-} \frac{(1 - |w|^2) \log \frac{2}{1 - |w|}}{(1 - |\varphi(w)|^2)^{\alpha-1}} |u'(w)| = 0.$$

This completes the proof of (i).

Next, we consider the case $\alpha = \frac{q+2}{p} < 1$. Assume that $u \in \beta_L^0$ and (14) holds. Suppose that $f \in F_0(p, q, s)$ is such that $\|f\|_{F(p,q,s)} \leq 1$. By Lemma 2.1 and 2.4, we get

$$\begin{aligned} & (1 - |z|^2) \log \frac{2}{1 - |z|} |(uC_\varphi f)'(z)| \\ \leq & C(1 - |z|^2) \log \frac{2}{1 - |z|} |u'(z)| + C \frac{(1 - |z|^2) \log \frac{2}{1 - |z|}}{(1 - |\varphi(z)|^2)^\alpha} |\varphi'(z)u(z)|, \end{aligned}$$

thus

$$\lim_{|z| \rightarrow 1^-} \sup\{(1 - |z|^2) \log \frac{2}{1 - |z|} |(uC_\varphi f)'(z)| : f \in F_0(p, q, s), \|f\|_{F(p,q,s)} \leq 1\} = 0.$$

So that uC_φ is compact from $F_0(p, q, s)$ to β_L^0 by Lemma 3.3. The fact that the conditions in (ii) are necessary for compactness of operator uC_φ as an operator from $F_0(p, q, s)$ to β_L^0 is proved in Theorem 3.1 and the case that $\alpha > 1$.

Finally, we will prove (iii). Similarly, suppose that u and φ satisfy the condition in (iii). By Theorem 2.2, we know that uC_φ is bounded from $F_0(p, q, s)$ to β_L^0 . Suppose that $f \in F_0(p, q, s)$ is such that $\|f\|_{F_0(p,q,s)} \leq 1$. By Lemma 2.1 and 2.4, we get

$$\begin{aligned} & (1 - |z|^2) \log \frac{2}{1 - |z|} |(uC_\varphi f)'(z)| \\ \leq & C(1 - |z|^2) \log \frac{2}{1 - |z|} \log \frac{2}{1 - |\varphi(z)|^2} |u'(z)| \\ & + C \frac{(1 - |z|^2) \log \frac{2}{1 - |z|}}{1 - |\varphi(z)|^2} |\varphi'(z)u(z)|, \end{aligned}$$

thus

$$\lim_{|z| \rightarrow 1^-} \sup\{(1 - |z|^2) \log \frac{2}{1 - |z|} |(uC_\varphi f)'(z)| : f \in F_0(p, q, s), \|f\|_{F_0(p,q,s)} \leq 1\} = 0,$$

so that uC_φ is compact from $F_0(p, q, s)$ to β_L^0 by Lemma 3.3.

Conversely, noting that the functions given in (6) are in $F_0(p, q, s)$ and have norms bounded uniformly in w , we get

$$\lim_{|w| \rightarrow 1^-} |u'(w)|(1 - |w|^2) \log\left(\frac{2}{1 - |w|}\right) \log\left(\frac{2}{1 - |\varphi(w)|^2}\right) = 0.$$

The another necessity in condition (iii) is similarly proved in (i). This completes the proof of Theorem 3.2 \square

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