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WEIGHTED EMPIRICAL AND QUANTILE PROCESSES

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We introduce a new Brownian bridge approximation to weighted empirical and quantile processes with rates in probability. This approximation leads to a number of general invariance theorems for empirical and quantile processes indexed by functions. Improved versions of the Chibisov–O'Reilly theorems, the Eicker–Jaeschke theorems for standardized empirical and quantile processes, the normal convergence criterion, and various other old and new asymptotic results on empirical and quantile processes are presented as consequences of our general theorems. In the process, we provide a new characterization of Erdős–Feller–Kolmogorov–Petrovski upper-class functions for the Brownian motion in an improved form.

0. Introduction. Let $U_{1,n} \le \cdots \le U_{n,n}$ denote the order statistics of the first n of independent uniform-(0,1) ($U(0,1)$) random variables (rv) $U_1, U_2, \dots$ with the corresponding uniform empirical distribution function $G_n(\cdot)$, defined to be right continuous, and uniform empirical quantile function

$$U_n(s) := U_{k,n}, \qquad (k-1)/n < s \le k/n \qquad (k = 1, \dots, n),$$

where $U_n(0) := U_{1,n}$. We define the *uniform empirical process*

$$\alpha_n(s) := n^{1/2}(G_n(s) - s), \qquad 0 \le s \le 1,$$

and the *uniform quantile process*

$$u_n(s) := n^{1/2}(s - U_n(s)), \qquad 0 \le s \le 1.$$

Komlós, Major, and Tusnády (1975a) showed that uniform empirical processes $\{\alpha_n\}$ can be constructed on the same probability space as a sequence of Brownian bridges $\{\hat{B}_n(s);\, 0 \le s \le 1\}$ in such a way that for all n and x we have

$$P\Big\{\sup_{0 \le s \le 1} |\alpha_n(s) - \hat{B}_n(s)| \ge n^{-1/2}(\hat{a}\log n + x)\Big\} \le \hat{b}e^{-\hat{c}x}, \tag{0.1}$$

where $\hat{a}$, $\hat{b}$, and $\hat{c}$ are positive absolute constants.

M. Csörgő and Révész (1978) showed that an analogous construction was possible for the uniform quantile process in terms of another sequence of

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Brownian bridges $\{\hat{\hat{B}}_n\}$ such that for all n and $|x| \le c_0 n^{1/2}$, $c_0 > 0$, we have

$$P\Big\{\sup_{0\le s\le 1} |u_n(s) - \hat{\hat{B}}_n(s)| > n^{-1/2}(\hat{a}\log n + x)\Big\} \le \hat{\hat{b}} e^{-\hat{c}x}, \tag{0.2}$$

where $\hat{a}$, $\hat{\hat{b}}$, and $\hat{c}$ are positive absolute constants.

These approximations have wide ranging applications in probability and statistics [cf. M. Csörgő and Révész (1981), M. Csörgő (1983), S. Csörgő and Hall (1984)]. They are, however, not quite strong enough to yield direct proofs of a number of limit theorems for weighted empirical and quantile processes, such as the Chibisov–O'Reilly theorems. In this paper we improve upon these approximations to get sharper results that can handle the weighted processes. In Section 1 we show that with an appropriate sequence of Brownian bridges $\{B_n\}$ we have

$$P\Big\{\sup_{0\le s\le d/n} |u_n(s) - B_n(s)| \ge n^{-1/2}(a\log d + x)\Big\} \le be^{-cx},$$

whenever $n_0 \le d \le n$, $0 \le x \le d^{1/2}$, where n_0, a, b, and c are suitably chosen positive constants. In Section 2 we turn the latter inequality into

$$\sup_{\lambda/n\le s\le 1} n^{\nu}|u_n(s) - B_n(s)|/s^{1/2-\nu} = O_P(1)$$

for every $0 \le \nu < \frac{1}{2}$ and $0 < \lambda < \infty$ as $n \to \infty$, while for every $0 \le \nu < \frac{1}{4}$ as $n \to \infty$ into

$$\sup_{U_{1,n}\le s\le 1} n^{\nu}|\alpha_n(s) - B_n(s)|/s^{1/2-\nu} = O_P(1).$$

Naturally we have similar results also in the neighbourhood of one (cf. Theorems 1.1, 2.1, and 2.2). In Section 3 our basic approximations become the essential tools to prove invariance principles for the empirical and quantile processes indexed by function classes or sequences of function classes. Sections 4.1–4.6 contain applications of these improvements upon (0.1) and (0.2).

Besides the notation introduced above we shall use F to denote any right-continuous distribution function and

$$Q(s) = \inf\{x\colon F(x) \ge s\}, \qquad 0 < s < 1,$$
$$Q(0) = Q(0+), \qquad Q(1) = Q(1-)$$

will denote its left-continuous inverse, the quantile function.

1. An improved construction for approximating the uniform quantile process by Brownian bridges. The aim of this section is to construct inequalities for the sake of describing how near the uniform sample quantile process $\{u_n(s);\ 0 \le s \le 1\}$ can be to a sequence of Brownian bridges $\{B_n(s);\ 0 \le s \le 1\}$. We have

THEOREM 1.1. *There exists a probability space $(\Omega, \mathscr{A}, P)$ with independent $U(0,1)$ rv $U_1, U_2, \dots$ and a sequence of Brownian bridges $\{B_i(s);\ 0 \le s \le 1\}$*

$(i = 1, 2, \dots)$ *such that*

$$(1.1) \qquad P\left\{ \sup_{0 \le s \le d/n} |u_n(s) - B_n(s)| \ge n^{-1/2}(a \log d + x) \right\} \le be^{-cx}$$

and

$$(1.2) \qquad P\left\{ \sup_{1 - d/n \le s \le 1} |u_n(s) - B_n(s)| \ge n^{-1/2}(a \log d + x) \right\} \le be^{-cx},$$

whenever $n_0 \le d \le n$, $0 \le x \le d^{1/2}$, *where* n_0, a, b, *and* c *are suitably chosen positive constants.*

When $d = n$, Theorem 1.1 reduces to Theorem 1 of M. Csörgő and Révész (1978) [cf. (0.2)] which, in turn, is based on a similar inequality of Komlós, Major, and Tusnády (1976) concerning the problem of approximating partial sums of i.i.d. rv by a Wiener process. The proof of Theorem 1.1 exploits the same route.

PROOF. Let $\{W^{(1)}(t);\ 0 \le t < \infty\}$ and $\{W^{(2)}(t);\ 0 \le t < \infty\}$ be independent copies of a standard Wiener process defined on a probability space with two independent sequences of i.i.d. exponential rv $Y_1^{(i)}, Y_2^{(i)}, \dots,$ $(i = 1, 2)$, with mean one, such that for all real x we have the Komlós, Major, and Tusnády (1975a, 1976) inequalities

$$(1.3) \qquad P\left\{ \max_{1 \le k \le m} \left| \left(S_k^{(i)} - k\right) - W^{(i)}(k) \right| \ge C \log m + x \right\} \le Ke^{-\lambda x},$$

where, for $i = 1, 2$,

$$S_m^{(i)} = \sum_{j=1}^{m} Y_j^{(i)}, \qquad m = 1, 2, \dots$$

and C, K, and λ are positive universal constants, independent of $i = 1, 2$ and $m = 1, 2, \dots$.

For each integer $n \ge 2$ let

$$Y_j(n) = \begin{cases} Y_j^{(1)} & \text{for } j = 1, \dots, \left[\frac{n}{2}\right], \\ Y_{n+2-j}^{(2)} & \text{for } j = \left[\frac{n}{2}\right] + 1, \dots, n+1. \end{cases}$$

Then the rv $Y_1(n), \dots, Y_{n+1}(n)$ are i.i.d. exponential rv with mean 1. Put

$$S_m(n) = \sum_{j=1}^{m} Y_j(n) \quad \text{for } m = 1, \dots, n+1.$$

For the sake of notational convenience we will write, from now on, S_m instead of $S_m(n)$, and also Y_j instead of $Y_j(n)$, and will also use the usual convention $S_0 = 0$.

For each integer $n \geq 2$ we define the stochastic process

$$W_n(s) = \begin{cases} W^{(1)}(s) & \text{for } 0 \leq s \leq \left[\frac{n}{2}\right], \\ W^{(1)}\left(\left[\frac{n}{2}\right]\right) + W^{(2)}\left(n+1-\left[\frac{n}{2}\right]\right) & \\ \quad - W^{(2)}(n+1-s) & \text{for } \left[\frac{n}{2}\right] < s \leq n+1. \end{cases}$$

Elementary calculations show that for each choice of $0 \leq s \leq t \leq n+1$ we have $EW_n(s) = 0$, $EW_n(s)W_n(t) = s$. Hence W_n is a standard Wiener process on $[0, n+1]$.

From (1.3) we get that for every $n \geq 2$, $1 \leq m \leq [n/2]$, and x

$$P\Big\{\max_{1 \leq k \leq m} |(S_k - k) - W_n(k)| \geq C \log m + x\Big\} \leq Ke^{-\lambda x}, \tag{1.4}$$

and, since for $1 \leq k \leq n - [n/2] + 1$, $S_{n+1} - S_{n+1-k} = S_k^{(2)}$, and $W_n(n+1) - W_n(n+1-k) = W^{(2)}(k)$, we also get that for every $1 \leq m \leq n - [n/2] + 1$ and x

$$P\Big\{\max_{1 \leq k \leq m} |(S_{n+1} - S_{n+1-k} - k) - (W_n(n+1) - W_n(n+1-k))| \geq C \log m + x\Big\} \leq Ke^{-\lambda x}. \tag{1.5}$$

An elementary argument based on (1.4) and (1.5) shows that there exist universal constants C_0, K_0, and λ_0 such that for every $n \geq 2$, $1 \leq m \leq n+1$, and x

$$P\Big\{\max_{1 \leq k \leq m} |(S_k - k) - W_n(k)| \geq C_0 \log m + x\Big\} \leq K_0 e^{-\lambda_0 x}, \tag{1.6}$$

and

$$P\Big\{\max_{1 \leq k \leq m} |(S_{n+1} - S_{n+1-k} - k) - (W_n(n+1) - W_n(n+1-k))| \geq C_0 \log m + x\Big\} \leq K_0 e^{-\lambda_0 x}. \tag{1.7}$$

Now let

$$\tilde{U}_{k,n} = S_k / S_{n+1} \quad \text{for } k = 1, \ldots, n \qquad (n \geq 2),$$

and define

$$\tilde{u}_n(s) = n^{1/2}(s - \tilde{U}_{k,n}) \quad \text{for } \frac{k-1}{n} < s \leq \frac{k}{n} \qquad (k = 1, \ldots, n),$$

$$\tilde{B}_n(s) = n^{-1/2}(sW_n(n) - W_n(sn)) \quad \text{for } 0 \leq s \leq 1.$$

We note in passing that $\tilde{B}_n(s)$ is a Brownian bridge for each n.

First we need some inequalities for our partial sums of i.i.d. exponential rv with mean 1 and Brownian bridges.

Lemma 3.1 in Devroye (1981) immediately implies that for any $0 \le x \le \frac{1}{2}$ we have

$$P\{|(S_n - n)/n| \ge x\} \le 2\exp(-nx^2/4). \tag{1.8}$$

The latter can be turned into

$$P\{|(S_{n+1} - n)/S_{n+1}| \ge x\} \le 7\exp(-nx^2/64) \tag{1.9}$$

for any $0 \le x \le \frac{1}{2}$ as follows:

$$\begin{aligned} P\{|(S_{n+1} - n)/S_{n+1}| \ge x\} &\le P\{|(S_n - n)/S_n| \ge x/2\} + P\{Y_{n+1}/S_n \ge x/2\} \\ &\le P\{|(S_n - n)/n| \ge x/4\} + P\{Y_{n+1}/n \ge x/4\} \\ &\quad + 2P\{S_n/n \le 1/2\} \\ &\le P\{|(S_n - n)/n| \ge x/4\} + P\{Y_{n+1}/n \ge x^2/2\} \\ &\quad + 2P\{(S_n - n)/n \le -1/2\} \\ &\le 3P\{|(S_n - n)/n| \ge x/4\} + P\{Y_{n+1} \ge nx^2/2\} \\ &\le 6\exp(-nx^2/64) + \exp(-nx^2/2) \\ &\le 7\exp(-nx^2/64), \end{aligned}$$

where in the third inequality $0 \le x \le \frac{1}{2}$, while in the fourth inequality $-\frac{1}{2} \le -x/4$ was utilized.

Next for any $4 \le y < \infty$ and integer $1 \le m < \infty$ such that $(y-2)/m^{1/2} \le \frac{1}{2}$ we have

$$P\Big\{\max_{1\le k\le m} |S_k - k| \ge ym^{1/2}\Big\} \le 8\exp(-y^2/16), \tag{1.10}$$

since by the P. Lévy inequalities [cf., e.g., pages 247–248 in Loève (1960)]

$$P\Big\{\max_{1\le k\le m} |S_k - k| \ge ym^{1/2}\Big\} \le 4P\{|S_m - m| \ge (y-2)m^{1/2}\}$$

and by (1.8)

$$P\{|(S_m - m)/m| \ge (y-2)/m^{1/2}\} \le 2\exp\big(-(y-2)^2/4\big) \le 2\exp(-y^2/16),$$

where, given our condition on y, the latter inequality is on account of $(y-2) \ge y/2$.

As to the needed inequality for a Brownian bridge $\{B(s); 0 \le s \le 1\}$, we have for any $0 < a < 1$, $h \ge 0$, and $0 < u < \infty$

$$\begin{aligned} P_{a,h}(u) &:= P\Big\{\sup_{s\in[a-h,a+h]\cap[0,1]} |B(a) - B(s)| \ge uh^{1/2}\Big\} \\ &\le Au^{-1}\exp(-u^2/8) \end{aligned} \tag{1.11}$$

with a suitably chosen universal constant A. The latter is immediate, for any Brownian bridge has the representation

$$\{B(s); 0 \le s \le 1\} =_{\mathscr{D}} \{W(s) - sW(1); 0 \le s \le 1\},$$

where W is a standard Wiener process, and hence

$$\begin{aligned}
P_{a,h}(u) &\le P\Big\{\sup_{a\le s\le a+h}|W(a)-W(s)|\ge (u/2)h^{1/2}\Big\}\\
&\quad +P\Big\{\sup_{a-h\le s\le a}|W(a)-W(s)|\ge (u/2)h^{1/2}\Big\}\\
&\quad +2P\big\{h|W(1)|\ge (u/2)h^{1/2}\big\}\\
&\le 2P\Big\{\sup_{0\le s\le h}|W(s)|\ge (u/2)h^{1/2}\Big\}+2P\{|W(1)|\ge u/2\}\\
&\le 10P\{|W(1)|\ge u/2\}\le 10\Big(\frac{2}{\pi}\Big)^{1/2}u^{-1}\exp(-u^2/8),
\end{aligned}$$

where for the last inequality we refer to Feller (1968, page 175).

Now, towards the proof of (1.1), we first note that it suffices to prove it with $d=m$, an integer. Set

$$y_m = n^{-1/2}(a\log m + x), \tag{1.12}$$

where $a = 15C_0$ with C_0 as in (1.6), and consider

$$\begin{aligned}
&P\Big\{\sup_{0\le s\le m/n}|\tilde{u}_n(s)-\tilde{B}_n(s)|\ge y_m\Big\}\\
&\quad\le P\Big\{\sup_{0\le s\le m/n}|\tilde{B}_n(s)-\tilde{B}_n(([ns]+1)/n)|\ge y_m/3\Big\}\\
&\qquad +P\Big\{\sup_{0\le s< m/n}|\tilde{u}_n(([ns]+1)/n)-\tilde{u}_n(s)|\ge y_m/3\Big\}\\
&\qquad +P\Big\{\max_{1\le k\le m}|\tilde{u}_n(k/n)-\tilde{B}_n(k/n)|\ge y_m/3\Big\}\\
&\quad := P_{1,n}+P_{2,n}+P_{3,n}.
\end{aligned}\tag{1.13}$$

An application of (1.11) gives

$$\begin{aligned}
P_{1,n} &\le mA\big(n^{1/2}y_m\big)^{-1}\exp\big(-ny_m^2/72\big)\\
&\le b_1\exp(-c_1x),
\end{aligned}\tag{1.14}$$

where $0<b_1,\ c_1<\infty$ are appropriately chosen absolute constants, which do not depend on $n\ge m\ge 2$ and $x>0$. Also

$$\sup_{0\le s<1}|\tilde{u}_n(([ns]+1)/n)-\tilde{u}_n(s)|\le 2n^{-1/2},$$

hence

$$P_{2,n}=0 \quad \text{whenever } n_0 \text{ is such that } a\log n_0>6. \tag{1.15}$$

Thus it remains only to estimate $P_{3,n}$.

First we observe that for any $1 \le k \le n$

$$
\begin{aligned}
|\tilde{u}_n(k/n) - \tilde{B}_n(k/n)| &\le n^{-1/2}|(S_k - k) - W_n(k)| \\
&\quad + n^{-1/2}\frac{k}{n}|(S_n - n) - W_n(n)| \\
&\quad + n^{-1/2}(1 + |\varepsilon_n|)\frac{k}{n}Y_{n+1} \\
&\quad + n^{-1/2}|\varepsilon_n|\,|S_k - k| + n^{-1/2}\frac{k}{n}|\varepsilon_n|\,|S_n - n| \\
&:= \sum_{i=1}^{5} \Delta_{i,n}(k),
\end{aligned}
\tag{1.16}
$$

where $\varepsilon_n = n/S_{n+1} - 1$. Hence

$$
P_{3,n} \le \sum_{i=1}^{5} P_{3,n,i}, \tag{1.17}
$$

where for each $1 \le i \le 5$

$$
P_{3,n,i} = P\Big\{\max_{1 \le k \le m} \Delta_{i,n}(k) \ge y_m/15\Big\}.
$$

Applying (1.6), we get

$$
P_{3,n,1} \le K_0 \exp(-\lambda_0 x/15). \tag{1.18}
$$

Next we observe that

$$
\begin{aligned}
P_{3,n,2} &\le P\Big\{|(S_n - n) - W_n(n)| \ge \frac{n}{m}\Big(C_0 \log m + \frac{x}{15}\Big)\Big\} \\
&\le P\Big\{|(S_n - n) - W_n(n)| \ge C_0\frac{n}{m}\log m + \frac{x}{15}\Big\} \\
&\le P\Big\{|(S_n - n) - W_n(n)| \ge C_0 \log n + \frac{x}{15}\Big\} \\
&\le K_0 \exp\Big(-\lambda_0 \frac{x}{15}\Big),
\end{aligned}
\tag{1.19}
$$

where the third inequality of (1.19) is due to

$$
\frac{n}{m}\log m \ge \log n \quad \text{for } n \ge m \ge n_0,
$$

for n_0 appropriately large, and the last one is by (1.6).

We have also

$$
\begin{aligned}
P_{3,n,3} &\le P\{n^{-1/2}(1 + |\varepsilon_n|)(m/n)Y_{n+1} \ge y_m/15\} \\
&\le P\{|\varepsilon_n| > \tfrac{1}{2}\} + P\{(m/n)Y_{n+1} \ge 2n^{1/2}y_m/45\} \\
&\le 7\exp(-n/256) + \exp(-2x/45) \\
&\le 7\exp(-x/256) + \exp(-2x/45) \\
&\le b_2 e^{-c_2 x},
\end{aligned}
\tag{1.20}
$$

where in the third inequality (1.9) was used, and then the assumption $x \le m^{1/2}$ and the fact that $m^{1/2} \le n$.

For the next term in the upper bound for $P_{3,n}$ we have

$$
\begin{aligned}
P_{3,n,4} &\le P\Big\{\max_{1\le k\le m} |S_k - k| \ge m^{1/2}\big(y_m n^{1/2}/15\big)^{1/2}\Big\} \\
&\quad + P\Big\{|\varepsilon_n| \ge m^{-1/2}\big(y_m n^{1/2}/15\big)^{1/2}\Big\} \\
&\le 8\exp\big(-n^{1/2}y_m/((16)(15))\big) \\
&\quad + 7\exp\big(-(n/m)y_m n^{1/2}/((15)(64))\big) \\
&\le b_3 e^{-c_3 x},
\end{aligned} \tag{1.21}
$$

where the second inequality is by (1.9) and (1.10) whenever $n \ge m$ and m is large enough so that for every $0 \le x \le m^{1/2}$

$$
m^{-1/2}\big(n^{1/2}y_m/15\big)^{1/2} \le \tfrac{1}{2}
$$

$$
4 \le \big(n^{1/2}y_m/15\big)^{1/2}
$$

and

$$
0 \le \Big(\big(n^{1/2}y_m/15\big)^{1/2} - 2\Big)/m^{1/2} \le \tfrac{1}{2},
$$

and hence the last inequality of (1.21) is also true for every $n_0 \le m \le n$ if n_0 is large enough.

Finally,

$$
\begin{aligned}
P_{3,n,5} &\le P\big\{n^{-1/2}(m/n)|\varepsilon_n|\,|S_n - n| \ge y_m/15\big\} \\
&\le P\big\{|\varepsilon_n|\,|(S_n - n)/n| \ge n^{-1/2}y_m/15\big\} \\
&\le P\Big\{|\varepsilon_n| \ge n^{-1/4}\big(y_m/15\big)^{1/2}\Big\} + P\Big\{|(S_n - n)/n| \ge n^{-1/4}\big(y_m/15\big)^{1/2}\Big\} \\
&\le 7\exp\big(-n^{1/2}y_m/((15)(64))\big) + 2\exp\big(-n^{1/2}y_m/((4)(15))\big) \\
&\le b_4 e^{-c_4 x},
\end{aligned} \tag{1.22}
$$

where the fourth inequality is by (1.8) and (1.9) whenever $n_0 \le m \le n$ with n_0 large enough so that for every $0 \le x \le m^{1/2}$

$$
n^{-1/4}\big(y_m/15\big)^{1/2} \le \tfrac{1}{2},
$$

and hence also the last inequality of (1.22).

Combining now (1.18), (1.19), (1.20), (1.21), and (1.22), we get an estimate of $P_{3,n}$ of (1.17), which, when combined with the estimate of $P_{1,n}$ and $P_{2,n}$ of (1.14) and (1.15), respectively, results in

$$
P\Big\{\sup_{0\le s\le d/n} |\tilde{u}_n(s) - \tilde{B}_n(s)| \ge n^{-1/2}(a\log d + x)\Big\} \le be^{-cx}, \tag{1.23}
$$

whenever $n_0 \le d \le n$, $0 \le x \le d^{1/2}$ with suitably chosen positive constants n_0, a, b, and c.

Towards the proof of (1.2) we can show similarly that under the same conditions as in (1.23) we have also

$$(1.24)\qquad P\left\{\sup_{1-d/n\le s\le 1}|\tilde{u}_n(s)-\tilde{B}_n(s)|\ge n^{-1/2}(a\log d+x)\right\}\le be^{-cx}.$$

At this stage we should note that, while the proven statements of (1.23) and (1.24) appear to have the desired appearance of (1.1) and (1.2), respectively, we have not yet proved the latter two statements. Indeed, the probability measure P and the stochastic processes $\tilde{u}_n$ and $\tilde{B}_n$ of (1.23), (1.24) are not the ones claimed in Theorem 1.1. On the other hand, we have for each n

$$(1.25)\qquad \{\tilde{u}_n(s);\,0\le s\le 1\} =_{\mathscr{D}} \{u_n(s);\,0\le s\le 1\},$$

and $\tilde{B}_n$ is a Brownian bridge for each n. Hence, by Lemma 4.4.4. of M. Csörgő and Révész (1981) [cf. also Lemma 3.1.1 in M. Csörgő (1983)], for each n one can construct a probability space $(\Omega_n, \mathscr{A}_n, P_n)$ with a sequence $U_1, \dots, U_n$ of independent $U(0,1)$ rv and with a Brownian bridge B_n^* such that

$$(1.26)\qquad P_n\left\{\sup_{0\le s\le d/n}|u_n(s)-B_n^*(s)|\ge n^{-1/2}(a\log d+x)\right\}\le be^{-cx},$$

and

$$(1.27)\qquad P_n\left\{\sup_{1-d/n\le s\le 1}|u_n(s)-B_n^*(s)|\ge n^{-1/2}(a\log d+x)\right\}\le be^{-cx},$$

whenever $n_0 \le d \le n$, $0 \le x \le d^{1/2}$.

Using now the construction of the proof of Lemma 3.1.2 in M. Csörgő (1983), (1.26) and (1.27) immediately imply (1.1) and (1.2) as claimed.

From now on throughout the remainder of this exposition, we assume that our processes and random variables are defined on the probability space constructed in Theorem 1.1.

2. Rates of convergence of the uniform quantile and empirical processes to a sequence of Brownian bridges. In this section Theorem 1.1 is used to deduce rates of convergence results for the sup-norm distance of the uniform sample quantile and empirical processes from a sequence of Brownian bridges. One of the main results here is

THEOREM 2.1. *On the probability space of Theorem* 1.1 *we have*

$$(2.1)\qquad \sup_{0\le s\le 1} n^{1/2}|u_n(s)-B_n(s)| = O(\log n),\quad a.s.,$$

while for every $0 \le \nu < \frac{1}{2}$ *and* $0 < \lambda < \infty$ *as* $n \to \infty$

$$\sup_{\lambda/n \le s \le 1-\lambda/n} n^{\nu}|u_n(s) - B_n(s)|/(s(1-s))^{1/2-\nu} = O_P(1). \tag{2.2}$$

PROOF. As to (2.1), we set $d = n$, $x = (2/c)\log n$ in (1.1) say, and (2.1) results [cf. (0.2)].

Now consider (2.2) which we will prove first with $\lambda = 1$. Choose any $0 \le \nu < \frac{1}{2}$, and write

$$\Delta_{n,\nu}^{(1)} = \sup_{1/n \le s \le 1} n^{\nu}|u_n(s) - B_n(s)|/s^{1/2-\nu}$$

and

$$\Delta_{n,\nu}^{(2)} = \sup_{0 \le s \le 1-1/n} n^{\nu}|u_n(s) - B_n(s)|/(1-s)^{1/2-\nu}.$$

In order to establish (2.2) with $\lambda = 1$, it is enough to show that

$$\Delta_{n,\nu}^{(1)} = O_P(1) \quad \text{for any } 0 \le \nu < \tfrac{1}{2}, \tag{2.3}$$

and

$$\Delta_{n,\nu}^{(2)} = O_P(1) \quad \text{for any } 0 \le \nu < \tfrac{1}{2}. \tag{2.4}$$

We prove only (2.3) and note that the proof of (2.4) is similar. For any $\max(e, n_0) < d < \infty$, let

$$e_1(d) = d, \qquad e_i(d) = \exp(e_{i-1}(d)) \qquad (i = 2, 3, \dots)$$

and

$$d_i = (e_{i+1}(d))^{2/(1-2\nu)} \quad \text{for } i = 0, \dots, i_n - 1,\ d_{i_n} = n,$$

where

$$i_n = \max\{i: d_{i-1} \le n\}.$$

Define the intervals

$$I_0 = [1/n, d_0/n] = [1/n, d^{2/(1-2\nu)}/n],$$
$$I_i = [d_{i-1}/n, d_i/n] \quad \text{for } i = 1, \dots, i_n - 1,$$

and

$$I_{i_n} = [d_{i_n-1}/n, d_{i_n}/n] = [d_{i_n-1}/n, 1].$$

For each $i = 0, \dots, i_n - 1$ let

$$\delta_{i,n} = \sup\{|u_n(s) - B_n(s)|: s \in [0, d_i/n]\}$$

and set

$$\delta_{i_n,n} = \sup\{|u_n(s) - B_n(s)|: s \in [0, 1]\}.$$

Let a, b, c be the constants of inequality (1.1). Then we have

$$P\left\{\Delta_{n,\nu}^{(1)} \geq \left(\frac{2a}{1-2\nu}+1\right)\log d\right\}$$

$$\leq P\left\{\sup_{s\in I_0} n^{\nu}|u_n(s)-B_n(s)|/s^{1/2-\nu} \geq \left(\frac{2a}{1-2\nu}+1\right)\log d\right\}$$

$$+\sum_{i=1}^{i_n} P\left\{\sup_{s\in I_i} n^{\nu}|u_n(s)-B_n(s)|/s^{1/2-\nu} \geq \left(\frac{2a}{1-2\nu}+1\right)\right\}$$

$$\leq P\left\{\delta_{0,n} \geq n^{-1/2}\left(\frac{2a}{1-2\nu}+1\right)\log d\right\}$$

$$+\sum_{i=1}^{i_n} P\left\{\delta_{i,n} \geq n^{-1/2}\left(\frac{2a}{1-2\nu}+1\right)e_i(d)\right\}$$

$$:= \sum_{i=0}^{i_n} P_{i,n}(d).$$

Next we should like to use inequality (1.1) in order to estimate the above probabilities $P_{i,n}(d)$ $(i=0,\dots,i_n)$. In order to be able to do so, in addition to requiring that $\max(e, n_0) < d < \infty$, we will also assume that d and n are chosen so that

(i) $$n_0 \leq d^{2/(1-2\nu)} \leq n.$$

The latter condition and definition of i_n will also ensure that

(ii) $$n_0 \leq d_i \leq n \quad \text{for } i=0,\dots,i_n,$$

where n_0 is as in Theorem 1.1. We have also

(iii) $$\log d \leq d_0^{1/2} = d^{1/(1-2\nu)}$$

and

(iv) $$e_i(d) \leq d_i^{1/2} \quad \text{for } i=1,\dots,i_n,$$

since in this case

$$e_i(d) = (\log d_i)\left(\tfrac{1}{2}-\nu\right) \leq d_i^{1/2} \quad \text{for } i=0,\dots,i_n-1,$$

and, by definition of i_n

$$e_{i_n}(d) \leq n^{1/2-\nu} \leq n^{1/2} = d_{i_n}^{1/2}.$$

With (i), (ii), (iii), and (iv) in mind we notice that by inequality (1.1) we have

$$P_{0,n}(d) = P\left\{\delta_{0,n} \geq n^{-1/2}(a\log d_0 + \log d)\right\}$$
$$\leq be^{-c\log d} = bd^{-c},$$
$$P_{i,n}(d) = P\left\{\delta_{i,n} \geq n^{-1/2}(a\log d_i + e_i(d))\right\}$$
$$\leq be^{-ce_i(d)} \quad \text{for } i=1,\dots,i_n-1,$$

and, since by definition of i_n,

$$\frac{2}{1-2\nu}e_{i_n}(d) > \log n = \log d_{i_n},$$

$$P_{i_n,n}(d) \le P\left\{\delta_{i_n,n} \ge n^{-1/2}\left(a\log d_{i_n} + e_{i_n}(d)\right)\right\}$$

$$\le be^{-ce_{i_n}(d)}.$$

Consequently,

$$P\left\{\Delta_{n,\nu}^{(1)} \ge \left(\frac{2a}{1-2\nu}+1\right)\log d\right\} \le bd^{-c} + \sum_{i=1}^{i_n} be^{-ce_i(d)} \tag{2.5}$$

$$\le bd^{-c}/(1-d^{-c}).$$

Now let $\varepsilon > 0$ be given. Then choose d such that $bd^{-c}/(1-d^{-c}) < \varepsilon$ and a $K = K(\varepsilon)$ such that $(2a/(1-2\nu)+1)\log d < K$. Then choosing n so large that (i) also holds true, by (2.5) we get

$$\limsup_{n\to\infty} P\left\{\Delta_{n,\nu}^{(1)} \ge K\right\} \le \varepsilon.$$

This completes the proof of (2.3).

To complete the proof of (2.2) with $0 < \lambda < \infty$, we have only to cover the ranges $[\lambda/n, 1/n]$ and $[1-1/n, 1-\lambda/n]$ for fixed λ with $0 < \lambda < 1$. It is easily seen that

$$\sup_{\lambda/n \le s \le 1/n} n^{\nu}|u_n(s) - B_n(s)|/s^{1/2-\nu}$$

$$\le \lambda^{\nu-1/2}n|U_{1,n} - 1/n| + \lambda^{\nu-1/2} + \lambda^{\nu-1/2}n^{1/2}\sup_{0\le s\le 1/n}|B_n(s)|$$

$$= O_P(1),$$

because both sequences of rv on the right-hand side of the above inequality converge in distribution to a proper rv. A similar argument holds over the interval $[1-1/n, 1-\lambda/n]$. This also completes the proof of Theorem 2.1.

At this stage a reasonable question is: What can we say about the empirical process $\{\alpha_n(s);\ 0 \le s \le 1\}$ along the lines of Theorem 2.1 on the probability space of Theorem 1.1? An answer to this question is:

THEOREM 2.2. *On the probability space of Theorem* 1.1 *we have*

(2.6)

$$\limsup_{n\to\infty}\sup_{0\le s\le 1} n^{1/4}|\alpha_n(s) - B_n(s)|/\left((\log n)^{1/2}(\log\log n)^{1/4}\right) = 2^{-1/4}, \quad a.s.,$$

$$\sup_{U_{1,n}\le s\le U_{n,n}} n^{1/4}|\alpha_n(s) - B_n(s)|/(s(1-s))^{1/4} = O_P(\log n), \tag{2.7}$$

while for every $0 \le \nu < \frac{1}{4}$ *as* $n \to \infty$

$$\sup_{U_{1,n} \le s \le U_{n,n}} n^{\nu}|\alpha_n(s) - B_n(s)|/(s(1-s))^{1/2-\nu} = O_P(1). \tag{2.8}$$

REMARK 2.1. Before proving this theorem we should note that if we started out proving a Theorem 1.1-like statement with, instead of u_n, α_n being approximated by another appropriate sequence of Brownian bridges, i.e., if instead of modifying the M. Csörgő and Révész (1978) quantile process construction, we would have modified the Komlós, Major, and Tusnády (1975) empirical process construction, then on another appropriate probability space with another appropriate sequence of Brownian bridges, we would have a Theorem 1.1- and also a Theorem 2.1-like result for α_n, and a Theorem 2.2-like result for u_n. Thus a dual approximation theory along these above lines to the one presented in this exposition for u_n and α_n (cf. Theorems 1.1, 2.1, and 2.2) is also feasible for α_n and u_n.

PROOF OF (2.6). Using the Brownian bridges B_n of Theorem 2.1, triangular inequalities, and a result of Kiefer (1970), (2.6) is immediate.

The proof of (2.7) and (2.8) will require the following results, listed as Facts 1, 2, and 3.

FACT 1 (Daniels, 1945). *For any* $0 < \lambda < 1$ *and* $n \ge 1$

$$P\Big\{\min_{1\le k\le n} nU_{k,n}/k \le \lambda\Big\} = P\Big\{\sup_{0\le s\le 1} G_n(s)/s \ge 1/\lambda\Big\} = \lambda, \tag{2.9}$$

and

$$\begin{aligned} &P\Big\{\min_{1\le k\le n} n(1-U_{n+1-k,n})/k \le \lambda\Big\} \\ &\quad = P\Big\{\sup_{0\le s\le 1} (1-G_n(s))/(1-s) \ge 1/\lambda\Big\} = \lambda. \end{aligned} \tag{2.10}$$

FACT 2 (Wellner, 1977). *For any* $1 \le \lambda < \infty$, $1 \le k \le n$, *and* $n \ge 1$

$$P\left\{\left|U_{k,n} - \frac{k}{n+1}\right| < \lambda\left(\frac{k}{n(n+1)}\left(1 - \frac{k}{n+1}\right)\right)^{1/2}\right\} \le 2e^{-\lambda/5}. \tag{2.11}$$

From Fact 2 we get

FACT 3. *For every* $\lambda \ge 1$ *and* $1 \le k \le n$

$$P\left\{\left|U_{k,n} - \frac{k}{n}\right| \ge \lambda k^{1/2}/n\right\} \le 2e^{-\lambda/10}. \tag{2.12}$$

PROOF. We have

$$P\left\{\left|U_{k,n}-\frac{k}{n}\right|\geq\frac{\lambda k^{1/2}}{n}\right\}\leq P\left\{\left|U_{k,n}-\frac{k}{n+1}\right|\geq\frac{\lambda k^{1/2}}{n}-\frac{k}{n(n+1)}\right\}$$
$$\leq P\left\{\left|U_{k,n}-\frac{k}{n+1}\right|\geq\frac{\lambda k^{1/2}}{n}\left(1-\frac{k^{1/2}}{n+1}\right)\right\}$$

where the last inequality is due to $\lambda\geq 1$. The proof of (2.12) now follows from (2.11) via the fact that for every $1\leq k\leq n$ and $n\geq 1$

$$(k^{1/2}/n)(1-k^{1/2}/(n+1))\geq(1/2)(1/n^{1/2})(k/(n+1))^{1/2}(1-k/(n+1))^{1/2}.$$

PROOF OF (2.7) AND (2.8). For any $0\leq\nu\leq\frac{1}{4}$ we write

$$A_{n,\nu}^{(1)}=\sup_{U_{1,n}\leq s\leq U_{n,n}}n^{\nu}|\alpha_n(s)-B_n(s)|/s^{1/2-\nu},$$

and

$$A_{n,\nu}^{(2)}=\sup_{U_{1,n}\leq s\leq U_{n,n}}n^{\nu}|\alpha_n(s)-B_n(s)|/(1-s)^{1/2-\nu}.$$

In order to establish (2.7) and (2.8) it is enough to show that

$$A_{n,\nu}^{(1)}=\begin{cases}O_P(\log n) & \text{when } \nu=\frac{1}{4},\\ O_P(1) & \text{when } 0\leq\nu<\frac{1}{4},\end{cases}\tag{2.13}$$

and

$$A_{n,\nu}^{(2)}=\begin{cases}O_P(\log n) & \text{when } \nu=\frac{1}{4},\\ O_P(1) & \text{when } 0\leq\nu<\frac{1}{4}.\end{cases}\tag{2.14}$$

We consider only (2.13). We notice that

$$A_{n,\nu}^{(1)}=\max_{1\leq k\leq n-1}\sup\left\{n^{\nu}|\alpha_n(s)-B_n(s)|/s^{1/2-\nu}\colon s\in[U_{k,n},U_{k+1,n})\right\}.$$

For any $0<\lambda\leq 1$ we set

$$\begin{aligned}A_{n,\nu}^{(1)}(\lambda)&=\max_{1\leq k\leq n-1}\sup\left\{\frac{n^{\nu}|\alpha_n(s)-B_n(s)|}{(\lambda k/n)^{1/2-\nu}}\colon s\in[U_{k,n},U_{k+1,n})\right\}\\&=\lambda^{\nu-1/2}A_{n,\nu}^{(1)}(1):=\lambda^{\nu-1/2}B_{n,\nu}^{(1)}\quad\text{with } B_{n,\nu}^{(1)}=A_{n,\nu}^{(1)}(1).\end{aligned}\tag{2.15}$$

We observe that by (2.9) we have for any $0<\lambda\leq 1$

$$\limsup_{n\to\infty}P\left\{A_{n,\nu}^{(1)}\leq\lambda^{\nu-1/2}B_{n,\nu}^{(1)}\right\}\geq 1-\lambda.\tag{2.16}$$

Consequently, since λ can be chosen arbitrarily close to zero, in order to verify

(2.13) it suffices to show that

$$B_{n,\nu}^{(1)} = \begin{cases} O_P(\log n) & \text{when } \nu = \frac{1}{4}, \\ O_P(1) & \text{when } 0 \le \nu < \frac{1}{4}. \end{cases} \tag{2.17}$$

We notice that for any $1 \le k \le n-1$

$$\begin{aligned}
&\sup\left\{n^\nu|\alpha_n(s) - B_n(s)|/(k/n)^{1/2-\nu}: s \in [U_{k,n}, U_{k+1,n})\right\} \\
&\quad \le n^\nu|u_n(k/n) - B_n(k/n)|/(k/n)^{1/2-\nu} \\
&\qquad + n^{\nu+1/2}|U_{k+1,n} - U_{k,n}|/(k/n)^{1/2-\nu} \\
&\qquad + \sup\left\{n^\nu|B_n(s) - B_n(k/n)|/(k/n)^{1/2-\nu}: s \in [U_{k,n}, U_{k+1,n})\right\} \\
&\quad \le 2n^\nu|u_n(k/n) - B_n(k/n)|/(k/n)^{1/2-\nu} \\
&\qquad + n^\nu|u_n((k+1)/n) - B_n((k+1)/n)|/(k/n)^{1/2-\nu} \\
&\qquad + \sup\left\{n^\nu|B_n(s) - B_n(k/n)|/(k/n)^{1/2-\nu}: s \in [U_{k,n}, U_{k+1,n})\right\} \\
&\qquad + n^{\nu-1/2}/(k/n)^{1/2-\nu} + n^\nu|B_n((k+1)/n) - B_n(k/n)|/(k/n)^{1/2-\nu}.
\end{aligned}$$

Also, since $(k+1)/2n \le k/n$, we have for any $1 \le k \le n-1$

$$\begin{aligned}
&n^\nu|u_n((k+1)/n) - B_n((k+1)/n)|/(k/n)^{1/2-\nu} \\
&\quad \le n^\nu|u_n((k+1)/n) - B_n((k+1)/n)|/((k+1)/2n)^{1/2-\nu}.
\end{aligned}$$

Hence, by the just presented inequalities, for $B_{n,\nu}^{(1)}$ of (2.15) we have

$$\begin{aligned}
B_{n,\nu}^{(1)} &\le 3 \cdot 2^{1/2-\nu} \sup_{1/n \le s \le 1} \frac{n^\nu|u_n(s) - B_n(s)|}{s^{1/2-\nu}} + 1 \\
&\quad + \max_{1 \le k \le n-1} \frac{n^\nu|B_n((k+1)/n) - B_n(k/n)|}{(k/n)^{1/2-\nu}} \\
&\quad + \max_{1 \le k \le n-1} \sup\left\{\frac{n^\nu|B_n(s) - B_n(k/n)|}{(k/n)^{1/2-\nu}}: s \in [U_{k,n}, U_{k+1,n})\right\} \\
&:= 3 \cdot 2^{1/2-\nu}\Delta_{n,\nu}^{(1)} + 1 + C_{n,\nu}^{(1)} + D_{n,\nu}^{(1)}.
\end{aligned} \tag{2.18}$$

By (2.3) we already have that

$$\Delta_{n,\nu}^{(1)} = O_P(1) \quad \text{when } 0 \le \nu < \tfrac{1}{2}. \tag{2.19}$$

Next we show that

$$C_{n,\nu}^{(1)} = O_P(1) \quad \text{when } 0 \le \nu \le \tfrac{1}{2}, \tag{2.20}$$

and

$$D_{n,\nu}^{(1)} = \begin{cases} O_P(\log n) & \text{when } \nu = \frac{1}{4}, \\ O_P(1) & \text{when } 0 \le \nu < \frac{1}{4}. \end{cases} \tag{2.21}$$

In order to verify (2.20), choose any $0 \leq \nu < \frac{1}{2}$, and $d > 0$, and set $0 < \frac{1}{2} - \nu = \alpha$. Then

$$\begin{aligned} P\{C_{n,\nu}^{(1)} \geq d\} &\leq \sum_{k=1}^{n-1} P\{n^{\nu}|B_n((k+1)/n) - B_n(k/n)|/(k/n)^{1/2-\nu} \geq d\} \\ &\leq \sum_{k=1}^{n-1} P\{|B_n((k+1)/n) - B_n(k/n)| \geq n^{-1/2} dk^{\alpha}\} \\ &\leq Ad^{-1} \sum_{k=1}^{\infty} k^{-\alpha} e^{-d^2 k^{2\alpha}/8} := P_\alpha(d), \end{aligned}$$

where the last inequality is by (1.11). Since $P_\alpha(d) \to 0$ as $d \to \infty$, we have (2.20).

Towards the proof of (2.21), choose any $0 \leq \nu \leq \frac{1}{4}$ and set $\delta = (\frac{1}{4} - \nu)/2$. For any $1 \leq k \leq n-1$ and $b > 0$ let

$$c_{k,n}^{(\delta)} = k^{2\delta + 1/2}/n,$$

and

$$I_{k,n}(b) = [k/n - 3bc_{k,n}^{(\delta)}, k/n + 3bc_{k,n}^{(\delta)}],$$

and write

$$E_{n,\nu}(b) = \max_{1 \leq k \leq n-1} \sup\{n^{\nu}|B_n(s) - B_n(k/n)|/(k/n)^{1/2-\nu} : s \in I_{k,n}(b)\}.$$

We will first show that for any sequence of positive constants b_n

$$E_{n,\nu}(b_n) = \begin{cases} O_P((b_n \log n)^{1/2}) & \text{when } \nu = \frac{1}{4}, \\ O_p(b_n^{1/2}) & \text{when } 0 \leq \nu < \frac{1}{4}. \end{cases} \tag{2.22}$$

We have for an $0 \leq \nu \leq \frac{1}{4}$ and $d > 0$

$$\begin{aligned} &P\{E_{n,\nu}(b_n) \geq d(3b_n)^{1/2}\} \\ &\leq \sum_{k=1}^{n-1} P\left\{ \sup_{s \in I_{k,n}(b_n)} |B_n(s) - B_n(k/n)| \geq d(3b_n)^{1/2} k^{1/2-\nu} n^{-1/2} \right\} \\ &= \sum_{k=1}^{n-1} P\left\{ \sup_{s \in I_{k,n}(b_n)} |B_n(s) - B_n(k/n)| \geq d(3b_n c_{k,n}^{(\delta)})^{1/2} k^{1/4-\nu-\delta} \right\} \\ &\leq Ad^{-1} \sum_{k=1}^{n-1} k^{-\alpha} e^{-d^2 k^{2\alpha}/8}, \end{aligned} \tag{2.23}$$

where, with $\alpha = \frac{1}{4} - \nu - \delta = \frac{1}{2}(\frac{1}{4} - \nu)$, the last inequality is by (1.11).

When $0 \leq \nu < \frac{1}{4}$, then $\alpha > 0$, and so the latter sum of (2.23) is bounded above by

$$Ad^{-1} \sum_{k=1}^{\infty} k^{-\alpha} e^{-d^2 k^{2\alpha}/8} := P_\alpha(d).$$

Since $P_\alpha(d) \to 0$ as $d \to \infty$, we have now verified (2.22) when $0 \leq \nu < \frac{1}{4}$.

When $\nu = \frac{1}{4}$, let $d = (8 \log n)^{1/2}$. In this case the sum of terms on the right-hand side of the last inequality of (2.23) is bounded above by

$$A(8 \log n)^{-1/2} n e^{-\log n} = A/(8 \log n)^{1/2},$$

which in turn verifies (2.22) when $\nu = \frac{1}{4}$.

Now we can complete the proof of (2.21). Choose any $0 \le \nu \le \frac{1}{4}$, $b \ge 1$, and set $\delta = (\frac{1}{4} - \nu)/2$ as above. For each $1 \le k < \infty$ set $\lambda_k = bk^{2\delta}$, and note that

$$k^{1/2}\lambda_k/n = bc_{k,n}^{(\delta)}.$$

By the latter line and (2.12) we get

$$\begin{aligned}\sum_{k=1}^{n} P\{|U_{k,n} - k/n| \ge bc_{k,n}^{(\delta)}\} &\le 2 \sum_{k=1}^{n} e^{-\lambda_k/10} \\ &= 2 \sum_{k=1}^{n} e^{-bk^{2\delta}/10} := Q_{n,\nu}(b).\end{aligned} \tag{2.24}$$

Thus, with probability larger than or equal to $1 - Q_{n,\nu}(b)$, we have for every $1 \le k \le n - 1$

$$k/n - bc_{k,n}^{(\delta)} \le U_{k,n} < U_{k+1,n} \le (k+1)/n + bc_{k+1,n}^{(\delta)}.$$

Since $0 < 2\delta + \frac{1}{2} \le \frac{3}{4}$, we have for every $1 \le k \le n - 1$ that

$$(k+1)^{2\delta + 1/2} \le k^{2\delta + 1/2} + 1 \le 2k^{2\delta + 1/2},$$

and since $b \ge 1$, also that $1/n \le bc_{k,n}^{(\delta)}$.

Therefore, with probability larger than or equal to $1 - Q_{n,\nu}(b)$,

$$[U_{k,n}, U_{k+1,n}) \subset I_{k,n}(b),$$

for every $1 \le k \le n - 1$. Hence, for every choice of $1 \le b < \infty$ and $0 \le \nu \le \frac{1}{4}$

$$D_{n,\nu}^{(1)} \le E_{n,\nu}(b) \tag{2.25}$$

with probability larger than or equal to $1 - Q_{n,\nu}(b)$.

Now when $0 \le \nu < \frac{1}{4}$

$$Q_{n,\nu}(b) \le 2 \sum_{k=1}^{\infty} e^{-bk^{2\delta}/10} := Q_\nu(b), \tag{2.26}$$

which converges to zero as $b \to \infty$. Thus, since by (2.22)

$$E_{n,\nu}(b) = O_P(1)$$

for each fixed b, for $0 \le \nu < \frac{1}{4}$ we can conclude that

$$D_{n,\nu}^{(1)} = O_P(1), \tag{2.27}$$

by (2.25) and (2.26).

Finally when $\nu = \frac{1}{4}$, for each $n \ge e$ we let $b = 20 \log n$. Then for each $n \ge 3$ [cf. (2.24)]

$$Q_{n,1/4}(20 \log n) \le 2ne^{-2 \log n} = 2n^{-1}. \tag{2.28}$$

Now by (2.22) we have

$$E_{n,1/4}(20\log n) = O_P(\log n).$$

Therefore, by (2.25) combined with (2.28) and the latter line, we get

$$D_{n,1/4}^{(1)} = O_P(\log n). \tag{2.29}$$

Next by (2.27) and (2.29) we get (2.21), and the latter combined with (2.19) and (2.20) gives (2.17) via (2.18). Having now proven (2.17), the latter via (2.15) and (2.16) renders also (2.13) true.

The proof of (2.14) is *mutatis mutandis* in notation, exactly the same as that of (2.13). The latter two in turn imply (2.7) and (2.8), and the proof of Theorem 2.2 is now complete.

The following corollary to Theorem 2.2 will be useful later on.

COROLLARY 2.1. *On the probability space of Theorem* 1.1 *we have for every* $0 < \lambda < \infty$ *as* $n \to \infty$

$$\sup_{\lambda/n \le s \le 1} \frac{n^{\nu}|\alpha_n(s) - B_n(s)|}{s^{1/2-\nu}} = \begin{cases} O_P(\log n) & \text{when } \nu = \frac{1}{4}, \\ O_P(1) & \text{when } 0 \le \nu < \frac{1}{4}, \end{cases} \tag{2.30}$$

$$\sup_{0 \le s \le 1-\lambda/n} \frac{n^{\nu}|\alpha_n(s) - B_n(s)|}{(1-s)^{1/2-\nu}} = \begin{cases} O_P(\log n) & \text{when } \nu = \frac{1}{4}, \\ O_P(1) & \text{when } 0 \le \nu < \frac{1}{4}, \end{cases} \tag{2.31}$$

and

$$\sup_{\lambda/n \le s \le 1-\lambda/n} \frac{n^{\nu}|\alpha_n(s) - B_n(s)|}{(s(1-s))^{1/2-\nu}} = \begin{cases} O_P(\log n) & \text{when } \nu = \frac{1}{4}, \\ O_P(1) & \text{when } 0 \le \nu < \frac{1}{4}. \end{cases} \tag{2.32}$$

PROOF. First we note that (2.30) and (2.31) together imply (2.32). We will only provide a proof of (2.30). The proof of (2.31) uses the same techniques. In order to prove (2.30), by (2.13) it is enough to show that for any $\lambda > 0$ and $0 \le \nu \le \frac{1}{4}$, we have as $n \to \infty$

$$\left| \sup_{\lambda/n \le s \le 1} \frac{n^{\nu}|\alpha_n(s) - B_n(s)|}{s^{1/2-\nu}} - A_{n,\nu}^{(1)} \right| = \begin{cases} O_P(\log n), & \nu = \frac{1}{4}, \\ O_P(1), & 0 \le \nu < \frac{1}{4}. \end{cases} \tag{2.33}$$

We observe that the left side of (2.33) is bounded above by

$$\begin{aligned} &\sup_{(\lambda/n)\wedge U_{1,n} \le s \le U_{1,n}} \frac{n^{\nu}|\alpha_n(s)|}{s^{1/2-\nu}} + \sup_{(\lambda/n)\wedge U_{1,n} \le s \le U_{1,n}} \frac{n^{\nu}|B_n(s)|}{s^{1/2-\nu}} \\ &\quad + \sup_{U_{n,n} \le s \le 1} \frac{n^{\nu}|\alpha_n(s) - B_n(s)|}{s^{1/2-\nu}} := A_{n,\nu} + B_{n,\nu} + C_{n,\nu}. \end{aligned} \tag{2.34}$$

We have

$$A_{n,\nu} \le n^{\nu-1/2}/U_{1,n}^{1/2-\nu} + n^{\nu+1/2}U_{1,n}^{1/2+\nu} + \lambda^{\nu-1/2}nU_{1,n}. \tag{2.35}$$

Since $nU_{1,n} \to_{\mathscr{D}}$ an exponential rv with mean 1, we get

$$U_{1,n} = O_P(1/n) \tag{2.36}$$

and

$$1/U_{1,n} = O_P(n). \tag{2.37}$$

Hence the right-hand side of (2.35) becomes

$$n^{\nu-1/2}O_P(n^{1/2-\nu}) + n^{\nu+1/2}O_P(n^{-1/2-\nu}) + \lambda^{\nu-1/2}O_P(1) = O_P(1).$$

Thus

$$A_{n,\nu} = O_P(1). \tag{2.38}$$

In order to show that

$$B_{n,\nu} = O_P(1), \tag{2.39}$$

on account of (2.36) and (2.37) it is enough to establish that for every $0 < a < b < \infty$

$$\sup_{a/n \le s \le b/n} n^{\nu}|B_n(s)|/s^{1/2-\nu} \le a^{\nu-1/2} \sup_{a/n\le s\le b/n} n^{1/2}|B_n(s)| = O_P(1).$$

The latter in turn follows from a straightforward argument based on (1.11).
Finally we note that

$$C_{n,\nu} \le U_{n,n}^{\nu-1/2} \sup_{0\le s\le 1} n^{\nu}|\alpha_n(s) - B_n(s)|,$$

which by the fact that $U_{n,n}^{\nu-1/2} = O_P(1)$ and by (2.6) implies

$$C_{n,\nu} = \begin{cases} O_P\big((\log n)^{1/2}(\log\log n)^{1/4}\big), & \nu = \frac{1}{4}, \\ o_P(1), & 0 \leqq \nu < \frac{1}{4}. \end{cases} \tag{2.40}$$

Combining now (2.38), (2.39), and (2.40), we get (2.33) via (2.34), and (2.30) itself is now proved by (2.33) and (2.13) combined. This also completes the proof of Corollary 2.1.

Concerning α_n, in addition to Theorem 2.2 and Corollary 2.2, the following result will be also useful later on.

COROLLARY 2.2. *On the probability space of Theorem* 1.1 *we have as $n \to \infty$*

$$\sup_{0\le s\le 1} n^{1/4}|\alpha_n(s) - B_n(s)|/(\Delta_n \log n)^{1/2} \to_P 1, \tag{2.41}$$

where $\Delta_n = \sup_{0\le s\le 1}|\alpha_n(s)|$.

PROOF. The proof follows directly from (2.1) in combination with the following result of Kiefer (1970)

$$\sup_{0\le s\le 1} n^{1/4}|\alpha_n(s)-u_n(s)|/(\Delta_n \log n)^{1/2} \to_P 1.$$

We note in particular that (2.41) implies

$$\sup_{0\le s\le 1} n^{1/4}|\alpha_n(s)-B_n(s)| = O_P\big((\log n)^{1/2}\big), \tag{2.42}$$

and that the $(\log n)^{1/2}$ rate in the O_P term is exact.

The results of this section immediately imply the following "weighted Kiefer phenomenon."

COROLLARY 2.3. *For any* $0 \le \nu < \frac{1}{4}$ *and* $\lambda > 0$ *we have*

$$\sup_{\lambda/n\le s\le 1-\lambda/n} n^{\nu}|\alpha_n(s)-u_n(s)|/(s(1-s))^{1/2-\nu} = O_P(1).$$

3. Sup–norm convergence of the uniform quantile and empirical processes indexed by functions. In the Abstract we have already mentioned that our improved approximation theory of the uniform quantile and empirical processes is also going to be useful for proving invariance principles for these processes indexed by functions. Indeed, the theorems of Section 2 will be essential to our approach to these types of problems.

Before we state and prove our theorems, we introduce the following conventions concerning integrals:

When $0 < a < b < 1$ and g is a left-continuous and f is a right-continuous function then

$$\int_a^b f\,dg = \int_{[a,b)} f\,dg \quad\text{and}\quad \int_a^b g\,df = \int_{(a,b]} g\,df,$$

whenever these integrals make sense as Lebesgue–Stieltjes integrals. In this case the usual integration by parts formula

$$\int_a^b f\,dg + \int_a^b g\,df = g(b)f(b) - g(a)f(a)$$

is valid.

For any Brownian bridge $\{B(s);\ 0\le s\le 1\}$, and with $0<a<b<1$ and the functions f and g as above we define the following stochastic integral

$$\int_a^b f(s)\,dB(s) := f(b)B(b) - f(a)B(a) - \int_a^b B(s)\,df(s)$$

and the same formula for g replacing f.

If g or f are not finite at at least one of the endpoints then the corresponding integrals are meant as improper integrals whenever they are finite in the nonstochastic case and almost surely finite in the stochastic case.

Towards introducing our functional invariance theorems for u_n, let $\mathscr{R}_n$ ($n = 1, 2, \dots$) denote any sequence of classes of functions $\imath$ defined on $[1/(n+1), n/(n+1)]$ such that

(R.1) each $\imath \in \mathscr{R}_n$ can be written as $\imath = \imath_1 - \imath_2$, where $\imath_1$ and $\imath_2$ are nondecreasing right-continuous functions defined on $[1/(n+1), n/(n+1)]$, and with $\varepsilon_n = (\log n)^2/n$

$$\text{(R.2)} \qquad M_n(\varepsilon_n) = o(1), \qquad n \to \infty,$$

where

$$M_n(\varepsilon_n) = \sup_{\imath \in \mathscr{R}_n} \sup_{1/(n+1) \le s \le \varepsilon_n} \left\{ \left(|\imath_1(s)| + |\imath_2(s)| + |\imath_1(1-s)| + |\imath_2(1-s)| \right) s^{1/2} \right\}.$$

THEOREM 3.1. *Let $\mathscr{R}_n$ ($n = 1, 2, \dots$) be any sequence of classes of functions $\imath$ as above, satisfying* (R.1) *and* (R.2). *Then on the probability space of Theorem* 1.1

$$\text{(3.1)} \quad D_n := \sup_{\imath \in \mathscr{R}_n} \left| \int_{1/(n+1)}^{n/(n+1)} \imath(s)\, du_n(s) - \int_{1/(n+1)}^{n/(n+1)} \imath(s)\, dB_n(s) \right| = o_P(1).$$

PROOF. We claim that (R.2) implies that

$$\text{(R.3)} \qquad \sup_{\imath \in \mathscr{R}_n} \sup_{\varepsilon_n \le s \le 1-\varepsilon_n} \left(|\imath_1(s)| + |\imath_2(s)| \right) = o\left(n^{1/2}/(\log n) \right).$$

In order to see this, we observe that, since $\imath_1$ and $\imath_2$ are nondecreasing, we have

$$\begin{aligned} \sup_{\varepsilon_n \le s \le 1-\varepsilon_n} \left(|\imath_1(s)| + |\imath_2(s)| \right) &\le \left(|\imath_1(\varepsilon_n)| + |\imath_2(\varepsilon_n)| + |\imath_1(1-\varepsilon_n)| + |\imath_2(1-\varepsilon_n)| \right) \\ &\le M_n(\varepsilon_n)/\varepsilon_n^{1/2} = o\left(n^{1/2}/(\log n) \right), \end{aligned}$$

and (R.3) is verified.

We will from now on assume without loss of generality that each $\imath \in \mathscr{R}_n$ is nondecreasing. We observe that by applying integration by parts we have

$$\begin{aligned} D_n &= \sup_{\imath \in \mathscr{R}_n} \left\{ \left| \left[\imath(s)(u_n(s) - B_n(s)) \right]_{1/(n+1)}^{n/(n+1)} - \int_{1/(n+1)}^{n/(n+1)} (u_n(s) - B_n(s))\, d\imath(s) \right| \right\} \\ &\le \sup_{\imath \in \mathscr{R}_n} \left\{ |\imath(n/(n+1))(u_n(n/(n+1)) - B_n(n/(n+1)))| \right. \\ &\qquad \left. + |\imath(1/(n+1))(u_n(1/(n+1)) - B_n(1/(n+1)))| \right\} \\ &\quad + \sup_{\imath \in \mathscr{R}_n} \left| \int_{1/(n+1)}^{n/(n+1)} (u_n(s) - B_n(s))\, d\imath(s) \right| \\ &:= A_n^{(1)} + A_n^{(2)}. \end{aligned}$$

First we note that

$$A_n^{(1)} \le M_n(\varepsilon_n) \sup_{1/(n+1) \le s \le n/(n+1)} |u_n(s) - B_n(s)|/(s(1-s))^{1/2}.$$

Now taking $\lambda = \frac{1}{2}$ in (2.2) and using (R.2) we get

$$M_n(\varepsilon_n)O_P(1) = o_P(1).$$

Hence

$$A_n^{(1)} = o_P(1).$$

Next consider

$$\begin{aligned} A_n^{(2)} &\le \sup_{\imath \in \mathscr{R}_n} \left| \int_{\varepsilon_n}^{1-\varepsilon_n} (u_n(s) - B_n(s))\, d\imath(s) \right| \\ &\quad + \sup_{\imath \in \mathscr{R}_n} \left| \int_{1/(n+1)}^{\varepsilon_n} (u_n(s) - B_n(s))\, d\imath(s) \right| \\ &\quad + \sup_{\imath \in \mathscr{R}_n} \left| \int_{1-\varepsilon_n}^{n/(n+1)} (u_n(s) - B_n(s))\, d\imath(s) \right| \\ &:= \Delta_n^{(1)} + \Delta_n^{(2)} + \Delta_n^{(3)}. \end{aligned}$$

First observe that

$$\Delta_n^{(1)} \le \sup_{\imath \in \mathscr{R}_n} \big(|\imath(\varepsilon_n)| + |\imath(1-\varepsilon_n)| \big) \sup_{0 \le s \le 1} |u_n(s) - B_n(s)|,$$

which by (2.1) and (R.3) is equal to

$$o\big(n^{1/2}/\log n\big) O_P\big(\log n / n^{1/2}\big) = o_P(1).$$

Hence

$$\Delta_n^{(1)} = o_P(1).$$

Choosing any $0 < \nu < \frac{1}{2}$, we see that

$$\begin{aligned} \Delta_n^{(2)} &\le \left(\sup_{\imath \in \mathscr{R}_n} \int_{1/(n+1)}^{\varepsilon_n} n^{-\nu} s^{1/2-\nu}\, d\imath(s) \right) \\ &\quad \times \sup_{1/(n+1) \le s \le n/(n+1)} n^{\nu} |u_n(s) - B_n(s)| / (s(1-s))^{1/2-\nu}, \end{aligned}$$

which by (2.2) with $\lambda = \frac{1}{2}$ equals

$$\left\{ \sup_{\imath \in \mathscr{R}_n} \int_{1/(n+1)}^{\varepsilon_n} n^{-\nu} s^{1/2-\nu}\, d\imath(s) \right\} O_P(1). \tag{3.2}$$

Applying integration by parts, we have

$$\begin{aligned} \sup_{\imath \in \mathscr{R}_n} \int_{1/(n+1)}^{\varepsilon_n} n^{-\nu} s^{1/2-\nu}\, d\imath(s) &= \sup_{\imath \in \mathscr{R}_n} \left\{ n^{-\nu} s^{1/2-\nu} \imath(s) \Big|_{1/(n+1)}^{\varepsilon_n} \right. \\ &\qquad \left. - \left(\tfrac{1}{2} - \nu\right) \int_{1/(n+1)}^{\varepsilon_n} s^{-1/2-\nu} n^{-\nu} \imath(s)\, ds \right\} \\ &\le 3M_n(\varepsilon_n) + \left(\tfrac{1}{2} - \nu\right) M_n(\varepsilon_n) \int_{1/(n+1)}^{\varepsilon_n} s^{-1-\nu} n^{-\nu}\, ds \\ &\le \big(3 + 2\left(\tfrac{1}{2} - \nu\right)/\nu\big) M_n(\varepsilon_n). \end{aligned}$$

Hence by (3.2) and (R.2)

$$\Delta_n^{(2)} = o_P(1).$$

It can be shown in a similar way that

$$\Delta_n^{(3)} = o_P(1),$$

and hence

$$A_n^{(2)} = o_P(1).$$

Thus we have established (3.1) and the proof of Theorem 3.1 is complete.

The next corollary to Theorem 3.1 is immediate, and it is going to be useful in situations where the class of functions r of $\mathscr{R}_n$ are defined on $(0,1)$ and $\mathscr{R}_n$ itself remains fixed in n.

Let $\mathscr{R}$ be a class of functions r defined on$(0,1)$ such that

(R.1′) each r can be written as $r = r_1 - r_2$, where r_1 and r_2 are nondecreasing right-continuous functions defined on $(0,1)$.

Let for $0 < \delta \le \frac{1}{2}$

$$M(\delta) = \sup_{r \in \mathscr{R}} \sup_{0 < s \le \delta} \left\{ \left(|r_1(s)| + |r_2(s)| + |r_1(1-s)| + |r_2(1-s)| \right) s^{1/2} \right\}.$$

COROLLARY 3.1. *Let $\mathscr{R}$ be any class of functions r as above, satisfying* (R.1′) *and such that*

(R.2′) $$\lim_{\delta \downarrow 0} M(\delta) = 0.$$

Then on the probability space of Theorem 1.1

(3.3) $$D_n := \sup_{r \in \mathscr{R}} \left| \int_{1/(n+1)}^{n/(n+1)} r(s)\, du_n(s) - \int_{1/(n+1)}^{n/(n+1)} r(s)\, dB_n(s) \right| = o_P(1).$$

Our next theorem is an analogue of Theorem 3.1 for the empirical process α_n. Towards introducing such an analogue, let $\mathscr{L}_n$ $(n = 1, 2, \ldots)$ denote any sequence of classes of functions ℓ defined on $(0,1)$ such that

(L.1) each $\ell \in \mathscr{L}_n$ can be written as $\ell = \ell_1 - \ell_2$, where ℓ_1 and ℓ_2 are nondecreasing left-continuous functions defined on $(0,1)$, and there exists a positive nonincreasing function L defined on $(0, \frac{1}{2}]$ slowly varying near zero such that with $\delta_n = (\log n)/n^{1/2}$

(L.2) $$N_n(\delta_n) = o(1), \qquad n \to \infty,$$

where

$$N_n(\delta_n) = \sup_{\ell \in \mathscr{L}_n} \sup_{0 \le s \le \delta_n} \left\{ \left(|\ell_1(s)| + |\ell_2(s)| + |\ell_1(1-s)| + |\ell_2(1-s)| \right) s^{1/2} / L(s) \right\}.$$

THEOREM 3.2. *Let $\mathscr{L}_n$ $(n = 1, 2, \ldots)$ be any sequence of classes of functions ℓ as above, satisfying* (L.1) *and* (L.2). *Then on the probability space of*

Theorem 1.1

$$(3.4)\qquad E_n := \frac{\sup_{\ell\in\mathscr{L}_n}\left|\int_0^1 \ell(s)\,d\alpha_n(s) - \int_{1/n}^{1-1/n}\ell(s)\,dB_n(s)\right|}{L(1/n)} = o_P(1).$$

PROOF. Proceeding as in the proof of Theorem 3.1, it is easy to show that (L.2) implies

$$(\text{L.3})\qquad \sup_{\ell\in\mathscr{L}_n}\ \sup_{\delta_n\le s\le 1-\delta_n}\left(|\ell_1(s)|+|\ell_2(s)|\right) = o\left(n^{1/4}/(\log n)^{1/2}\right).$$

We will from now on assume without loss of generality that each $\ell\in\mathscr{L}_n$ is nondecreasing. Consider

$$\begin{aligned} E_n \le &\sup_{\ell\in\mathscr{L}_n}\left|\int_{1/n}^{1-1/n}\ell(s)\,d\alpha_n(s) - \int_{1/n}^{1-1/n}\ell(s)\,dB_n(s)\right|\Big/L(1/n)\\ &+\sup_{\ell\in\mathscr{L}_n}\left|\int_0^{1/n}\ell(s)\,d\alpha_n(s)\right|\Big/L(1/n) + \sup_{l\in\mathscr{L}_n}\left|\int_{1-1/n}^{1}\ell(s)\,d\alpha_n(s)\right|\Big/L(1/n)\\ &:= E_n^{(1)} + E_n^{(2)} + E_n^{(3)}. \end{aligned}$$

Now the proof of the fact that $E_n^{(1)} = o_P(1)$ proceeds, with minor changes of details, much as the proof that $D_n = o_P(1)$, except that (2.42), (L.3), and (2.32) are used instead of (2.1), (R.3), and (2.2). For the sake of brevity we omit the details.

We will now show that $E_n^{(2)} = o_P(1)$. We have

$$\begin{aligned} E_n^{(2)} \le &\frac{\sup_{\ell\in\mathscr{L}_n} n^{1/2}\int_0^{1/n}|\ell(s)|\,ds}{L(1/n)} + \frac{\sup_{\ell\in\mathscr{L}_n} n^{1/2}\int_0^{1/n}|\ell(s)|\,dG_n(s)}{L(1/n)}\\ &:= D_n^{(1)} + D_n^{(2)}. \end{aligned}$$

First we note that

$$D_n^{(1)} \le N(\delta_n)n^{1/2}\int_0^{1/n}L(s)s^{-1/2}\,ds/L(1/n).$$

Since L is slowly varying at zero,

$$n^{1/2}\int_0^{1/n}L(s)s^{-1/2}\,ds/L(1/n) \to 2 \quad \text{as } n\to\infty$$

[cf., e.g., Theorem 1.2.1 on page 15 of deHaan (1970)]. Hence by (L.2)

$$D_n^{(1)} = o_P(1).$$

Next we observe that

$$\begin{aligned} D_n^{(2)} &\le N_n(\delta_n)\sum_{i=1}^{n} L(U_{i,n})U_{i,n}^{-1/2}I(U_{i,n}\le 1/n)/L(1/n)\\ &\le N_n(\delta_n)K_nL(U_{1,n})(nU_{1,n})^{-1/2}/L(1/n), \end{aligned}$$

where the last inequality is due to $u^{-1/2}L(u)$ being nonincreasing, and $K_n = nG_n(1/n)$.

We claim that

$$L(U_{1,n})/L(1/n) = O_P(1).$$

Since L is nonnegative and nonincreasing, and since by (2.37) $P\{U_{1,n} \geq \lambda/n\}$ tends to one as $n \to \infty$ and $\lambda \downarrow 0$, it is enough to show that for every $0 < \lambda < 1$

$$L(\lambda/n)/L(1/n) = O(1).$$

This last statement follows from the assumption that L is slowly varying near zero and the above claim is proven. Noticing also that $K_n = O_P(1)$, we have in combination with (2.37) again that

$$D_n^{(2)} = N(\delta_n)O_P(1) = o_P(1).$$

Hence, as claimed above,

$$E_n^{(2)} = o_P(1).$$

In the same way it is shown that $E_n^{(3)} = o_P(1)$. Thus the proof of Theorem 3.2 is now complete.

In many practical situations the classes of functions $\mathscr{L}_n$ remain fixed for all n. In these situations the following corollary is going to be useful.

Let $\mathscr{L}$ denote any class of functions ℓ defined on (0, 1) such that

(L.1′) each ℓ can be written as $\ell = \ell_1 - \ell_2$, where ℓ_1 and ℓ_2 are nondecreasing left-continuous functions defined on (0, 1).

Let L be a positive nonincreasing function defined on $(0, \frac{1}{2}]$ slowly varying near zero and define

$$N(\delta) = \sup_{\ell \in \mathscr{L}} \sup_{0 < s \leq \delta} \left\{ \left(|\ell_1(s)| + |\ell_2(s)| + |\ell_1(1-s)| + |\ell_2(1-s)| \right) s^{1/2}/L(s) \right\}.$$

The following corollary follows immediately from Theorem 3.2.

COROLLARY 3.2. *Let $\mathscr{L}$ be any class of functions ℓ as above, satisfying* (L.1′) *and such that*

$$\text{(L.2′)} \qquad \lim_{\delta \downarrow 0} N(\delta) = 0.$$

Then on the probability space of Theorem 1.1

$$\text{(3.5)} \qquad \begin{aligned} E_n := \sup_{\ell \in \mathscr{L}} \left| \int_0^1 \ell(s)\, d\alpha_n(s) - \int_{1/n}^{1-1/n} \ell(s)\, dB_n(s) \right| \Big/ L(1/n) \\ = o_P(1). \end{aligned}$$

Our next goal is to replace the limits of integration $1/(n+1)$ and $n/(n+1)$, respectively $1/n$ and $(n-1)/n$, by 0 and 1 when integrating with respect to the Brownian bridges B_n in (3.3), respectively in (3.5).

A *function* q defined on $(0, \frac{1}{2}]$ is going to be called *positive* if $\inf_{\delta \leq s \leq 1/2} q(s) > 0$ for all $0 < \delta < \frac{1}{2}$.

A *function* w defined on $(0,1)$ is going to be called *positive* if $\inf_{\delta \le s \le 1-\delta} w(s) > 0$ for all $0 < \delta < \frac{1}{2}$.

Let q be any *positive* function defined on $(0, \frac{1}{2}]$, nondecreasing in a neighbourhood of zero. Such a function q will be called an Erdös–Feller–Kolmogorov–Petrovski (EFKP) upper-class function of a Brownian bridge $\{B(s); 0 \le s \le 1\}$ if

$$\limsup_{s \downarrow 0} |B(s)|/q(s) < \infty, \quad \text{a.s.} \tag{3.6}$$

REMARK 3.1. We note that, by the usual representation of a Brownian bridge in terms of a standard Wiener process [cf., e.g., the proof of (1.11)], q is an EFKP upper-class function of a Brownian bridge if and only if it is an EFKP upper-class function of a standard Wiener process.

REMARK 3.2. A routine application of Blumenthal's 0–1 law [cf. Itô and McKean (1965)] shows that (3.6) holds if and only if there exists a constant $0 \le \beta < \infty$ such that

$$\limsup_{s \downarrow 0} |B(s)|/q(s) = \beta, \quad \text{a.s.} \tag{3.7}$$

An EFKP upper-class function q of a Brownian bridge will be called a Chibisov–O'Reilly function if $\beta = 0$ in (3.7).

We introduce the following integrals:

$$E(q, c) := \int_0^{1/2} s^{-3/2} q(s) \exp\left(-cs^{-1}q^2(s)\right) ds, \tag{3.8}$$

and

$$I(q, c) := \int_0^{1/2} s^{-1} \exp\left(-cs^{-1}q^2(s)\right) ds \tag{3.9}$$

for some constant $0 < c < \infty$.

The integral $E(q, c)$ appeared in the works of Kolmogorov, Petrovski, Erdös, and Feller. For details we refer to Itô and McKean (1965, Section 1.8).

The integral $I(q, c)$ appeared in the works of Chibisov (1964) and O'Reilly (1974).

PROPOSITION 3.1. (i) *Whenever the integral* $I(q, c) < \infty$, *then* $E(q, c + \varepsilon) < \infty$ *for any* $\varepsilon > 0$ *and* $q(s)/s^{1/2} \to \infty$ *as* $s \downarrow 0$.

(ii) *Whenever* $E(q, c) < \infty$ *and* $q(s)/s^{1/2} \to \infty$ *as* $s \downarrow 0$, *then* $I(q, c) < \infty$.

This proposition and the next two theorems are proved in the Appendix of this paper.

It is well known that the classical EFKP test gives a criterion, in terms of the integral $E(q, c)$, for q to be an upper-class or lower-class function of a Brownian bridge, provided we assume that $q(s)/s^{1/2}$ is nonincreasing in a neighbourhood of zero. The latter condition is unnatural when talking about upper-class results only (it might, in fact, be superfluous even for the lower-class result; see our conjecture at the end of the paper), and this very condition has created some confusion in the literature and has led to separating the EFKP upper-class result from that of O'Reilly (1974). Part of this confusion, and errors in the literature in connection with it, were cleared up in Section 2 of M. Csörgő, S. Csörgő, Horváth, and Mason (1985). However, at that time, we could not connect the said two upper-class results. Proposition 3.1 above makes the connection between the two integrals figuring in the mentioned two results and leads to the next two theorems. The real message of these two theorems is that there is only one characterization of upper-class functions. In addition to this message, the first of them (Theorem 3.3) in itself provides an interesting addendum to the classic EFKP upper-class result through neither requiring the continuity of q nor the mentioned monotonicity of $q(s)/s^{1/2}$ near zero, while the second one of them (Theorem 3.4) is an extension of O'Reilly's (1974) Proposition 2.1 for possibly discontinuous q functions.

THEOREM 3.3. *A function q is an EFKP upper-class function of a Brownian bridge if and only if the integral $I(q, c) < \infty$ for some $c > 0$ or, equivalently, if and only if the integral $E(q, c) < \infty$ for some $c > 0$ and $\lim_{s\downarrow 0} q(s)/s^{1/2} = \infty$.*

THEOREM 3.4. *A function q is a Chibisov–O'Reilly function if and only if the integral $I(q, c) < \infty$ for all $c > 0$ or, equivalently, if and only if the integral $E(q, c) < \infty$ for all $c > 0$ and $\lim_{s\downarrow 0} q(s)/s^{1/2} = \infty$.*

Now we are in the position of further studying the statement of (3.3) and that of (3.5) as promised above.

Let $\mathscr{R}$ be a class of functions $\imath$ defined on $(0,1)$ such that they satisfy (R.1′), and let q_{11}, q_{12}, q_{21}, and q_{22} be any positive functions defined on $(0, \frac{1}{2}]$, nondecreasing in a neighbourhood of zero, and assumed to be left-continuous. For $\delta > 0$ and small enough so that the $\{q_{ij}\}$ $(i = 1,2;\ j = 1,2)$ are already nondecreasing on $(0, \delta]$, we define

$$M_i^{(1)}(\delta) = \sup_{\imath\in\mathscr{R}} \int_0^{\delta} |\imath_i(s)|\, dq_{1i}(s), \tag{3.10}$$

$$M_i^{(2)}(\delta) = \sup_{\imath\in\mathscr{R}} \int_0^{\delta} |\imath_i(1-s)|\, dq_{2i}(s), \tag{3.11}$$

$i = 1,2$, $\imath = \imath_1 - \imath_2 \in$ (R.1′).

COROLLARY 3.3. *Let $\mathscr{R}$ be any class of functions $\imath$ as above, satisfying* (R.1′), *and let the $\{q_{ij}\}$ $(i = 1,2;\ j = 1,2)$ functions be also as above. If these $\{q_{ij}\}$ functions*

(3.12) *are EFKP upper-class functions of a Brownian bridge and*

$$\lim_{\delta\downarrow 0} \max_{1\le i,\, j\le 2} M_i^{(j)}(\delta) = 0, \tag{3.13}$$

or if they

(3.14) *are Chibisov–O'Reilly functions and*

$$\lim_{\delta \downarrow 0} \max_{1 \le i,\, j \le 2} M_i^{(j)}(\delta) \le M < \infty, \tag{3.15}$$

then on the probability space of Theorem 1.1 *we have, as $n \to \infty$,*

$$\tilde{D}_n := \sup_{\ell \in \mathscr{R}} \left| \int_{1/(n+1)}^{n/(n+1)} \ell(s)\, du_n(s) - \int_0^1 \ell(s)\, dB_n(s) \right| = o_P(1). \tag{3.16}$$

PROOF. We first note that we have

$$\limsup_{\delta \downarrow 0} \sup_{\ell \in \mathscr{R}} \max_{1 \le i \le 2} \sup_{0 < s \le \delta} \big(|\ell_i(s)| q_{1i}(s) + |\ell_i(1-s)| q_{2i}(s) \big) = \gamma, \tag{3.17}$$

where $\gamma = 0$ if (3.13) holds true, and $0 \le \gamma < \infty$ if (3.15) holds true. Hence (3.17) in combination with $\lim_{s \downarrow 0} q_{ij}(s)/s^{1/2} = \infty$ implies (R.2′), which in turn implies that Corollary 3.1 is true under our present conditions. [We note that condition (3.12) is to be interpreted via Theorem 3.3 and that of (3.14) via Theorem 3.4.]

Thus, in order to prove (3.16), it suffices to show that, as $n \to \infty$,

$$\sup_{\ell \in \mathscr{R}} \left(\left| \int_0^{1/(n+1)} \ell(s)\, dB_n(s) \right| + \left| \int_{n/(n+1)}^1 \ell(s)\, dB_n(s) \right| \right) = o_P(1). \tag{3.18}$$

We will only demonstrate that

$$\Delta_n := \sup_{\ell \in \mathscr{R}} \left| \int_0^{1/(n+1)} \ell(s)\, dB_n(s) \right| = o_P(1), \tag{3.19}$$

for a similar argument yields also that

$$\sup_{\ell \in \mathscr{R}} \left| \int_{n/(n+1)}^1 \ell(s)\, dB_n(s) \right| = o_P(1).$$

From now on we will assume without loss of generality that each $\ell \in \mathscr{R}$ is nondecreasing. Consequently, when demonstrating (3.19) one generic q function will suffice in its proof. Towards (3.19) now, we observe that by applying integration by parts we get

$$\begin{aligned} \Delta_n &= \sup_{\ell \in \mathscr{R}} \left| \left\{ \ell(s) q(s) \frac{B_n(s)}{q(s)} \Big|_0^{1/(n+1)} - \int_0^{1/(n+1)} B_n(s)\, d\ell(s) \right\} \right| \\ &\le \left(\sup_{0 < s \le (n+1)^{-1}} \frac{|B_n(s)|}{q(s)} \right) \left(\sup_{\ell \in \mathscr{R}} \sup_{0 < s \le (n+1)^{-1}} |\ell(s)| q(s) \right) \\ &\quad + \sup_{\ell \in \mathscr{R}} \left| \int_0^{(n+1)^{-1}} B_n(s)\, d\ell(s) \right| \\ &:= \Delta_n^{(1)} + \Delta_n^{(2)}. \end{aligned} \tag{3.20}$$

Now

$$\Delta_n^{(2)} \le \Big(\sup_{0<s\le(n+1)^{-1}} |B_n(s)|/q(s) \Big) \Big(\sup_{\imath\in\mathscr{R}} \Big| \int_0^{(n+1)^{-1}} q(s)\, d\imath(s) \Big| \Big)$$

$$= \Big(\sup_{0<s\le(n+1)^{-1}} \frac{|B_n(s)|}{q(s)} \Big) \Big(\sup_{\imath\in\mathscr{R}} \Big| \Big\{ q(s)\imath(s)|_0^{(n+1)^{-1}} - \int_0^{(n+1)^{-1}} \imath(s)\, dq(s) \Big\} \Big| \Big),$$

where the latter integration by parts equality is on account of taking n large enough so that q is already nondecreasing on $(0, 1/(n+1)]$. Hence, for large enough n,

$$\Delta_n^{(2)} \le \Big(\sup_{0<s\le 1/(n+1)} |B_n(s)|/q(s) \Big) \Big\{ \sup_{\imath\in\mathscr{R}} \sup_{0<s\le 1/(n+1)} |\imath(s)| q(s) + \sup_{\imath\in\mathscr{R}} \int_0^{1/(n+1)} |\imath(s)|\, dq(s) \Big\}.$$

Assuming now (3.12) and (3.13), by (3.7) we conclude that

$$\sup_{0<s\le 1/(n+1)} |B_n(s)|/q(s) = O_P(1), \qquad n\to\infty. \tag{3.21}$$

Condition (3.13) implies that [cf. (3.17)]

$$\sup_{\imath\in\mathscr{R}} \sup_{0<s\le 1/(n+1)} |\imath(s)| q(s) = o(1), \qquad n\to\infty, \tag{3.22}$$

and that

$$\sup_{\imath\in\mathscr{R}} \int_0^{1/(n+1)} |\imath(s)|\, dq(s) = o(1), \qquad n\to\infty. \tag{3.23}$$

Hence by (3.21) and (3.22) $\Delta_n^{(1)} = o_P(1)$, and by (3.21), (3.22), and (3.23) $\Delta_n^{(2)} = o_P(1)$. Thus (3.19) is now proved under the conditions (3.12) and (3.13).

When assuming (3.14) and (3.15), by (3.7) with $\beta = 0$, we conclude that

$$\sup_{0<s\le 1/(n+1)} |B_n(s)|/q(s) = o_P(1), \qquad n\to\infty. \tag{3.24}$$

Condition (3.15) implies that [cf. (3.17)]

$$\sup_{\imath\in\mathscr{R}} \sup_{0<s\le 1/(n+1)} |\imath(s)| q(s) = O(1), \qquad n\to\infty, \tag{3.25}$$

and that

$$\sup_{\imath\in\mathscr{R}} \int_0^{1/(n+1)} |\imath(s)|\, dq(s) = O(1), \qquad n\to\infty. \tag{3.26}$$

Consequently by (3.24) and (3.25), $\Delta_n^{(1)} = o_P(1)$, and by (3.24), (3.25), and (3.26), $\Delta_n^{(2)} = o_P(1)$. Thus (3.19) is now proved under the conditions (3.14) and (3.15). This also completes the proof of (3.18) and also that of Corollary 3.3.

Towards our goal to replace the limits of integration $1/n$ and $(n-1)/n$ by 0 and 1 in (3.5), let $\mathscr{L}$ be a class of functions ℓ defined on $(0,1)$ such that they satisfy (L.1′), and let q_{11}, q_{12}, q_{21}, and q_{22} be any positive functions defined on $(0, \frac{1}{2}]$, nondecreasing in a neighbourhood of zero and assumed to be right-continuous. For $\delta > 0$ and small enough so that the $\{q_{ij}\}$ $(i = 1,2;\ j = 1,2)$ are already nondecreasing on $(0, \delta]$, we define

$$N_i^{(1)}(\delta) = \sup_{\ell \in \mathscr{L}} \int_0^{\delta} |\ell_i(s)|\, dq_{1i}(s), \tag{3.27}$$

$$N_i^{(2)}(\delta) = \sup_{\ell \in \mathscr{L}} \int_0^{\delta} |\ell_i(1-s)|\, dq_{2i}(s), \tag{3.28}$$

$i = 1,2$, $\ell = \ell_1 - \ell_2 \in$ (L.1′).

COROLLARY 3.4. *Let $\mathscr{L}$ be any class of functions ℓ as above, satisfying* (L.1′), *and let the $\{q_{ij}\}$ $(i = 1,2;\ j = 1,2)$ functions be also as above. If these $\{q_{ij}\}$ functions*

(3.29) *are EFKP upper-class function of a Brownian bridge and*

$$\lim_{\delta \downarrow 0} \max_{1 \le i,\, j \le 2} N_i^{(j)}(\delta) = 0, \tag{3.30}$$

or if they

(3.31) *are Chibisov–O'Reilly functions and*

$$\lim_{\delta \downarrow 0} \max_{1 \le i,\, j \le 2} N_i^{(j)}(\delta) \le N < \infty, \tag{3.32}$$

then on the probability space of Theorem 1.1, *we have, as $n \to \infty$,*

$$\tilde{E}_n := \sup_{\ell \in \mathscr{L}} \left| \int_0^1 \ell(s)\, d\alpha_n(s) - \int_0^1 \ell(s)\, dB_n(s) \right| = o_P(1). \tag{3.33}$$

PROOF. We note that we have

$$\limsup_{\delta \downarrow 0} \sup_{\ell \in \mathscr{L}} \max_{1 \le i \le 2} \sup_{0 < s \le \delta} \big(|\ell_i(s)| q_{1i}(s) + |\ell_i(1-s)| q_{2i}(s) \big) = \gamma, \tag{3.34}$$

where $\gamma = 0$ if (3.30) holds true, and $0 \le \gamma < \infty$ if (3.32) holds true. Hence (3.34) in combination with $\lim_{\delta \downarrow 0} q(s)/s^{1/2} = \infty$ implies (L.2′) with its $L(s) \equiv 1$. This in turn implies that, under our present conditions, Corollary 3.2 is true with $L(1/n) \equiv 1$ in (3.5). Thus, in order to prove (3.33), it suffices to show that, as $n \to \infty$,

$$\sup_{\ell \in \mathscr{L}} \left(\left| \int_0^{1/n} \ell(s)\, dB_n(s) \right| + \left| \int_{(n-1)/n}^{1} \ell(s)\, dB_n(s) \right| \right) = o_P(1).$$

The rest of this proof is similar to that of Corollary 3.3 and we omit the details.

4. Applications

4.1. A probabilistic proof of the sufficiency part of the normal convergence criterion. One of the classical central limit theorems of probability theory is the following result of Feller, Hinchin, and Lévy [cf. Gnedenko and Kolmogorov (1954, page 172)]:

THEOREM 4.1.1. *Let $X, X_1, X_2, \ldots$ be independent nondegenerate rv with common distribution function F. There exist sequences of constants $\{A_n\}$ and $\{C_n\}$ $(n = 1, 2, \ldots)$ such that*

$$A_n\left(\sum_{i=1}^{n} X_i - C_n\right) \to_{\mathscr{D}} N(0,1) \tag{4.1.1}$$

if and only if

$$x^2 P\{|X| \geq x\}/E\big(|X|^2 I(|X| < x)\big) \to 0 \quad \text{as } x \to \infty. \tag{4.1.2}$$

We show here the fact that (4.1.2) implies (4.1.1) is a consequence of Corollary 3.2.

Assume (4.1.2). In this case $E|X| < \infty$ [cf. Rényi (1970), page 455)]. Write $\mu = EX$ and let Q be the quantile function of F. We observe that

$$\int_0^1 Q(s)\, d\alpha_n(s) = n^{1/2}\left(n^{-1}\sum_{i=1}^{n} Q(U_i) - \mu\right),$$

which is equal in distribution to

$$n^{1/2}\left(n^{-1}\sum_{i=1}^{n} X_i - \mu\right).$$

Let $\mathscr{L} = \{Q\}$. Since Q is a nondecreasing left-continuous function defined on $(0,1)$, we see that $\mathscr{L}$ satisfies (L.1′). For $0 < s < \frac{1}{2}$ write

$$\sigma^2(s, Q) = \int_s^{1-s}\int_s^{1-s} (u \wedge v - uv)\, dQ(u)\, dQ(v).$$

Since F is a nondegenerate distribution function, we can choose a $0 < b < \frac{1}{2}$ such that $\sigma^2(b, Q) > 0$. Let

$$L(s) = \begin{cases} \sigma(s, Q) & \text{for } 0 < s \leq b, \\ \sigma(b, Q) & \text{for } b < s < \frac{1}{2}. \end{cases}$$

It is shown in M. Csörgő, S. Csörgő, Horváth, and Mason (1986) that (4.1.2) implies that

$$L \text{ is slowly varying near zero} \tag{4.1.3}$$

and

$$s^{1/2}\{|Q(s)| + |Q(1-s)|\}/L(s) \to 0 \quad \text{as } s \downarrow 0. \tag{4.1.4}$$

Thus $\mathscr{L}$ also satisfies (L.2′) of Corollary 3.2 with the above L. Hence by the latter corollary's (3.5) we have

$$\left|\int_0^1 Q(s)\,d\alpha_n(s) - \int_{1/n}^{1-1/n} Q(s)\,dB_n(s)\right| / L(1/n) = o_P(1), \qquad n \to \infty.$$

Next we show that

$$\int_{1/n}^{1-1/n} Q(s)\,dB_n(s)/L(1/n) \to_{\mathscr{D}} N(0,1). \tag{4.1.5}$$

Write

$$Z_n = Q(U)I(1/n \le U < 1 - 1/n),$$

where U is a uniform (0, 1) rv. Let

$$K_n(s) = \begin{cases} Q(1-1/n), & 1 - 1/n \le s < 1, \\ Q(s), & 1/n \le s < 1 - 1/n, \\ Q(1/n), & 0 < s < 1/n, \end{cases}$$

and set $Y_n = K_n(U)$. We note that

$$\operatorname{Var}(Z_n/L(1/n)) = \operatorname{Var}\left(\int_{1/n}^{1-1/n} Q(s)\,dB_n(s)/L(1/n)\right),$$

and, since K_n is a nondecreasing left-continuous function on (0, 1),

$$\begin{aligned} \operatorname{Var} Y_n &= \int_0^1\int_0^1 (u \wedge v - uv)\,dK_n(u)\,dK_n(v) \\ &= \sigma^2(1/n, Q), \end{aligned}$$

which equals to $L^2(1/n)$ for all sufficiently large n. In order to show (4.1.5), it suffices to demonstrate that

$$\operatorname{Var}\left(\frac{\int_{1/n}^{1-1/n} Q(s)\,dB_n(s)}{L(1/n)}\right) = \frac{\operatorname{Var} Z_n}{\operatorname{Var} Y_n} \to 1, \qquad n \to \infty. \tag{4.1.6}$$

We note that by Minkowski's inequality

$$\begin{aligned} &\left|(\operatorname{Var} Z_n)^{1/2} - (\operatorname{Var} Y_n)^{1/2}\right| / (\operatorname{Var} Y_n)^{1/2} \\ &\quad\le \left(E(Y_n - Z_n)^2\right)^{1/2}/L(1/n) \\ &\quad= \left((1/n)\left(Q^2(1/n) + Q^2(1-1/n)\right)\right)^{1/2}/L(1/n), \end{aligned} \tag{4.1.7}$$

which by (4.1.4) converges to zero as $n \to \infty$. Thus we have (4.1.6), which in turn implies (4.1.5). Hence (4.1.1) is true with $A_n = 1/(n^{1/2}\dot{L}(1/n))$ and $C_n = n\mu$.

The reader is referred to Root and Rubin (1973) for an alternative probabilistic proof of the normal convergence criterion.

4.2. Weak convergence of the uniform empirical process in weighted sup–norm *metrics and beyond.* Let g be any such real valued positive function on (0, 1) for which we have

$$\lim_{s \downarrow 0} g(s) = \lim_{s \uparrow 1} g(s) = \infty. \tag{4.2.1}$$

Our first result is an immediate corollary to Theorem 2.2.

COROLLARY 4.2.1. *On the probability space of Theorem* 1.1 *we have, as* $n \to \infty$,

$$\sup_{U_{1,n} \le s \le U_{n,n}} |\alpha_n(s) - B_n(s)| / \big((s(1-s))^{1/2} g(s)\big) = o_P(1) \tag{4.2.2}$$

with any function g as in (4.2.1).

PROOF. By (2.8) of Theorem 2.2 with $\nu = 0$ the statement of (4.2.2) follows.

Our next statement is an analogue of Corollary 4.2.1, fashioned after Corollary 3.2. For each $t \in (0, 1)$, let

$$\ell_t(s) = \begin{cases} I(s \le t) / \big((t(1-t))^{1/2} g(t)\big) & \text{if } t \in \big(0, \tfrac{1}{2}\big), \\ (1 - I(s \le t)) / \big((t(1-t))^{1/2} g(t)\big) & \text{if } t \in \big[\tfrac{1}{2}, 1\big), \end{cases} \tag{4.2.3}$$

where the function g is as in (4.2.1). We write

$$\overline{B}_n(t) = \begin{cases} B_n(t) & \text{for } t \in [1/n, 1 - 1/n), \\ 0 & \text{elsewhere,} \end{cases} \tag{4.2.4}$$

where B_n is a Brownian bridge for each n.

COROLLARY 4.2.2. *On the probability space of Theorem* 1.1 *we have, as* $n \to \infty$,

$$\sup_{0<t<1} |\alpha_n(t) - \overline{B}_n(t)| / \big((t(1-t))^{1/2} g(t)\big) = o_P(1), \tag{4.2.5a}$$

and for any $0 \leqq \nu < \frac{1}{4}$

$$n^{\nu} \sup_{0<t<1} |\alpha_n(t) - \overline{B}_n(t)| / (t(1-t))^{1/2-\nu} = O_P(1). \tag{4.2.5b}$$

PROOF. Let $w(t) := (t(1-t))^{1/2} g(t)$ and consider

$$\Delta_n(t) := \left| \int_0^1 \ell_t(s)\, d\alpha_n(s) - \int_{1/n}^{1-1/n} \ell_t(s)\, dB_n(s) \right|,$$

where

$$\int_0^1 \ell_t(s)\, d\alpha_n(s) = \begin{cases} \alpha_n(t)/w(t) & \text{if } t \in \big(0, \tfrac{1}{2}\big), \\ -\alpha_n(t)/w(t) & \text{if } t \in \big[\tfrac{1}{2}, 1\big), \end{cases}$$

and

$$\int_{1/n}^{1-1/n} \ell_t(s)\,dB_n(s) = \begin{cases} 0 & \text{if } 0 < t < 1/n, \\ (B_n(t) - B_n(1/n))/w(t) & \text{if } 1/n \le t < \frac{1}{2}, \\ (B_n(1-1/n) - B_n(t))/w(t) & \text{if } \frac{1}{2} \le t < 1 - 1/n, \\ 0 & \text{if } 1 - 1/n \le t < 1. \end{cases}$$

By condition (4.2.1) the (L.2′) assumption of Corollary 3.2 is satisfied and by (3.5) of the latter corollary we get

$$\sup_{0<t<1} \Delta_n(t) = o_P(1), \qquad n \to \infty.$$

Hence, in order to verify (4.2.5a), it suffices to show that, as $n \to \infty$,

$$\sup_{1/n \le t < 1/2} |B_n(1/n)|/w(t) = o_P(1) \tag{4.2.6}$$

and

$$\sup_{1/2 \le t < 1-1/n} |B_n(1-1/n)|/w(t) = o_P(1). \tag{4.2.7}$$

By condition (4.2.1) we have that for any given $\varepsilon > 0$ there exists a $\theta = \theta(\varepsilon)$ such that $1/g(t) < \varepsilon$ whenever $t \in (0, \theta)$. Since $\sup_{\theta < t < 1/2} |B_n(1/n)|/w(t) = o_P(1)$ for any $\theta > 0$ as $n \to \infty$, and with $\theta > 0$ and $\varepsilon > 0$ as above

$$\sup_{1/n \le t \le \theta} |B_n(1/n)|/w(t) \le n^{1/2}(1 - n^{-1})^{-1/2} |B_n(1/n)| \varepsilon,$$

we have (4.2.6). The proof of (4.2.7) goes along similar lines and (5.2.5a) is proved.

To prove (4.2.5b), in light of (2.32) of Corollary 2.1, it is enough to show that for any $0 \leqq \nu < \frac{1}{4}$

$$A_n^{(1)} = n^{\nu} \sup_{0<t<1/n} |\alpha_n(t)|/(t(1-t))^{1/2-\nu} = O_P(1)$$

and

$$A_n^{(2)} = n^{\nu} \sup_{1-1/n \leqq t < 1} |\alpha_n(t)|/(t(1-t))^{1/2-\nu} = O_P(1).$$

In particular, notice that

$$A_n^{(1)} \le \sup_{0<t<1/n} (nt)^{1/2+\nu}/(1-t)^{1/2-\nu} + nG_n\left(\frac{1}{n}\right) \Big/ \left((nU_{1,n})^{1/2-\nu} \left(1 - \frac{1}{n}\right)^{1/2-\nu} \right),$$

which is obviously $O_P(1)$. Similarly, $A_n^{(2)} = O_P(1)$ and the theorem is proved.

While Corollary 4.2.2 is already a $o_P(1)$ sup–norm convergence statement over the whole interval $(0,1)$ with $w(t)$, $t \in (0,1)$, as in its proof, we should of course remember that $\overline{B}_n$ in (4.2.5) is zero outside $[1/n, 1 - 1/n]$. Hence if we want to replace $\overline{B}_n$ by B_n in (4.2.5a), it is clear that the weight function $w(\cdot)$ on $(0,1)$ will have to be such that

$$\sup_{0<t<1/n} |B_n(t)|/w(t) = o_P(1), \qquad n \to \infty,$$

and

$$\sup_{1-1/n \le t < 1} |B_n(t)|/w(t) = o_P(1), \qquad n \to \infty.$$

Now the latter two statements are true if and only if their weight function w is a Chibisov–O'Reilly function. A function w on $(0,1)$ is called a Chibisov–O'Reilly weight function if $w(t)$ and $w(1-t)$ are Chibisov–O'Reilly functions on $[0, \frac{1}{2}]$ according to our previous definition following (3.6). The definition of an EFKP upper-class function in (3.6) is extended to $(0,1)$ in a similar way. Consequently, we proved the sufficiency part of the following:

THEOREM 4.2.1. *Let w be a positive function on $(0,1)$ such that it is nondecreasing in a neighbourhood of zero and nonincreasing in a neighbourhood of one. On the probability space of Theorem* 1.1 *we have, as $n \to \infty$,*

$$\sup_{0<t<1} |\alpha_n(t) - B_n(t)|/w(t) = o_P(1) \tag{4.2.8}$$

if and only if w is a Chibisov–O'Reilly function on $(0,1)$, *that is, if and only if*

$$\hat{I}(w,c) := \int_0^1 (t(1-t))^{-1} \exp\left(-c(t(1-t))^{-1} w^2(t)\right) dt < \infty \tag{4.2.9}$$

for all $c > 0$.

REMARK 4.2.1. According to our Theorem 3.4, the statement (4.2.8) of Theorem 4.2.1 is also equivalent to

$$\hat{E}(w,c) := \int_0^1 \frac{w(t)}{(t(1-t))^{3/2}} \exp\left(-c(t(1-t))^{-1} w^2(t)\right) dt < \infty \tag{4.2.10}$$

for all $c > 0$, and

$$\lim_{t \downarrow 0} w(t) t^{-1/2} = \lim_{t \uparrow 1} w(t)(1-t)^{-1/2} = \infty. \tag{4.2.11}$$

Theorem 4.2.1 has a long history. It was first proved by Chibisov (1964), assuming some regularity conditions on w, and then by O'Reilly (1974), assuming only the continuity of w, which is not assumed any more in Theorem 4.2.1. There have been unsuccessful attempts in the literature to reprove the Chibisov–O'Reilly theorem. For comments on these we refer to Section 2 in M. Csörgő et al. (1985). Our Corollary 3.4 may be viewed as an indexed by functions generalization of the Chibisov–O'Reilly theorem, or, rather, that of Theorem 4.2.1. In order to demonstrate this point, we give two further proofs of the sufficiency part of Theorem 4.2.1 based on Corollary 3.4.

PROOF (b). Let w be a Chibisov–O'Reilly function on $(0,1)$ and write

$$\ell_t(s) = \begin{cases} I(s \le t)/w(t) & \text{if } t \in \left(0, \frac{1}{2}\right), \\ (1 - I(s \le t))/w(t) & \text{if } t \in \left[\frac{1}{2}, 1\right), \end{cases} \tag{4.2.12}$$

and let

$$q_{11}(s) = q_{12}(s) = w(s), \qquad s \in (0, \tfrac{1}{2}],$$
$$q_{21}(s) = q_{22}(s) = w(1-s), \qquad s \in (0, \tfrac{1}{2}]$$

be the $\{q_{ij}\}$ $(i = 1,2;\ j = 1,2)$ functions of Corollary 3.4. Then it can be easily verified that the appropriate version of condition (3.32) in our present context holds true with the upper bound $N = 1$. Hence the sufficiency part of Theorem 4.2.1 is proved again.

The third proof of the sufficiency part of Theorem 4.2.1 requires also the following lemma, which is of interest on its own.

LEMMA 4.2.1. *For any Chibisov–O'Reilly function q there exists an EFKP upper-class function q^* such that* $\lim_{s\downarrow 0} q^*(s)/q(s) = 0$.

PROOF. Consider $I(q,c) < \infty$ for all $c > 0$. The integrand of $I(q,c)$ is a continuous monotone nonincreasing function of c. Hence by the monotone convergence theorem we have $\lim_{c\downarrow 0} I(q,c) = \infty$. Thus we may proceed with the following construction:

Let $c_1 = 1$, and $\lambda_1 = \frac{1}{2}$. Let $c_2 = \frac{1}{2}$, and define λ_2 by

$$\tfrac{1}{2}I(q,1) = \int_0^{\lambda_2} t^{-1}\exp\left(-(1/2)^2 q^2(t)/t\right) dt$$

and let

$$q^*(t) = q(t), \qquad \lambda_2 < t \le \tfrac{1}{2}.$$

Let $c_3 = \min(1/2^2, q^*(\lambda_2 +)/q(\lambda_2 -))$, and define λ_3 by

$$(1/2^2)I(q,1) = \int_0^{\lambda_3} t^{-1}\exp\left(-c_3^2 q^2(t)/t\right) dt$$

and let

$$q^*(t) = c_2 q(t), \qquad \lambda_3 < t \le \lambda_2.$$

Continuing similarly, in general, we let

$$c_k = \min\left(1/2^{k-1}, q^*(\lambda_{k-1} +)/q(\lambda_{k-1} -)\right)$$

and define λ_k by

$$(1/2^{k-1})I(q,1) = \int_0^{\lambda_k} t^{-1}\exp\left(-c_k^2 q^2(t)/t\right) dt$$

and let

$$q^*(t) = c_{k-1} q(t), \qquad \lambda_k < t \le \lambda_{k-1}.$$

Then $q^*(t)$ is an EFKP upper-class function, for, by definition, it is a nondecreasing function in a neighbourhood of zero, and

$$\begin{aligned} I(q^*,1) &= \sum_{k=2}^{\infty} \int_{\lambda_k}^{\lambda_{k-1}} t^{-1}\exp\big(-(q^*(t))^2/t\big)\,dt \\ &= \sum_{k=2}^{\infty} \int_{\lambda_k}^{\lambda_{k-1}} t^{-1}\exp\big(-c_{k-1}^2 q^2(t)/t\big)\,dt \\ &\le \sum_{k=2}^{\infty} \int_{0}^{\lambda_{k-1}} t^{-1}\exp\big(-c_{k-1}^2 q^2(t)/t\big)\,dt \\ &= \sum_{k=2}^{\infty} 2^{2-k} I(q,1) = 2I(q,1) < \infty. \end{aligned}$$

Also,

$$q^*(t)/q(t) = c_{k-1} \le 2^{-(k-2)} \quad \text{if } \lambda_k < t \le \lambda_{k-1},$$

and this also concludes the proof of Lemma 4.2.1.

PROOF (c) OF THE SUFFICIENCY PART OF THEOREM 4.2.1. Let w be a Chibisov–O'Reilly function on $(0,1)$ $\ell_t(\cdot)$ be defined as in (4.2.12). Since w is a Chibisov–O'Reilly function, by Lemma 4.2.1 there exists an EFKP upper-class function w^* such that

$$\lim_{s \downarrow 0} w^*(s)/w(s) = \lim_{s \downarrow 0} w^*(1-s)/w(1-s) = 0. \tag{4.2.13}$$

Let

$$\begin{aligned} q_{11}(s) &= q_{12}(s) = w^*(s), & s &\in (0, \tfrac{1}{2}], \\ q_{12}(s) &= q_{22}(s) = w^*(1-s), & s &\in (0, \tfrac{1}{2}] \end{aligned}$$

be the $\{q_{ij}\}$ $(i = 1,2;\ j = 1,2)$ functions of Corollary 3.4. Using these functions we wish to show now that condition (3.30) of Corollary 3.4 is satisfied. We demonstrate the latter via showing that $\lim_{\delta \downarrow 0} N_1^{(1)}(\delta) = 0$. The proof of $\lim_{\delta \downarrow 0} N_2^{(2)}(\delta) = 0$ is similar, while $N_1^{(2)}(\delta) = N_2^{(1)}(\delta) = 0$ by definition in our present context. Without loss of generality we assume that w is a nondecreasing function on $(0, \frac{1}{2})$ and consider

$$\begin{aligned} N_1^{(1)}(\delta) &= \sup_{0<t<1/2} \int_0^{\delta} \big(I(s \le t)/w(t)\big)\,dw^*(s) \\ &= \sup_{0<t<1/2} \big(w^*(\delta \wedge t)/w(t)\big) \\ &= \max\Big\{ \sup_{0<t<\delta} \big(w^*(t)/w(t)\big),\ \sup_{\delta \le t < 1/2} \big(w^*(\delta)/w(t)\big)\Big\} \\ &\le \sup_{0<t\le\delta} \big(w^*(t)/w(t)\big) \to 0 \quad \text{as } \delta \downarrow 0, \end{aligned}$$

where the latter limit is by (4.2.13). Hence the sufficiency part of Theorem 4.2.1 is proved again.

PROOF OF THE NECESSARY PART OF THEOREM 4.2.1. Let w be any positive function on $(0,1)$ such that it is nondecreasing in a neighbourhood of zero and nonincreasing in a neighbourhood of one, and assume that (4.2.8) holds true with any sequence of Brownian bridges. First we show that, if (4.2.8) holds true, then we must have

$$\lim_{s\downarrow 0} w(s)/s^{1/2} = \lim_{s\uparrow 1} w(s)/(1-s)^{1/2} = \infty. \tag{4.2.14}$$

We will demonstrate (4.2.14) via showing that

$$\lim_{s\downarrow 0} w(s)/s^{1/2} = \infty. \tag{4.2.15}$$

The proof for the other half of (4.2.14) goes along similar lines. Clearly,

$$P\{G_n(c/n)=0\} = P\{U_{1,n} > c/n\} \to e^{-c}, \qquad n\to\infty,$$

for all $c>0$. Then by (4.2.8) and the latter line we have

$$\liminf_{n\to\infty} P\left\{\left|B_n(c/n) + c/n^{1/2}\right|/w(c/n) < \varepsilon\right\} \geq e^{-c} \tag{4.2.16}$$

for all $\varepsilon > 0$ and $c>0$. Let $c>0$ be given and assume that we have $\lim_{n\to\infty} w(c/n)/(c/n)^{1/2} \neq \infty$. Hence there must exist a sequence of integers $\{n_k\}$, $n_k\to\infty$ as $k\to\infty$, such that, for the given $c>0$,

$$\lim_{k\to\infty} w(c/n_k)/(c/n_k)^{1/2} = \eta < \infty.$$

Then by (4.2.16) we have

$$\lim_{k\to\infty} P\left\{\left|B_{n_k}(c/n_k) + c/n_k^{1/2}\right|/w(c/n_k) < \varepsilon\right\}$$
$$= \Phi\left(\varepsilon\eta - c^{1/2}\right) - \Phi\left(-\varepsilon\eta - c^{1/2}\right) \geq e^{-c}$$

for all $\varepsilon>0$ and the given $c>0$. Choosing now $\varepsilon>0$ small enough, the latter inequality leads to a contradiction. Hence, given (4.2.8), we must have (4.2.15) and also (4.2.14).

Given now that we must have (4.2.14) if (4.2.8) holds true, we first conclude by (4.2.5a) that

$$\sup_{0<t<1/n} |\alpha_n(t)|/w(t) = o_P(1), \qquad n\to\infty,$$

hence we must have also that

$$\sup_{0<t<1/n} |B_n(t)|/w(t) = o_P(1). \tag{4.2.17}$$

Since B_n is a Brownian bridge for each n, $w(\cdot)$ must be a Chibisov–O'Reilly function. Hence by Theorem 3.4 we have also (4.2.9), or, equivalently, (4.2.10) and (4.2.11).

Our next theorem is the EFKP upper-class functions analogue of Theorem 4.2.1.

THEOREM 4.2.2. *Let w be a positive function on $(0,1)$ such that it is nondecreasing in a neighbourhood of zero and nonincreasing in a neighbourhood of one. On the probability space of Theorem* 1.1 *we have as $n \to \infty$,*

$$\sup_{0<t<1} |\alpha_n(t) - B_n(t)|/w(t) = O_P(1) \tag{4.2.18}$$

if and only if w is an EFKP upper-class function on $(0,1)$, *that is, if and only if the integral $\hat{I}(w,c)$ is finite for some $c > 0$, or, equivalently, if and only if the integral $\hat{E}(w,c)$ is finite for some $c > 0$ and* (4.2.11) *holds true.*

PROOF. Let w be an EFKP upper-class function on $(0,1)$. Then we have (4.2.5a) of Corollary 4.2.2. Hence by

$$\sup_{0<t\le 1/n} |B_n(t)|/w(t) = O_P(1), \qquad n \to \infty,$$

and

$$\sup_{1-1/n\le t<1} |B_n(t)|/w(t) = O_P(1), \qquad n \to \infty,$$

being true if and only if their weight function w is an EFKP function of a Brownian bridge, we get (4.2.18).

Conversely, let w be any positive function on $(0,1)$ such that it is nondecreasing in a neighbourhood of zero and nonincreasing in a neighbourhood of one, and assume that (4.2.18) holds true with any sequence of Brownian bridges. First we show that, given (4.2.18), then we must have also (4.2.14). We show only that (4.2.15) holds, for the proof of the other half of (4.2.14) is similar. Given (4.2.18), we have

$$P\left\{\sup_{0<t<1} |\alpha_n(t) - B_n(t)|/w(t) < K\right\} \ge \tfrac{1}{2}$$

for some K if n is large enough. Consider

$$\begin{aligned}\sup_{0<t<1} |\alpha_n(t) - B_n(t)|/w(t) &\geqq \sup_{0<t\leqq 1/2n} (\alpha_n(t) - B_n(t))/w(t)\\ &\geqq \sup_{0<t\leqq 1/2n} \left(-n^{1/2}t - B_n(t)\right)/w(t)\\ &\geqq \sup_{0<t\leqq 1/2n} \left(-2^{-1/2}t^{1/2} - B_n(t)\right)/w(t).\end{aligned}$$

Hence, with $\{-B_n(t);\ 0 \leqq t \leqq 1\} =_{\mathscr{D}} \{B(t);\ 0 \leqq t \leqq 1\}$, a Brownian bridge for each n, we have

$$P\left\{\sup_{0<t\leqq 1/2n} \left(-2^{-1/2}t^{1/2} + B(t)\right)/w(t) < K\right\} \geqq \tfrac{1}{2},$$

for n large enough, and consequently,

$$P\left\{\limsup_{t\downarrow 0} B(t)/\left(Kw(t) + 2^{-1/2}t^{1/2}\right) \le 1\right\} \ge \tfrac{1}{2}.$$

Now an application of Blumenthal's 0–1 law implies that

$$P\Big\{\limsup_{t\downarrow 0} B(t)/\big(Kw(t)+2^{-1/2}t^{1/2}\big)\le 1\Big\}=1.$$

Consequently, the function $Kw(t)+2^{-1/2}t^{1/2}$ is an EFKP upper-class function of a Brownian bridge, which in turn implies that we have (4.2.15).

Given now that we must have (4.2.14) if (4.2.18) is true, we first conclude that

$$\sup_{0<t<1/n}|\alpha_n(t)|/w(t)=o_P(1),\qquad n\to\infty,$$

hence we must have

$$\sup_{0<t<1/n}|B_n(t)|/w(t)=O_P(1),\qquad n\to\infty,$$

by (4.2.18). Since B_n is a Brownian bridge for each n, $w(\cdot)$ must be an EFKP upper-class function. This also concludes the proof of Theorem 4.2.2.

REMARK 4.2.2. An immediate generalization of the sufficiency part of Theorem 4.2.2 is that if in Corollary 3.4 we assume (3.29) and (3.32), then we obtain (3.33) with $O_P(1)$ instead of $o_P(1)$.

Our next lemma is to augment Corollary 4.2.2, Theorems 4.2.1 and 4.2.2, and will be useful in the sequel.

LEMMA 4.2.2. *Whenever w is an EFKP upper-class function, then for each $-\infty<x<\infty$ and any Brownian bridge B*

$$\begin{aligned}R_n(x)&:=P\Big\{\sup_{1/n\le s\le 1-1/n}|B(s)|/w(s)\le x\Big\}\\ &\to R(x):=P\Big\{\sup_{0<s<1}|B(s)|/w(s)\le x\Big\},\qquad n\to\infty.\end{aligned}\tag{4.2.19}$$

PROOF. Choose any $-\infty<x<\infty$ For $n\ge 2$, let

$$A_n:=\Big\{\sup_{1/n\le s\le 1-1/n}|B(s)|/w(s)\le x\Big\}$$

and

$$A:=\Big\{\sup_{0<s<1}|B(s)|/w(s)\le x\Big\}.$$

We notice that $A_{n+1}\subset A_n$ $(n=2,3,\dots)$ and $\bigcap_{n=2}^{\infty}A_n=A$. Hence $R_n(x)=P(A_n)\downarrow R(x)=P(A)$, and (4.2.19) is proved.

REMARK 4.2.3. We observe that (3.7) implies

$$\liminf_{n\to\infty}\sup_{1/n\le s\le 1-1/n}|B(s)|/w(s)\ge\delta=\max(\beta,\gamma)\quad\text{a.s.},$$

where $\gamma < \infty$ is the corresponding lim sup as $s \uparrow 1$. This in turn implies that for each $x < \delta$

$$(4.2.20) \qquad R_n(x) \to R(x) = 0, \qquad n \to \infty.$$

We note also that if w is a continuous EFKP upper-class function, then R is continuous on $(-\infty, \delta) \cup (\delta, \infty)$.

THEOREM 4.2.3. *Let w be a positive function on $(0,1)$ such that it is nondecreasing in a neighbourhood of zero and nonincreasing in a neighbourhood of one. The sequence of rv $\sup_{0<s<1}|\alpha_n(s)|/w(s)$ converges in distribution to a nondegenerate rv if and only if w is an EFKP upper-class function. The latter nondegenerate rv must be the rv $\sup_{0<s<1}|B(s)|/w(s)$.*

PROOF. Assume first that w is an EFKP upper-class function. Then by Corollary 4.2.2 we have

$$(4.2.21) \qquad \lim_{n\to\infty}\left|P\left\{\sup_{0<s<1}\frac{|\alpha_n(s)|}{w(s)} \le x\right\} - P\left\{\sup_{1/n\le s\le 1-1/n}\frac{|B(s)|}{w(s)} \le x\right\}\right| = 0$$

for any Brownian bridge and $-\infty < x < \infty$. Hence by Lemma 4.2.2 the sufficiency part of Theorem 4.2.3 is proved.

Conversely, assume that the rv $\sup_{0<s<1}|\alpha_n(s)|/w(s)$ converge in distribution to a nondegenerate rv. We show first that then we have

$$(4.2.22) \qquad \liminf_{s\downarrow 0} w(s)/s^{1/2} \ge K > 0,$$

and

$$(4.2.23) \qquad \liminf_{s\uparrow 1} w(s)/(1-s)^{1/2} \ge K > 0.$$

We demonstrate only (4.2.22) and note that the proof of (4.2.23) is the same. Choose any $0 < \varepsilon < 1$. Then there exists a $0 < c < 1$ and a $0 < C < \infty$ such that for all n sufficiently large,

$$P\{U_{1,n} > c/n\} \ge 1 - \varepsilon/2$$

and

$$P\left\{\sup_{0<s<1}|\alpha_n(s)|/w(s) < C\right\} \ge 1 - \varepsilon/2.$$

Hence with probability greater than or equal to $1-\varepsilon$

$$(4.2.24) \qquad \sup_{0<s\le c/n}|\alpha_n(s)|/w(s) \le \sup_{0<s\le U_{1,n}}|\alpha_n(s)|/w(s) < C.$$

The left-hand side of (4.2.24) equals

$$(4.2.25) \qquad \sup_{0<s\le c/n} n^{1/2}s/w(s) \ge c^{1/2}(c/n)^{1/2}/w(c/n).$$

Thus, on account of (4.2.24) and (4.2.25), we have

$$w(c/n)/(c/n)^{1/2} > c^{1/2}/C \tag{4.2.26}$$

for all n sufficiently large. The latter relationship in turn implies (4.2.22). By (2.32) and (4.2.22) combined we get

$$\sup_{1/n \le s \le 1-1/n} |\alpha_n(s) - B_n(s)|/w(s) = O_P(1).$$

Consequently, $\sup_{1/n \le s \le 1-1/n}|B_n(s)|/w(s) = O_P(1)$, $n \to \infty$, and hence $P\{\limsup_{s\downarrow 0}|B(s)|/w(s) < \infty\} = 1$ on account of $B_n =_{\mathscr{D}} B$, a Brownian bridge for each n. The latter implies that w must be an EFKP upper-class function.

REMARK 4.2.4. While obvious, we should nevertheless like to note that the result of Theorem 4.2.1 cannot imply that of Theorem 4.2.3. That is to say, as far as convergence in distribution of sup-functionals of α_n/w is concerned, a Chibisov–O'Reilly type theorem like Theorem 4.2.1 is far from being optimal. It excludes all those EFKP upper-class functions w from the game which are not necessarily Chibisov–O'Reilly functions. For example, w of Theorem 4.2.3 can be taken to be the function $(s(1-s)\log\log(1/(s(1-s))))^{1/2}$. However, the latter function cannot be the w of Theorem 4.2.1.

4.3. Weak convergence of the uniform quantile process in weighted sup–norm *metrics and beyond.* Here we consider the problem of approximating the uniform quantile process u_n by B_n along the lines of Section 4.2. Our first statement is an immediate corollary to Corollary 2.1.

COROLLARY 4.3.1. *On the probability space of Theorem* 1.1 *we have, as* $n \to \infty$,

$$\sup_{1/(n+1) \le s \le n/(n+1)} |u_n(s) - B_n(s)|/\big((s(1-s))^{1/2} g(s)\big) = o_P(1) \tag{4.3.1}$$

with any function g as in (4.2.1).

A direct analogue of Theorem 4.2.1 is impossible for the uniform quantile process u_n. This is due to the fact that

$$\sup_{0 < s \le (U_{1,n}/2)\wedge(1/n)} |u_n(s)|/w(s) = \sup_{0 < s \le (U_{1,n}/2)\wedge(1/n)} \big(n^{1/2}|U_{1,n} - s|\big)/w(s)$$

$$\ge (1/2)n^{1/2}U_{1,n}/w(0+) = \infty \quad \text{for all } n,$$

for any function w for which we have $\lim_{s\downarrow 0} w(s) = 0$.

Naturally enough, if we *redefine* u_n to be zero on the outside of the interval $[1/(n+2), (n+1)/(n+2)]$, then Theorems 4.2.1, 4.2.2, and 4.2.3 remain true with the thus redefined u_n replacing α_n in their statements.

Following O'Reilly (1974), we can also work with the following modification $\bar{u}_n$ of u_n. Let

$$u_n^{(1)}(s) = \begin{cases} n^{1/2}s, & 0 < s \le 1/(n+2), \\ n^{1/2}(s - U_{k,n}), & \dfrac{k}{n+2} < s \le \dfrac{k+1}{n+2} \ (k = 1,\dots,n), \\ n^{1/2}(s-1), & (n+1)/(n+2) < s \le 1, \end{cases}$$

$$u_n^{(2)}(s) = \begin{cases} 0, & 0 < s \le 1/(n+2), \\ u_n^{(1)}(s), & k/(n+2) < s \le (k+1)/(n+2) \ (k = 1,\dots,n), \\ 0, & (n+1)/(n+2) < s \le 1, \end{cases}$$

and *define* $\bar{u}_n$ to be any one of $u_n^{(1)}$ and $u_n^{(2)}$. Then Theorems 4.2.1, 4.2.2, 4.2.3, and their relevant remarks remain true with $\bar{u}_n$ replacing α_n in their statements.

4.4. The Eicker (1979) and Jaeschke (1979) results on the asymptotic distribution of the supremum of standardized empirical processes based entirely on invariance. Let

$$a(x) = (2\log x)^{1/2}, \qquad b(x) = 2\log x + 2^{-1}\log\log x - 2^{-1}\log\pi,$$
$$a_n = a(\log n) \quad \text{and} \quad b_n = b(\log n).$$

Also, for $-\infty < t < \infty$, let E denote the extreme value distribution

$$E(t) = \exp(-\exp(-t)).$$

THEOREM 4.4.1. *For any* $-\infty < t < \infty$, *as* $n \to \infty$

$$P\Big\{a_n \sup_{0<s<1} \alpha_n(s)/(s(1-s))^{1/2} - b_n \le t\Big\} \to E(t), \tag{4.4.1}$$

$$P\Big\{a_n \sup_{0<s<1} |\alpha_n(s)|/(s(1-s))^{1/2} - b_n \le t\Big\} \to E^2(t), \tag{4.4.2}$$

$$P\Big\{a_n \sup_{1/(n+1)\le s\le n/(n+1)} u_n(s)/(s(1-s))^{1/2} - b_n \le t\Big\} \to E(t), \tag{4.4.3}$$

$$P\Big\{a_n \sup_{1/(n+1)\le s\le n/(n+1)} |u_n(s)|/(s(1-s))^{1/2} - b_n \le t\Big\} \to E^2(t). \tag{4.4.4}$$

We will show here that the four statements of Theorem 4.4.1 follow entirely from our approximation. For this purpose, we isolate several facts as lemmas leading up to the said proofs.

LEMMA 4.4.1. *Let* $\varepsilon_n = n^{-1}(\log n)^3$. *For any* $-\infty < t < \infty$, *as* $n \to \infty$, *we have for any Brownian bridge* $\{B(s);\ 0 \le s \le 1\}$

$$P\Big\{a_n \sup_{\varepsilon_n \le s \le 1-\varepsilon_n} B(s)/(s(1-s))^{1/2} - b_n \le t\Big\} \to E(t), \tag{4.4.5}$$

and

$$P\left\{a_n \sup_{\varepsilon_n \le s \le 1-\varepsilon_n} |B(s)|/(s(1-s))^{1/2} - b_n \le t\right\} \to E^2(t). \tag{4.4.6}$$

This lemma follows directly from the results of Darling and Erdős (1956) [cf., e.g., Theorem 1.9.1 and Corollary 1.9.1 in M. Csörgő and Révész (1981) or Lemma 1 in Jaeschke (1979)]. See, however, the correction of a misprint in M. Csörgő and Révész (1981) as corrected in the proof of Lemma 4.4.3 below.

LEMMA 4.4.2. *For any* $-\infty < t < \infty$ *and* ε_n *as in Lemma* 4.4.1, *as* $n \to \infty$

$$P\left\{a_n \sup_{\varepsilon_n \le s \le 1-\varepsilon_n} \alpha_n(s)/(s(1-s))^{1/2} - b_n \le t\right\} \to E(t), \tag{4.4.7}$$

$$P\left\{a_n \sup_{\varepsilon_n \le s \le 1-\varepsilon_n} |\alpha_n(s)|/(s(1-s))^{1/2} - b_n \le t\right\} \to E^2(t), \tag{4.4.8}$$

$$P\left\{a_n \sup_{\varepsilon_n \le s \le 1-\varepsilon_n} u_n(s)/(s(1-s))^{1/2} - b_n \le t\right\} \to E(t), \tag{4.4.9}$$

and

$$P\left\{a_n \sup_{\varepsilon_n \le s \le 1-\varepsilon_n} |u_n(s)|/(s(1-s))^{1/2} - b_n \le t\right\} \to E^2(t). \tag{4.4.10}$$

PROOF. Consider (4.4.7) and (4.4.8). Choose any $0 < \nu < \frac{1}{4}$. We have for large enough n

$$a_n \sup_{\varepsilon_n \le s \le 1-\varepsilon_n} \frac{|\alpha_n(s) - B_n(s)|}{(s(1-s))^{1/2}} \le \frac{2a_n}{(\log n)^{3\nu}} n^{\nu} \sup_{1/n \le s \le 1-1/n} \frac{|\alpha_n(s) - B_n(s)|}{(s(1-s))^{1/2-\nu}}$$

which by Corollary 2.2 equals $O_P(a_n/(\log n)^{3\nu}) = o_P(1)$. Hence (4.4.7) and (4.4.8) follow by Lemma 4.4.1. The other two statements follow in the same way using Theorem 2.1 instead of Corollary 2.2.

LEMMA 4.4.3. *Let* $\{\delta_n\}$ *be any sequence of positive numbers such that* $1 \le \delta_n \le n$, $\delta_n \to \infty$, *and* $\delta_n/n \to 0$ *as* $n \to \infty$. *Let* $B^*(s) = B(s)/(s(1-s))^{1/2}$. *Then for any Brownian bridge* B, *every* $0 < c < \infty$ *and* $-\infty < t < \infty$ *as* $n \to \infty$

$$P\left\{a(\log \delta_n) \sup_{c/n \le s \le \delta_n/n} B^*(s) - b(\log \delta_n^{1/2}) \le t\right\} \to E(t), \tag{4.4.11}$$

$$P\left\{a(\log \delta_n) \sup_{c/n \le s \le \delta_n/n} |B^*(s)| - b(\log \delta_n^{1/2}) \le t\right\} \to E^2(t), \tag{4.4.12}$$

$$P\left\{a(\log \delta_n) \sup_{1-\delta_n/n \le s \le 1-c/n} B^*(s) - b(\log \delta_n^{1/2}) \le t\right\} \to E(t), \tag{4.4.13}$$

and

(4.4.14) $$P\left\{a(\log \delta_n) \sup_{1-\delta_n/n \le s \le 1-c/n} |B^*(s)| - b(\log \delta_n^{1/2}) \le t\right\} \to E^2(t).$$

PROOF. We prove only (4.4.11) here. The proofs for the other statements follow the same route. Let U denote the Ornstein–Uhlenbeck process as described on pages 55–57 of M. Csörgő and Révész (1981). Since

$$\{U(t);\ -\infty < t < \infty\} =_{\mathscr{D}} \{(1 + e^{2t})e^{-t}B(e^{2t}/(1 + e^{2t}));\ -\infty < t < \infty\}$$

[this is the correct version of (1.9.7) In M. Csörgő and Révész (1981), resulting in corresponding trivial changes in their Corollary 1.9.1], we have

$$\sup_{c/n \le s \le \delta_n/n} B^*(s) = \sup\left\{U(t);\ \frac{1}{2}\log\left(\frac{c/n}{1 - c/n}\right) \le t \le \frac{1}{2}\log\left(\frac{\delta_n/n}{1 - \delta_n/n}\right)\right\},$$

which by stationarity of U is equal in distribution to

$$\sup_{0 < s \le \nu_n} U(s),$$

where

$$\nu_n = (\log \delta_n - \log c + \log(n - c) - \log(n - \delta_n))/2.$$

Now Theorem 1.9.1 (Darling and Erdős, 1956) in M. Csörgő and Révész (1981) implies that for any $-\infty < t < \infty$

(4.4.15) $$P\left\{a(\nu_n) \sup_{0 < s \le \nu_n} U(s) - b(\nu_n) \le t\right\} \to E(t), \qquad n \to \infty.$$

Since by our condition on δ_n we have $\nu_n/\log \delta_n^{1/2} \to 1$ as $n \to \infty$, (4.4.11) follows from (4.4.15).

LEMMA 4.4.4. *Let $\{\delta_n\}$ be any sequence of positive numbers such that $1 \le \delta_n \le n$, $\delta_n \to \infty$, and $\delta_n/n \to 0$ as $n \to \infty$. Then, as $n \to \infty$,*

(4.4.16) $$\sup_{0 < s \le \delta_n/n} \alpha_n(s)/\left((s(1 - s))^{1/2} a(\log \delta_n)\right) \to_P 1,$$

(4.4.17) $$\sup_{1-\delta_n/n \le s < 1} \alpha_n(s)/\left((s(1 - s))^{1/2} a(\log \delta_n)\right) \to_P 1,$$

(4.4.18) $$\sup_{1/(n+1) \le s \le \delta_n/n} u_n(s)/\left((s(1 - s))^{1/2} a(\log \delta_n)\right) \to_P 1,$$

and

(4.4.19) $$\sup_{1-\delta_n/n \le s \le n/(n+1)} u_n(s)/\left((s(1 - s))^{1/2} a(\log \delta_n)\right) \to_P 1.$$

Also, the same statements hold with $\alpha_n(s)$ replaced by $|\alpha_n(s)|$ in (4.4.16) *and* (4.4.17), *and $u_n(s)$ replaced by $|u_n(s)|$ in* (4.4.18) *and* (4.4.19).

PROOF. We will only provide a proof for (4.4.16). The other statements are proven in the same way. Consider

$$\left| \sup_{0<s\le\delta_n/n} \frac{\alpha_n(s)}{(s(1-s))^{1/2}a(\log\delta_n)} - \sup_{1/n\le s\le\delta_n/n} \frac{B_n(s)}{(s(1-s))^{1/2}a(\log\delta_n)} \right|$$

$$(4.4.20)\qquad \le \sup_{0\le s\le 1} \frac{|\alpha_n(s) - \overline{B}_n(s)|}{(s(1-s))^{1/2}a(\log\delta_n)}$$

$$= O_P(1/a(\log\delta_n)),$$

where the latter equality is due to (4.2.5b) of Corollary 4.2.2. Since $a(\log\delta_n) \to \infty$ as $n \to \infty$, the first expression of (4.4.20) is a $o_P(1)$ rv as $n \to \infty$. Now statement (4.4.11) with $c = 1$ implies that, as $n \to \infty$,

$$(4.4.21)\qquad \sup_{1/n\le s\le\delta_n/n} B_n(s)/\big((s(1-s))^{1/2}a(\log\delta_n)\big) \to_P 1.$$

Hence (4.4.20) and (4.4.21) imply (4.4.16).

PROOF OF THEOREM 4.4.1. We now show how statement (4.4.1) follows from the above lemmas. Statements (4.4.2), (4.4.3), and (4.4.4) are proven in exactly the same way. We write

$$T_n := a_n \sup_{0<s<1} \alpha_n(s)/(s(1-s))^{1/2} - b_n,$$

$$T_n^{(1)} := a_n \sup_{0<s\le\varepsilon_n} \alpha_n(s)/(s(1-s))^{1/2} - b_n,$$

$$T_n^{(2)} := a_n \sup_{\varepsilon_n\le s\le 1-\varepsilon_n} \alpha_n(s)/(s(1-s))^{1/2} - b_n,$$

and

$$T_n^{(3)} := a_n \sup_{1-\varepsilon_n\le s<1} \alpha_n(s)/(s(1-s))^{1/2} - b_n.$$

Then

$$T_n = \max\{T_n^{(1)}, T_n^{(2)}, T_n^{(3)}\}.$$

By (4.4.7) of Lemma 4.4.2 we see that (4.4.1) will be proven if we can show that

$$(4.4.22)\qquad T_n^{(1)} \to_P -\infty \quad \text{as } n\to\infty$$

and

$$(4.4.23)\qquad T_n^{(3)} \to_P -\infty \quad \text{as } n\to\infty.$$

Consider (4.4.22). Let $\delta_n = (\log n)^3$. Applying (4.4.16) we have

$$(4.4.24)\qquad T_n^{(1)} = a_n O_P(a(\log\delta_n)) - b_n.$$

It is easily checked that for any constant $0 < K < \infty$

$$Ka_n a(\log \delta_n) - b_n \to -\infty \quad \text{as } n \to \infty. \tag{4.4.25}$$

Hence by (4.4.24) and (4.4.25) we have (4.4.22). Statement (4.4.23) is proven in the same way. Thus by the above comments we have (4.4.1).

REMARK 4.4.1. The Eicker (1979) and Jaeschke (1979) versions of (4.4.3) and (4.4.4) read as follows:

$$P\left\{a_n \sup_{U_{1,n} \le s < U_{n,n}} \frac{\alpha_n(s)}{\left(G_n(s)(1 - G_n(s))\right)^{1/2}} - b_n \le t\right\} \to E(t) \tag{4.4.26}$$

and

$$P\left\{a_n \sup_{U_{1,n} \le s < U_{n,n}} \frac{|\alpha_n(s)|}{\left(G_n(s)(1 - G_n(s))\right)^{1/2}} - b_n \le t\right\} \to E^2(t) \tag{4.4.27}$$

as $n \to \infty$. With very little difficulty it can be shown that (4.4.3) and (4.4.4) imply (4.4.26) and (4.4.27).

4.5. Rényi-type statistics. In this section we prove the following result.

THEOREM 4.5.1. *Let a_n be any sequence of positive constants such that for some $0 < \beta < 1$, we have $0 < a_n \le \beta$ for all large enough n, and $na_n \to \infty$. Then, as $n \to \infty$,*

$$P\left\{\left(\frac{a_n}{1 - a_n}\right)^{1/2} \sup_{a_n \le s \le 1} \frac{\alpha_n(s)}{s} \le x\right\} \to P\left\{\sup_{0 \le t \le 1} W(t) \le x\right\} \tag{4.5.1}$$

and

$$P\left\{\left(\frac{a_n}{1 - a_n}\right)^{1/2} \sup_{a_n \le s \le 1} \frac{|\alpha_n(s)|}{s} \le x\right\} \to P\left\{\sup_{0 \le t \le 1} |W(t)| \le x\right\} \tag{4.5.2}$$

for any real x, where W is the standard Wiener process.

PROOF. Choose any $0 < \nu < \frac{1}{4}$. For all n sufficiently large,

$$\begin{aligned}
&\left(\frac{a_n}{1 - a_n}\right)^{1/2} \sup_{a_n \le s \le 1} \frac{|\alpha_n(s) - B_n(s)|}{s} \\
&\quad \le \left(\frac{a_n}{1 - a_n}\right)^{1/2} a_n^{-1/2-\nu} n^{-\nu} \sup_{1/n \le s \le 1} n^{\nu} \frac{|\alpha_n(s) - B_n(s)|}{s^{1/2-\nu}} \\
&\quad = (na_n)^{-\nu}(1 - a_n)^{-1/2} O_P(1) \\
&\quad = o_P(1)
\end{aligned}$$

by Corollary 2.1. But for each $n = 1, 2, \dots$

$$\left\{\left(\frac{a_n}{1-a_n}\right)^{1/2}\frac{B_n(s)}{s}: a_n \le s \le 1\right\} =_{\mathscr{D}} \left\{W\left(\frac{a_n}{1-a_n}\frac{1-s}{s}\right): a_n \le s \le 1\right\},$$

and therefore, for each $-\infty < x < \infty$,

$$P\left\{\sup_{a_n \le s \le 1}\left(\frac{a_n}{1-a_n}\right)^{1/2}\frac{B_n(s)}{s} \le x\right\} = P\left\{\sup_{a_n \le s \le 1} W\left(\frac{a_n}{1-a_n}\frac{1-s}{s}\right) \le x\right\}$$

$$= P\left\{\sup_{0 \le t \le 1} W(t) \le x\right\}.$$

Thus we have (4.5.1), and (4.5.2) follows in the same way.

In the special case, when $a_n \equiv \beta$, (4.51) and (4.52) are due to Rényi (1953). [See also M. Csörgő (1967) and page 165 of M. Csörgő and Révész (1981).] The special case of (4.5.1) when $a_n \to 0$ as $n \to \infty$ is Theorem 2.8 of Csáki (1974), while (4.5.2) with varying a_n appears to be new. The left-sided versions of (4.5.1) and (4.5.2) for $\alpha_n(s)/(1-s)$ can be easily formulated. The corresponding results for $u_n(s)/s$ and $u_n(s)/(1-s)$, which are basically for $\alpha_n(s)/G_n(s)$ and $\alpha_n(s)/(1 - G_n(s))$, can be formulated again very easily.

4.6. Invariance principles for the increments of the uniform quantile and empirical processes. For any $0 < c < \infty$, let

$$\Delta_n(c) = \sup\left\{\frac{|u_n(b) - u_n(a) - (B_n(b) - B_n(a))|}{\tilde{w}(b-a)} : 0 \le a < b \le 1 \text{ and } b - a \ge \frac{(c\log n)}{n}\right\}, \tag{4.6.1}$$

where $\tilde{w}$ is a positive function on $(0,1)$ such that it is nondecreasing in a neighbourhood of zero and symmetric about the point $1/2$.

THEOREM 4.6.1. *On the probability space of Theorem* 1.1 *we have for any* $0 < c < \infty$

$$\Delta_n(c) = o(1) \quad \text{a.s.}, \qquad n \to \infty, \tag{4.6.2}$$

if and only if

$$\tilde{w}(s)/(s\log(1/s))^{1/2} \to \infty \quad \text{as } s \downarrow 0. \tag{4.6.3}$$

PROOF. First assume (4.6.3). We observe that for any $0 < c < \infty$ and n large enough

$$\Delta_n(c) \le 2\sup_{0<s<1}|u_n(s) - B_n(s)|/\tilde{w}(n^{-1}c\log n)$$

$$= O((n^{-1/2}\log n)/\tilde{w}(cn^{-1}\log n)) \quad \text{a.s.}$$

by (2.1) of Theorem 2.1. Condition (4.6.3) implies that

$$1/\tilde{w}(n^{-1}c\log n) = o(n^{1/2}/\log n),$$

and hence we have (4.6.2).

Now assume (4.6.2) for some $0 < c < \infty$. Let

$$\Delta_n^{(1)}(c) = \sup\{n^{1/2}|u_n(b) - u_n(a)|/c\log n : 0 < a < b \le 1,\ b - a = n^{-1}c\log n\}$$

and

$$\Delta_n^{(2)}(c) = \sup\{n^{1/2}|B_n(b) - B_n(a)|/c\log n : 0 \le a < b \le 1,\ b - a = n^{-1}c\log n\}.$$

Then

$$\Delta_n(c) \ge ((n^{-1/2}c\log n)/\tilde{w}(cn^{-1}\log n))|\Delta_n^{(1)}(c) - \Delta_n^{(2)}(c)|. \tag{4.6.4}$$

Now

$$\Delta_n^{(1)}(c) \to (\lambda_c - 1) \quad \text{a.s.}, \qquad n \to \infty,$$

where $\lambda_c > 1$ is the unique root of the equation $\lambda + \log(1/\lambda) = 1 + 1/c$ [cf. Mason (1984) and Komlós et al. (1975b)]. By the P. Lévy modulus of continuity for the Brownian bridge [cf., e.g., Theorem 4.1 in M. Csörgő and Révész (1981)] we have

$$\Delta_n^{(2)}(c) \to_P 2^{1/2}/c, \qquad n \to \infty.$$

It is easily checked that for any $0 < c < \infty$, $1 + 2^{1/2}/c$ is not equal to λ_c. Thus we have

$$|\Delta_n^{(1)}(c) - \Delta_n^{(2)}(c)| \to_P |\lambda_c - 1 - 2^{1/2}/c| \ne 0, \qquad n \to \infty. \tag{4.6.5}$$

Hence combining (4.6.2) with (4.6.4) and (4.6.5), we get

$$(n^{-1/2}c\log n)/\tilde{w}(cn^{-1}\log n) \to 0, \qquad n \to \infty,$$

which in turn implies that

$$\frac{(n^{-1}c\log n)^{1/2}(\log(n/(c\log n)))^{1/2}}{w(n^{-1}c\log n)} \to 0, \qquad n \to \infty. \tag{4.6.6}$$

Using now the tail monotonicity of $\tilde{w}$, an elementary argument shows that (4.6.6) implies (4.6.3).

REMARK 4.6.1. Komlós, Major, and Tusnády (1975a) show that on an appropriate probability space with an appropriate sequence of Brownian bridges $\hat{B}_n$ we have, as $n \to \infty$ [cf. (0.1)],

$$\sup_{0 \le s \le 1} n^{1/2}|\alpha_n(s) - \hat{B}_n(s)| = O(\log n), \quad \text{a.s.} \tag{4.6.7}$$

Using the latter result in combination with Komlós et al. (1975b), we note that on the probability space of (4.6.7), we have also (4.6.2) with α_n replacing u_n if and only if (4.6.3) holds true just as easily, *mutatis mutandis*, as the proof of Theorem 4.6.1 above. For another proof of this statement, which uses the Skorohod construction and assumes that $\tilde{w}$ is continuous and strictly increasing on $(0, \frac{1}{2}]$, we refer to Theorem 1.2 in Shorack and Wellner (1982), although Einmahl (1983) has recently pointed out to us that the necessity part of their proof is in error. Wellner has afterwards informed us that they have corrected their proof.

APPENDIX

Proof of Proposition 3.1. (i) If $I(q, c) < \infty$, then $I(q, \hat{c}) < \infty$ for all $\hat{c} \geq c > 0$. Hence we can assume that $c > 1$. Any q function is nondecreasing in a neighbourhood of zero. Let $(0, \theta]$ be such a neighbourhood, and consider the following integrals on the interval $(0, c^{-1}\theta]$:

$$\begin{aligned} \int_t^{ct} s^{-1}\exp(-cs^{-1}q^2(s))\,ds &\geq \int_t^{ct} s^{-1}\exp(-ct^{-1}q^2(s))\,ds \\ &\geq \int_t^{ct} s^{-1}\exp(-ct^{-1}q^2(ct))\,ds \\ &= \log c \exp(-c^2q^2(ct)/(ct)). \end{aligned} \tag{A.1}$$

The left-hand side of the inequality (A.1) tends to zero as $t \downarrow 0$, and so we get that

$$\lim_{t \downarrow 0} q(t)/t^{1/2} = \infty. \tag{A.2}$$

By (A.2), in turn, we get that for any $\varepsilon > 0$,

$$q(t)/t^{1/2} \leq \exp(\varepsilon q^2(t)/t) \quad \text{if } 0 < t \leq \theta^*,$$

where θ^* is an appropriately chosen constant. Consequently,

$$\int_0^{\theta^*} t^{-1}(q(t)/t^{1/2})\exp(-(c+\varepsilon)t^{-1}q^2(t))\,dt \leq \int_0^{\theta^*} t^{-1}\exp(-ct^{-1}q^2(t))\,dt < \infty,$$

i.e., $I(q, c) < \infty$ implies $E(q, c + \varepsilon) < \infty$ for any $\varepsilon > 0$, and this together with (A.2) completes the proof of (i) of Proposition 3.1.

(ii) Assuming now (A.2) we have

$$\inf_{0 < t \leq 1/2} t^{-1/2}q(t) = K > 0,$$

and so

$$K^{-1}E(q, c) \geq I(q, c),$$

i.e., finiteness of $E(q, c)$ with (A.2) also assumed implies that of $I(q, c)$.

Proof of Theorem 3.3. By Remark 3.1 it suffices to consider the problem of upper-class functions a Wiener process W.

Let us assume that $I(q, c) < \infty$. Then by Proposition 3.1 we have also (A.2). Let $\phi(\cdot)$ be a nondecreasing positive function on the interval $(0, b]$. Then for an *arbitrary* division $0 < a = t_0 < t_1 < \cdots < t_n = b \leq 1$ of the interval $[a, b]$ we have [cf. Itô and McKean ((1965), page 34]

$$\begin{aligned} &P\{W(t) > \phi(t) \text{ for some } t \text{ in } [a, b]\} \\ &\leqq \int_0^a (2\pi t^3)^{-1/2}\phi(a)\exp(-\phi^2(a)/(2t))\,dt \\ &\quad + \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} (2\pi t^3)^{-1/2}\phi(t_{m-1})\exp(-\phi^2(t_{m-1})/(2t))\,dt. \end{aligned}$$

Consider now the following *specific* division of the interval $[b\lambda^{-n}, b]$:

$$t_m = (1/\lambda)^{n-m}b, \qquad m = 0, \dots, n,$$

where $\lambda > 1$ and n is an arbitrary fixed positive integer. On account of $t_m/t_{m-1} = \lambda$, we have for $t \in [t_{m-1}, t_m]$

$$\text{(A.3)} \quad t^{-1/2}\phi(t_{m-1})\exp\left(\frac{-\phi^2(t_{m-1})}{(2t)}\right) \le t_{m-1}^{-1/2}\phi(t_{m-1})\exp\left(\frac{-\phi^2(t_{m-1})}{(2\lambda t_{m-1})}\right).$$

Let ρ be any number such that $\rho > \lambda$. Then we have $x\exp(-x^2/(2\lambda)) \le \exp(-x^2/(2\rho))$ for $x \ge K > 0$ if K is sufficiently large. Let

$$\phi(t) = (2\rho^2 c)^{1/2} q(t).$$

We can assume that b of $[b\lambda^{-n}, b]$ is so small that

$$\phi(t)/t^{1/2} \ge K$$

on $(0, b]$ because of (A.2). Hence for $t \in [t_{m-1}, t_m]$ the right-hand side of the inequality of (A.3) is bounded above by

$$\begin{aligned} \exp(-\phi^2(t_{m-1})/(2\rho t_{m-1})) &\le \exp(-\phi^2(t_{m-1})/(2\rho t)) \\ &\le \exp(-\phi^2(t/\lambda)/(2\rho t)) \\ &\le \exp(-\phi^2(t/\rho)/(2\rho t)). \end{aligned}$$

Thus if b of $[b\lambda^{-n}, b]$ is picked small enough, then we have arrived at

$$\begin{aligned} &P\{W(t) > \phi(t) \text{ for some } t \text{ in } [b\lambda^{-n}, b]\} \\ &\le \int_0^{t_0/\phi^2(t_0)} (2\pi t^3)^{-1/2}\exp(-1/(2t))\,dt \\ &\quad + \int_{t_0}^{b} (2\pi t^2)^{-1/2}\exp(-\phi^2(t/\rho)/(2\rho t))\,dt. \end{aligned}$$

Using again (A.2), the latter inequality yields by letting $n \to \infty$,

$$\begin{aligned} &P\left\{W(t) > (2\rho^2 c)^{1/2} q(t) \text{ for some } t \text{ in } (0, b]\right\} \\ &\le (2\pi)^{-1/2}\int_0^{b/\rho} t^{-1}\exp(-cq^2(t)/t)\,dt. \end{aligned}$$

The first part of the statement of Theorem 3.3 follows by letting $b \downarrow 0$ on account of having assumed $I(q, c) < \infty$. The latter inequality actually proves more than what the first part of the statement of Theorem 3.3 claims. Namely, since

$\rho > \lambda > 1$ may be taken as close to one as we wish, we concluded that if the integral $I(q, c)$ is finite, then

(A.4) $$\limsup_{t \downarrow 0} |W(t)|/q(t) \le (2c)^{1/2} \quad \text{a.s.}$$

Towards proving the converse now, we assume that q is an EFKP upper-class function of a Wiener process. We can assume without loss of generality that $\beta \leqq 1$ in (3.7). On the basis of Problem 1 of page 35 of Itô and McKean (1965) we have

(A.5) $$\begin{aligned} P\{W(t) > 2q(t) \text{ for some } t \text{ in } (0, b]\} &\ge P\{W(b) > 2q(b)\} \\ &= 1 - \Phi(2b^{-1/2}q(b)). \end{aligned}$$

The left-hand side of the latter inequality tends to zero as $b \downarrow 0$, hence we have that (A.2) holds true for any EFKP upper-class function q of W. Let us assume that b is so small that for $\varepsilon > 4$

(A.6) $$P\left\{ \sup_{0 \le t \le b} |W(t)|/q(t) \le \varepsilon^{1/2}/2 \right\} \ge \tfrac{1}{2},$$

and that q is nondecreasing on $(0, b]$. We introduce the following notations:

$$\begin{aligned} b_j &= b2^{-j}, \\ L_j &= (b_{j+1}, b_j] = (b2^{-(j+1)}, b2^{-j}], \\ B_{jx} &= \{ -\tfrac{1}{2}\varepsilon^{1/2}q(t) - x \le W(t) - W(b_j) \le \tfrac{1}{2}\varepsilon^{1/2}q(t) - x \\ &\qquad \text{for all } t \text{ in } L_k \text{ and for all } k = 0, 1, 2, \dots, j-1 \}, \\ A_{jx} &= \{ b_j W(t) - tW(b_j) > \tfrac{1}{2}\varepsilon^{1/2} b_j q(b_j) - tx \text{ for some } t \text{ in } L_j \}, \\ &\qquad j = 0, 1, 2, \dots, \end{aligned}$$

and let F_j be the distribution function of the rv $W(b_j)$. Hence

$$\begin{aligned} &P\{W(t) > \tfrac{1}{2}\varepsilon^{1/2}q(t) \text{ for some } 0 < t \le b\} \\ &\ge P\left\{ \bigcup_{j=1}^{\infty} \left(W(t) > \tfrac{1}{2}\varepsilon^{1/2}q(b_j) \text{ for some } t \text{ in } L_j \right) \right\} \\ &= \sum_{j=1}^{\infty} P\{W(t) > \tfrac{1}{2}\varepsilon^{1/2}q(b_j) \text{ for some } t \text{ in } L_j \text{ and} \\ &\qquad W(t) \leqq \tfrac{1}{2}\varepsilon^{1/2}q(b_k) \text{ for all } t \text{ in } L_k,\ k = 0, 1, \dots, j-1\} \\ &\ge \sum_{j=1}^{\infty} P\{W(t) > \tfrac{1}{2}\varepsilon^{1/2}q(b_j) \text{ for some } t \text{ in } L_j \text{ and} \\ &\qquad -\tfrac{1}{2}\varepsilon^{1/2}q(b_k) \le W(t) \le \tfrac{1}{2}\varepsilon^{1/2}q(b_k) \\ &\qquad \text{for all } t \text{ in } L_k,\ k = 0, 1, \dots, j-1\}. \end{aligned}$$

Conditioning each term in $W(b_j)$, we get

$$P\{W(t) > \tfrac{1}{2}\varepsilon^{1/2}q(b_j) \text{ for some } t \text{ in } L_j \text{ and } -\tfrac{1}{2}\varepsilon^{1/2}q(b_k) \le W(t) \le \tfrac{1}{2}\varepsilon^{1/2}q(b_k) \text{ for all } t \text{ in } L_k,\ k = 0,1,\dots, j-1\}$$
$$= \int_{-\infty}^{\infty} P\{A_{jx} \text{ and } B_{jx}\}\, dF_j(x).$$

The stochastic process $\{b_jW(t) - tW(b_j);\ t \in L_j\}$ is independent of the stochastic processes $\{W(t) - W(b_j);\ t \in L_k\}$ $(k = 0,1,2,\dots, j-1)$. This can be easily justified by computing covariance functions. Hence

$$\int_{-\infty}^{\infty} P\{A_{jx} \text{ and } B_{jx}\}\, dF_j(x) = \int_{-\infty}^{\infty} P\{A_{jx}\}P\{B_{jx}\}\, dF_j(x)$$
$$\ge \int_{0}^{\infty} P\{A_{jx}\}P\{B_{jx}\}\, dF_j(x).$$

It can be easily seen that

$$\{W(t/b_j) - (t/b_j)W(1);\ b2^{-(j+1)} < t \le b2^{-j}\} =_{\mathscr{D}} \{B(t);\ \tfrac{1}{2} < t \le 1\},$$

where $B(\cdot)$ is any Brownian bridge. Hence for $x \geqq 0$

$$P\{A_{jx}\} \ge P\{b_jW(t) - tW(b_j) > \tfrac{1}{2}\varepsilon^{1/2}b_j^{1/2}q(b_j) \text{ for some } t \text{ in } L_j\}$$
$$= P\{B(t) > \tfrac{1}{2}\varepsilon^{1/2}b_j^{-1/2}q(b_j) \text{ for some } t \text{ in } \tfrac{1}{2} \le t \le 1\}$$
$$\ge P\{B(\tfrac{1}{2}) > \tfrac{1}{2}\varepsilon^{1/2}b_j^{-1/2}q(b_j)\}$$
$$\ge \exp(-\varepsilon b_j^{-1}q^2(b_j)),$$

where the latter inequality is by Feller (1968, page 175) if b is small enough, because of (A.2), which we already verified by (A.5). Consider now

$$\int_{0}^{\infty} P(B_{jx})\, dF_j(x) = \tfrac{1}{2}\int_{-\infty}^{\infty} P(B_{jx})\, dF_j(x)$$
$$= \tfrac{1}{2}P\{-\tfrac{1}{2}\varepsilon^{1/2}q(t) \le W(t) \le \tfrac{1}{2}\varepsilon^{1/2}q(t) \text{ for all } t \text{ in } L_k \text{ and for all } k = 0,1,2,\dots, j-1\}$$
$$= \tfrac{1}{2}P\{-\tfrac{1}{2}\varepsilon^{1/2}q(t) \le W(t) \le \tfrac{1}{2}\varepsilon^{1/2}q(t),\ b_j \le t \le b\}$$
$$\ge \tfrac{1}{4},$$

where in the last step (A.6) was used. Hence we have

$$P\{W(t) > \tfrac{1}{2}\varepsilon^{1/2}q(t) \text{ for some } 0 < t \le b\}$$
$$\ge \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} P\{A_{jx} \text{ and } B_{jx}\}\, dF_j(x)$$
$$\ge \tfrac{1}{4}\sum_{j=1}^{\infty} \exp(-\varepsilon b_j^{-1}q^2(b_j))$$
$$\ge \tfrac{1}{4}\sum_{j=1}^{\infty} \int_{b_j}^{b_{j-1}} t^{-1}\exp(-2\varepsilon q^2(t)/t)\, dt$$
$$= \tfrac{1}{4}\int_{0}^{b} t^{-1}\exp(-2\varepsilon q^2(t)/t)\, dt.$$

The latter inequality proves more than what the second part of Theorem 3.3 claims. Namely, we showed that if

$$\text{(A.7)} \qquad \limsup_{t \downarrow 0} |W(t)|/q(t) = \beta \quad \text{a.s.,} \quad \text{for some } 0 \leqq \beta < \infty$$

then

$$\text{(A.8)} \qquad I(q, c) < \infty \quad \text{for any } c > 8\beta^2.$$

PROOF OF THEOREM 3.4. The proof of Theorem 3.3 yields (A.4) and (A.8). Combining the latter two conclusions with Proposition 3.1 we get Theorem 3.4.

We should note that our proof of the second part of the statement of Theorem 3.3 is a modified version of that of Proposition 2.1 of O'Reilly (1974).

CONJECTURE. We conjecture that (A.7) implies that $I(q, c) < \infty$ for any $c > \beta^2/2$. We already know that $I(q, c) < \infty$ implies (A.4). Hence, on account of Proposition 3.1, our conjecture amounts to saying that in the EFKP upper–lower functions integral test, as given on page 33 of Itô and McKean (1965), the assumption that their h be continuous and be such that $h(t)/t^{1/2}$ is nonincreasing can be dropped.

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