J. Korean Math. Soc. **45** (2008), No. 5, pp. 1203–1220

WEIGHTED COMPOSITION OPERATORS ON WEIGHTED SPACES OF VECTOR-VALUED ANALYTIC FUNCTIONS

Jasbir Singh Manhas

Abstract. Let V be an arbitrary system of weights on an open connected subset G of $\mathbb{C}^N$ $(N \geq 1)$ and let $B(E)$ be the Banach algebra of all bounded linear operators on a Banach space E. Let $HV_b(G,E)$ and $HV_0(G,E)$ be the weighted locally convex spaces of vector-valued analytic functions. In this paper, we characterize self-analytic mappings $\phi : G \to G$ and operator-valued analytic mappings $\Psi : G \to B(E)$ which generate weighted composition operators and invertible weighted composition operators on the spaces $HV_b(G,E)$ and $HV_0(G,E)$ for different systems of weights V on G. Also, we obtained compact weighted composition operators on these spaces for some nice classes of weights.

1. Introduction

Weighted composition operators have been appearing in a natural way on different spaces of analytic functions. For example, De Leeuw, Rudin and Wermer [12] and Nagasawa [27] have shown that the isometries of Hardy spaces $H^1(\mathbb{D})$ and $H^\infty(\mathbb{D})$ are weighted composition operators whereas Forelli [13] has shown the same behaviour on the Hardy spaces H^p for $1 < p < \infty, p \neq 2$. Also, the same type of results were obtained by Kolaski [17] and Mazur [24] on Bergman spaces. These results have further been extended to the spaces of vector-valued analytic functions by Cambern and Jarosz [6] and Lin [18]. In recent years, the theory of weighted composition operators on spaces of analytic functions is gaining importance as it includes two nice classes of operators such as composition operators and multiplication operators and a very less information is known about weighted composition operators as compared to the theory of composition operators which has been presented in three monographs (see Cowen and MacCluer [11], Shapiro [33] and Singh and Manhas [34]).

Received January 11, 2006.

2000 *Mathematics Subject Classification.* Primary 47B37, 47B38, 47B07, 46E40, 46E10; Secondary 47D03, 37B05, 32A10, 30H05.

Key words and phrases. system of weights, Banach algebra, weighted locally convex spaces of vector-valued analytic functions, weighted composition operators, invertible and compact operators.

The research of the author was supported by SQU grant No. IG/SCI/DOMS/07/02.

Contreras and Hernandez-Diaz [9] have made a study of weighted composition operators on Hardy spaces whereas Mirzakarimi and Siddighi [25] have studied these operators on Bergman and Dirichlet spaces. On Bloch-type spaces, these operators are explored by MacCluer and Zhao [21], Ohno [28], Ohno and Zhao [31] and Ohno, Stroethoff and Zhao [29]. Also Ohno and Takagi [30] have obtained a study of these operators on the disc algebra and the Hardy space $H^{\infty}(\mathbb{D})$. Recently, Montes-Rodriguez [26] and Contreras and Hernandez-Diaz [8] have obtained some nice properties of these operators on weighted Banach spaces of analytic functions. The applications of these operators can be found in the theory of semigroups and dynamical systems (see [16] and [36]).

We have organized this paper into six sections. The preliminaries required for proving the results in the remaining sections are presented in Section 2. In Section 3, we characterize the weighted composition operators on the weighted locally convex spaces $HV_b(G,E)$ and $HV_0(G,E)$ of vector-valued analytic functions generalizing the boundedness results of Contreras and Hernandez-Diaz [8], Monte-Rodriguez [26] and Manhas [23] obtained on the spaces of scalar-valued analytic functions. We have investigated the invertibility of these operators on the spaces $HV_b(G,E)$ and $HV_0(G,E)$ in Section 4 which generalize the results of Ohno and Takagi [30] obtained on the Hardy space $H^{\infty}(\mathbb{D})$. In Section 5, we have presented some results on compact weighted composition operators on weighted Banach spaces of vector-valued analytic functions generalizing some of the results obtained in [7] and [8]. Finally, in Section 6, we have given some examples to illustrate the theory.

2. Preliminaries

Let G be an open connected subset of $\mathbb{C}^N (N \geq 1)$ and $H(G,E)$ be the space of all vector-valued analytic functions from G into the Banach space E. Let V be a set of non-negative upper semicontinuous functions on G. Then V is said to be *directed upward* if for every pair $u_1, u_2 \in V$ and $\lambda > 0$, there exists $v \in V$ such that $\lambda u_i \leq v$ (pointwise on G) for $i = 1, 2$. If V is directed upward and for each $z \in G$, there exists $v \in V$ such that $v(z) > 0$, then we call V as an *arbitrary system* of weights on G. If U and V are two arbitrary systems of weights on G such that for each $u \in U$, there exists $v \in V$ for which $u \leq v$, then we write $U \leq V$. If $U \leq V$ and $V \leq U$, then we write $U \cong V$. Let V be an arbitrary system of weights on G. Then we define

$$HV_b(G,E) = \{f \in H(G,E) : vf(G) \text{ is bounded in } E \text{ for each } v \in V\}$$

and

$$HV_0(G,E) = \{f \in H(G,E) : vf \text{ vanishes at infinity on } G \text{ for each } v \in V\}.$$

For $v \in V$ and $f \in H(G,E)$, we define

$$\|f\|_{v,E} = \sup\{v(z)\, \|f(z)\| : z \in G\}.$$

Clearly the family $\{\|\cdot\|_{v,E} : v \in V\}$ of seminorms defines a Hausdorff locally convex topology on each of theses spaces $HV_b(G,E)$ and $HV_0(G,E)$. With this topology the spaces $HV_b(G,E)$ and $HV_0(G,E)$ are called the weighted locally convex spaces of vector-valued analytic functions. These spaces have a basis of closed absolutely convex neighborhoods of the form

$$B_{v,E} = \{f \in HV_b(G,E) \text{ (resp. } HV_0(G,E)) : \|f\|_{v,E} \leq 1\}.$$

If $E = \mathbb{C}$, then we write $HV_b(G,E) = HV_b(G), HV_0(G,E) = HV_0(G)$ and

$$B_v = \{f \in HV_b(G) \text{ (resp. } HV_0(G)) : \|f\|_v \leq 1\}.$$

Throughout the paper, we assume for each $z \in G$, there exists $f_z \in HV_0(G)$ such that $f_z(z) \neq 0$.

Now using the definitions of weights given in ([2], [4], [5]), we give definitions of some systems of weights which are required for characterizing some results in the remaining sections.

Let V be an arbitrary system of weights on G and let $v \in V$. Then define $\tilde{w} : G \to \mathbb{R}^+$ as

$$\tilde{w}(z) = \sup\{|f(z)| : \|f\|_v \leq 1\} \leq \frac{1}{v(z)}$$

and

$$\tilde{v}(z) = \frac{1}{\tilde{w}(z)} \text{ for every } z \in G.$$

In case $\tilde{w}(z) \neq 0, \tilde{v}$ is an upper semicontinuous and we call it an associated weight of v. Let $\tilde{V}$ denote the system of all associated weights of V. Then an arbitrary system of weights V is called a *reasonable system* if it satisfies the following properties:

(2.a) for each $v \in V$, there exists $\tilde{v} \in \tilde{V}$ such that $v \leq \tilde{v}$;

(2.b) for each $v \in V$, $\|f\|_v \leq 1$ if and only if $\|f\|_{\tilde{v}} \leq 1$, for every $f \in HV_b(G)$;

(2.c) if $v \in V$, then for every $z \in G$, there exists $f_z \in B_v$ such that $|f_z(z)| = \dfrac{1}{\tilde{v}(z)}$.

Let $v \in V$. Then v is called *essential* if there exists a constant $\lambda > 0$ such that $v(z) \leq \tilde{v}(z) \leq \lambda v(z)$ for each $z \in G$. A reasonable system of weights V is called an *essential system* if each $v \in V$ is an essential weight. If V is an essential system of weights, then we have $V \cong \tilde{V}$. For example, let $G = \mathbb{D}$, the open unit disc and let $f \in H(\mathbb{D})$ be non-zero. Then define $v_f(z) = [\sup\{|f(z)| : |z| = r\}]^{-1}$ for every $z \in \mathbb{D}$. Clearly each v_f is a weight satisfying $\tilde{v}_f = v_f$ and the family $V = \{v_f : f \in H(\mathbb{D}), f \text{ is non-zero}\}$ is an essential system of weights on $\mathbb{D}$. For more details on these weights, we refer to [2]. Let G be any balanced (i.e., $\lambda z \in G$, whenever $z \in G$ and $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$) open subset of $\mathbb{C}^N (N \geq 1)$. Then a weight $v \in V$ is called radial and typical if $v(z) = v(\lambda z)$ for all $z \in G$

and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and vanishes at the boundary ∂G. In particular, a weight v on $\mathbb{D}$ is radial and typical if $v(z) = v(|z|)$ and $\lim_{|z|\to 1} v(z) = 0$. For instance in [8], it is shown that $v_p(z) = (1-|z|^2)^p, (0 < p < \infty)$ for every $z \in \mathbb{D}$, are essential typical weights. For more details on the weighted Banach spaces of analytic functions and the weighted locally convex spaces of analytic functions associated with these weights, we refer to [1], [2], [3], [4], [5], [14], [19], [20]. For basic definitions and facts in complex analysis and functional analysis, we refer to [10], [15], [32].

Let $F(G, E)$ be a topological vector space of vector-valued analytic functions from G into E and let $L(G, E)$ be the vector space of all vector-valued functions from G into E. Let $B(E)$ be the Banach algebra of all bounded linear operators on E. Then for an operator-valued map $\Psi : G \to B(E)$ and self-map $\phi : G \to G$, we define the linear map $W_{\Psi,\phi} : F(G, E) \to L(G, E)$ as $W_{\Psi,\phi}(f) = \Psi \cdot f \circ \phi$, for every $f \in F(G, E)$, where the product $\Psi \cdot f \circ \phi$ is defined pointwise on G as $(\Psi \cdot f \circ \phi)(z) = \Psi_z(f(\phi(z)))$ for every $z \in G$. In case $W_{\Psi,\phi}$ takes $F(G, E)$ into itself and is continuous, we call $W_{\Psi,\phi}$, the *weighted composition operator* on $F(G, E)$ induced by the symbols Ψ and ϕ. If $\Psi(z) = I$, the identity operator on E for every $z \in G$, then $W_{\Psi,\phi}$ is called the *composition operator* and in case $\phi(z) = z$ for every $z \in G, W_{\Psi,\phi}$ is called the *multiplication operator.*

3. Analytic mappings inducing weighted composition operators

In this section we characterize the operator-valued analytic mappings $\Psi : G \to B(E)$ and the self-analytic mappings $\phi : G \to G$ which induce weighted composition operators on the spaces $HV_b(G, E)$ and $HV_0(G, E)$ for different systems of weights V on G. We begin with the following proposition.

Proposition 3.1. *Let U and V be arbitrary systems of weights on G. Let $\Psi : G \to B(E)$ and $\phi : G \to G$ be analytic mappings. Then $W_{\Psi,\phi} : HU_b(G, E) \to HV_b(G, E)$ is a weighted composition operator if for every $v \in V$, there exists $u \in U$ such that $v(z)\|\Psi_z\| \leq u(\phi(z))$ for every $z \in G$.*

Proof. We shall show that $W_{\Psi,\phi}$ is continuous at the origin. Let $v \in V$. Then by the given condition there exists $u \in U$ such that $v(z)\|\Psi_z\| \leq u(\phi(z))$ for every $z \in G$. We claim that $W_{\Psi,\phi}(B_{u,E}) \subseteq B_{v,E}$. For $f \in B_{u,E}$, we have

$$\begin{aligned}\|W_{\Psi,\phi}f\|_{v,E} &= \sup\{v(z)\|\Psi_z(f(\phi(z)))\| : z \in G\} \\ &\leq \sup\{v(z)\|\Psi_z\|\,\|f(\phi(z))\| : z \in G\} \\ &\leq \sup\{u(\phi(z))\|f(\phi(z))\| : z \in G\} \\ &\leq \sup\{u(z)\|f(z)\| : z \in G\} \\ &= \|f\|_{u,E} \leq 1.\end{aligned}$$

This proves that $W_{\Psi,\phi}$ is continuous at the origin and hence $W_{\Psi,\phi}$ is a weighted composition operator. □

Remark 3.1. The condition in Proposition 3.1 is not sufficient for $W_{\Psi,\phi}$ to induce a weighted composition operator from $HU_0(G,E) \to HV_0(G,E)$. For instance, let $G = \{z \in G : z = x + iy, y > 0\}$ be the upper half-plane. Let $U = V$ be the system of constant weights on G. Let $E = H^{\infty}(\mathbb{D})$ be the Banach algebra of bonded analytic functions on the unit disc $\mathbb{D}$. Let $z_0 \in G$ be fixed and let $\Psi : G \to B(E)$ be defined as $\Psi(z) = M_{z_0}$ for every $z \in G$, where $M_{z_0} : E \to E$ is given by $M_{z_0} f = z_0 f$ for every $f \in E$. Define $\phi : G \to G$ as $\phi(z) = z_0$ for every $z \in G$. Now, it is easy to see that for every $v \in V$, there exists $u \in V$ such that $v(z)\|\Psi_z\| \leq u(\phi(z))$ for every $z \in G$. But $W_{\Psi,\phi}$ is not a weighted composition operator on $HV_0(G,E)$. For instance, let $f \in E$ be such that $\|f\|_{\infty} = 1$ and define $F : G \to E$ as $F(z) = \frac{1}{z} f$ for every $z \in G$. Clearly $F \in HV_0(G,E)$. But $W_{\Psi,\phi}(F) \notin HV_0(G,E)$. So, we need an additional condition for $W_{\Psi,\phi}$ to be a weighted composition operator from $HU_0(G,E)$ to $HV_0(G,E)$. Now, we define a set which we need for our additional condition. Let $v \in V$ and $\varepsilon > 0$. Then define the set $N(v,\varepsilon) = \{z \in G : v(z) \geq \varepsilon\}$.

Proposition 3.2. *Let U and V be arbitrary systems of weights on G. Let $\Psi : G \to B(E)$ and $\phi : G \to G$ be analytic mappings. Then $W_{\Psi,\phi} : HU_0(G,E) \to HV_0(G,E)$ is a weighted composition operator if*

(i) *for every $v \in V$, there exists $u \in U$ such that $v(z)\|\Psi_z\| \leq u(\phi(z))$ for every $z \in G$;*
(ii) *for every $v \in V$, $\varepsilon > 0$ and compact set $K \subseteq G$, the set $\phi^{-1}(K) \cap N(v\|\Psi\|,\varepsilon)$ is compact, where $N(v\|\Psi\|,\varepsilon) = \{z \in G : v(z)\|\Psi_z\| \geq \varepsilon\}$.*

Proof. According to Proposition 3.1, Condition (i) implies that

$$W_{\Psi,\phi} : HU_b(G,E) \to HV_b(G,E)$$

is a weighted composition operator. Now, it is enough to show that $W_{\Psi,\phi} : HU_0(G,E) \to HV_0(G,E)$ is an into map. For, let $g \in HU_0(G,E)$. Fix $v \in V$ and $\varepsilon > 0$. We shall show that the set $K = \{z \in G : v(z)\|\Psi_z(f(\phi(z)))\| \geq \varepsilon\}$ is a compact subset of G. Now, condition (i) implies that there exists $u \in U$ such that $v(z)\|\Psi_z\| \leq u(\phi(z))$ for every $z \in G$. Clearly the set $S = \{z \in G : u(z)\|f(z)\| \geq \varepsilon\}$ is compact in G and $\phi(K) \subseteq S$. If we put $m = \sup\{\|f(z)\| : z \in S\}$, then $m > 0$ and $S \subseteq N\left(u, \frac{\varepsilon}{m}\right)$. From condition (ii), it follows that the set $H = \phi^{-1}(S) \cap N(v\|\Psi\|, \frac{\varepsilon}{m})$ is compact and hence K being a closed subset of H is compact. This completes the proof. □

The following corollaries are immediate consequences of Proposition 3.1 and Proposition 3.2.

Corollary 3.1. *Let U and V be arbitrary systems of weights on G. Let $\Psi \in H^{\infty}(G, B(E))$ and $\phi \in H(G)$ be such that $\phi(G) \subseteq G$. Then*

(i) *$W_{\Psi,\phi} : HU_b(G,E) \to HV_b(G,E)$ is a weighted composition operator if $V \leq U \circ \phi$.*

(ii) $W_{\Psi,\phi} : HU_0(G,E) \to HV_0(G,E)$ *is a weighted composition operator if* $V \leq U \circ \phi$ *and* ϕ *is a conformal mapping of* G *onto itself.*

Corollary 3.2. *Let* $V = \{\lambda\chi_K : \lambda \geq 0, K \subseteq G, K \text{ is compact}\}$. *Then every operator-valued analytic map* $\Psi : G \to B(E)$ *and self-analytic map* $\phi : G \to G$ *induces a weighted composition operator* $W_{\Psi,\phi}$ *on* $HV_0(G,E)$.

Remark 3.2. Corollary 3.2 makes it clear that the converse of Corollary 3.1 may not be true. That is, if $W_{\Psi,\phi}$ is a weighted composition operator on $HV_b(G,E)$, then $\Psi : G \to B(E)$ may not be bounded and $\phi : G \to G$ may not be conformal mapping. Thus the behaviour of the weighted composition operator is very much influenced by different systems of weights on G.

Now in the following theorems, we obtained a necessary and sufficient conditions for $W_{\Psi,\phi}$ to be a weighted composition operator on the spaces $HV_b(G,E)$ and $HV_0(G,E)$.

Theorem 3.1. *Let* V *be an arbitrary system of weights on* G *and let* U *be a reasonable system of weights on* G. *Let* $\Psi : G \to B(E)$ *and* $\phi : G \to G$ *be analytic mappings. Then* $W_{\Psi,\phi} : H\tilde{U}_b(G,E) \to HV_b(G,E)$ *is a weighted composition operator if and only if for every* $v \in V$, *there exists* $u \in U$ *such that* $v(z)\|\Psi_z\| \leq \tilde{u}(\phi(z))$ *for every* $z \in G$.

Proof. We first show that $W_{\Psi,\phi}$ is a weighted composition operator. For, let $v \in V$ and let $B_{v,E}$ be a neighborhood of the origin in $HV_b(G,E)$. Then by the given condition, there exists $u \in U$ such that $v(z)\|\Psi_z\| \leq \tilde{u}(\phi(z))$ for every $z \in G$. We claim that $W_{\Psi,\phi}(B_{\tilde{u},E}) \subseteq B_{v,E}$. Let $f \in B_{\tilde{u},E}$. Then $\|f\|_{\tilde{u},E} \leq 1$. Now

$$\begin{aligned}
\|W_{\Psi,\phi}f\|_{v,E} &= \sup\{v(z)\|\Psi_z(f(\phi(z)))\| : z \in G\} \\
&\leq \sup\{v(z)\|\Psi_z\|\|f(\phi(z))\| : z \in G\} \\
&\leq \sup\{\tilde{u}(\phi(z))\|f(\phi(z))\| : z \in G\} \\
&\leq \sup\{\tilde{u}(z)\|f(z)\| : z \in G\} \\
&= \|f\|_{\tilde{u},E} \leq 1.
\end{aligned}$$

This shows that $W_{\Psi,\phi}$ is a weighted composition operator.

Conversely, suppose that $W_{\Psi,\phi}$ is a weighted composition operator. To establish the given condition, let $v \in V$. Then by the continuity of $W_{\Psi,\phi}$ at the origin, there exists $\tilde{u} \in \tilde{U}$ with $u \in U$ such that $u \leq \tilde{u}$ and $W_{\Psi,\phi}(B_{\tilde{u},E}) \subseteq B_{v,E}$. Now, we claim that $v(z)\|\Psi_z\| \leq \tilde{u}(\phi(z))$ for every $z \in G$. Fix $z_0 \in G$. Then by (2.c), there exists $f_{z_0} \in B_u$ such that $|f_{z_0}(\phi(z_0))| = \dfrac{1}{\tilde{u}(\phi(z_0))}$. Let $y \in E$ be such that $\|y\| = 1$ and let $g_{z_0} : G \to E$ be defined as $g_{z_0}(z) = f_{z_0}(z)y$ for every $z \in G$. Then clearly $g_{z_0} \in B_{u,E}$ and $\|g_{z_0}(\phi(z_0))\| = \dfrac{1}{\tilde{u}(\phi(z_0))}$. Also, according to (2.b), $f_{z_0} \in B_{\tilde{u}}$ and therefore $g_{z_0} \in B_{\tilde{u},E}$. Thus $W_{\Psi,\phi}(g_{z_0}) \in B_{v,E}$. That is, $v(z)\|\Psi_z(g_{z_0}(\phi(z)))\| \leq 1$ for every $z \in G$. For $z = z_0$, we have

$v(z_0)\|\Psi_{z_0}(y)\| \leq \tilde{u}(\phi(z_0))$. Further, it implies that $v(z_0)\|\Psi_{z_0}\| \leq \tilde{u}(\phi(z_0))$. This completes the proof of the theorem. □

Corollary 3.3. *Let V be an arbitrary system of weights on G and let U be an essential system of weights on G. Let $\Psi : G \to B(E)$ and $\phi : G \to G$ be analytic mappings. Then $W_{\Psi,\phi} : HU_b(G,E) \to HV_b(G,E)$ is a weighted composition operator if and only if for every $v \in V$, there exists $u \in U$ such that $v(z)\|\Psi_z\| \leq u(\phi(z))$ for every $z \in G$.*

Proof. Follows from Theorem 3.1 since $U \cong \tilde{U}$. □

Corollary 3.4. *Let U and V be reasonable systems of weights on G. Let $\Psi : G \to B(E)$ and $\phi : G \to G$ be analytic mappings. Then the following statements are equivalent:*

(i) *$W_{\Psi,\phi} : H\tilde{U}_b(G,E) \to H\tilde{V}_b(G,E)$ is a weighted composition operator;*
(ii) *for every $v \in V$, there exists $u \in U$ such that $v(z)\|\Psi_z\| \leq \tilde{u}(\phi(z))$ for every $z \in G$;*
(iii) *for every $v \in V$, there exists $u \in U$ such that $\tilde{v}(z)\|\Psi_z\| \leq \tilde{u}(\phi(z))$ for every $z \in G$.*

Proof. The proof can be established by using the same technique as in Theorem 3.1. □

Theorem 3.2. *Let V be an arbitrary system of weights on G and let U be an essential system of weights on G such that each weight of U vanishes at infinity. Let $\Psi : G \to B(E)$ and $\phi : G \to G$ be analytic mappings. Then $W_{\Psi,\phi} : HU_0(G,E) \to HV_0(G,E)$ is a weighted composition operator if and only if*

(i) *for every $v \in V$, there exists $u \in U$ such that $v(z)\|\Psi_z(y)\| \leq u(\phi(z))\|y\|$ for every $z \in G$ and $y \in E$;*
(ii) *for each $v \in V$ and $\varepsilon > 0$, the set $\{z \in G : v(z)\|\Psi_z(y)\| \geq \varepsilon\}$ is compact for every $0 \neq y \in E$.*

Proof. Suppose that conditions (i) and (ii) hold. From condition (i), it follows that $v(z)\|\Psi_z\| \leq u(\phi(z))$ for every $z \in G$. According to Proposition 3.1, it readily follows that $W_{\Psi,\phi}$ is a weighted composition operator from $HU_b(G,E)$ to $HV_b(G,E)$. To show that $W_{\Psi,\phi} : HU_0(G,E) \to HV_0(G,E)$ is a weighted composition operator, it is enough to prove that $W_{\Psi,\phi}(HU_0(G,E)) \subseteq HV_0(G,E)$. Let $f \in HU_0(G,E)$ and let $v \in V$, $\varepsilon > 0$. We shall show that the set $S = \{z \in G : v(z)\|\Psi_z(f(\phi(z)))\| \geq \varepsilon\}$ is compact in G. Further, condition (i) implies that there exists $u \in U$ such that $v(z)\|\Psi_z(y)\| \leq u(\phi(z))\|y\|$ for every $z \in G$ and $y \in E$. Let $H = \{z \in G : u(z)\|f(z)\| \geq \varepsilon\}$. Then H is compact and $\phi(S) \subseteq H$. Let $m = \sup\{u(z) : z \in H\}$ and let $N_\varepsilon = \{y \in E : \|y\| < \frac{\varepsilon}{2m}\}$. Clearly N_ε is an open neighborhood of zero in E. Now, since $f(\phi(S))$ is totally bounded in E, there exists a finite set $\{f(\phi(z_i))\}_{i=1}^{n}$ in $f(\phi(S))$ such that $f(\phi(S)) \subseteq \bigcup_{i=1}^{n}[f(\phi(z_i)) + N_\varepsilon]$. Further, condition (ii) implies that the set $K_i = \left\{z \in G : v(z) \|\Psi_z(f(\phi(z_i)))\| \geq \frac{\varepsilon}{2}\right\}$ is compact for each $i \in \{1, 2, \ldots, n\}$. Let $z_0 \in S$. Then we may choose $j \in \{1, 2, \ldots, n\}$ such that $f(\phi(z_0)) - f(\phi(z_j)) \in N_\varepsilon$. Also, we have

$$\begin{aligned}
\varepsilon &\leq v(z_0) \|\Psi_{z_0}(f(\phi(z_0)))\| \\
&\leq v(z_0) \|\Psi_{z_0}(f(\phi(z_0))) - \Psi_{z_0}(f(\phi(z_j)))\| + v(z_0) \|\Psi_{z_0}(f(\phi(z_j)))\| \\
&\leq u(\phi(z_0)) \|f(\phi(z_0)) - f(\phi(z_j))\| + v(z_0) \|\Psi_{z_0}(f(\phi(z_j)))\| \\
&\leq \frac{\varepsilon}{2} + v(z_0) \|\Psi_{z_0}(f(\phi(z_j)))\| .
\end{aligned}$$

Thus it follows that $z_0 \in K_j$. Hence S is compact being a closed subset of the union $\bigcup_{i=1}^{n} K_i$. This shows that $W_{\Psi,\phi}(f) \in HV_0(G, E)$.

Conversely, suppose that $W_{\Psi,\phi} : HU_0(G, E) \to HV_0(G, E)$ is a weighted composition operator. Since $U \cong \tilde{U}$, therefore, by using the same argument as in Theorem 3.1, we can prove that for every $v \in V$, there exists $u \in U$ such that $v(z) \|\Psi_z\| \leq u(\phi(z))$ for every $z \in G$. Further, it is immediate that $v(z) \|\Psi_z(y)\| \leq u(\phi(z)) \|y\|$ for every $z \in G$ and $y \in E$. This proves condition (i). To prove condition (ii), we fix $v \in V$ and $\varepsilon > 0$. Let $0 \neq y \in E$. Consider the constant function $1_y : G \to E$ as $1_y(z) = y$ for every $z \in G$. Then clearly $1_y \in HU_0(G, E)$ and hence, $W_{\Psi,\phi}(1_y) \in HV_0(G, E)$. Clearly the set $C = \{z \in G : v(z) \|\Psi_z(y)\| \geq \varepsilon\}$ is compact in G. This proves condition (ii). With this the proof of the theorem is complete. □

Remark 3.3. Theorem 3.1, Theorem 3.2, Corollary 3.3, and Corollary 3.4 are generalizations of boundedness results of weighted composition operators given in ([8], [23], [26]). If $G = \mathbb{D}, E = \mathbb{C}, \Psi(z) = 1$ for every $z \in \mathbb{D}$ and U and V consists of single continuous weights only, then Corollary 3.3, Corollary 3.4, and Theorem 3.2 reduces to the boundedness results of composition operators given in [5]. If $\phi : G \to G$ is the identity map, then Proposition 3.1, Corollary 3.1, Corollary 3.2, Corollary 3.3, and Theorem 3.1 reduces to the boundedness results of multiplication operators given in [22].

4. Invertible weighted composition operators

In [30], Ohno and Takagi have characterized the invertible weighted composition operators on the disc algebra $A(\bar{\mathbb{D}})$ and the Hardy space $H^\infty(\mathbb{D})$. Further, in [23], the author has generalized these invertible results to the weighted spaces $HV_0(G)$ and $HV_b(G)$ of scalar-valued analytic functions. In this section, we characterize the operator-valued analytic mappings $\Psi : G \to B(E)$ and the self-analytic mappings $\phi : G \to G$ which generate invertible weighted composition operators on the spaces $HV_0(G, E)$ and $HV_b(G, E)$ of vector-valued analytic functions. We begin with stating a generalized definition of bounded below operator and an invertibility criterion on a Hausdorff topological vector space [35] which we shall use for characterizing invertible weighted composition operators.

A continuous linear operator T on a Hausdorff topological vector space E is said to be *bounded below* if for every neighborhood N of the origin in E, there exists a neighborhood M of the origin in E such that $T(N^c) \subseteq M^c$, where the symbol "c" stands for the complement of the neighborhood in E.

Theorem 4.1. (a) *Let E be a complete Hausdorff topological vector space and let $T : E \to E$ be a continuous linear operator. Then T is invertible if and only if T is bounded below and has dense range.*

(b) *Let E be a Hausdorff topological vector space and let $T : E \to E$ be a continuous linear operator. Then T is invertible if and only if T is bounded below and onto.*

Theorem 4.2. *Let V be an arbitrary system of weights on G. Let $\Psi : G \to B(E)$ and $\phi : G \to G$ be analytic mappings such that $W_{\Psi,\phi}$ is a weighted composition operator on $HV_b(G,E)$. Then $W_{\Psi,\phi}$ is invertible if*

(i) *for each $v \in V$, there exists $u \in V$ such that $v(\phi(z))\|y\| \leq u(z)\|\Psi_z(y)\|$ for every $z \in G$ and $y \in E$;*

(ii) *for each $z \in G, \Psi(z) : E \to E$ is onto and ϕ is a conformal mapping of G onto itself.*

Proof. According to Theorem 4.1, it is enough to show that $W_{\Psi,\phi}$ is bounded below and onto. For, let $v \in V$ and $B_{v,E}$ be a neighborhood of the origin in $HV_b(G,E)$. Then by condition (i), there exists $u \in V$ such that $v(\phi(z))\|y\| \leq u(z)\|\Psi_z(y)\|$ for every $z \in G$ and $y \in E$.

We claim that $W_{\Psi,\phi}\left(B^c_{v,E}\right) \subseteq B^c_{u,E}$. Let $f \in B^c_{v,E}$. Then

$$\begin{aligned}1 < \|f\|_{v,E} &= \sup\{v(z)\|f(z)\| : z \in G\} = \sup\{v(\phi(z))\|f(\phi(z))\| : z \in G\}\\ &\leq \sup\{u(z)\|\Psi_z(f(\phi(z)))\| : z \in G\}\\ &= \|W_{\Psi,\phi}f\|_{u,E}\,.\end{aligned}$$

This shows that $W_{\Psi,\phi}(f) \in B^c_{u,E}$ and hence $W_{\Psi,\phi}$ is bounded below. Now, we shall show that $W_{\Psi,\phi}$ is onto. For, let $g \in HV_b(G,E)$. Fix $z_0 \in G$ and let $v \in V$ be such that $v(\phi(z_0)) > 0$. Then by condition (i), there exists $u \in V$ such that $v(\phi(z_0))\|y\| \leq u(z_0)\|\Psi_{z_0}(y)\|$ for every $y \in E$. That is, $\|\Psi_{z_0}(y)\| \geq \lambda\|y\|$ for every $y \in E$, where $\lambda = \dfrac{v(\phi(z_0))}{u(z_0)} > 0$. This proves that Ψ_{z_0} is bounded below on E and hence, by condition (ii), Ψ_{z_0} is invertible in $B(E)$. We denote the inverse of Ψ_{z_0} by $\Psi^{-1}_{z_0}$. We define an analytic map $\Psi^{-1} : G \to B(E)$ as $\Psi^{-1}(z) = \Psi^{-1}_z$ for every $z \in G$. Also, by condition (ii), we have inverse analytic map $\phi^{-1} : G \to G$. Now, we define an analytic map $h : G \to E$ as $h = \Psi^{-1}\circ\phi^{-1}\cdot g\circ\phi^{-1}$. Then we show that $h \in HV_b(G,E)$. For, let $v \in V$. Then by condition (i), there exists $u \in V$ such that $v(\phi(z))\|y\| \leq u(z)\|\Psi_z(y)\|$ for every $z \in G$ and $y \in E$.

Now, we have

$$\begin{aligned}\|h\|_{v,E} &= \sup\{v(z)\|h(z)\| : z\in G\}\\ &= \sup\{v(\phi(z))\|h(\phi(z))\| : z\in G\}\\ &\le \sup\{u(z)\|\Psi_z(h(\phi(z)))\| : z\in G\}\\ &= \sup\left\{u(z)\left\|\Psi_z\left(\Psi_z^{-1}(g(z))\right)\right\| : z\in G\right\}\\ &= \sup\{u(z)\|g(z)\| : z\in G\} = \|g\|_{u,E} < \infty.\end{aligned}$$

This proves that $h \in HV_b(G,E)$ and clearly $W_{\Psi,\phi}(h) = g$. Hence $W_{\Psi,\phi}$ is invertible. □

Corollary 4.1. *Let V be an arbitrary system of weights on G. Let $\Psi : G \to B(E)$ and $\phi : G \to G$ be analytic mappings. Then $W_{\Psi,\phi} : HV_0(G,E) \to HV_0(G,E)$ is an invertible weighted composition operator if*

(i) *Ψ is bounded, bounded away from zero and for each $z \in G, \Psi(z) : E \to E$ is onto;*

(ii) *ϕ is a conformal mapping of G onto itself such that $V \circ \phi \cong V$.*

Proof. Since $\Psi : G \to B(E)$ is bounded analytic function and $\phi : G \to G$ is a conformal mapping such that $V \le V \circ \phi$, therefore Corollary 3.1 implies that $W_{\Psi,\phi}$ is a weighted composition operator on $HV_0(G,E)$. Also, since $\Psi : G \to B(E)$ is bounded away from zero, there exists $m > 0$ such that $\|\Psi_z(y)\| \ge m\|y\|$ for every $z \in G$ and $y \in E$. Let $v \in V$. Then, since $V \circ \phi \le V$, there exists $u \in V$ such that $v(\phi(z)) \le u(z)$ for every $z \in G$. Further, there exists $w \in V$ such that $\frac{1}{m}v \le w$ and $v(\phi(z))\|y\| \le w(z)\|\Psi_z(y)\|$ for every $z \in G$ and $y \in E$. Thus, according to the given conditions and Theorem 4.2, it follows that $W_{\Psi,\phi}$ is an invertible weighted composition operator on $HV_b(G,E)$. □

Corollary 4.2. *Let G be a simply connected open subset of $\mathbb{C}$ and let $V = \{\lambda\chi_K : \lambda \ge 0, K \subseteq G, K \text{ is compact}\}$. Let $\Psi : G \to B(E)$ and $\phi : G \to G$ be analytic mappings. Then $W_{\Psi,\phi}$ is an invertible weighted composition operator on $HV_b(G,E)$ if and only if $\Psi(z)$ is invertible in $B(E)$ for every $z \in G$ and ϕ is a conformal mapping of G onto itself.*

Proof. In view of Corollary 3.2, we can conclude that $W_{\Psi,\phi}$ is a weighted composition operator on $HV_0(G,E)$. Since $\Psi(z)$ is invertible in $B(E)$ for every $z \in G$, we define an analytic map $\Psi^{-1} : G \to B(E)$ as $\Psi^{-1}(z) = (\Psi(z))^{-1}$ for every $z \in G$. Also, since $\phi : G \to G$ is a conformal mapping, it follows that $\phi^{-1} : G \to G$ is analytic. Again, according to Corollary 3.2, it follows that $W_{\Psi^{-1}\circ\phi^{-1},\phi^{-1}}$ is a weighted composition operator on $HV_0(G,E)$ and it is the inverse of $W_{\Psi,\phi}$. This proves that $W_{\Psi,\phi}$ is invertible.

Conversely, suppose that $W_{\Psi,\phi}$ is an invertible weighted composition operator on $HV_0(G,E)$. To show that $\Psi(z)$ is invertible in $B(E)$ for every $z \in G$, it is enough to prove that $\Psi(z)$ is bounded below and onto. For, let $z_0 \in G$ and let $K = \{z_0\}$. Let $v = \chi_K$ and let $y \in E$. Define the constant function

$1_y : G \to E$ as $1_y(z) = y$ for every $z \in G$. Clearly $1_y \in HV_0(G,E)$. Since $W_{\Psi,\phi}$ is invertible on $HV_0(G,E)$ and hence by Theorem 4.1, $W_{\Psi,\phi}$ is bounded below and onto. Now, there exists $m > 0$ such that $\|W_{\Psi,\phi} f\|_v \geq \frac{1}{m}\|f\|_v$ for every $f \in HV_0(G,E)$. In particular, for $f = 1_y$, we have

$$\|W_{\Psi,\phi} 1_y\|_v \geq \frac{1}{m}\|1_y\|_v .$$

That is, $\|\Psi_{z_0}(y)\| \geq \frac{1}{m}\|y\|$ for every $y \in E$. Thus, Ψ_{z_0} is bounded below. To show that $\Psi_{z_0} : E \to E$ is onto, we fix $y \in E$. Then for $1_y \in HV_0(G,E)$, there exists $g \in HV_0(G,E)$ such that $W_{\Psi,\phi} g = 1_y$. That is, $\Psi_{z_0}(g(\phi(z_0))) = y$. This proves that Ψ_{z_0} is invertible. In order to show that ϕ is a conformal mapping, it is enough to prove that ϕ is bijective. Let $z_1, z_2 \in G$ be such that $z_1 \neq z_2$. Then by the Riemann Mapping Theorem, there exists one-one analytic function $g \in H(G)$ such that $g(z_1) \neq g(z_2)$. Now, if we define an analytic function $h : G \to \mathbb{C}$ as $h(z) = g(z) - g(z_1)$, then clearly $h \in HV_0(G)$ such that and $h(z_1) = 0$ and $h(z_2) \neq 0$. Let $y \in E$ be such that $\|y\| = 1$. Then define $h_y : G \to E$ as $h_y(z) = h(z)y$ for every $z \in G$. Then clearly $h_y \in HV_0(G,E)$ such that $h_y(z_1) = 0$ and $h_y(z_2) \neq 0$. Since Ψ_{z_1} and Ψ_{z_2} are bounded below operators on E, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\|\Psi_{z_1}(y)\| \geq \frac{1}{\delta_1}\|y\| \text{ and } \|\Psi_{z_2}(y)\| \geq \frac{1}{\delta_2}\|y\|$$

for every $y \in E$. Let

$$F = \frac{1}{\delta_2 \|h_y(z_2)\|} h_y.$$

Then $F \in HV_0(G,E)$. Let $\delta = \delta_1 + \delta_2$ and let $v = \delta\chi_K$, where $K = \{z_1, z_2\}$. Since $W_{\Psi,\phi}(HV_0(G,E))$ is dense in $HV_0(G,E)$, there exists $f \in HV_0(G,E)$ such that $\|W_{\Psi,\phi} f - F\|_v < \frac{1}{2}$. That is, we have

$$\delta\|\Psi_{z_1}(f(\phi(z_1)))\| < \frac{1}{2} \text{ and } \delta\left|\|\Psi_{z_2}(f(\phi(z_2)))\| - \frac{1}{\delta_2}\right| < \frac{1}{2}.$$

Further, it implies that $\frac{\delta}{\delta_1}\|f(\phi(z_1))\| < \frac{1}{2}$ and $\frac{\delta}{\delta_2}|\|f(\phi(z_2))\| - 1| < \frac{1}{2}$. Thus,

$$\|f(\phi(z_1))\| < \frac{1}{2} \text{ and } |\|f(\phi(z_2))\| - 1| < \frac{1}{2}.$$

Now, if $\phi(z_1) = \phi(z_2)$, then from the last two inequalities we get a contradiction. Hence, $\phi(z_1) \neq \phi(z_2)$. Also, using the injectivity argument of $W_{\Psi,\phi}$, we can prove that ϕ is onto. With this the proof is complete. □

Remark 4.1. Corollary 4.2 makes it clear that the converse of Corollary 4.1 may not be true. For instance, let $G = \{z = x + iy : y > 0\}$ be the upper half plane and let $V = \{\lambda\chi_K : \lambda \geq 0, K \subseteq G, K \text{ is compact}\}$. Let $E = H^\infty(G)$ and let $\Psi : G \to B(E)$ be defined as $\Psi(z) = M_z$ for every $z \in G$, where

$M_z : H^\infty(G) \to H^\infty(G)$ is defined as $M_z f = zf$ for every $f \in H^\infty(G)$. Then clearly $\|\Psi(z)\| = \|M_z\| = |z|$ for every $z \in G$. Let $\phi : G \to G$ be defined as $\phi(z) = z + 1$ for every $z \in G$. Then clearly ϕ is a conformal mapping of G onto itself and $\Psi(z)$ is invertible in $B(E)$ for every $z \in G$. Thus according to Corollary 4.2, $W_{\Psi,\phi}$ is an invertible weighted composition operator. But Ψ is neither bounded nor bounded away from zero. This shows that the theory of weighted composition operators is very much influenced by different systems of weights.

Theorem 4.3. *Let G be a simply connected open subset of $\mathbb{C}$ and let V be a reasonable system of weights on G. Let $\phi : G \to G$ and $\Psi : G \to B(E)$ be analytic maps such that each $\Psi(z)$ is one-one and $W_{\Psi,\phi}$ is a weighted composition operator on $H\tilde{V}_b(G, E)$. Then $W_{\Psi,\phi}$ is invertible if and only if*

(i) *ϕ is a conformal mapping of G onto itself and $\Psi(z) : E \to E$ is onto for every $z \in G$;*

(ii) *for each $v \in V$, there exists $u \in V$ such that $\tilde{v}(\phi(z))\|y\| \le \tilde{u}(z)\|\Psi_z(y)\|$ for every $z \in G$ and $y \in E$.*

Proof. Suppose conditions (i) and (ii) hold. Let $z_0 \in G$ and let $v \in V$ be such that $v(\phi(z_0)) > 0$. Then $\tilde{v}(\phi(z_0)) > 0$ and by condition (ii), there exists $u \in V$ such that $\tilde{v}(\phi(z_0))\|y\| \le \tilde{u}(z_0)\|\Psi_{z_0}(y)\|$ for every $y \in E$. Thus $\|\Psi_{z_0}(y)\| \ge \lambda\|y\|$ for every $y \in E$, where $\lambda = \frac{\tilde{v}(\phi(z_0))}{\tilde{u}(z_0)} > 0$. This implies that each Ψ_{z_0} is bounded below and hence, by condition (i), it follows that each Ψ_{z_0} is invertible and we denote its inverse by $\Psi_{z_0}^{-1}$. Define an analytic map $\Psi^{-1} : G \to B(E)$ as $\Psi^{-1}(z) = \Psi_z^{-1}$ for every $z \in G$. From condition (ii). It follows that $\tilde{v}(\phi(z))\left\|\Psi_z^{-1}(y)\right\| \le \tilde{u}(z)\|y\|$. Further by condition (i), we have that $\tilde{v}(z)\left\|\Psi_{\phi^{-1}(z)}^{-1}(y)\right\| \le \tilde{u}(\phi^{-1}(z))\|y\|$ for every $z \in G$ and $y \in E$. That is, $\tilde{v}(z)\left\|\left(\Psi^{-1} \circ \phi^{-1}\right)(z)\right\| \le \tilde{u}\left(\phi^{-1}(z)\right)$ for every $z \in G$. According to Theorem 3.1, $W_{\Psi^{-1}\circ\phi^{-1},\phi^{-1}}$ is a weighted composition operator on $H\tilde{V}_b(G, E)$ such that $W_{\Psi,\phi}W_{\Psi^{-1}\circ\phi^{-1},\phi^{-1}} = W_{\Psi^{-1}\circ\phi^{-1},\phi^{-1}}W_{\Psi,\phi} = I$, the identity operator. Hence, $W_{\Psi,\phi}$ is invertible.

Conversely, suppose that $W_{\Psi,\phi}$ is invertible on $H\tilde{V}_b(G, E)$. We first show that for each $z_0 \in G, \Psi_{z_0} : E \to E$ is onto. Let $f_{z_0} \in \tilde{H}V_b(G)$ be such that $f_{z_0}(z_0) = 1$. Choose $y \in E$ such that $\|y\| = 1$. If we define $g_y : G \to E$ as $g_y(z) = f_{z_0}(z)y$ for every $z \in G$, then clearly $g_y \in H\tilde{V}_b(G, E)$. Since $W_{\Psi,\phi}$ is onto, there exists $f \in \tilde{H}V_b(G, E)$ such that $W_{\Psi,\phi}(f) = g_y$. That is, $\Psi_{z_0}(f(\phi(z_0))) = y$. This proves that Ψ_{z_0} is onto. Now, to show that ϕ is a conformal mapping, we need to prove that ϕ is bijective. Let $z_1, z_2 \in G$ be such that $z_1 \ne z_2$. Then by the Riemann Mapping Theorem, there exists one-one analytic function $g \in H^\infty(G)$ such that $g(z_1) \ne g(z_2)$. Define $h : G \to \mathbb{C}$ as $h(z) = g(z) - g(z_1)$ for every $z \in G$. Let $g_{z_2} \in H\tilde{V}_b(G)$ be such that $g_{z_2}(z_2) \ne 0$ and let $h_0 = hg_{z_2}$. Since $H\tilde{V}_b(G)$ is a module over $H^\infty(G)$, it follows that $h_0 \in H\tilde{V}_b(G)$ such that $h_0(z_1) = 0$ and $h_0(z_2) \ne 0$. Let $y \in E$

be such that $\|y\| = 1$. Then define $h_y : G \to E$ as $h_y(z) = h_0(z)\, y$ for every $z \in G$. Then clearly $h_y \in H\tilde{V}_b(G, E)$ such that by $h_y(z_1) = 0$ and $h_y(z_2) \neq 0$. Since Ψ_{z_1} and Ψ_{z_2} are invertible operators and hence bounded below, there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$\|\Psi_{z_1}(y)\| \geq \frac{1}{\varepsilon_1}\|y\| \text{ and } \|\Psi_{z_2}(y)\| \geq \frac{1}{\varepsilon_2}\|y\|$$

for every $y \in E$. Let

$$H = \frac{1}{\varepsilon_2 \|h_y(z_2)\|} h_y.$$

Then $H \in H\tilde{V}_b(G, E)$. Let $\varepsilon = \varepsilon_1 + \varepsilon_2$ and let $v \in V$ be such that $v(z_1) \geq 1$ and $v(z_2) \geq 1$. Then there exists $\tilde{v} \in \tilde{V}$ such that $\tilde{v}(z_1) \geq 1$ and $\tilde{v}(z_2) \geq 1$. Since $W_{\Psi,\phi}\left(H\tilde{V}_b(G, E)\right)$ is dense in $H\tilde{V}_b(G, E)$, there exists $F \in H\tilde{V}_b(G, E)$ such that $\|W_{\Psi,\phi}F - H\|_{\tilde{v}} < \frac{1}{2\varepsilon}$. That is,

$$\tilde{v}(z_1)\|\Psi_{z_1}(F(\phi(z_1)))\| < \frac{1}{2\varepsilon} \text{ and } \tilde{v}(z_2)\left|\|\Psi_{z_2}(F(\phi(z_2)))\| - \frac{1}{\varepsilon_2}\right| < \frac{1}{2\varepsilon}.$$

Further, it implies that

$$\|\Psi_{z_1}(F(\phi(z_1)))\| < \frac{\varepsilon_1}{2\varepsilon} < \frac{1}{2} \text{ and } |\|\Psi_{z_2}(F(\phi(z_2)))\| - 1| < \frac{\varepsilon_2}{2\varepsilon} < \frac{1}{2}.$$

From the last two inequalities, it is easy to conclude that $\phi(z_1) \neq \phi(z_2)$. Thus ϕ is injective and also using the injectivity argument of $W_{\Psi,\phi}$, we can conclude that ϕ is onto. Now to prove condition (ii), we fix $v \in V$. Then there exists $\tilde{v} \in \tilde{V}$ such that $v \leq \tilde{v}$. According to Theorem 4.1, $W_{\Psi,\phi}$ is bounded below on $H\tilde{V}_b(G, E)$. Thus, there exists $\tilde{u} \in \tilde{V}$ with $u \in V$ such that $W_{\Psi,\phi}\left(B^c_{\tilde{v},E}\right) \subseteq B^c_{\tilde{u},E}$. We claim that $\tilde{v}(\phi(z))\left\|\Psi_z^{-1}(y)\right\| \leq \tilde{u}(z)\|y\|$ for every $z \in G$ and $y \in E$. Let $z_0 \in G$ and $y_0 \in E$. According to (2.c), there exists $f_{z_0} \in B_u$ such that $|f_{z_0}(z_0)| = \frac{1}{\tilde{u}(z_0)}$. Define $h_0 : G \to E$ as $h_0(z) = \frac{y_0}{\|y_0\|} f_{z_0}(z)$ for every $z \in G$. Thus $h_0 \in B_{u,E}$ and $\|h_0(z_0)\| = \frac{1}{\tilde{u}(z_0)}$. Also (2.b) implies that $f_{z_0} \in B_{\tilde{u}}$ and hence $h_0 \in B_{\tilde{u},E}$. Since $W_{\Psi,\phi}$ is onto, there exists $g_0 \in H\tilde{V}_b(G, E)$ such that $W_{\Psi,\phi}(g_0) = h_0$. That is,

$$\Psi_{z_0}(g_0(\phi(z_0))) = h_0(z_0).$$

Since each Ψ_{z_0} is invertible, we have

$$g_0(\phi(z_0)) = \Psi_{z_0}^{-1}(h_0(z_0)) = \Psi_{z_0}^{-1}(y_0)\frac{f_{z_0}(z_0)}{\|y_0\|}.$$

Since $h_0 \in B_{\tilde{u},E}$, we have $W_{\Psi,\phi}(g_0) \notin B^c_{\tilde{u},E}$. This implies that $g_0 \notin B^c_{\tilde{v},E}$. That is, $\tilde{v}(z)\|g_0(z)\| \leq 1$ for every $z \in G$. For $z = \phi(z_0)$, we have

$$\tilde{v}(\phi(z_0))\|g_0(\phi(z_0))\| \leq 1.$$

That is, $\tilde{v}(\phi(z_0)) \left\|\Psi_{z_0}^{-1}(y_0)\right\| \leq \tilde{u}(z_0)\|y_0\|$. This proves our claim. Since each $\Psi(z)$ is invertible, we have $\tilde{v}(\phi(z))\|y\| \leq \tilde{u}(z)\|\Psi_z(y)\|$ for $z \in G$ and $y \in E$. This proves condition (ii). With this the proof is complete. □

Corollary 4.3. *Let G be a simply connected open subset of $\mathbb{C}$ and let V be an essential system of weights on G. Let $\Psi : G \to B(E)$ and $\phi : G \to G$ be analytic mappings such that each $\Psi(z)$ is one-one and $W_{\Psi,\phi}$ is a weighted composition operator on $HV_b(G, E)$. Then $W_{\Psi,\phi}$ is invertible if and only if*

(i) *ϕ is a conformal mapping of G onto itself and $\Psi(z) : E \to E$ is onto for every $z \in G$;*
(ii) *for every $v \in V$, there exists $u \in V$ such that $v(\phi(z))\|y\| \leq u(z)\|\Psi_z(y)\|$ for every $z \in G$ and $y \in E$.*

Proof. Follows from Theorem 4.3 since each $v \in V$ is essential. □

5. Compact weighted composition operators

In [7], Contreras and Diaz-Madrigal have characterized compact weighted composition operators on the Hardy space $H^\infty(\mathbb{D})$. Further in [30], Ohno and Takagi gave a different proof for characterizing compactness of weighted composition operators on $H^\infty(\mathbb{D})$. Also, Montes-Rodriguez [26] and Contreras and Hernandez-Diaz [8] have further extended these results to the weighted Banach spaces of analytic functions by computing the essential norm of these operators. In this section, our efforts are to generalize some of the compactness results to the spaces of vector-valued analytic functions.

Theorem 5.1. *Let $V = \{v\}$ and $U = \{u\}$ be consist of single continuous weights on $\mathbb{D}$, where u is essential. Let $\Psi : \mathbb{D} \to B(E)$ and $\phi : \mathbb{D} \to \mathbb{D}$ be analytic mappings. If the operator $W_{\Psi,\phi} : HU_b(\mathbb{D}, E) \to HV_b(\mathbb{D}, E)$ is compact, then*

(i) *$v(z)\|\Psi_z\| \leq u(\phi(z))$ for every $z \in \mathbb{D}$;*
(ii) $\displaystyle\lim_{r\to 1} \sup_{|\phi(z)|>r} \frac{v(z)\|\Psi_z\|}{u(\phi(z))} = 0.$

Proof. Since $W_{\Psi,\phi}$ is compact, it is continuous and hence by Corollary 3.3, condition (i) follows. Now, we assume that condition (ii) does not holds. Then there exists $\varepsilon > 0$ and a sequence $\{z_n\}$ in $\mathbb{D}$ such that

$$|\phi(z_n)| \to 1 \quad \text{and} \quad \frac{v(z_n)\|\Psi_{z_n}\|}{u(\phi(z_n))} \geq \varepsilon \quad \text{for all } n.$$

Further, it implies that there exists a sequence $\{y_n\}$ in E such that

$$\|y_n\| \leq 1 \quad \text{and} \quad \frac{v(z_n)\|\Psi_{z_n}(y_n)\|}{u(\phi(z_n))} \geq \varepsilon \quad \text{for all } n.$$

For each n, we select $f_n \in B_u$ such that $|f_n(\phi(z_n))|\, u(\phi(z_n)) \geq \frac{1}{2}$ and $\alpha(n) \in \mathbb{N}$ such that $|\phi(z_n)|^{\alpha(n)} \geq \frac{1}{2}$ and $\alpha(n) \to \infty$. Now, for each n, if we define $h_n(z) = z^{\alpha(n)} f_n(z)\, y_n$ for all $z \in \mathbb{D}$, then clearly the sequence $\{h_n\}$ is in

$B_{u,E}$ and converges to zero uniformly on the compact subsets of $\mathbb{D}$. Also, compactness of $W_{\Psi,\phi}$ implies that $\|W_{\Psi,\phi}h_n\|_v \to 0$. But

$$\begin{aligned}\|W_{\Psi,\phi}h_n\|_v &= \sup\{v(z)\|\Psi_z(h_n(\phi(z)))\| : z \in \mathbb{D}\}\\ &\geq v(z_n)\|\Psi_{z_n}(h_n(\phi(z_n)))\|\\ &\geq v(z_n)|\phi(z_n)|^{\alpha(n)}|f_n(\phi(z_n))|\,\|\Psi_{z_n}(y_n)\|\\ &\geq \frac{1}{4}\cdot\frac{v(z_n)\|\Psi_{z_n}(y_n)\|}{u(\phi(z_n))} \geq \frac{\varepsilon}{4},\end{aligned}$$

which is a contradiction. Hence, condition (ii) holds. This completes the proof. □

Theorem 5.2. *Let V be the system of constant weights on $\mathbb{D}$ and let E be a finite dimensional Banach space. Let $\Psi : \mathbb{D} \to B(E)$ and $\phi : \mathbb{D} \to \mathbb{D}$ be analytic mappings. Then $W_{\Psi,\phi}$ is a compact weighted composition operator on $HV_b(\mathbb{D}, E)$ if and only if*

(i) *Ψ is bounded,*

(ii) *for every sequence $\{z_n\}$ in $\mathbb{D}$ such that $|\phi(z_n)| \to 1$ as $n \to \infty$, we have $\|\Psi_{z_n}\| \to 0$.*

Proof. Suppose that $W_{\Psi,\phi}$ is a compact weighted composition operator on $HV_b(\mathbb{D}, E)$. For $y \in E$, we define the constant function $1_y(z) = y$ for every $z \in \mathbb{D}$. Obviously $1_y \in HV_b(\mathbb{D}, E)$. Also, there exists $M > 0$ such that $\|W_{\Psi,\phi}1_y\|_\infty \leq M\|1_y\|_\infty$ for every $y \in E$. That is, $\|\Psi_z(y)\| \leq M\|y\|$ for every $z \in \mathbb{D}$ and $y \in E$. Thus $\|\Psi_z\| \leq M$ for every $z \in \mathbb{D}$ and hence Ψ is bounded. Condition (ii) follows from Theorem 5.1.

Conversely, suppose that conditions (i) and (ii) are true. Condition (i) clearly implies that $W_{\Psi,\phi}$ is a weighted composition operator on $HV_b(\mathbb{D}, E)$. Now, using condition (ii), we shall prove that $W_{\Psi,\phi}$ is compact. Let $K \subseteq HV_b(\mathbb{D}, E)$ be a bounded subset. Then we shall show that $W_{\Psi,\phi}(B)$ is relatively compact in $HV_b(\mathbb{D}, E)$. For, let $\{f_n\}$ be a sequence in K. Since K is bounded, it is relatively compact with respect to the topology of the uniform convergence on compact subsets of $\mathbb{D}$. By Montel's Theorem, there exists a subsequence $\{f_{n_k}\}$ which converges uniformly on compact subsets of $\mathbb{D}$ to some holomorphic function f. Clearly $f \in HV_b(\mathbb{D}, E)$ We shall show that $\|W_{\Psi,\phi}f_{n_k} - W_{\Psi,\phi}f\|_\infty \to 0$, as $k \to \infty$. Suppose this is not true. Then there exists $\varepsilon > 0$ and a subsequence of $\{f_{n_k}\}$ which we still denote by $\{f_{n_k}\}$ such that $\|W_{\Psi,\phi}f_{n_k} - W_{\Psi,\phi}f\|_\infty > \varepsilon$ for all k. That is,

$$\sup\{\|\Psi_z(f_{n_k}(\phi(z))) - \Psi_z(f(\phi(z)))\| : z \in \mathbb{D}\} > \varepsilon$$

for all k. Further, it implies that there exists a sequence $\{z_{n_k}\}$ in $\mathbb{D}$ such that $\left\|\Psi_{z_{n_k}}(f_{n_k}(\phi(z_{n_k}))) - \Psi_{z_{n_k}}(f(\phi(z_{n_k})))\right\| > \varepsilon$. That is,

$$\varepsilon < \left\|\Psi_{z_{n_k}}\right\| \|f_{n_k}(\phi(z_{n_k})) - f(\phi(z_{n_k}))\|.$$

We can assume that $\phi(z_{n_k}) \to z_0$. Since $f_{n_k} \to f$ uniformly on every compact subset of $\mathbb{D}$, it follows that $z_0 \in \mathbb{T}$. Thus $|\phi(z_{n_k})| \to 1$, as $k \to \infty$. Further, by the given condition, we have $\left\|\Psi_{z_{n_k}}\right\| \to 0$ and hence, we get a contradiction. This proves that $W_{\Psi,\phi}$ is a compact weighted composition operator. □

6. Examples

Example 6.1. Let $G = \mathbb{D}$, the open unit disc and let $E = H^\infty(\mathbb{D})$. Let $V = \{\lambda\chi_K : \lambda \geq 0, K \subseteq \mathbb{D}, K \text{ a compact set}\}$. Then $HV_0(\mathbb{D}, E) = (H(\mathbb{D}, E), k)$, where k denotes the compact open topology. Now, we define $\Psi: \mathbb{D} \to B(H^\infty(\mathbb{D}))$ as $\Psi(z) = M_z$ for every $z \in \mathbb{D}$, where $M_z : H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D})$ is defined as $M_z f = e^z f$ for every $f \in H^\infty(\mathbb{D})$. Let $\phi : \mathbb{D} \to \mathbb{D}$ be defined as $\phi(z) = \frac{1-2z}{2-z}$ for every $z \in \mathbb{D}$. Since $\Psi : \mathbb{D} \to B(E)$ is an operator-valued analytic map such that each $\Psi(z)$ is invertible in $B(E)$ and ϕ is a conformal mapping of $\mathbb{D}$ onto itself, according to Corollary 4.2, it follows that $W_{\Psi,\phi}$ is an invertible weighted composition operator on $HV_0(\mathbb{D}, E)$.

Example 6.2. Let $G = \mathbb{D}$ and let v be a weight defined as $v(z) = 1 - |z|^2$ for every $z \in \mathbb{D}$. Let $V = \{\lambda v : \lambda \geq 0\}$ and let $E = H^\infty(\mathbb{D})$. Then clearly V is an essential system of weights. Let $\phi : \mathbb{D} \to \mathbb{D}$ be defined as $\phi(z) = z^2$ for every $z \in \mathbb{D}$ and let $\Psi : \mathbb{D} \to B(H^\infty(\mathbb{D}))$ be defined as $\Psi(z) = M_z$ for every $z \in \mathbb{D}$, where $M_z : H^\infty(\mathbb{D}) \to H^\infty(\mathbb{D})$ is defined as $M_z f = 2zf$ for every $f \in H^\infty(\mathbb{D})$. Now, by the Pick-Schwarz Lemma, we have $v(z)\|\Psi(z)\| = \left(1 - |z|^2\right)|\phi'(z)| \leq 1 - |\phi(z)|^2 = v(\phi(z))$ for every $z \in \mathbb{D}$. Hence, by Corollary 3.3, $W_{\Psi,\phi}$ is a weighted composition operator on $HV_b(\mathbb{D}, E)$. But, according to Corollary 4.3, $W_{\Psi,\phi}$ is not an invertible weighted composition operator.

References

[1] K. D. Bierstedt, J. Bonet, and A. Galbis, *Weighted spaces of holomorphic functions on balanced domains*, Michigan Math. J. **40** (1993), no. 2, 271–297.

[2] K. D. Bierstedt, J. Bonet, and J. Taskinen, *Associated weights and spaces of holomorphic functions*, Studia Math. **127** (1998), no. 2, 137–168.

[3] K. D. Bierstedt, R. Meise, and W. H. Summers, *A projective description of weighted inductive limits*, Trans. Amer. Math. Soc. **272** (1982), no. 1, 107–160.

[4] K. D. Bierstedt and W. H. Summers, *Biduals of weighted Banach spaces of analytic functions*, J. Austral. Math. Soc. Ser. A **54** (1993), no. 1, 70–79.

[5] J. Bonet, P. Domanski, M. Lindström, and J. Taskinen, *Composition operators between weighted Banach spaces of analytic functions*, J. Austral. Math. Soc. Ser. A **64** (1998), no. 1, 101–118.

[6] M. Cambern and K. Jarosz, *The isometries of H^1_H*, Proc. Amer. Math. Soc. **107** (1989), no. 1, 205–214.

[7] M. D. Contreras and S. Diaz-Madrigal, *Compact-type operators defined on H^∞*, Function spaces (Edwardsville, IL, 1998), 111–118, Contemp. Math., 232, Amer. Math. Soc., Providence, RI, 1999.

[8] M. D. Contreras and A. G. Hernandez-Diaz, *Weighted composition operators in weighted Banach spaces of analytic functions*, J. Austral. Math. Soc. Ser. A **69** (2000), no. 1, 41–60.

[9] ———, *Weighted composition operators on Hardy spaces*, J. Math. Anal. Appl. **263** (2001), no. 1, 224–233.

[10] J. B. Conway, *Functions of One Complex Variable*, Second edition. Graduate Texts in Mathematics, 11. Springer-Verlag, New York-Berlin, 1978.

[11] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[12] K. De Leeuw, W. Rudin, and J. Wermer, *The isometries of some function spaces*, Proc. Amer. Math. Soc. **11** (1960), 694–698.

[13] F. Forelli, *The isometries of H^p*, Canad. J. Math. **16** (1964), 721–728.

[14] A. Galbis, *Weighted Banach spaces of entire functions*, Arch. Math. (Basel) **62** (1994), no. 1, 58–64.

[15] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, Inc. 1981.

[16] F. Jafari, T. Tonev, E. Toneva, and K. Yale, *Holomorphic flows, cocycles, and coboundaries*, Michigan Math. J. **44** (1997), no. 2, 239–253.

[17] C. J. Kolaski, *Surjective isometries of weighted Bergman spaces*, Proc. Amer. Math. Soc. **105** (1989), no. 3, 652–657.

[18] P.-K. Lin, *The isometries of $H^\infty(E)$*, Pacific J. Math. **143** (1990), no. 1, 69–77.

[19] W. Lusky, *On the structure of $Hv_0(D)$ and $hv_0(D)$*, Math. Nachr. **159** (1992), 279–289.

[20] ———, *On weighted spaces of harmonic and holomorphic functions*, J. London Math. Soc. (2) **51** (1995), no. 2, 309–320.

[21] B. D. MacCluer and R. Zhao, *Essential norms of weighted composition operators between Bloch-type spaces*, Rocky Mountain J. Math. **33** (2003), no. 4, 1437–1458.

[22] J. S. Manhas, *Multiplication operators on weighted locally convex spaces of vector-valued analytic functions*, Southeast Asian Bull. Math. **27** (2003), no. 4, 649–660.

[23] ———, *Weighted composition operators on weighted locally convex spaces of analytic functions*, Southeast Asian Bull. Math. **29** (2005), no. 1, 127–140.

[24] T. Mazur, *Canonical isometry on weighted Bergman spaces*, Pacific J. Math. **136** (1989), no. 2, 303–310.

[25] G. Mirzakarimi and K. Siddighi, *Weighted composition operators on Bergman and Dirichlet spaces*, Georgian Math. J. **4** (1997), no. 4, 373–383.

[26] A. Montes-Rodriguez, *Weighted composition operators on weighted Banach spaces of analytic functions*, J. London Math. Soc. (2) **61** (2000), no. 3, 872–884.

[27] M. Nagasawa, *Isomorphisms between commutative Banach algebras with an application to rings of analytic functions*, Kōdai Math. Sem. Rep. **11** (1959), 182–188.

[28] S. Ohno, *Weighted composition operators between H^∞ and the Bloch space*, Taiwanese J. Math. **5** (2001), no. 3, 555–563.

[29] S. Ohno, K. Stroethoff, and R. Zhao, *Weighted composition operators between Bloch-type spaces*, Rocky Mountain J. Math. **33** (2003), no. 1, 191–215.

[30] S. Ohno and H. Takagi, *Some properties of weighted composition operators on algebras of analytic functions*, J. Nonlinear Convex Anal. **2** (2001), no. 3, 369–380.

[31] S. Ohno and R. Zhao, *Weighted composition operators on the Bloch space*, Bull. Austral. Math. Soc. **63** (2001), no. 2, 177–185.

[32] W. Rudin, *Functional Analysis*, McGraw-Hill Book Co., 1973.

[33] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag, New York, 1993.

[34] R. K. Singh and J. S. Manhas, *Composition Operators on Function Spaces*, North-Holland Mathematics Studies, 179. North-Holland Publishing Co., Amsterdam, 1993.

[35] ———, *Invertible composition operators on weighted function spaces*, Acta Sci. Math. (Szeged) **59** (1994), no. 3-4, 489–501.

[36] A. Siskakis, *Weighted composition semigroups on Hardy spaces*, Proceedings of the symposium on operator theory (Athens, 1985). Linear Algebra Appl. **84** (1986), 359–371.

Department of Mathematics and Statistics
College of Science, Sultan Qaboos University
P.O. Box 36, P. C. 123 Al-Khod, Sultanate of Oman
E-mail address: manhas@squ.edu.om