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WEIGHTED ESTIMATES FOR THE HANKEL TRANSFORM TRANSPLANTATION OPERATOR

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Abstract. The Hankel transform transplantation operator is investigated by means of a suitably established local version of the Calderón-Zygmund operator theory. This approach produces weighted norm inequalities with weights more general than previously considered power weights. Moreover, it also allows to obtain weighted weak type $(1, 1)$ inequalities, which seem to be new even in the unweighted setting. As a typical application of the transplantation, multiplier results in weighted L^p spaces with general weights are obtained for the Hankel transform of any order greater than -1 by transplanting cosine transform multiplier results.

1. Introduction. Given $\alpha > -1$ and a suitable function f on $(0, \infty)$, its Hankel transform is defined by

$$\mathcal{H}_\alpha f(x) = \int_0^\infty (xy)^{1/2} J_\alpha(xy) f(y) dy , \quad x > 0 .$$

Here $J_\alpha(x)$ denotes the Bessel function of the first kind of order α, see [7] or [14]. Then $(\mathcal{H}_\alpha \circ \mathcal{H}_\alpha) f = f$ and $\|\mathcal{H}_\alpha f\|_{L^2} = \|f\|_{L^2}$, for any $f \in C_c^\infty(0, \infty)$, the space of C^∞ functions with compact support in $(0, \infty)$. These two facts are known in the literature for $\alpha \geq -1/2$; in [2, Lemma 2.6] a proof valid for any $\alpha > -1$ was furnished. If $\alpha = -1/2$, then $J_{-1/2}(t) = (2/\pi t)^{1/2} \cos t$, therefore $\mathcal{H}_{-1/2}$ becomes the cosine transform on $(0, \infty)$.

Guy [5] showed that the size of the Hankel transform of any suitable function, when measured in the power weight L^p norm, remains the same whatever the order of the Hankel transform is. More precisely, given $\alpha, \gamma \geq -1/2$, $1 < p < \infty$ and $-1/p < a < 1 - 1/p$, there is a constant $C = C(\alpha, \gamma, p, a)$ such that for every appropriate function f

$$C^{-1} \|\mathcal{H}_\gamma f\|_{p,a} \leq \|\mathcal{H}_\alpha f\|_{p,a} \leq C \|\mathcal{H}_\gamma f\|_{p,a} . \tag{1.1}$$

In another way, (1.1) may be expressed as

$$\|(\mathcal{H}_\alpha \circ \mathcal{H}_\gamma) f\|_{p,a} \leq C \|f\|_{p,a} ,$$

where, for $1 \leq p < \infty$ and any real number a,

$$\|g\|_{p,a} = \left(\int_0^\infty |g(x) x^a|^p dx \right)^{1/p} .$$

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Another proof of Guy's transplantation theorem was delivered by Schindler [12]. She found an explicit expression of integral kernel of the transplantation operator

$$T_{\alpha\gamma} = \mathcal{H}_\alpha \circ \mathcal{H}_\gamma \,.$$

Due to a singularity along the diagonal, the corresponding integral was understood in the principal value sense.

In [13] one of the authors extended Guy's result by enlarging the range of admissible parameters α and γ to $\alpha > -1$ and $\gamma > -1$, and extending the range of power weight exponent a to $-(\alpha + 1/2) - 1/p < a < (\gamma + 3/2) - 1/p$. The result was obtained by transferring Muckenhoupt's transplantation theorem for Jacobi expansions to the Hankel transform setting. In the restricted range $\alpha \geq -1/2$, $\gamma \geq -1/2$ Schindler's explicit kernel representation was used to obtain the same conclusion. This was done by splitting the integration into the three regions: $0 < y < x/2$, $x/2 < y < 3x/2$ and $3x/2 < y < \infty$. The splitting brought an advantage: while on both outer regions Hardy's integral inequalities were applied, the integration on the inner region was treated by using local versions of the Hardy-Littlewood maximal function and the Hilbert transform.

The present paper deals with the transplantation operator $T_{\alpha\gamma}$, $\alpha, \gamma > -1$, initially defined as a bounded operator on L^2, from the (one-dimensional) Calderón-Zygmund theory point of view, and the main purpose is to study weighted L^p, $1 \leq p < \infty$, mapping properties of $T_{\alpha\gamma}$ with general weights allowed. The associated (Schindler's) kernel $K_{\alpha\gamma}(x, y)$ is a Calderón-Zygmund kernel if $\alpha, \gamma \geq 1/2$, but it fails to satisfy the appropriate Hörmander condition when either $\alpha < 1/2$ or $\gamma < 1/2$. In these cases problems occur on the regions $0 < y < x/2$ and $3x/2 < y < \infty$. Therefore we split the operator $T_{\alpha\gamma}$ according to these regions:

$$T_{\alpha\gamma} = T^1_{\alpha\gamma} + T^2_{\alpha\gamma} + T^3_{\alpha\gamma} \,,$$

where the kernels $K^i_{\alpha\gamma}$ defining the integral operators $T^i_{\alpha\gamma}$, $i = 1, 2$ are given by

$$K^1_{\alpha\gamma}(x, y) = \chi_{\{(x,y)\,;\,0<y<x/2\}} K_{\alpha\gamma}(x, y)\,,$$
$$K^2_{\alpha\gamma}(x, y) = \chi_{\{(x,y)\,;\,0<3x/2<y\}} K_{\alpha\gamma}(x, y)\,.$$

Then $T^1_{\alpha\gamma}$ and $T^2_{\alpha\gamma}$ are easy to handle by means of weighted Hardy's inequalities.

To treat $T^3_{\alpha\gamma}$ we introduce a notion of a local Calderón-Zygmund operator, which may be of independent interest. A canonical example of such an operator is a local analogue of the Hilbert transform

$$H_o f(x) = \mathrm{P.V.} \int_{x/2}^{3x/2} \frac{f(y)}{y - x} dy\,, \quad x > 0\,,$$

considered (with a slight modification) by Andersen and Muckenhoupt in [1, Lemma 1]. We prove that local Calderón-Zygmund operators, as well as the associated maximal truncated integral operators, are bounded in weighted L^p spaces, $1 < p < \infty$, and satisfy weighted weak type $(1, 1)$ inequalities, with weights meeting a local A_p condition (which is weaker than the usual A_p condition). Finally, we show that $T^3_{\alpha\gamma}$ is, in fact, a local Calderón-Zygmund operator, hence its mapping properties follow.

Throughout the paper we use a fairly standard notation. Thus, for a nonnegative weight w on $(0,\infty)$ we write $L^p(w)$ and $L^{1,\infty}(w)$ to denote the weighted L^p and the weighted weak L^1 spaces (with respect to the Lebesgue measure dx) that consist of all functions f on $(0,\infty)$ for which

$$\|f\|_{p,w} = \left(\int_0^\infty |f(x)w(x)|^p\,dx\right)^{1/p} < \infty$$

or

$$\|f\|_{L^{1,\infty}(w)} = \sup_{t>0}\left(t\int_{\{|f|>t\}} w(x)\,dx\right) < \infty\,,$$

respectively. If $w \equiv 1$ we simplify the notation by writing L^p and $\|\cdot\|_p$, or $L^{1,\infty}$ and $\|\cdot\|_{L^{1,\infty}}$. Given $1 \le p \le \infty$, p' denotes its conjugate, $1/p + 1/p' = 1$. By $\langle f, g\rangle$ we mean $\int_0^\infty f(x)\overline{g(x)}\,dx$ whenever the integral makes sense. We will frequently write CZ to abbreviate the term “Calderón-Zygmund”. The symbol $\boldsymbol{N}$ is used to denote the set of positive integers $\{1,2,\dots\}$.

The structure of the paper is as follows. In Section 2 we state the main results; these are contained in Theorems 2.1 and 2.2. Section 3 is devoted to a study of the integral kernel associated with the Hankel transplantation operator. In Section 4 we introduce a notion of a local Calderón-Zygmund operator and prove relevant mapping properties in weighted setting. Finally, in Section 5 we provide proofs of the main results and make some additional observations, including a refinement of Schindler’s singular integral representation of $T_{\alpha\gamma}$ (Proposition 5.1). Definition and some basic properties of local A_p weights are contained in the Appendix (Section 6), which is essentially self-contained.

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2. Preliminaries and statement of results. We will use the bounds

$$J_\alpha(t) = O(t^\alpha)\,, \quad t \to 0^+\,, \tag{2.1}$$

and

$$J_\alpha(t) = O(t^{-1/2})\,, \quad t \to \infty\,. \tag{2.2}$$

A more precise description of behavior of the Bessel function $J_\alpha(t)$ at infinity is given by the asymptotic formula (cf. [7, (5.11.6)])

$$\sqrt{t}\,J_\alpha(t) = \sqrt{2/\pi}\,(\cos(t+a_\alpha) + b_\alpha t^{-1}\sin(t+a_\alpha) + O(t^{-2}))\,, \quad t \to \infty\,. \tag{2.3}$$

A bit of comment is, perhaps, necessary on the question why $(\mathcal{H}_\alpha \circ \mathcal{H}_\gamma)f$ is well-defined for $f \in C_c^\infty(0,\infty)$. If $\alpha \ge -1/2$, then a natural assumption to make the integral defining $\mathcal{H}_\alpha g(x)$ convergent is to assume g to be Lebesgue integrable (the integral kernels $(xy)^{1/2}J_\alpha(xy)$, $x > 0$, are (uniformly) bounded on $0 < y < \infty$). Assume that $\alpha, \gamma > -1$ and $f \in C_c^\infty(0,\infty)$. Then $\mathcal{H}_\gamma f(y)$ is a continuous function of $0 < y < \infty$ and, by using (2.1),

$$\mathcal{H}_\gamma f(y) = O(y^{\gamma+1/2})\,, \quad y \to 0^+\,. \tag{2.4}$$

Moreover, by applying (2.3),

$$\mathcal{H}_\gamma f(y) = O(y^{-2}), \quad y \to \infty \tag{2.5}$$

(using higher order asymptotics, better than (2.3), allows to get $\mathcal{H}_\gamma f(y) = O(y^{-k})$ with arbitrarily large k). Note that (2.4) and (2.5) ensure $\mathcal{H}_\gamma f(y)$ to be integrable and hence, for $\alpha \geq -1/2$, $\mathcal{H}_\alpha(\mathcal{H}_\gamma f)(x)$, $0 < x < \infty$, makes sense. In the general case $\alpha, \gamma > -1$, (2.4) and (2.5) show that the function $y \to (xy)^{1/2} J_\alpha(xy)\mathcal{H}_\gamma f(y)$ is integrable and again the integral defining $\mathcal{H}_\alpha(\mathcal{H}_\gamma f)(x)$, $0 < x < \infty$, makes sense. Thus, from now on, by $T_{\alpha\gamma}$ we understand the (unique) isometrical extension on L^2 of the operator which for $f \in C_c^\infty(0, \infty)$ has the integral representation

$$T_{\alpha\gamma} f(x) = \int_0^\infty \int_0^\infty (xy)^{1/2} J_\alpha(xy)(yt)^{1/2} J_\gamma(yt) f(t)\,dt\,dy, \quad x > 0. \tag{2.6}$$

Given a nonnegative weight function $w(x)$ on $(0, \infty)$, consider the following set of conditions:

$$\sup_{r>0} \left(\int_r^\infty w(x)^p x^{-p(\gamma+3/2)} dx \right)^{1/p} \left(\int_0^r w(x)^{-p'} x^{p'(\gamma+1/2)} dx \right)^{1/p'} < \infty, \tag{2.7}$$

$$\sup_{r>0} \left(\int_0^r w(x)^p x^{p(\alpha+1/2)} dx \right)^{1/p} \left(\int_r^\infty w(x)^{-p'} x^{-p'(\alpha+3/2)} dx \right)^{1/p'} < \infty, \tag{2.8}$$

$$\sup_{0<u<v<2u} \frac{1}{v-u} \left(\int_u^v w(x)^p dx \right)^{1/p} \left(\int_u^v w(x)^{-p'} dx \right)^{1/p'} < \infty. \tag{2.9}$$

We admit $1 \leq p \leq \infty$ when considering (2.7) and (2.8), and $1 \leq p < \infty$ when considering (2.9). Here and later on, for $p = 1$ or $p = \infty$, integrals of the form appearing in (2.7)–(2.9) have the usual interpretation. For example, when $p = 1$, the second factor in (2.7) is taken as $\operatorname{ess\,sup}_{x \in (0,r)}[w(x)^{-1} x^{\gamma+1/2}]$. Note that if a nonnegative weight w on $(0, \infty)$ satisfies any of the conditions (2.7)–(2.9) (or the condition (4.6)), then w is either identically 0 or $w > 0$ a.e. (here the convention $0 \cdot \infty = 0$ is used).

It is easily seen that for a power weight function $w(x) = x^a$, $a \in \boldsymbol{R}$, (2.7) is satisfied if and only if $a < -1/p + (\gamma + 3/2)$, (2.8) is satisfied if and only if $a > -(\alpha + 1/2) - 1/p$, and (2.9) is satisfied for each $a \in \boldsymbol{R}$. Condition (2.7) is necessary and sufficient for weighted Hardy's inequality

$$\left(\int_0^\infty \left| w(x) x^{-(\gamma+3/2)} \int_0^x f(t)\,dt \right|^p dx \right)^{1/p} \leq C \left(\int_0^\infty \left| w(x) x^{-(\gamma+1/2)} f(x) \right|^p dx \right)^{1/p} \tag{2.10}$$

to hold, while the condition (2.8) is necessary and sufficient for its dual version

$$\left(\int_0^\infty \left| w(x) x^{\alpha+1/2} \int_x^\infty f(t)\,dt \right|^p dx \right)^{1/p} \leq C \left(\int_0^\infty \left| w(x) x^{\alpha+3/2} f(x) \right|^p dx \right)^{1/p} \tag{2.11}$$

to be satisfied, cf. [9]. The local A_p condition (2.9) for w^p is, for $1 < p < \infty$, necessary and sufficient for the estimate

$$\int_0^\infty |T_o f(x)w(x)|^p dx \le C \int_0^\infty |f(x)w(x)|^p dx$$

to hold, where T_o represents one of the two operators: either M_o, the local version of the one-dimensional Hardy-Littlewood maximal operator

$$M_o f(x) = \sup_{|x-y|\le x/2} \frac{1}{y-x} \int_x^y |f(t)|dt\,, \quad x > 0\,,$$

or H_o, the local version of the Hilbert transform. The sufficiency part in the above is just a version of [10, Lemma (9.6)], see also remarks following [13, Lemma 6.1]. Necessity of (2.9) in case of H_o is stated in [1, Lemma 1] and in case of M_o is provided in Section 6, see Remark 6.4 below. In the case $p = 1$ the condition (2.9) is necessary and sufficient for the weighted weak type (1,1) inequality

$$\int_{\{x>0\ ;\ |H_o f(x)|>\lambda\}} w(x)dx \le \frac{C}{\lambda} \int_0^\infty |f(x)|w(x)\,dx, \quad \lambda > 0\,,$$

to hold, cf. [1, Lemma 1], and the same is true if we replace H_o by M_o, see Section 6.

The main results of the paper are contained in the following two theorems.

THEOREM 2.1. *Let $\alpha, \gamma > -1$, $\alpha \ne \gamma$, and $1 < p < \infty$ if $|\alpha - \gamma| \ne 2k$ for every $k \in \boldsymbol{N}$, or $1 \le p \le \infty$ if $|\alpha - \gamma| = 2k$ for some $k \in \boldsymbol{N}$. Let $w(x)$ be a nonnegative weight that satisfies*: *Condition* (2.7) *if $\alpha = \gamma + 2k$ for some $k \in \boldsymbol{N}$*; *Condition* (2.8) *if $\gamma = \alpha + 2k$ for some $k \in \boldsymbol{N}$*; *Conditions* (2.7), (2.8) *and* (2.9) *if $|\alpha - \gamma| \ne 2k$ for every $k \in \boldsymbol{N}$. Then*

$$\left(\int_0^\infty |T_{\alpha\gamma} f(x)w(x)|^p dx \right)^{1/p} \le C \left(\int_0^\infty |f(x)w(x)|^p dx \right)^{1/p}$$

for all $f \in L^2 \cap L^p(w)$. Consequently, $T_{\alpha\gamma}$ extends to a bounded linear operator on $L^p(w)$.

In order to treat the weak type (1,1) inequalities for the transplantation operator, for a given nonnegative weight function $w(x)$ on $(0, \infty)$, consider the following set of conditions:

$$\sup_{r>0} \left(\int_r^\infty \left(\frac{r}{x}\right)^\delta \frac{w(x)}{x^{\gamma+3/2}} dx \right) \left(\operatorname*{ess\,sup}_{x\in(0,r)} \frac{x^{\gamma+1/2}}{w(x)} \right) < \infty\,, \tag{2.12}$$

$$\sup_{r>0} r^{\alpha+1/2} \left(\int_0^r w(x)dx \right) \left(\operatorname*{ess\,sup}_{x\in(r,\infty)} \frac{1}{x^{\alpha+3/2} w(x)} \right) < \infty\,, \tag{2.13}$$

$$\sup_{r>0} \left(\int_0^r \left(\frac{x}{r}\right)^\delta x^{\alpha+1/2} w(x)dx \right) \left(\operatorname*{ess\,sup}_{x\in(r,\infty)} \frac{1}{x^{\alpha+3/2} w(x)} \right) < \infty\,. \tag{2.14}$$

In (2.12) and (2.14) we assume that there exists a positive δ such that the corresponding quantities are finite. Moreover, (2.13) is considered for $\alpha \in (-1, -1/2]$ while (2.14) is taken into account for $\alpha \in (-1/2, \infty)$.

It is easily seen that for a power weight function $w(x) = x^a$, $a \in \boldsymbol{R}$, (2.12) is satisfied if and only if $a \le \gamma + 1/2$, (2.13) and (2.14) are satisfied if and only if $a \ge -(\alpha + 3/2)$ ($>$ if $\alpha = -1/2$).

Let P_η, Q_η, η real, denote the Hardy operators

$$P_\eta f(x) = x^{-\eta} \int_0^x f(t)dt\,, \quad Q_\eta f(x) = x^{-\eta} \int_x^\infty f(t)dt\,.$$

Condition (2.12) is necessary and sufficient for the inequality

$$\int_{\{x>0\ ;\ |P_{\gamma+3/2}f(x)|>\lambda\}} w(x)dx \le \frac{C}{\lambda} \int_0^\infty |f(x)| x^{-(\gamma+1/2)} w(x)dx\,, \quad \lambda > 0\,, \tag{2.15}$$

to hold, cf. [1, Theorem 2] taken with $p = q = 1$, $\eta = \gamma + 3/2 > 0$, $U(x) = w(x)$ and $V(x) = x^{-(\gamma+1/2)} w(x)$. Condition (2.13) in the case $\alpha \in (-1, -1/2]$, or Condition (2.14) in the case $\alpha \in (-1/2, \infty)$, are necessary and sufficient for the inequality

$$\int_{\{x>0\ ;\ |Q_{-(\alpha+1/2)}f(x)|>\lambda\}} w(x)dx \le \frac{C}{\lambda} \int_0^\infty |f(x)| x^{\alpha+3/2} w(x)dx\,, \quad \lambda > 0\,, \tag{2.16}$$

to hold, cf. [1, Theorem 4] and [1, Theorem 5] taken with $p = q = 1$, $\eta = -(\alpha + 1/2)$, $U(x) = w(x)$ and $V(x) = x^{(\alpha+3/2)} w(x)$.

THEOREM 2.2. *Assume that* $\alpha, \gamma > -1$ *and* $\alpha \ne \gamma$. *Let* $w(x)$ *be a nonnegative weight that satisfies*: *Condition* (2.12) *if* $\alpha = \gamma + 2k$ *for some* $k \in \boldsymbol{N}$; *Condition* (2.13) *if* $\gamma = \alpha + 2k$ *for some* $k \in \boldsymbol{N}$ *and* $\alpha \in (-1, -1/2]$; *Condition* (2.14) *if* $\gamma = \alpha + 2k$ *for some* $k \in \boldsymbol{N}$ *and* $\alpha \in (-1/2, \infty)$; *Conditions* (2.12), (2.9) *with* $p = 1$, *and either* (2.13) *or* (2.14) *depending on whether* $\alpha \in (-1, -1/2]$ *or* $\alpha \in (-1/2, \infty)$, *if* $|\alpha - \gamma| \ne 2k$ *for every* $k \in \boldsymbol{N}$. *Then*

$$\int_{\{x>0\ ;\ |T_{\alpha\gamma}f(x)|>\lambda\}} w(x)dx \le \frac{C}{\lambda} \int_0^\infty |f(x)| w(x)dx\,, \quad \lambda > 0\,,$$

for all $f \in L^2 \cap L^1(w)$. *Consequently*, $T_{\alpha\gamma}$ *extends to a bounded linear operator from* $L^1(w)$ *to* $L^{1,\infty}(w)$.

A typical application of transplantation theorems is that for multipliers. We say that a bounded measurable function m on $(0, \infty)$ is an $L^p(w)$ multiplier for $\mathcal{H}_\alpha$ provided

$$\|\mathcal{H}_\alpha(m\mathcal{H}_\alpha f)\|_{p,w} \le D\|f\|_{p,w}\,, \quad f \in L^2 \cap L^p(w)\,.$$

Given an $L^p(w)$ multiplier m for $\mathcal{H}_\alpha$, assume that w is also such that for a $\gamma > -1$ the transplantation inequality of Theorem 2.1 is satisfied and, in addition, the same is true for the transplantation operator $T_{\gamma\alpha}$ replacing $T_{\alpha\gamma}$. Let $C_{\alpha\gamma}$ and $C_{\gamma\alpha}$ denote constants appearing there. Then, for $f \in L^2 \cap L^p(w)$ we can write

$$\begin{aligned}
\|\mathcal{H}_\gamma(m\mathcal{H}_\gamma f)\|_{p,w} &= \|T_{\gamma\alpha}\mathcal{H}_\alpha(m\mathcal{H}_\gamma f)\|_{p,w} \\
&\le C_{\gamma\alpha}\|\mathcal{H}_\alpha(m\mathcal{H}_\alpha(T_{\alpha\gamma} f))\|_{p,w} \\
&\le C_{\gamma\alpha} D\|T_{\alpha\gamma} f\|_{p,w} \\
&\le C_{\gamma\alpha} D C_{\alpha\gamma}\|f\|_{p,w}\,.
\end{aligned}$$

This means that m is an $L^p(w)$ multiplier for $\mathcal{H}_\gamma$.

The above specified to $\alpha = -1/2$ and $\gamma > -1$ gives, by Theorem 2.1, the following.

COROLLARY 2.3. *Let $\gamma > -1,\ 1 < p < \infty$ and w be a given weight that satisfies*

$$\sup_{r>0}\left(\int_0^r w(x)^p dx\right)^{1/p}\left(\int_r^\infty (xw(x))^{-p'}dx\right)^{1/p'} < \infty \tag{2.17}$$

and

$$\sup_{r>0}\left(\int_r^\infty (x^{-1}w(x))^p dx\right)^{1/p}\left(\int_0^r w(x)^{-p'}dx\right)^{1/p'} < \infty \tag{2.18}$$

if $\gamma = -1/2 + 2k$ for some $k \in \boldsymbol{N}$, or (2.17), (2.18), (2.7), (2.8) *with α replaced by γ and* (2.9) *if $\gamma \neq -1/2 + 2k$ for every $k \in \boldsymbol{N}$. If m is an $L^p(w)$ multiplier for the cosine transform $\mathcal{H}_{-1/2}$, then m is also an $L^p(w)$ multiplier for $\mathcal{H}_\gamma$.*

Consequently, we are enabled to derive weighted L^p boundedness multiplier results, with general weights, for the Hankel transform of arbitrary order $\gamma > -1$ by applying known results (for instance those in [11]) for the Fourier transform, modified in an obvious manner to the cosine transform. This improves previous Hankel multiplier results existing in the literature, see references in [2].

For example, for the weight $w_{a,b}(x) = x^a\chi_{(0,1]}(x) + x^b\chi_{(1,\infty)}(x)$ both (2.17) and (2.18) are satisfied provided $-1/p < a, b < 1 - 1/p$; it is easy to check, that (2.7) and (2.8) with $\alpha = \gamma$ are satisfied simultaneously whenever $-1/p-(\gamma+1/2) < a, b < -1/p+(\gamma+3/2)$, whereas (2.9) is satisfied with any $a, b \in \boldsymbol{R}$.

3. The integral kernel $K_{\alpha\gamma}$.

In the case $\alpha, \gamma \geq -1/2,\ \alpha \neq \gamma$, Schindler [12] found an explicit (singular) integral representation of the transplantation operator $T_{\alpha\gamma} = \mathcal{H}_\alpha \circ \mathcal{H}_\gamma$: for any $f \in C_c^\infty(0, \infty)$,

$$T_{\alpha\gamma}f(x) = \text{P. V.}\int_0^\infty K_{\alpha\gamma}(x, y)f(y)dy + C_{\alpha\gamma}f(x)\,, \tag{3.1}$$

where $C_{\alpha\gamma} = \cos((\alpha - \gamma)\pi/2)$ and, for $0 < y < x$, $K_{\alpha\gamma}(x, y)$ is given by

$$\frac{2\Gamma((\alpha+\gamma+2)/2)}{\Gamma(\gamma+1)\Gamma((\alpha-\gamma)/2)}x^{-(\gamma+3/2)}y^{\gamma+1/2}\cdot {}_2F_1\left(\frac{\alpha+\gamma+2}{2}, \frac{\gamma-\alpha+2}{2}; \gamma+1; \left(\frac{y}{x}\right)^2\right),$$

while, for $0 < x < y$, $K_{\alpha\gamma}(x, y)$ equals

$$\frac{2\Gamma((\alpha+\gamma+2)/2)}{\Gamma(\alpha+1)\Gamma((\gamma-\alpha)/2)}x^{\alpha+1/2}y^{-(\alpha+3/2)}\cdot {}_2F_1\left(\frac{\alpha+\gamma+2}{2}, \frac{\alpha-\gamma+2}{2}; \alpha+1; \left(\frac{x}{y}\right)^2\right)$$

(in Section 5 we will show that the formula (3.1) is valid for a wider range of α, γ and for much more general functions f, see Proposition 5.1 below). Moreover, it was shown that the singularity along the diagonal is of the following form: with the constant $D_{\alpha\gamma} =$

$4/\big(\Gamma((\alpha-\gamma)/2)\Gamma((\gamma-\alpha)/2)(\gamma-\alpha)\big)$,

$$K_{\alpha\gamma}(x,y) = D_{\alpha\gamma}\frac{x}{x^2-y^2} + O\left(\frac{1}{x}\log\frac{x}{x-y}\right), \quad x/2 \le y < x\,, \tag{3.2}$$

and

$$K_{\alpha\gamma}(x,y) = D_{\gamma\alpha}\frac{y}{y^2-x^2} + O\left(\frac{1}{y}\log\frac{y}{y-x}\right), \quad x < y \le 3x/2\,. \tag{3.3}$$

(In fact (3.2) and (3.3) remain valid for $\alpha,\gamma > -1$, see the proof of Proposition 5.1). In the above formulas we consider $\Gamma(z)^{-1}$ to be a continuous function with the sequence of isolated zeroes in $0,-1,-2,\dots$. Hence, if $\alpha = \gamma+2k$, $k=1,2,\dots$, then $K_{\alpha\gamma}(x,y)=0$ on $0<x<y$ and, moreover, $K_{\alpha\gamma}(x,y)$ is continuous as a function considered on the region $0<y\le x$. Similarly, if $\gamma=\alpha+2k$, $k=1,2,\dots$, then $K_{\alpha\gamma}(x,y)=0$ on $0<y<x$ and $K_{\alpha\gamma}(x,y)$ is continuous on $0<x\le y$. This is because, in the first case, $(\gamma-\alpha+2)/2 \in \{0,-1,-2,\dots\}$ which means that ${}_2F_1((\alpha+\gamma+2)/2, (\gamma-\alpha+2)/2; \gamma+1; t)$ is a polynomial in t and the same is true, in the second case, for ${}_2F_1((\alpha+\gamma+2)/2, (\alpha-\gamma+2)/2; \alpha+1; t)$. It is clear, therefore, that the significance of P. V. in (3.1) is only for $|\alpha-\gamma| \ne 2k$, $k=1,2,\dots$.

From now on, we assume $K_{\alpha\gamma}(x,y)$ to be defined (by the above formulas) for $\alpha,\gamma > -1$. It seems that the restriction $\alpha,\gamma \ge -1/2$ in [12] was caused only by assuming the inversion and Plancherel's formulas to be valid for $\alpha \ge -1/2$; as we have already mentioned, they are valid for $-1<\alpha<-1/2$ as well.

The result that follows shows that the kernel $K_{\alpha\gamma}$ is indeed associated with the operator $T_{\alpha\gamma}$ (in the CZ operator theory sense, cf. [3] or [4]). Our proof of this fact contains ideas and arguments from [12]; we present it for the sake of completeness.

PROPOSITION 3.1. *Let $\alpha,\gamma > -1$, $\alpha \ne \gamma$, and suppose that $f,g \in C_c^\infty(0,\infty)$ have disjoint supports. Then*

$$\langle(\mathcal{H}_\alpha \circ \mathcal{H}_\gamma) f, g\rangle = \int_0^\infty\int_0^\infty K_{\alpha\gamma}(x,y) f(y)\overline{g(x)}\,dx\,dy\,. \tag{3.4}$$

PROOF. First, note that

$$\langle(\mathcal{H}_\alpha \circ \mathcal{H}_\gamma) f, g\rangle = \langle \mathcal{H}_\gamma f, \mathcal{H}_\alpha g\rangle = \lim_{\rho\to 0^+}\int_0^\infty \mathcal{H}_\gamma f(t)\overline{\mathcal{H}_\alpha g(t)}\,\frac{dt}{t^\rho}\,. \tag{3.5}$$

This is because for $\rho > 0$ sufficiently small, such that $\alpha+\gamma+2-\rho > 0$ to be precise, by using (2.4), (2.5) and choosing $\varepsilon > 0$ sufficiently small, we have

$$|\mathcal{H}_\gamma f(t)\mathcal{H}_\alpha g(t)|t^{-\rho} \le C\begin{cases} t^{-1+\varepsilon}\,, & \text{for } 0<t\le 1\,,\\ t^{-4}\,, & \text{for } t>1\end{cases} \tag{3.6}$$

with C independent of ρ. Hence the dominated convergence theorem is applicable. Since for ρ sufficiently small the function $\mathcal{H}_\gamma f(t)\mathcal{H}_\alpha g(t)t^{-\rho}$ is integrable, we have

$$\int_0^\infty \mathcal{H}_\gamma f(t)\overline{\mathcal{H}_\alpha g(t)}\,\frac{dt}{t^\rho} = \lim_{c\to 0^+}\int_0^\infty e^{-ct}\mathcal{H}_\gamma f(t)\overline{\mathcal{H}_\alpha g(t)}\,\frac{dt}{t^\rho}\,. \tag{3.7}$$

Combining (3.5) and (3.7), we get

$$\langle \mathcal{H}_\gamma f, \mathcal{H}_\alpha g\rangle = \lim_{\rho\to 0^+}\lim_{c\to 0^+}\int_0^\infty\int_0^\infty f(y)\overline{g(x)} \times\left(\int_0^\infty e^{-ct}(xt)^{1/2}J_\alpha(xt)(yt)^{1/2}J_\gamma(yt)\,\frac{dt}{t^\rho}\right)dx\,dy\,. \tag{3.8}$$

An application of Fubini's theorem was possible since

$$\int_0^\infty\int_0^\infty\int_0^\infty |f(y)g(x)|e^{-ct}(xy)^{1/2}((xt)^\alpha+(xt)^{-1/2})((yt)^\gamma+(yt)^{-1/2})t^{1-\rho}\,dt\,dx\,dy<\infty\,.$$

A detailed analysis (to be performed in a moment) then shows that entering with $\lim_{c\to 0^+}$ under the double integral in (3.8) is possible. Moreover, for $x\neq y$,

$$\lim_{c\to 0^+}\int_0^\infty e^{-ct}(xt)^{1/2}J_\alpha(xt)(yt)^{1/2}J_\gamma(yt)\,\frac{dt}{t^\rho} = (xy)^{1/2}\int_0^\infty \frac{J_\alpha(xt)J_\gamma(yt)}{t^{\rho-1}}\,dt\,,$$

since the last integral is convergent in the Riemann sense if $x\neq y,\ \alpha,\gamma>-1$ and $\rho>0$. Rewriting [14, p. 401 (2)], we see that the Weber-Schafheitlin integral

$$\int_0^\infty \frac{J_\alpha(xt)J_\gamma(yt)}{t^{\rho-1}}\,dt$$

equals

$$\frac{y^\gamma\,\Gamma\left(\frac{\alpha+\gamma+2-\rho}{2}\right)}{2^{\rho-1}x^{\gamma+2-\rho}\,\Gamma(\gamma+1)\,\Gamma\left(\frac{\alpha-\gamma+\rho}{2}\right)}\,{}_2F_1\left(\frac{\alpha+\gamma+2-\rho}{2},\frac{\gamma-\alpha+2-\rho}{2};\gamma+1;\left(\frac{y}{x}\right)^2\right)$$

if $0<y<x$, or

$$\frac{x^\alpha\,\Gamma\left(\frac{\alpha+\gamma+2-\rho}{2}\right)}{2^{\rho-1}y^{\alpha+2-\rho}\,\Gamma(\alpha+1)\,\Gamma\left(\frac{\gamma-\alpha+\rho}{2}\right)}\,{}_2F_1\left(\frac{\alpha+\gamma+2-\rho}{2},\frac{\alpha-\gamma+2-\rho}{2};\alpha+1;\left(\frac{x}{y}\right)^2\right)$$

if $0<x<y$. Thus, multiplying the above expressions by $(xy)^{1/2}$ and denoting the outcome by $K_{\alpha\gamma}(\rho;x,y)$ we see that the right side of (3.8) reduces to

$$\lim_{\rho\to 0^+}\int_0^\infty\int_0^\infty f(y)\overline{g(x)}K_{\alpha\gamma}(\rho;x,y)\,dx\,dy\,.$$

Finally, the assumption made on the supports of f and g and parameter continuity of the hypergeometric function ${}_2F_1$ easily justify an application of the dominated convergence theorem in the last expression. This finishes proving (3.4).

We now return to justifying the possibility of entering with $\lim_{c\to 0^+}$ under the double integral in (3.8). Recall that the supports of f and g are bounded, separated from zero and such that the distance between them is greater than zero. Our task will be done once we check that

$$\left|\int_0^\infty e^{-ct}(xt)^{1/2}J_\alpha(xt)(yt)^{1/2}J_\gamma(yt)\,\frac{dt}{t^\rho}\right|\le M$$

with M independent of $c\in(0,1),\ \ x\in\operatorname{supp} f$ and $y\in\operatorname{supp} g$. Splitting the integration into $(0,1)$ and $(1,\infty)$ reduces the aim to an analogous estimate with the region of integration $(1,\infty)$ in place of $(0,\infty)$.

Since $t \geq 1,\ x \geq \varepsilon,\ y \geq \varepsilon$ for an $\varepsilon > 0$, we may use the asymptotic (2.3) to expand both $(xt)^{1/2} J_\alpha(xt)$ and $(yt)^{1/2} J_\gamma(yt)$. It is then readily seen that after multiplying both expansions, out of the six resulting terms only the integral including the main terms makes a problem. For any other integral we enter with the absolute value inside, bound e^{-ct}, the sine and the cosine by 1 and end up with a convergent integral not depending on c, $x \in \operatorname{supp} f$ and $y \in \operatorname{supp} g$.

Thus, we are reduced to proving the uniform bound

$$\left| \int_1^\infty e^{-ct} \cos(xt + a_\alpha) \cos(yt + a_\gamma) \frac{dt}{t^\rho} \right| \leq C\,,$$

which further reduces to showing that the integrals

$$\int_1^\infty e^{-ct} \cos((x \pm y)t) \frac{dt}{t^\rho}\,, \quad \int_1^\infty e^{-ct} \sin((x \pm y)t) \frac{dt}{t^\rho}\,,$$

are bounded independently of $c \in (0,1),\ \ x \in \operatorname{supp} f,\ \ y \in \operatorname{supp} g$. We consider the first integral only; the reasoning for the remaining three integrals is analogous.

A change of variable $u = (x+y)t$ shows that the integral equals

$$(x+y)^{\rho-1} \int_{x+y}^\infty e^{-uc/(x+y)} \cos u \, \frac{du}{u^\rho}\,.$$

Recalling that $m \leq x + y \leq M$ for some $0 < m < M < \infty$ and using the uniform boundedness of $\int_{x+y}^M \exp(-uc/(x+y)) \cos(u) u^{-\rho}\, du$, we simplify our task to checking that the integral

$$\int_M^\infty e^{-uc'} \cos u \, \frac{du}{u^\rho}$$

is a bounded function of $c' \to 0^+$. This, however, follows from the right continuity at $c' = 0^+$, since the integral

$$\int_M^\infty \frac{\cos u}{u^\rho}\, du$$

is convergent in the Riemann sense. □

We end this section by establishing essential growth and smoothness regularity estimates for $K_{\alpha\gamma}$.

PROPOSITION 3.2. *Let $\alpha, \gamma > -1$ and $|\alpha - \gamma| \neq 2k$, $k = 0, 1, 2, \ldots$. Then*

$$|K_{\alpha\gamma}(x,y)| \leq \frac{C}{|x-y|}\,, \tag{3.9}$$

$$|\nabla K_{\alpha\gamma}(x,y)| \leq \frac{C}{(x-y)^2} \tag{3.10}$$

hold in the local region $0 < x/2 \leq y \leq 3x/2,\ \ x \neq y$. Moreover, if $\alpha, \gamma \geq 1/2$, then the above estimates hold for all $x, y > 0,\ \ x \neq y$ (in fact (3.9) *holds for all $x, y > 0,\ \ x \neq y$ if $\alpha, \gamma \geq -1/2$).*

PROOF. The estimate (3.9) is a straightforward consequence of (3.2) and (3.3). Indeed, if $x/2 \le y < x$, then using (3.2) and the inequality $\log t < t,\ t > 1$, we get

$$|K_{\alpha\gamma}(x, y)| \le |D_{\alpha\gamma}| \frac{x}{(x+y)(x-y)} + \frac{C}{x-y} \le \frac{C}{x-y} .$$

The case $x < y \le 3x/2$ is treated in the same way, with the aid of (3.3).

Now consider the second estimate. In view of the symmetry $K_{\alpha\gamma}(x, y) = K_{\gamma\alpha}(y, x)$ it is sufficient to prove (3.10) with ∇ replaced by $\partial/\partial x$. We will examine two cases, depending on whether (x, y) is below or above the diagonal.

Case 1. $0 < x/2 \le y < x$.

By the differentiation rule [7, (9.2.2)]

$$\frac{\partial}{\partial z}\, {}_2F_1(a, b; c; z) = \frac{ab}{c}\, {}_2F_1(a+1, b+1; c+1; z) \tag{3.11}$$

we get

$$\begin{aligned}\frac{\partial}{\partial x} K_{\alpha\gamma}(x, y) &= D_1(\alpha, \gamma) x^{-\gamma-5/2} y^{\gamma+1/2}\, {}_2F_1\left(\frac{\alpha+\gamma+2}{2}, \frac{\gamma-\alpha+2}{2}; \gamma+1; \left(\frac{y}{x}\right)^2\right) \\ &\quad + D_2(\alpha, \gamma) x^{-\gamma-9/2} y^{\gamma+5/2}\, {}_2F_1\left(\frac{\alpha+\gamma+4}{2}, \frac{\gamma-\alpha+4}{2}; \gamma+2; \left(\frac{y}{x}\right)^2\right) \\ &\equiv D_1(\alpha, \gamma) P_{\alpha\gamma}(x, y) + D_2(\alpha, \gamma) Q_{\alpha\gamma}(x, y) .\end{aligned}$$

An application of (3.9) gives

$$|P_{\alpha\gamma}(x, y)| = \frac{C}{x} |K_{\alpha\gamma}(x, y)| \le \frac{C}{x-y} |K_{\alpha\gamma}(x, y)| \le \frac{C}{(x-y)^2} .$$

Using the formula (see [7, (9.2.1), (9.2.6)])

$${}_2F_1(a, b; c; z) = \frac{1}{c(1-z)} [c\, {}_2F_1(a, b-1; c; z) + (a-c)z\, {}_2F_1(a, b; c+1; z)] , \tag{3.12}$$

we obtain

$$\begin{aligned}|Q_{\alpha\gamma}(x, y)| &\le C_1 \left(\frac{y}{x}\right)^{\gamma+5/2} \frac{1}{x^2-y^2} \left| {}_2F_1\left(\frac{\alpha+\gamma+4}{2}, \frac{\gamma-\alpha+2}{2}; \gamma+2; \left(\frac{y}{x}\right)^2\right)\right| \\ &\quad + C_2 \left(\frac{y}{x}\right)^{\gamma+9/2} \frac{1}{x^2-y^2} \left| {}_2F_1\left(\frac{\alpha+\gamma+4}{2}, \frac{\gamma-\alpha+4}{2}; \gamma+3; \left(\frac{y}{x}\right)^2\right)\right| \\ &= \tilde{C}_1 \frac{y}{x^2-y^2} |K_{\alpha+1,\gamma+1}(x, y)| + \tilde{C}_2 \frac{y^2}{x(x^2-y^2)} |K_{\alpha,\gamma+2}(x, y)| \\ &\le \frac{C}{x-y} (|K_{\alpha+1,\gamma+1}(x, y)| + |K_{\alpha,\gamma+2}(x, y)|) .\end{aligned}$$

By (3.9), the last expression is estimated from above by $C(x-y)^{-2}$.

Case 2. $0 < x < y \le 3x/2$.

Using (3.11), we get

$$\begin{aligned}\frac{\partial}{\partial x}K_{\alpha\gamma}(x,y) &= E_1(\alpha,\gamma)x^{\alpha-1/2}y^{-\alpha-3/2}{}_2F_1\left(\frac{\alpha+\gamma+2}{2},\frac{\alpha-\gamma+2}{2};\alpha+1;\left(\frac{x}{y}\right)^2\right)\\ &\quad + E_2(\alpha,\gamma)x^{\alpha+3/2}y^{-\alpha-7/2}{}_2F_1\left(\frac{\alpha+\gamma+4}{2},\frac{\alpha-\gamma+4}{2};\alpha+2;\left(\frac{x}{y}\right)^2\right)\\ &\equiv E_1(\alpha,\gamma)R_{\alpha\gamma}(x,y)+E_2(\alpha,\gamma)S_{\alpha\gamma}(x,y)\,.\end{aligned}$$

Now, the estimate (3.9) implies

$$|R_{\alpha\gamma}(x,y)| = \frac{C}{x}|K_{\alpha\gamma}(x,y)| \le \frac{C}{(x-y)^2}\,.$$

The remaining part is treated with the aid of (3.12):

$$\begin{aligned}|S_{\alpha\gamma}(x,y)| &\le C_1\left(\frac{x}{y}\right)^{\alpha+3/2}\frac{1}{y^2-x^2}\left|{}_2F_1\left(\frac{\alpha+\gamma+4}{2},\frac{\alpha-\gamma+2}{2};\alpha+2;\left(\frac{x}{y}\right)^2\right)\right|\\ &\quad + C_2\left(\frac{x}{y}\right)^{\alpha+7/2}\frac{1}{y^2-x^2}\left|{}_2F_1\left(\frac{\alpha+\gamma+4}{2},\frac{\alpha-\gamma+4}{2};\alpha+3;\left(\frac{x}{y}\right)^2\right)\right|\\ &= \tilde{C}_1\frac{y}{y^2-x^2}|K_{\alpha+1,\gamma+1}(x,y)| + \tilde{C}_2\frac{x}{y^2-x^2}|K_{\alpha+2,\gamma}(x,y)|\\ &\le \frac{C}{(x-y)^2}\,.\end{aligned}$$

Finally, it is not difficult to show that if $\alpha,\gamma \ge -1/2$, then (3.9) and, if $\alpha,\gamma \ge 1/2$, then (3.10) hold also in the regions $0 < y \le x/2$ and $0 < 3x/2 \le y$; we simply use the fact that ${}_2F_1$ is bounded on [0,1/2]. □

4. Local Calderón-Zygmund operators. It is clear that the CZ theory (specified to $\boldsymbol{R}$) works, with appropriate adjustments, when the underlying space is $(0,\infty)$ equipped with Lebesgue measure dx. Thus we use properly adjusted facts from the classic CZ theory (presented, for instance, in [4]) to the aforementioned setting without further comments.

Let $\Delta = \{(x,x)\,;\, x\in \boldsymbol{R}_+\}$, $\boldsymbol{R}_+ = (0,\infty)$, be the diagonal of $\boldsymbol{R}_+\times\boldsymbol{R}_+$. We say, cf. [4, p. 99], that $K:\boldsymbol{R}_+\times\boldsymbol{R}_+\setminus\Delta\to\boldsymbol{C}$ is a standard kernel if, for $x,y,z>0$,

$$|K(x,y)| \le C|x-y|^{-1}\,, \tag{4.1}$$

$$|K(x,y)-K(x,z)| \le C|y-z||x-y|^{-2} \quad \text{if} \quad |x-y|>2|y-z|\,, \tag{4.2}$$

$$|K(x,y)-K(z,y)| \le C|x-z||x-y|^{-2} \quad \text{if} \quad |x-y|>2|x-z|\,. \tag{4.3}$$

Note that by (4.2) and (4.3) standard kernels are continuous. Clearly, they also satisfy the Hörmander conditions

$$\int_{\{x>0\,;\,|x-y|>2|y-z|\}}|K(x,y)-K(x,z)|dx \le C\,,$$

$$\int_{\{y>0\,;\,|x-y|>2|x-w|\}}|K(x,y)-K(w,y)|dy \le C\,,$$

for all $x,y,w,z>0$.

DEFINITION 4.1. A local standard kernel is a kernel $K : \boldsymbol{R}_+ \times \boldsymbol{R}_+ \setminus \Delta \to \boldsymbol{C}$ supported in the region

$$\mathcal{D} = \{(x, y)\,;\ 0 < x/2 \le y \le 3x/2\}\,,$$

and satisfying (4.1)–(4.3) on $\mathcal{D}$.

DEFINITION 4.2. An operator T is a local Calderón-Zygmund operator if:

(1) T is bounded on $L^2(0, \infty)$;

(2) there exists a local standard kernel K associated with T such that

$$\langle Tf, g\rangle = \int_0^\infty \int_{x/2}^{3x/2} K(x, y) f(y)\overline{g(x)}dydx$$

for all $f, g \in C_c^\infty(0, \infty)$ with disjoint supports.

PROPOSITION 4.1. *Let $K(x, y)$ be a local standard kernel. Then K satisfies the following Hörmander type conditions*:

$$\int_{(0,\infty)\setminus 2I} |K(x, y) - K(x, z)|\,|f(x)|\,dx \le CM_+ f(y)\,, \quad y, z \in I\,, \tag{4.4}$$

$$\int_{(0,\infty)\setminus 2I} |K(x, y) - K(w, y)|\,|f(y)|dy \le CM_+ f(x)\,, \quad x, w \in I\,, \tag{4.5}$$

for all intervals $I \subset (0, \infty)$. Here M_+ denotes the (non-centered) Hardy-Littlewood maximal function on $(0, \infty)$,

$$M_+ f(x) = \sup_{0\le u<x<v} \frac{1}{v-u}\int_u^v |f(y)|dy$$

and $2I$ is the interval with the same center as I and such that $|2I| = 2|I|$.

PROOF. We focus on proving (4.4) since the proof of (4.5) is entirely analogous. Let $I = (u, v) \subset (0, \infty)$. We may assume that $u < y < z < v$; the analysis of the case $z < y$ is similar.

Since $2I = ((3u - v)/2, (3v - u)/2)$ the region of integration in (4.4), due to the assumption on the support of K, is the set

$$(y/2, 3z/2)\setminus((3u - v)/2, (3v - u)/2)\,.$$

Note that the supports of $K(\cdot, y)$ and $K(\cdot, z)$ overlap only on $(z/2, 3y/2)$. Thus, proving (4.4) reduces to showing that each of the three integrals

$$I_1 = \int_{B_1} |K(x, y)||f(x)|dx\,, \quad B_1 = (y/2, \min\{z/2, (3u - v)/2\})\,,$$

$$I_2 = \int_{B_2} |K(x, y) - K(x, z)|\,|f(x)|\,dx\,, \quad B_2 = (z/2, (3u - v)/2) \cup ((3v - u)/2, 3y/2)\,,$$

$$I_3 = \int_{B3} |K(x, z)||f(x)|dx\,, \quad B_3 = (\max\{3y/2, (3v - u)/2\}, 3z/2)\,,$$

is bounded by the right side of (4.4). Here we use the convention that $(a, b) = \emptyset$ if $a \ge b$.

Consider I_1 first. If $v/u \le 3/2$, then $B_1 = (y/2, z/2)$ and for $x \in (y/2, z/2)$ we have $y - x > y - z/2 > y/4$ (the last inequality follows from the fact that $z < v \le 3u/2 < 3y/2$). Thus

$$I_1 \le C \int_{y/2}^{z/2} \frac{|f(x)|}{|y-x|} dx \le \frac{C}{y} \int_{y/2}^{z/2} |f(x)| dx \le \frac{C}{y} \int_{y/2}^{3y/2} |f(x)| dx \le C M_+ f(y) .$$

If $v/u > 3/2$, then $3/4 > (3 - v/u)/2$. Hence, for $x \in (y/2, \min\{z/2, (3u - v)/2\})$, $y - x > y/4$. This is because $3y/4 > 3u/4 > (3u - v)/2$, therefore $y - x > y - (3u - v)/2 > y/4$. Consequently,

$$I_1 \le C \int_{y/2}^{(3u-v)/2} \frac{|f(x)|}{|y-x|}\, dx \le \frac{C}{y} \int_{y/2}^{3y/2} |f(x)|\, dx \le C M_+ f(y) .$$

Considering I_2, we denote $l = v - u$ and use the growth and smoothness conditions (4.1), (4.2) to get

$$I_2 \le C \int_{B_2} \frac{|y-z|}{|x-y|^2}\, |f(x)|\, dx \le C l \int_{B_2} \frac{|f(x)|}{|x-y|^2} dx .$$

The last integral multiplied by l is less than (the series below, in fact, terminates)

$$\begin{aligned} l \sum_{k=-1}^{\infty} \int_{\{2^k l < |x-y| < 2^{k+1} l\}} & \frac{|f(x)|}{|x-y|^2} \chi_{(z/2, 3y/2)}(x) dx \\ & \le l \sum_{k=-1}^{\infty} \frac{1}{(2^k l)^2} \int_{|x-y|<2^{k+1} l} |f(x)| \chi_{(z/2,3y/2)}(x) dx \\ & \le \sum_{k=-1}^{\infty} \frac{1}{2^k} \frac{1}{2^k l} \int_{\{x\, ;\, |x-y|<2^{k+1} l\} \cap (z/2, 3y/2)} |f(x)| dx \\ & \le 4 \Bigg(\sum_{k=-1}^{\infty} 2^{-k} \Bigg) M_+ f(y) . \end{aligned}$$

Finally, consider I_3. If $v/u \le 4/3$, then $B_3 = (3y/2, 3z/2)$ and for $x \in (3y/2, 3z/2)$ we have $x - z > 3y/2 - z > y/6$ (the last inequality follows from the fact that $z < v \le 4u/3 < 4y/3$). Thus

$$I_3 \le C \int_{3y/2}^{3z/2} \frac{|f(x)|}{|x-z|}\, dx \le \frac{C}{y} \int_{y/2}^{2y} |f(x)| dx \le C M_+ f(y) .$$

If $v/u > 4/3$, then $(3 - u/v)/2 > 9/8$. Hence, for $x \in (\max\{3y/2, (3v - u)/2\}, 3z/2)$, $x - z > z/8$. This is because $9z/8 < 9v/8 < (3v - u)/2$, therefore $x - z > (3v - u)/2 - z > z/8$. Accordingly,

$$I_3 \le C \int_{3y/2}^{3z/2} \frac{|f(x)|}{|x-z|}\, dx \le \frac{C}{z} \int_{0}^{3z/2} |f(x)| dx \le C M_+ f(y) . \qquad \square$$

DEFINITION 4.3. Let $1 \le p < \infty$ and w be a nonnegative weight on $(0, \infty)$. We say that w^p satisfies the (global) A_p condition if

$$\sup_{0 \le u < v < \infty} \frac{1}{v-u} \left(\int_u^v w^p \right)^{1/p} \left(\int_u^v w^{-p'} \right)^{1/p'} < \infty \tag{4.6}$$

(if $p = 1$, then the second integral is understood as $\operatorname{ess\,sup}_{(u,v)} w^{-1}$). We then write $w^p \in A_p(0, \infty)$ and denote the left side of (4.6) by $\|w^p\|_{A_p}$.

Given a local CZ operator T with the associated kernel $K(x, y)$ and a nonnegative weight w such that $w^p \in A_p(0, \infty)$, consider the truncated integrals

$$T_\varepsilon f(x) = \int_{\{y>0\,;\,|x-y|>\varepsilon\}} K(x, y) f(y) dy\,, \quad x > 0$$

(well-defined for every $\varepsilon > 0$, $f \in L^p(w)$, $1 \le p < \infty$, $w^p \in A_p(0, \infty)$) and the corresponding maximal operator

$$T^* f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|\,.$$

An important consequence of Proposition 4.1 and the general CZ theory is the following.

PROPOSITION 4.2. *Let T be a local Calderón-Zygmund operator and w a nonnegative weight such that $w^p \in A_p(0, \infty)$. Then T extends to a bounded operator on $L^p(w)$ if $1 < p < \infty$, and to a w-weighted weak type $(1, 1)$ operator, if $p = 1$. Moreover,*

$$\|Tf\|_{p,w} \le C_p \|w^p\|_{A_p} \|f\|_{p,w}\,, \quad f \in L^p(w)\,, \quad 1 < p < \infty\,,$$
$$\|Tf\|_{L^{1,\infty}(w)} \le C_1 \|w\|_{A_1} \|f\|_{1,w}\,, \quad f \in L^1(w)\,,$$

with C_p independent of w. Analogous inequalities are also valid for T^*.

PROOF. A careful analysis of the corresponding reasoning for the usual (global) CZ operators (cf. for instance [4, Chapters 5, 7]) shows that the (global) standard estimates (4.2) and (4.3) are exploited only to conclude (4.4) and (4.5). Hence, one can apply the estimates from Proposition 4.1 directly. First, to obtain the unweighted L^p estimates or weak type $(1, 1)$ for T it is sufficient to use (4.4) and (4.5) with $f \equiv 1$, cf. [4, Theorem 5.10]. Next, to obtain weighted L^p and weak type $(1, 1)$ estimates for T and T^* we use (4.4) and (4.5) and imitate the argument contained in the proofs of [4, Lemmas 5.15 and 7.9, Theorem 7.12]. The conclusions of [4, Theorem 7.11 and 7.12, Corollary 7.13] then follow. □

It occurs that the results contained in Proposition 4.2 may be strengthened by allowing more general weights. This is the essence of the following theorem, which is the main result of this section (for the definition and properties of local A_p classes, A^p_{loc}, see Section 6).

THEOREM 4.3. *Assume that T is a local Calderón-Zygmund operator and let w be a nonnegative weight on $(0, \infty)$ such that $w^p \in A^p_{\mathrm{loc}}$.*

(a) *If $1 < p < \infty$, then T extends to a bounded linear operator on $L^p(w)$;*

(b) *if $p = 1$, then T extends to a bounded linear operator from $L^1(w)$ to $L^{1,\infty}(w)$;*

(c) *the maximal operator T^* is bounded on $L^p(w)$ if $1 < p < \infty$; when $p = 1$, T^* satisfies w-weighted weak type $(1,1)$ inequality.*
Moreover, the corresponding L^p and weak type constants depend on w only through the local A_p norm of w^p.

PROOF. We shall use the argument from [1, Section 5], see also [10, p. 31], together with Proposition 4.2. Let $1 < p < \infty$, $f \in L^2 \cap L^p(w)$, $w^p \in A^p_{\text{loc}}$ and consider the intervals $I_n = [2^n, 2^{n+3})$, $n \in \boldsymbol{Z}$. Define the weight w_n on $(0, \infty)$ to be a restriction of the weight on $(-\infty, \infty)$ which is equal to w on I_n, periodic with period $2|I_n|$, and symmetric around the point 2^n. Then one verifies that the (global) A_p norm of w_n^p is estimated from above, up to a multiplicative constant independent of w and n, by the local A_p norm of w^p (more precisely, $\|w_n^p\|_{A_p} \le 2\|w^p\|_{A^p_{8,\text{loc}}} \le 2c_p\|w^p\|_{A^p_{\text{loc}}}$, see Section 6 for the notation and the last inequality). Thus, denoting $f_n = f\chi_{I_n}$, $J_n = [2^{n+1}, 2^{n+2})$ and using the fact that $Tf(x) = Tf_n(x)$ a.e. $x \in J_n$ (explained below) together with Proposition 4.2 we obtain

$$
\begin{aligned}
\int_0^\infty |Tf(x)w(x)|^p\,dx &= \sum_{n\in\boldsymbol{Z}} \int_{J_n} |Tf_n(x)w_n(x)|^p dx \\
&\le C_p\|w^p\|_{A^p_{\text{loc}}} \sum_{n\in\boldsymbol{Z}} \int_0^\infty |f_n(x)w_n(x)|^p dx \\
&= 3C_p\|w^p\|_{A^p_{\text{loc}}} \int_0^\infty |f(x)w(x)|^p dx\,.
\end{aligned}
$$

The identity $Tf(x) = Tf_n(x)$ a.e. $x \in J_n$ is a consequence of the weak association of T with the kernel supported in the region $\mathcal{D}$. Indeed, write $f = f_n + f\chi_{(I_n)^c}$; since the functions χ_{J_n} and $f\chi_{(I_n)^c}$ have disjoint supports, and for $x \in J_n$ the condition $x/2 \le y \le 3x/2$ implies $y \in I_n$, it follows that

$$
\begin{aligned}
\int_{J_n} T(f\chi_{(I_n)^c})(x)dx &= \int_0^\infty T(f\chi_{(I_n)^c})(x)\chi_{J_n}(x)dx \\
&= \int_0^\infty \int_{x/2}^{3x/2} K(x,y)f(y)\chi_{(I_n)^c}(y)\chi_{J_n}(x)dydx \\
&= 0\,.
\end{aligned}
$$

Clearly, the same is true if we replace J_n by its arbitrary subinterval, thus $T(f\chi_{(I_n)^c})(x) = 0$ a.e. $x \in J_n$. A careful reader surely observed that we have applied the weak association condition to functions which are not C_c^∞; nevertheless, this is not an obstacle, because of Proposition 4.2 and an approximation argument.

Treatment of the case $p = 1$ is analogous. Given $\lambda > 0$, define the level sets

$$
E_\lambda = \{x > 0\,;\ |Tf(x)| > \lambda\}\,, \qquad E_\lambda^n = \{x > 0\,;\ |Tf_n(x)| > \lambda\}
$$

and write

$$\begin{aligned}\int_{E_\lambda} w(x)dx &= \sum_{n\in\mathbf{Z}} \int_{J_n} \chi_{E_\lambda^n}(x) w_n(x)dx \\ &\le C_1\|w\|_{A^1_{\mathrm{loc}}} \lambda^{-1} \sum_{n\in\mathbf{Z}} \int_0^\infty |f_n(x)| w_n(x)dx \\ &= 3C_1\|w\|_{A^1_{\mathrm{loc}}} \lambda^{-1} \int_0^\infty |f(x)| w(x)dx\,.\end{aligned}$$

In a similar way we deal with T^*. □

As it was already remarked in Section 2, the local A_p condition is also necessary, at least as the local Hilbert transform is concerned: if w is a nonnegative weight on $(0,\infty)$ such that H_o is bounded on $L^p(w)$ for some $1 < p < \infty$, or satisfies w-weighted weak type $(1,1)$ inequality if $p = 1$, then w^p must be a local A_p weight.

5. Proofs of the main results and final remarks. Recall that $T^1_{\alpha\gamma}$ and $T^2_{\alpha\gamma}$ denote the integral operators

$$T^1_{\alpha\gamma} f(x) = \int_0^{x/2} K_{\alpha\gamma}(x,y) f(y)dy\,, \quad T^2_{\alpha\gamma} f(x) = \int_{3x/2}^\infty K_{\alpha\gamma}(x,y) f(y)dy\,.$$

Note that due to the boundedness of ${}_2F_1$ on $(0, 1/2)$ we have

$$|K_{\alpha\gamma}(x,y)| \le Cx^{-(\gamma+3/2)} y^{\gamma+1/2}\,, \quad 0 < y < x/2\,, \tag{5.1}$$

$$|K_{\alpha\gamma}(x,y)| \le Cx^{\alpha+1/2} y^{-(\alpha+3/2)}\,, \quad 3x/2 < y < \infty\,. \tag{5.2}$$

By taking $p = 2$ and $w(x) \equiv 1$ in (2.10) and (2.11) it follows that $T^1_{\alpha\gamma}$ and $T^2_{\alpha\gamma}$ are bounded on $L^2(0,\infty)$, see the estimates in the proof of Theorem 2.1 below. Thus

$$T^3_{\alpha\gamma} = T_{\alpha\gamma} - T^1_{\alpha\gamma} - T^2_{\alpha\gamma}$$

is also bounded on $L^2(0,\infty)$. Moreover, by Proposition 3.1, $T^3_{\alpha\gamma}$ is associated with the kernel

$$K^3_{\alpha\gamma}(x,y) = \chi_{\mathcal{D}}(x,y) K_{\alpha\gamma}(x,y)\,,$$

which by Proposition 3.2 is a local CZ kernel (the gradient estimate (3.10) implies the smoothness conditions (4.2) and (4.3)). Thus $T^3_{\alpha\gamma}$ is a local CZ operator.

PROOF OF THEOREM 2.1. Assume that $1<p<\infty$ and $|\alpha-\gamma| \ne 2k,\ k=0,1,2,\dots$. An application of (5.1) and Hardy's inequality (2.10), under the condition (2.7), gives

$$\begin{aligned}\int_0^\infty \left|T^1_{\alpha\gamma} f(x) w(x)\right|^p dx &\le C\int_0^\infty \left(w(x)\int_0^{x/2} x^{-(\gamma+3/2)} y^{\gamma+1/2}|f(y)|\,dy\right)^p dx \\ &\le C\int_0^\infty \left(w(x) x^{-(\gamma+1/2)} x^{\gamma+1/2}|f(x)|\right)^p dx \\ &= C\int_0^\infty \left|f(x)w(x)\right|^p dx\,.\end{aligned}$$

Similarly, using (5.2) and Hardy's inequality (2.11), under the condition (2.8), we get

$$\int_0^\infty \left|T^2_{\alpha\gamma} f(x)w(x)\right|^p dx \le C \int_0^\infty \left|f(x)w(x)\right|^p dx \,.$$

The corresponding L^p inequality for $T^3_{\alpha\gamma}$ is a consequence of Theorem 4.3.

Now, consider $1 \le p \le \infty$ and $k = 1, 2, \dots$. If $\alpha = \gamma + 2k$, then $K_{\alpha\gamma}(x, y)$ vanishes on the region $0 < x < y$ and, moreover, the hypergeometric function defining $K_{\alpha\gamma}(x, y)$ is bounded on $0 < y \le x$. Hence the estimate (5.1) holds for $0 < y < x$ and the desired result follows by Hardy's inequality (2.10). When $\gamma = \alpha + 2k$, then $K_{\alpha\gamma}(x, y) = 0$ on $0 < y < x$ and ${}_2F_1$ defining it is bounded on $0 < x \le y$, so (5.2) holds for $0 < x < y$ and the conclusion follows with the aid of Hardy's inequality (2.11). □

PROOF OF THEOREM 2.2. Argue as in the proof of Theorem 2.1, using weighted weak type inequalities (2.15) and (2.16), instead of weighted Hardy's inequalities (2.10) and (2.11). □

The next result refines and enlarges Schindler's singular integral representation of $T_{\alpha\gamma}$.

PROPOSITION 5.1. *Let* $\alpha, \gamma > -1,\ |\alpha - \gamma| \ne 2k,\ k = 0, 1, \dots$. *Then the extension of* $T_{\alpha\gamma}$ (*obtained in Theorems* 2.1 *and* 2.2) *acting on weighted* L^p *spaces*, $1 \le p < \infty$, *has the singular integral representation* (3.1) *valid for all* $f \in L^p(w)$, w *being a weight satisfying the conditions described in Theorem* 2.1 *if* $p > 1$ *or in Theorem* 2.2 *if* $p = 1$.

PROOF. Recall that Schindler [12] proved that (3.1) holds for $f \in C_c^\infty$, provided $\alpha, \gamma \ge -1/2$. Here we claim that a detailed and careful analysis of Schindler's argument [12, pp. 368–379] shows that (3.1) remains valid for $f \in C_c^\infty(0, \infty)$ when $\alpha, \gamma > -1$ (in particular, (3.2) and (3.3) hold also if α or γ is in $(-1, -1/2)$). The conclusion of Proposition 5.1 then follows by standard arguments (see [4, Chapter 5, Section 4]), Theorem 4.3 (c), the fact that $T^3_{\alpha\gamma}$ is a local CZ operator and the density of $C_c^\infty(0, \infty)$ in all weighted L^p spaces under consideration.

We now give merely an outline of necessary modifications of Schindler's reasoning for proving (3.1) under the aforementioned assumptions. The starting point is the formula (2.6) (understood as iterative integral, with both inner and outer integrals Lebesgue integrable, no application of Fubini's theorem possible), which replaces the second formula in [12, Section 2]. The next point to be explained is the fact that the contribution of

$$\int_0^1 \int_{|x-t|\le\delta} (xy)^{1/2} J_\alpha(xy)(yt)^{1/2} J_\gamma(yt) f(t)\,dt\,dy$$

tends to zero as $\delta \to 0^+$. In view of (2.1) the absolute value of this expression is bounded by

$$Cx^{\alpha+1/2} \int_0^1 \int_{|x-t|\le\delta} |f(t)t^{\gamma+1/2}|\,dt\,y^{\alpha+\gamma+1} dy$$

and the required claim follows since $f(t)t^{\gamma+1/2}$ is bounded on $(0,\infty)$ and $\alpha+\gamma+1>-1$. Another place to be modified is the result contained in [12, Lemma 2], which in our setting is

$$\lim_{\rho\to 0^+}\int_0^\infty y^{-\rho}\mathcal{H}_\gamma f(y)(xy)^{1/2}J_\alpha(xy)dy=\mathcal{H}_\alpha(\mathcal{H}_\gamma f)(x)\,.$$

The problem is in justifying the use of the dominated convergence theorem; this was already done in the proof of Proposition 3.1. In all other places that need a modification a similar reasoning is used. □

REMARK 5.2. By Proposition 3.2 the kernel $K_{\alpha\gamma}(x,y)$ is a (global) standard kernel, provided $\alpha,\gamma\ge 1/2,\ |\alpha-\gamma|\ne 2k,\ k=0,1,\dots$. As it may be easily seen, the argument used in the proof of Proposition 3.2 for showing (3.10) (the estimate used in proving (4.2) and (4.3)) fails outside $\mathcal{D}$ either if $\alpha<1/2$ or $\gamma<1/2$. It is important to stress here, however, that even in the case $\alpha,\gamma\ge 1/2$ restricting the kernel to $\mathcal{D}$ brings an advantage: outside $\mathcal{D}$ we use Hardy's inequalities while inside $\mathcal{D}$ we deal with a local CZ operator which results in admitting more weights. To be more precise, we shall show that for $1<p<\infty$ the condition (4.6) implies (2.7) for $\gamma\ge -1/2$ and (2.8) for $\alpha\ge -1/2$ (we thank Óscar Ciaurri for assistance in proving this) while in the case $p=1$ the condition (4.6) implies (2.12) for $\gamma\ge -1/2$ and either (2.14) for $\alpha>-1/2$ or (2.13) for $\alpha=-1/2$. Indeed, in the case $1<p<\infty$, if w satisfies (4.6) and $\gamma\ge -1/2$, $\alpha\ge -1/2$, then $w^p\in A_p(0,\infty)$ and hence

$$\|M_+g\|_{p,w}\le C\|g\|_{p,w}\,. \tag{5.3}$$

Since M_+ dominates the Hardy operator P_1,

$$\left|\frac{1}{x}\int_0^x g(t)dt\right|\le 2M_+g(x),\qquad x\in(0,\infty)\,,$$

it follows that

$$|P_{\gamma+3/2}f(x)|\le |P_1\big(f(t)t^{-(\gamma+1/2)}\big)(x)|\le 2M_+\big(f(t)t^{-(\gamma+1/2)}\big)(x)\,,$$

therefore

$$\|P_{\gamma+3/2}f\|_{p,w}\le C\|f(x)x^{-(\gamma+1/2)}\|_{p,w}\,. \tag{5.4}$$

This is (2.10) hence, necessarily, w satisfies (2.7). On the other hand, if w satisfies (4.6), then $w^{-p'}\in A_{p'}(0,\infty)$ and hence (5.3) holds with w^{-1} and p' replacing w and p. Thus (5.4) holds with the analogous replacement and, in addition, with γ replaced by α. It is easily seen that the dual inequality to (5.4) with the aforementioned replacements is (2.11) hence, necessarily, w satisfies (2.8).

In the case $p=1$ the argument is similar. If w satisfies (4.6) with $p=1$, then $w\in A_1(0,\infty)$ and hence

$$\|M_+g\|_{L^{1,\infty}(w)}\le C\|g\|_{1,w}\,.$$

Consequently, given $\gamma\ge -1/2$ it follows that

$$\|P_{\gamma+3/2}f\|_{L^{1,\infty}(w)}\le C\|f(x)x^{-(\gamma+1/2)}\|_{1,w}\,.$$

This is (2.15) hence, necessarily, w satisfies (2.12). On the other hand, if w satisfies (4.6) with $p = 1$, then

$$\frac{1}{v-u}\int_u^v w \le C \operatorname*{ess\,inf}_{x\in(u,v)} w(x)\,, \qquad 0 \le u < v < \infty\,.$$

This is the A_1 condition, readily seen to be equivalent with $M_+ w(x) \le C w(x)$ a.e., cf. [4, p. 134], which is necessary and sufficient for

$$\|M_+ g w^{-1}\|_\infty \le C\|g w^{-1}\|_\infty$$

to hold, cf. [8, Theorem 4]. (Here is a short argument of this fact: for sufficiency we can assume that $\|gw^{-1}\|_\infty < \infty$ and then

$$\|M_+(gw^{-1}w)w^{-1}\|_\infty \le \|gw^{-1}\|_\infty \|(M_+w)w^{-1}\|_\infty \le C\|gw^{-1}\|_\infty\,;$$

necessity is immediate.) Hence, given $\alpha \ge -1/2$ we obtain

(5.5) $$\|P_{\alpha+3/2} f w^{-1}\|_\infty \le C\|f(x)x^{-(\alpha+1/2)}w(x)^{-1}\|_\infty\,.$$

Since the dual to $L^1(w)$ is $L^\infty(w^{-1})$ (with the pairing $h \mapsto \int_0^\infty h\varphi$, $\varphi \in L^\infty(w^{-1})$!), it is easily seen that the dual inequality to (5.5) is

$$\|Q_{-(\alpha+1/2)} f\|_{1,w} \le C\|f(x)x^{\alpha+3/2}\|_{1,w}\,,$$

which implies (2.16) hence, necessarily, either (2.14) if $\alpha > -1/2$ or (2.13) if $\alpha = -1/2$ follows.

REMARK 5.3. Mapping properties of $T^3_{\alpha\gamma}$ can be obtained in another way, by proving the estimate

$$|T^3_{\alpha\gamma} f(x)| \le C(M_o f(x) + H_o f(x))\,,$$

see [13]. Nevertheless, the CZ approach is more insightful and results in some additional profits, one of them being Proposition 5.1.

REMARK 5.4. As it was already pointed out, $K_{\alpha\gamma}$ is a standard kernel whenever $\alpha, \gamma \ge 1/2$, $|\alpha - \gamma| \ne 2k$, $k = 0, 1, 2, \ldots$. Consequently, in such a case further mapping properties of $T_{\alpha\gamma}$ follow by a general theory, cf. [4, Chapter 6]. For instance, $T_{\alpha\gamma}$ extends to a bounded operator from H^1 to L^1. It is worth noting that Kanjin [6] has recently proved a stronger result: $T_{\alpha\gamma}$ extends to a bounded operator on H^1 whenever $\alpha \ge -1/2$ and $\gamma > -1/2$.

6. Appendix: local A_p theory. In this section we show that most of the basic properties of A_p weights carry over local A_p weights. Although some of the facts are not indispensable for the rest of the paper, they seem to be worthy of attention and therefore are presented. The proofs are direct modifications of the corresponding proofs for (global) A_p weights. We give a part of them for the sake of convenience and completeness, mainly according to [4, Chapter 7].

In what follows we shall denote

$$\mathcal{I}_k = \{[u, v)\,;\ 0 < u < v < ku\}\,, \qquad k > 1\,.$$

Observe, that if $I \in \mathcal{I}_k$ and $[u, v) \subset I$, then also $[u, v) \in \mathcal{I}_k$. Given $1 \le p < \infty$ and $k > 1$, let $A^p_{k,\mathrm{loc}}$ be the class of all nonnegative weights w on $(0, \infty)$ satisfying

$$\sup_{I \in \mathcal{I}_k} \frac{1}{|I|}\Big(\int_I w\Big)^{1/p}\Big(\int_I w^{-p'/p}\Big)^{1/p'} < \infty\,. \tag{6.1}$$

If $p = 1$, then we understand the second integral as $\operatorname{ess\,sup}_I w^{-1}$. Note, that w satisfies (2.9) if and only if $w^p \in A^p_{2,\mathrm{loc}}$.

PROPOSITION 6.1. *Let $1 \le p < \infty$. Then $A^p_{k,\mathrm{loc}} = A^p_{2,\mathrm{loc}}$ for any $k > 1$.*

PROOF. Clearly, $A^p_{k_2,\mathrm{loc}} \subset A^p_{k_1,\mathrm{loc}}$ whenever $1 < k_1 \le k_2$. To show the converse inclusion we shall use the reasoning from the proof of [1, Lemma 1]. Let $1 < p < \infty$. Notice first that Hölder's inequality gives

$$(\sqrt{k_1} - 1)u \le \Big(\int_u^{\sqrt{k_1}u} w\Big)^{1/p}\Big(\int_u^{\sqrt{k_1}u} w^{-p'/p}\Big)^{1/p'}$$

and, consequently,

$$\Big(\int_u^{\sqrt{k_1}u} w^{-1/(p-1)}\Big)^{1-p} \le (\sqrt{k_1} - 1)^{-p} u^{-p} \int_u^{\sqrt{k_1}u} w\,.$$

Now, let $w \in A^p_{k_1,\mathrm{loc}}$. Since the condition (6.1) for w is equivalent to

$$\Big(\int_u^v w\Big)\Big(\int_u^v w^{-1/(p-1)}\Big)^{p-1} \le C(v-u)^p\,, \quad 0 < u < v < k_1 u\,, \tag{6.2}$$

we obtain

$$\begin{aligned}\int_{\sqrt{k_1}u}^{k_1 u} w \ \le\ \int_u^{k_1 u} w &\le C(k_1 - 1)^p u^p \Big(\int_u^{k_1 u} w^{-1/(p-1)}\Big)^{1-p}\\ &\le C(k_1 - 1)^p u^p \Big(\int_u^{\sqrt{k_1}u} w^{-1/(p-1)}\Big)^{1-p}\\ &\le C(\sqrt{k_1} + 1)^p \int_u^{\sqrt{k_1}u} w\,.\end{aligned}$$

Iterating this process we conclude that

$$\int_u^v w \le C_{p,k_1,k_2} \int_u^{\sqrt{k_1}u} w\,, \quad k_1 u \le v < k_2 u\,.$$

Similar arguments show an analogous inequality for $\int_u^v w^{-1/(p-1)}$, therefore (6.2) holds with k_1 replaced by k_2 (and a new constant C depending also on k_1 and k_2).

In the case $p = 1$ essentially the same reasoning works hence we do not provide the details. □

Thus the class $A^p_{k,\mathrm{loc}}$ is in fact independent of $k > 1$, hence it will be denoted by A^p_{loc}. The condition (6.1) will be referred to as a local A_p condition, and weights from A^p_{loc} will be called local A_p weights. By the $A^p_{k,\mathrm{loc}}$ constant (or norm) of $w \in A^p_{\mathrm{loc}}$ we mean the value of the left side in (6.1) and denote it by $\|w\|_{A^p_{k,\mathrm{loc}}}$. Note that by the proof of Proposition 6.1 the $A^p_{k,\mathrm{loc}}$ and $A^p_{l,\mathrm{loc}}$ norms are comparable, the corresponding constants being dependent only on k, l and p. The $A^p_{2,\mathrm{loc}}$ norm of $w \in A^p_{\mathrm{loc}}$ will be simply called local A_p norm and denoted by $\|w\|_{A^p_{\mathrm{loc}}}$.

Define for $k > 1$ a local version of the Hardy-Littlewood maximal function

$$M_{k,\mathrm{loc}} f(x) = \sup_{x \in I \in \mathcal{I}_k} \frac{1}{|I|} \int_I |f(y)| dy .$$

PROPOSITION 6.2. *Let $1 \leq p < \infty$ and $k > 1$. The condition $w \in A^p_{\mathrm{loc}}$ is necessary for the local maximal function $M_{k,\mathrm{loc}}$ to satisfy the weighted weak type (p, p) inequality*

$$w(\{x > 0 \, ; \, M_{k,\mathrm{loc}} f(x) > \lambda\}) \leq \frac{C}{\lambda^p} \int_0^\infty |f(x)|^p w(x) dx , \quad \lambda > 0 .$$

PROOF. Let $f \geq 0$ and, with the notation $f(I) = \int_I f$, let $I \in \mathcal{I}_k$ be such that $f(I) > 0$. Observe, that if $0 < \lambda < f(I)/|I|$, then

$$I \subset \{x > 0 \, ; \, M_{k,\mathrm{loc}}(\chi_I f)(x) > \lambda\} .$$

Therefore, using the weighted weak type (p, p) inequality for $M_{k,\mathrm{loc}}$, we see that

$$w(I) \leq \frac{C}{\lambda^p} \int_I f^p w , \quad 0 < \lambda < f(I)/|I| ,$$

with a constant C independent of f and λ. This gives

$$w(I) \left(\frac{f(I)}{|I|} \right)^p \leq C \int_I f^p w , \tag{6.3}$$

which after substituting $f = \chi_S, \; S \subset I$, specializes to

$$w(I) \left(\frac{|S|}{|I|} \right)^p \leq C w(S) . \tag{6.4}$$

Note that, after excluding the trivial cases $w \equiv 0$ and $w \equiv \infty$ a.e., the above inequality implies that $0 < w < \infty$ a.e.

Assume first that $p = 1$ and let $A = \operatorname{ess\,inf}_{x \in I} w(x)$. For each $\varepsilon > 0$ there exists a set $S_\varepsilon \subset I$ of positive measure such that $w(x) \leq A + \varepsilon$ for $x \in S_\varepsilon$. Now, (6.4) gives $w(I)/|I| \leq C(A + \varepsilon)$ and hence

$$\frac{w(I)}{|I|} \leq C w(x) , \quad \text{a.e. } x \in I ,$$

which is equivalent to the local A_1 condition.

When $1 < p < \infty$ we take $f = w^{1-p'} \chi_I$ in (6.3) to get

$$w(I) \left(\frac{1}{|I|} \int_I w^{1-p'} \right)^p \leq C \int_I w^{1-p'} ,$$

which is easily seen to be equivalent to the local A_p condition. □

PROPOSITION 6.3. *Let $k > 1$. If $1 < p < \infty$ and $w^p \in A^p_{\text{loc}}$, then the local maximal function $M_{k,\text{loc}}$ is bounded from $L^p(w)$ to $L^p(w)$. Moreover, if $w \in A^1_{\text{loc}}$, then $M_{k,\text{loc}}$ is bounded from $L^1(w)$ to $L^{1,\infty}(w)$. The corresponding L^p and weak type $(1,1)$ constants depend on w only through the local A_p norm of w^p.*

PROOF. We shall use the argument, which was already applied in the proof of Theorem 4.3. Let $I_n = [k^n, k^{n+3})$, $n \in \mathbf{Z}$. Define the weight w_n on $(-\infty, \infty)$ to be equal w on I_n, periodic with period $2|I_n|$, and symmetric around the point k^n. Then one verifies that the (global) A_p norm of w_n^p is estimated from above by the local A_p norm of w^p times a constant independent of w and n. Thus, denoting $f_n = f\chi_{I_n}$, $J_n = [k^{n+1}, k^{n+2})$ and observing that if $x \in J_n$ then the condition $0 < u < x < v < ku$ implies $u, v \in I_n$, we obtain for $p > 1$

$$\begin{aligned}
\int_0^\infty (M_{k,\text{loc}} f(x) w(x))^p \, dx &= \sum_{n\in\mathbf{Z}} \int_{J_n} (M_{k,\text{loc}} f_n(x) w_n(x))^p \, dx \\
&\le \sum_{n\in\mathbf{Z}} \int (M f_n(x) w_n(x))^p \, dx \\
&\le c \sum_{n\in\mathbf{Z}} \int |f_n(x) w_n(x)|^p \, dx \\
&= 3c \int_0^\infty |f(x) w(x)|^p \, dx \, .
\end{aligned}$$

In the above we used weighted L^p inequality for the Hardy-Littlewood maximal function M. Treatment of the case $p = 1$ is analogous, see the proof of Theorem 4.3. □

REMARK 6.4. Since $M_{3/2,\text{loc}} f \le 2M_o f \le 2M_{2,\text{loc}} f$, Propositions 6.2 and 6.3 remain true with $M_{k,\text{loc}}$ replaced by M_o.

COROLLARY 6.5. *Let $1 \le p < \infty$, $w \in A^p_{\text{loc}}$ and $I \in \mathcal{I}_k$ for some $k > 1$. There exists a constant C depending only on k, p and the A^p_{loc} norm of w such that for each measurable set $S \subset I$*

$$\left(\frac{|S|}{|I|}\right)^p \le C \frac{w(S)}{w(I)} \, .$$

PROOF. In virtue of Proposition 6.3 the constant C in (6.4) depends only on p and the $A^p_{k,\text{loc}}$ norm of w. □

PROPOSITION 6.6. *The local A_1 condition is equivalent to*

$$M_{k,\text{loc}} w(x) \le C w(x) \, , \quad \text{a.e. } x \in (0, \infty) \, , \tag{6.5}$$

with $k > 1$ fixed.

PROOF. It is straightforward that (6.5) implies the local A_1 condition. To prove the converse, suppose that for every $I \in \mathcal{I}_k$

$$\frac{w(I)}{|I|} \le C w(x) \, , \quad \text{a.e. } x \in I \, .$$

Observe, that if x is such that $M_{k,\mathrm{loc}}w(x) > Cw(x)$, then there exists $I \in \mathcal{I}_k$ with rational endpoints such that $w(I)/|I| > Cw(x)$, so x lies in a subset of I of measure 0. Taking the union of all such exceptional sets for all intervals $I \in \mathcal{I}_k$ with rational endpoints we conclude that $M_{k,\mathrm{loc}}w(x) > Cw(x)$ holds only on a set of measure zero. □

PROPOSITION 6.7.
(a) $A^p_{\mathrm{loc}} \subset A^q_{\mathrm{loc}},\ 1 \le p < q$;
(b) *$w \in A^p_{\mathrm{loc}}$ if and only if $w^{1-p'} \in A^{p'}_{\mathrm{loc}},\ 1 < p < \infty$;*
(c) *If $w_0, w_1 \in A^1_{\mathrm{loc}}$, then $w_0 w_1^{1-p} \in A^p_{\mathrm{loc}},\ 1 < p < \infty$.*

PROOF. All the statements are rather direct consequences of the local A_p condition; for example, to check (a) we take $I \in \mathcal{I}_2$ and write for $p = 1$

$$\left(\frac{1}{|I|}\int_I w^{1-q'}\right)^{q-1} \le \operatorname*{ess\,sup}_{x\in I} w(x)^{-1} \le C\left(\frac{w(I)}{|I|}\right)^{-1},$$

and when $p > 1$ we use Hölder's inequality to get

$$\left(\frac{1}{|I|}\int_I w^{1-q'}\right)^{q-1} \le \left(\frac{1}{|I|}\int_I w^{1-p'}\right)^{p-1} \le C\left(\frac{w(I)}{|I|}\right)^{-1}.$$

The remaining items are verified in a similar manner. □

PROPOSITION 6.8 (Reverse Hölder Inequality). *Let $1 \le p < \infty$, $w \in A^p_{\mathrm{loc}}$ and $k > 1$. There exist constants C and ε, depending only on p, k and the local A_p norm of w, such that*

$$\left(\frac{1}{|I|}\int_I w^{1+\varepsilon}\right)^{1/(1+\varepsilon)} \le \frac{C}{|I|}\int_I w\,, \quad I \in \mathcal{I}_k\,.$$

PROOF. The reasoning is essentially the same as that for global A_p weights, see [4, Chapter 7, Section 2]. □

The reverse Hölder inequality has the following notable consequences.

COROLLARY 6.9.
(a) $A^p_{\mathrm{loc}} = \bigcup_{q<p} A^q_{\mathrm{loc}},\ \ 1 < p < \infty$;
(b) *if $w \in A^p_{\mathrm{loc}},\ 1 \le p < \infty$, then $w^{1+\varepsilon} \in A^p_{\mathrm{loc}}$ for some $\varepsilon > 0$;*
(c) *if $k > 1$ and $w \in A^p_{\mathrm{loc}},\ 1 \le p < \infty$, then there exists $\delta > 0$ such that*

$$\frac{w(S)}{w(I)} \le C\left(\frac{|S|}{|I|}\right)^{\delta}$$

for all $I \in \mathcal{I}_k$ and $S \subset I$.

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