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Research Article

Weighted Estimates of a Class of Integral Operators with Three Parameters

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We characterize the validity of a Hardy-type inequality with a kernel and three parameters $1 < p, q, r < \infty$ under some conditions on three weight functions u, v, and w.

1. Introduction

Let $1 < p, r < \infty$, $0 < q < \infty$, and $-\infty \le a < b \le +\infty$. Let $u(\cdot)$, $v(\cdot)$, and $w(\cdot)$ be positive functions locally integrable on (a, b), hereinafter referred to as weights. Suppose that for two nonnegative quantities A and B the expression $A \ll B$ means $A \le CB$ with some constant C that through the paper depends only on the parameters r, p, and q. The notation $A \approx B$ means $A \ll B \ll A$. Moreover, $1/p + 1/p' = 1$.

We consider the following inequalities:

$$\left(\int_a^b u(x) \cdot \left(\int_a^x \left(\int_t^x K(s,t) f(s)\,ds\right)^r w(t)\,dt\right)^{q/r} dx\right)^{1/q} \le C\left(\int_a^b v(x) f^p(x)\,dx\right)^{1/p}, \quad (1)$$

$$\left(\int_a^b u(x) \cdot \left(\int_x^b \left(\int_x^t K(t,s) f(s)\,ds\right)^r w(t)\,dt\right)^{q/r} dx\right)^{1/q} \le C\left(\int_a^b v(x) f^p(x)\,dx\right)^{1/p} \quad (2)$$

for all $f \ge 0$, where the kernel $K(\cdot,\cdot)$ satisfies the conditions

$$K(s,t) \ge 0,$$
$$a < t \le s < b, \ K(s,t) \text{ is increasing in } s \text{ and decreasing in } t, \quad (3)$$
$$K(s,t) \approx K(s,z) + K(z,t)$$

for all t, z, and s such that $a < t \le z \le s < b$.

A class of Volterra type integral operators with kernels $K(\cdot,\cdot)$ satisfying condition (3) was introduced in [1] and independently in [2]. Later such kernels were considered in many works (see, e.g., [3–8]).

The main aim of this paper is to find necessary and sufficient conditions on the weights u, v, and w for the validity of inequalities (1) and (2) in the case $1 < p, q, r < \infty$. The same problem for $K(\cdot,\cdot) = 1$ was considered in [9, 10].

Assume

$$A_0^-(\alpha,\beta) = \sup_{\alpha<x<\beta} \left(\int_\alpha^x K^r(x,s) w(s)\,ds\right)^{1/r} \cdot \left(\int_x^\beta v^{1-p'}(s)\,ds\right)^{1/p'},$$

$$A_1^-(\alpha,\beta) = \sup_{\alpha<x<\beta} \left(\int_\alpha^x w(s)\,ds\right)^{1/r} \left(\int_x^\beta K^{p'}(s,x) \cdot v^{1-p'}(s)\,ds\right)^{1/p'},$$

$$A_0^+(\alpha,\beta)=\sup_{\alpha<x<\beta}\left(\int_x^\beta K^r(s,x)\,w(s)\,ds\right)^{1/r}\cdot\left(\int_\alpha^x v^{1-p'}(s)\,ds\right)^{1/p'},$$

$$A_1^+(\alpha,\beta)=\sup_{\alpha<x<\beta}\left(\int_x^\beta w(s)\,ds\right)^{1/r}\left(\int_\alpha^x K^{p'}(x,s)\cdot v^{1-p'}(s)\,ds\right)^{1/p'},$$

$$B_0^-(\alpha,\beta)=\left(\int_\alpha^\beta\left(\int_\alpha^x K^r(x,s)\,w(s)\,ds\right)^{p/(p-r)}\cdot\left(\int_x^\beta v^{1-p'}(s)\,ds\right)^{p(r-1)/(p-r)}v^{1-p'}(x)\,dx\right)^{(p-r)/pr},$$

$$B_1^-(\alpha,\beta)=\left(\int_\alpha^\beta\left(\int_\alpha^x w(s)\,ds\right)^{r/(p-r)}\cdot\left(\int_x^\beta K^{p'}(s,x)\,v^{1-p'}(s)\,ds\right)^{r(p-1)/(p-r)}w(x)\,dx\right)^{(p-r)/pr},$$

$$B_0^+(\alpha,\beta)=\left(\int_\alpha^\beta\left(\int_x^\beta K^r(s,x)\,w(s)\,ds\right)^{p/(p-r)}\cdot\left(\int_\alpha^x v^{1-p'}(s)\,ds\right)^{p(r-1)/(p-r)}v^{1-p'}(x)\,dx\right)^{(p-r)/pr},$$

$$B_1^+(\alpha,\beta)=\left(\int_\alpha^\beta\left(\int_x^\beta w(s)\,ds\right)^{r/(p-r)}\cdot\left(\int_\alpha^x K^{p'}(x,s)\,v^{1-p'}(s)\,ds\right)^{r(p-1)/(p-r)}w(x)\,dx\right)^{(p-r)/pr},$$

$$U(\alpha,\beta)=\left(\int_\alpha^\beta u(x)\,dx\right)^{1/q},$$

$$A^-=\max\{A_0^-,A_1^-\},$$

$$A^+=\max\{A_0^+,A_1^+\},$$

$$B^-=\max\{B_0^-,B_1^-\},$$

$$B^+=\max\{B_0^+,B_1^+\},$$

$$J^-(\alpha,\beta)=\sup_{f\geq 0}\frac{\left(\int_\alpha^\beta\left(\int_t^\beta K(s,t)\,f(s)\,ds\right)^r w(t)\,dt\right)^{1/r}}{\left(\int_\alpha^\beta v(x)\,f^p(x)\,dx\right)^{1/p}},$$

$$J^+(\alpha,\beta)=\sup_{f\geq 0}\frac{\left(\int_\alpha^\beta\left(\int_\alpha^t K(t,s)\,f(s)\,ds\right)^r w(t)\,dt\right)^{1/r}}{\left(\int_\alpha^\beta v(x)\,f^p(x)\,dx\right)^{1/p}}. \qquad (4)$$

Two-sided estimates of the values J^- and J^+ with kernels satisfying condition (3) were found in [11]. Moreover, when $K(\cdot,\cdot)=1$ we get standard Hardy-type estimates that have been extensively investigated by many authors. A complete review of Hardy-type estimates and generalized Hardy-type estimates can be found in books [12, 13] and references given there.

The following theorem will be used for the main results.

Theorem A (see [11]). *(1) If $1<p\leq r<\infty$, then for all $f\geq 0$ we have*

$$J^-\approx A^-,\qquad J^+\approx A^+. \qquad (5)$$

(2) If $1<r<p<\infty$, then for all $f\geq 0$ we have

$$J^-\approx B^-,\qquad J^+\approx B^+. \qquad (6)$$

Remark 1. Since the expressions $A_0^\pm$, $A_1^\pm$, $B_0^\pm$, and $B_1^\pm$ are decreasing in α and increasing in β, then from (5) and (6) we have that $J^\pm(\alpha,\beta)$ are equivalent to a decreasing function in α and an increasing function in β. This means that there exists a constant $C>0$ depending only on p and r such that $J^\pm(\alpha,\beta)\leq CJ^\pm(\alpha_1,\beta_1)$ for $\alpha_1\leq\alpha<\beta\leq\beta_1$.

2. Main Results

2.1. Case $p\leq q$

Theorem 2. *Let $1<p\leq q<\infty$ and $1<r<\infty$. Inequality (1) holds for all $f\geq 0$ if and only if $E^-:=\sup_{a<z<b}J^-(a,z)U(z,b)<\infty$. Moreover, $E^-\approx C$, where C is the best constant in (1).*

Theorem 3. *Let $1<p\leq q<\infty$ and $1<r<\infty$. Inequality (2) holds for all $f\geq 0$ if and only if $E^+:=\sup_{a<z<b}J^+(z,b)U(a,z)<\infty$. Moreover, $E^+\approx C$, where C is the best constant in (2).*

Remark 4. Let us prove only Theorem 2 since the proof of Theorem 3 is similar.

Proof of Theorem 2.

Sufficiency. Let $E^-<\infty$. For any integer k we introduce

$$x_k=\sup\left\{a<x<b:\int_a^x\left(\int_t^x K(s,t)\,f(s)\,ds\right)^r w(t)\,dt\leq 2^{rk}\right\}. \qquad (7)$$

It is obvious that for any k we have $x_k \le x_{k+1}$. However, when $x_k < b$ we have $x_k < x_{k+1} \le b$. Therefore,

$$\int_a^{x_k} \left(\int_t^{x_k} K(s,t) f(s)\, ds \right)^r w(t)\, dt = 2^{rk}, \tag{8}$$

$$\int_a^{x_{k+1}} \left(\int_t^{x_{k+1}} K(s,t) f(s)\, ds \right)^r w(t)\, dt \le 2^{r(k+1)}. \tag{9}$$

Let $I_k = [x_k, x_{k+1})$. Then

$$(a,b) = \bigcup_k [x_k, x_{k+1}). \tag{10}$$

Suppose that $x_k < b$; then from (8), twice applying Minkowski's inequality, we get

$$\begin{aligned}
2^{k-1} &= 2^k - 2^{k-1} = \left(\int_a^{x_k} \left(\int_t^{x_k} K(s,t) f(s)\, ds \right)^r \cdot w(t)\, dt \right)^{1/r} \\
&- \left(\int_a^{x_{k-1}} \left(\int_t^{x_{k-1}} K(s,t) f(s)\, ds \right)^r w(t)\, dt \right)^{1/r} \\
&\le \left(\int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} K(s,t) f(s)\, ds \right)^r w(t)\, dt \right)^{1/r} \\
&+ \left(\int_a^{x_{k-1}} \left(\int_t^{x_{k-1}} K(s,t) f(s)\, ds + \int_{x_{k-1}}^{x_k} K(s,t) f(s)\, ds \right)^r w(t)\, dt \right)^{1/r} \\
&- \left(\int_a^{x_{k-1}} \left(\int_t^{x_{k-1}} K(s,t) f(s)\, ds \right)^r w(t)\, dt \right)^{1/r} \\
&\le \left(\int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} K(s,t) f(s)\, ds \right)^r w(t)\, dt \right)^{1/r} \\
&+ \left(\int_a^{x_{k-1}} \left(\int_{x_{k-1}}^{x_k} K(s,t) f(s)\, ds \right)^r w(t)\, dt \right)^{1/r}.
\end{aligned} \tag{11}$$

Since $a < t \le x_{k-1} \le s < b$, we can use (3) so that the last gives

$$\begin{aligned}
2^{k-1} &\ll \left(\int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} K(s,t) f(s)\, ds \right)^r w(t)\, dt \right)^{1/r} \\
&+ \left(\int_a^{x_{k-1}} \left(\int_{x_{k-1}}^{x_k} K(s,x_{k-1}) f(s)\, ds \right)^r w(t)\, dt \right)^{1/r} \\
&+ \left(\int_a^{x_{k-1}} \left(\int_{x_{k-1}}^{x_k} K(x_{k-1},t) f(s)\, ds \right)^r w(t)\, dt \right)^{1/r} \\
&= \left(\int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} K(s,t) f(s)\, ds \right)^r w(t)\, dt \right)^{1/r} \\
&+ \int_{x_{k-1}}^{x_k} K(s,x_{k-1}) f(s)\, ds \left(\int_a^{x_{k-1}} w(t)\, dt \right)^{1/r} \\
&+ \int_{x_{k-1}}^{x_k} f(s)\, ds \left(\int_a^{x_{k-1}} K^r(x_{k-1},t)\, w(t)\, dt \right)^{1/r}.
\end{aligned} \tag{12}$$

From (9) and (10) we have

$$\begin{aligned}
T &:= \int_a^b u(x) \cdot \left(\int_a^x \left(\int_t^x K(s,t) f(s)\, ds \right)^r w(t)\, dt \right)^{q/r} dx \\
&\le \sum_k \int_{x_k}^{x_{k+1}} u(x)\, dx \cdot \left(\int_a^{x_{k+1}} \left(\int_t^{x_{k+1}} K(s,t) f(s)\, ds \right)^r w(t)\, dt \right)^{q/r} \\
&\le \sum_k 2^{q(k+1)} \int_{x_k}^{x_{k+1}} u(x)\, dx \\
&= 2^{2q} \sum_k 2^{q(k-1)} \int_{x_k}^{x_{k+1}} u(x)\, dx.
\end{aligned} \tag{13}$$

From (12) and (13) it follows that

$$\begin{aligned}
T &\ll 2^{2q} \sum_k \left(\left(\int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} K(s,t) f(s)\, ds \right)^r w(t)\, dt \right)^{1/r} \right. \\
&+ \int_{x_{k-1}}^{x_k} K(s,x_{k-1}) f(s)\, ds \left(\int_a^{x_{k-1}} w(t)\, dt \right)^{1/r} \\
&\left. + \int_{x_{k-1}}^{x_k} f(s)\, ds \left(\int_a^{x_{k-1}} K^r(x_{k-1},t)\, w(t)\, dt \right)^{1/r} \right)^q \cdot \int_{x_k}^{x_{k+1}} u(x)\, dx \\
&\ll \sum_k \left(\int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} K(s,t) f(s)\, ds \right)^r w(t)\, dt \right)^{q/r} \cdot \int_{x_k}^{x_{k+1}} u(x)\, dx + \sum_k \left(\int_{x_{k-1}}^{x_k} K(s,x_{k-1}) f(s)\, ds \right)^q \\
&\cdot \left(\int_a^{x_{k-1}} w(t)\, dt \right)^{q/r} \int_{x_k}^{x_{k+1}} u(x)\, dx \\
&+ \sum_k \left(\int_{x_{k-1}}^{x_k} f(s)\, ds \right)^q \left(\int_a^{x_{k-1}} K^r(x_{k-1},t)\, w(t)\, dt \right)^{q/r} \\
&\cdot \int_{x_k}^{x_{k+1}} u(x)\, dx = T_1 + T_2 + T_3.
\end{aligned} \tag{14}$$

Next, we separately estimate T_1, T_2, and T_3 for $1 < p \le \min\{r, q\} < \infty$ and $1 < r < p \le q < \infty$.

Let $1 < p \le \min\{r, q\} < \infty$. From (5) we get

$$\begin{aligned} T_1 &\ll \sum_k \left(J^-(x_{k-1}, x_k) U(x_k, x_{k+1})\right)^q \\ &\quad \cdot \left(\int_{x_{k-1}}^{x_k} v(t) f^p(t)\, dt\right)^{q/p} \\ &\ll \sum_k \left(J^-(a, x_k) U(x_k, b)\right)^q \\ &\quad \cdot \left(\int_{x_{k-1}}^{x_k} v(t) f^p(t)\, dt\right)^{q/p} \le (E^-)^q \\ &\quad \cdot \left(\sum_k \int_{x_{k-1}}^{x_k} v(t) f^p(t)\, dt\right)^{q/p} \le (E^-)^q \\ &\quad \cdot \left(\int_a^b v(t) f^p(t)\, dt\right)^{q/p}. \end{aligned} \tag{15}$$

To estimate T_2 we use Hölder's inequality:

$$\begin{aligned} T_2 &= \sum_k \left(\int_{x_{k-1}}^{x_k} K(s, x_{k-1}) f(s) v^{1/p}(s) v^{-1/p}(s)\, ds\right)^q \\ &\quad \cdot \left(\int_a^{x_{k-1}} w(t)\, dt\right)^{q/r} \int_{x_k}^{x_{k+1}} u(x)\, dx, \end{aligned} \tag{16}$$

$$\begin{aligned} T_2 &\le \sum_k \left(\int_a^{x_{k-1}} w(t)\, dt\right)^{q/r} \\ &\quad \cdot \left(\int_{x_{k-1}}^{x_k} K^{p'}(s, x_{k-1}) v^{1-p'}(s)\, ds\right)^{q/p'} \int_{x_k}^{x_{k+1}} u(x)\, dx \\ &\quad \cdot \left(\int_{x_{k-1}}^{x_k} v(s) f^p(s)\, ds\right)^{q/p} \\ &\le \sum_k \left(A_1^-(a, x_k) U(x_k, b)\right)^q \\ &\quad \cdot \left(\int_{x_{k-1}}^{x_k} v(s) f^p(s)\, ds\right)^{q/p}, \end{aligned} \tag{17}$$

$$\begin{aligned} T_2 &\ll \sup_{a<z<b} \left(J^-(a, z) U(z, b)\right)^q \left(\int_a^b v(s) f^p(s)\, ds\right)^{q/p} \\ &= (E^-)^q \left(\int_a^b v(t) f^p(t)\, dt\right)^{q/p}. \end{aligned} \tag{18}$$

To estimate T_3 we again use Hölder's inequality and get

$$\begin{aligned} T_3 &= \sum_k \left(\int_{x_{k-1}}^{x_k} f(s) v^{1/p}(s) v^{-1/p}(s)\, ds\right)^q \\ &\quad \cdot \left(\int_a^{x_{k-1}} K^r(x_{k-1}, t) w(t)\, dt\right)^{q/r} \int_{x_k}^{x_{k+1}} u(x)\, dx, \end{aligned} \tag{19}$$

$$\begin{aligned} T_3 &\le \sum_k \left(\int_a^{x_{k-1}} K^r(x_{k-1}, t) w(t)\, dt\right)^{q/r} \\ &\quad \cdot \left(\int_{x_{k-1}}^{x_k} v^{1-p'}(s)\, ds\right)^{q/p'} \int_{x_k}^{x_{k+1}} u(x)\, dx \\ &\quad \cdot \left(\int_{x_{k-1}}^{x_k} v(s) f^p(s)\, ds\right)^{q/p} \\ &\le \sum_k \left(A_0^-(a, x_k) U(x_k, b)\right)^q \\ &\quad \cdot \left(\int_{x_{k-1}}^{x_k} v(s) f^p(s)\, ds\right)^{q/p}, \end{aligned} \tag{20}$$

$$\begin{aligned} T_3 &\ll \sup_{a<z<b} \left(J^-(a, z) U(z, b)\right)^q \left(\int_a^b v(s) f^p(s)\, ds\right)^{q/p} \\ &= (E^-)^q \left(\int_a^b v(t) f^p(t)\, dt\right)^{q/p}. \end{aligned} \tag{21}$$

From (14), (15), (18), and (21) it follows that for $1 < p \le \min\{r, q\} < \infty$ inequality (1) is correct. Moreover,

$$C \ll E^-, \tag{22}$$

where C is the best constant in (1).

Let us turn to the case $1 < r < p \le q < \infty$. In the same way as above from (6) we get

$$\begin{aligned} T_1 &\ll \sum_k \left(J^-(x_{k-1}, x_k) U(x_k, x_{k+1})\right)^q \\ &\quad \cdot \left(\int_{x_{k-1}}^{x_k} v(t) f^p(t)\, dt\right)^{q/p} \\ &\ll \sum_k \left(J^-(a, x_k) U(x_k, b)\right)^q \\ &\quad \cdot \left(\int_{x_{k-1}}^{x_k} v(t) f^p(t)\, dt\right)^{q/p} \le (E^-)^q \\ &\quad \cdot \left(\sum_k \int_{x_{k-1}}^{x_k} v(t) f^p(t)\, dt\right)^{q/p} \le (E^-)^q \\ &\quad \cdot \left(\int_a^b v(t) f^p(t)\, dt\right)^{q/p}. \end{aligned} \tag{23}$$

To estimate T_2 we work with (17). Since

$$\begin{aligned}&\left(\int_a^{x_{k-1}} w(t)\,dt\right)^{1/r}\\&\cdot\left(\int_{x_{k-1}}^{x_k} K^{p'}(s,x_{k-1})\,v^{1-p'}(s)\,ds\right)^{1/p'}\le\left(\frac{p-r}{p}\right.\\&\left.\cdot\int_a^{x_{k-1}}\left(\int_a^x w(t)\,dt\right)^{r/(p-r)} w(t)\,dt\right)^{(p-r)/pr}\\&\cdot\left(\int_{x_{k-1}}^{x_k} K^{p'}(s,x_{k-1})\,v^{1-p'}(s)\,ds\right)^{1/p'}\ll B_1^-(a,x_k),\end{aligned}\tag{24}$$

we have

$$\begin{aligned}T_2&\ll\sum_k\left(B_1^-(a,x_k)\,U(x_k,b)\right)^q\\&\cdot\left(\int_{x_{k-1}}^{x_k} v(s)\,f^p(s)\,ds\right)^{q/p}\\&\ll\sup_{a<z<b}\left(J^-(a,z)\,U(z,b)\right)^q\left(\int_a^b v(s)\,f^p(s)\,ds\right)^{q/p}\\&=(E^-)^q\left(\int_a^b v(t)\,f^p(t)\,dt\right)^{q/p}.\end{aligned}\tag{25}$$

Similarly, working with (20) we have

$$\begin{aligned}&\left(\int_a^{x_{k-1}} K^r(x_{k-1},t)\,w(t)\,dt\right)^{1/r}\left(\int_{x_{k-1}}^{x_k} v^{1-p'}(s)\,ds\right)^{1/p'}\\&\le\left(\int_a^{x_{k-1}} K^r(x_{k-1},t)\,w(t)\,dt\right)^{1/r}\left(\frac{p-r}{r(p-1)}\right.\\&\cdot\int_{x_{k-1}}^{x_k}\left(\int_x^{x_k} v^{1-p'}(s)\,ds\right)^{p(r-1)/(p-r)}\\&\left.\cdot v^{1-p'}(s)\,ds\right)^{(p-r)/pr}\ll B_0^-(a,x_k)\end{aligned}\tag{26}$$

that yields

$$\begin{aligned}T_3&\ll\sum_k\left(B_0^-(a,x_k)\,U(x_k,b)\right)^q\\&\cdot\left(\int_{x_{k-1}}^{x_k} v(s)\,f^p(s)\,ds\right)^{q/p}\\&\ll\sup_{a<z<b}\left(J^-(a,z)\,U(z,b)\right)^q\left(\int_a^b v(s)\,f^p(s)\,ds\right)^{q/p}\\&=(E^-)^q\left(\int_a^b v(t)\,f^p(t)\,dt\right)^{q/p}.\end{aligned}\tag{27}$$

Combining (14), (23), (25), and (27), we have that for $1<r<p\le q<\infty$ inequality (1) is correct. Moreover,

$$C\ll E^-,\tag{28}$$

where C is the best constant in (1).

Necessity. Let (1) be valid. Let $z\in(a,b)$ and $f:(a,z)\to R$ be an arbitrary function such that $\int_a^z v(x)f^p(x)dx<\infty$. Suppose that

$$f_z(s)=\begin{cases}f(s), & a<s<z,\\ 0, & z\le s<b.\end{cases}\tag{29}$$

If we substitute the function f_z in (1) we have

$$\begin{aligned}&\left(\int_z^b u(x)\,dx\right)^{1/q}\\&\cdot\left(\int_a^z\left(\int_t^z K(s,t)\,f(s)\,ds\right)^r w(t)\,dt\right)^{1/r}\\&\le C\left(\int_a^z v(x)\,f^p(x)\,dx\right)^{1/p}.\end{aligned}\tag{30}$$

From (30) we have

$$U(z,b)\,J^-(a,z)\le C\quad\forall z\in(a,b).\tag{31}$$

Therefore,

$$E^-\le C.\tag{32}$$

Moreover, from (22), (28), and (32) we have $C\approx E^-$, where C is the best constant in (1). The proof of Theorem 2 is complete. □

2.2. Case $q<p$. In this section we consider the case $0<q<p<\infty$, $p>1$, and $1<r<\infty$ and present sufficient conditions for the validity of inequalities (1) and (2).

Let

$$\begin{aligned}F^-&=\left(\int_a^b u(x)\left(\int_x^b u(s)\,ds\right)^{q/(p-q)}\right.\\&\left.\cdot\left(J^-(a,x)\right)^{pq/(p-q)}dx\right)^{(p-q)/pq},\\F^+&=\left(\int_a^b u(x)\left(\int_a^x u(s)\,ds\right)^{q/(p-q)}\right.\\&\left.\cdot\left(J^+(x,b)\right)^{pq/(p-q)}dx\right)^{(p-q)/pq}.\end{aligned}\tag{33}$$

Theorem 5. *Let $0<q<p<\infty$, $p>1$, and $1<r<\infty$. Inequality (1) holds if $F^-<\infty$. Moreover, $C\ll F^-$, where C is the best constant in (1).*

Theorem 6. *Let $0<q<p<\infty$, $p>1$, and $1<r<\infty$. Inequality (2) holds if $F^+<\infty$. Moreover, $C\ll F^+$, where C is the best constant in (2).*

Remark 7. Let us prove only Theorem 5 since the proof of Theorem 6 is similar.

Proof of Theorem 5. The first steps of the proof are similar to those in Theorem 2 up to (14), where

$$T \ll T_1 + T_2 + T_3. \tag{34}$$

This means that we need to separately estimate T_1, T_2, and T_3.

Let us start with T_1. Let us notice that we use Hölder's inequality:

$$\begin{aligned}
T_1 &= \sum_k \left(\int_{x_{k-1}}^{x_k} \left(\int_t^{x_k} K(s,t) f(s)\, ds \right)^r w(t)\, dt \right)^{q/r} \\
&\quad \cdot \int_{x_k}^{x_{k+1}} u(x)\, dx \le \sum_k \left(J^-(x_{k-1}, x_k)\right)^q \\
&\quad \cdot \int_{x_k}^{x_{k+1}} u(x)\, dx \left(\int_{x_{k-1}}^{x_k} v(t) f^p(t)\, dt \right)^{q/p} \\
&\le \left(\sum_k \left(\int_{x_k}^{x_{k+1}} u(x)\, dx \right)^{p/(p-q)} \right. \\
&\quad \cdot \left. \left(J^-(x_{k-1}, x_k)\right)^{pq/(p-q)} \right)^{(p-q)/p} \left(\sum_k \int_{x_{k-1}}^{x_k} v(t) \right. \\
&\quad \cdot \left. f^p(t)\, dt \right)^{q/p} \ll \left(\sum_k \int_{x_k}^{x_{k+1}} u(x) \right. \\
&\quad \cdot \left(\int_x^{x_{k+1}} u(s)\, ds \right)^{q/(p-q)} dx \\
&\quad \cdot \left. \left(J^-(x_{k-1}, x_k)\right)^{pq/(p-q)} \right)^{(p-q)/p} \left(\int_a^b v(t) \right. \\
&\quad \cdot \left. f^p(t)\, dt \right)^{q/p} \ll \left(\sum_k \int_{x_k}^{x_{k+1}} u(x) \right. \\
&\quad \cdot \left(\int_x^b u(s)\, ds \right)^{q/(p-q)} \\
&\quad \cdot \left. \left(J^-(a, x)\right)^{pq/(p-q)} dx \right)^{(p-q)/p} \left(\int_a^b v(t) \right. \\
&\quad \cdot \left. f^p(t)\, dt \right)^{q/p} \le \left(F^-\right)^q \left(\int_a^b v(t) \right. \\
&\quad \cdot \left. f^p(t)\, dt \right)^{q/p}.
\end{aligned} \tag{35}$$

Now, we turn to the estimation of T_2. Again Hölder's inequality is used:

$$\begin{aligned}
T_2 &= \sum_k \left(\int_{x_{k-1}}^{x_k} K(s, x_{k-1}) f(s)\, ds \right)^q \left(\int_a^{x_{k-1}} w(t)\, dt \right)^{q/r} \int_{x_k}^{x_{k+1}} u(x)\, dx \le \sum_k \left(\int_{x_{k-1}}^{x_k} K^{p'}(s, x_{k-1})\, v^{1-p'}(s)\, ds \right)^{q/p'} \\
&\quad \cdot \left(\int_a^{x_{k-1}} w(t)\, dt \right)^{q/r} \int_{x_k}^{x_{k+1}} u(x)\, dx \left(\int_{x_{k-1}}^{x_k} v(t) f^p(t)\, dt \right)^{q/p} \le \left(\sum_k \left(\int_{x_k}^{x_{k+1}} u(x)\, dx \right)^{p/(p-q)} \right. \\
&\quad \cdot \left. \left(\int_{x_{k-1}}^{x_k} K^{p'}(s, x_{k-1})\, v^{1-p'}(s)\, ds \right)^{q(p-1)/(p-q)} \left(\int_a^{x_{k-1}} w(t)\, dt \right)^{qp/r(p-q)} \right)^{(p-q)/p} \left(\sum_k \int_{x_{k-1}}^{x_k} v(t) f^p(t)\, dt \right)^{q/p} \\
&\le \left(\sum_k \left(\int_{x_k}^{x_{k+1}} u(x)\, dx \right)^{p/(p-q)} \right. \\
&\quad \cdot \left. \left[\left(\int_a^{x_{k-1}} w(t)\, dt \right)^{p/(p-r)} \left(\int_{x_{k-1}}^{x_k} K^{p'}(s, x_{k-1})\, v^{1-p'}(s)\, ds \right)^{r(p-1)/(p-r)} \right]^{q(p-r)/r(p-q)} \right)^{(p-q)/p} \left(\int_a^b v(t) f^p(t)\, dt \right)^{q/p} \\
&\ll \left(\sum_k \int_{x_k}^{x_{k+1}} u(x) \left(\int_x^{x_{k+1}} u(s)\, ds \right)^{q/(p-q)} dx \right. \\
&\quad \cdot \left. \left[\int_a^{x_{k-1}} w(t) \left(\int_a^t w(\tau)\, d\tau \right)^{r/(p-r)} dt \left(\int_{x_{k-1}}^{x_k} K^{p'}(s, x_{k-1})\, v^{1-p'}(s)\, ds \right)^{r(p-1)/(p-r)} \right]^{q(p-r)/r(p-q)} \right)^{(p-q)/p} \\
&\quad \cdot \left(\int_a^b v(t) f^p(t)\, dt \right)^{q/p} \le \left(\sum_k \int_{x_k}^{x_{k+1}} u(x) \left(\int_x^b u(s)\, ds \right)^{q/(p-q)} dx \right.
\end{aligned}$$

$$\cdot\left(\left[\int_a^{x_{k-1}} w(t)\left(\int_a^t w(\tau)\,d\tau\right)^{r/(p-r)}\left(\int_t^{x_k} K^{p'}(s,t)\,v^{1-p'}(s)\,ds\right)^{r(p-1)/(p-r)} dt\right]^{(p-r)/pr}\right)^{pq/(p-q)}\Bigg)^{(p-q)/p}$$
$$\cdot\left(\int_a^b v(t)\,f^p(t)\,dt\right)^{q/p} \le \left(\int_a^b u(x)\left(\int_x^b u(s)\,ds\right)^{q/(p-q)}\left(B_1^-(a,x)\right)^{pq/(p-q)} dx\right)^{(p-q)/p}\left(\int_a^b v(t)\,f^p(t)\,dt\right)^{q/p}$$
$$\ll \left(\int_a^b u(x)\left(\int_x^b u(s)\,ds\right)^{q/(p-q)}\left(J^-(a,x)\right)^{pq/(p-q)} dx\right)^{(p-q)/p}\left(\int_a^b v(t)\,f^p(t)\,dt\right)^{q/p} = \left(F^-\right)^q$$
$$\cdot\left(\int_a^b v(t)\,f^p(t)\,dt\right)^{q/p}. \tag{36}$$

The last step is to estimate T_3:

$$T_3 = \sum_k\left(\int_{x_{k-1}}^{x_k} f(s)\,ds\right)^q\left(\int_a^{x_{k-1}} K^r(x_{k-1},t)\,w(t)\,dt\right)^{q/r}\int_{x_k}^{x_{k+1}} u(x)\,dx \le \sum_k\left(\int_{x_{k-1}}^{x_k} v^{1-p'}(s)\,ds\right)^{q/p'}\left(\int_a^{x_{k-1}} K^r(x_{k-1},t)\right.$$
$$\left.\cdot\, w(t)\,dt\right)^{q/r}\int_{x_k}^{x_{k+1}} u(x)\,dx\left(\int_{x_{k-1}}^{x_k} v(s)\,f^p(s)\,ds\right)^{q/p} \le \left(\sum_k\left(\int_{x_k}^{x_{k+1}} u(x)\,dx\right)^{p/(p-q)}\right.$$
$$\left.\cdot\left[\left(\int_a^{x_{k-1}} K^r(x_{k-1},t)\,w(t)\,dt\right)^{q/r}\left(\int_{x_{k-1}}^{x_k} v^{1-p'}(s)\,ds\right)^{q/p'}\right]^{p/(p-q)}\right)^{(p-q)/p}\left(\int_a^b v(s)\,f^p(s)\,ds\right)^{q/p}$$
$$\le \left(\sum_k\left(\int_{x_k}^{x_{k+1}} u(x)\,dx\right)^{p/(p-q)}\left(\left[\left(\int_{x_{k-1}}^{x_k} v^{1-p'}(s)\,ds\right)^{r(p-1)/(p-r)}\right.\right.\right.$$
$$\left.\left.\left.\cdot\left(\int_a^{x_{k-1}} K^r(x_{k-1},t)\,w(t)\,dt\right)^{p/(p-r)}\right]^{(p-r)/pr}\right)^{pq/(p-q)}\right)^{(p-q)/p}\left(\int_a^b v(s)\,f^p(s)\,ds\right)^{q/p} \ll \left(\sum_k\int_{x_k}^{x_{k+1}} u(x)\right.$$
$$\cdot\left(\int_x^b u(\tau)\,d\tau\right)^{q/(p-q)} dx\left(\left[\int_{x_{k-1}}^{x_k} v^{1-p'}(s)\left(\int_s^{x_k} v^{1-p'}(t)\,dt\right)^{p(r-1)/(p-r)} ds\right.\right.$$
$$\left.\left.\cdot\left(\int_a^{x_{k-1}} K^r(x_{k-1},t)\,w(t)\,dt\right)^{p/(p-r)}\right]^{(p-r)/pr}\right)^{pq/(p-q)}\Bigg)^{(p-q)/p}\left(\int_a^b v(s)\,f^p(s)\,ds\right)^{q/p} \le \left(\sum_k\int_{x_k}^{x_{k+1}} u(x)\right.$$
$$\cdot\left(\int_x^b u(\tau)\,d\tau\right)^{q/(p-q)}$$
$$\left.\cdot\left(\left[\int_a^x v^{1-p'}(s)\left(\int_s^x v^{1-p'}(t)\,dt\right)^{p(r-1)/(p-r)}\left(\int_a^s K^r(s,t)\,w(t)\,dt\right)^{p/(p-r)} ds\right]^{(p-r)/pr}\right)^{pq/(p-q)} dx\right)^{(p-q)/p}$$
$$\cdot\left(\int_a^b v(s)\,f^p(s)\,ds\right)^{q/p} \le \left(\int_a^b u(x)\left(\int_x^b u(\tau)\,d\tau\right)^{q/(p-q)}\left(B_0^-(a,x)\right)^{pq/(p-q)} dx\right)^{(p-q)/p}\left(\int_a^b v(s)\,f^p(s)\,ds\right)^{q/p}$$
$$\ll \left(\int_a^b u(x)\left(\int_x^b u(\tau)\,d\tau\right)^{q/(p-q)}\left(J^-(a,x)\right)^{pq/(p-q)} dx\right)^{(p-q)/p}\left(\int_a^b v(s)\,f^p(s)\,ds\right)^{q/p} = \left(F^-\right)^q\left(\int_a^b v(s)\right.$$
$$\left.\cdot\, f^p(s)\,ds\right)^{q/p}. \tag{37}$$

Combining (14), (35), (36), and (37), we have that inequality (1) is correct. Moreover, $C \ll F^{-}$, where C is the best constant in (1). □

Remark 8. Let us consider the following inequalities:

$$\left(\int_a^b u(x) \cdot \left(\int_a^x \left(\int_t^b K(s,t) f(s)\, ds\right)^r w(t)\, dt\right)^{q/r} dx\right)^{1/q} \le \widetilde{C}\left(\int_a^b v(x) f^p(x)\, dx\right)^{1/p}, \tag{38}$$

$$\left(\int_a^b u(x) \cdot \left(\int_x^b \left(\int_a^t K(t,s) f(s)\, ds\right)^r w(t)\, dt\right)^{q/r} dx\right)^{1/q} \le \widetilde{C}\left(\int_a^b v(x) f^p(x)\, dx\right)^{1/p}. \tag{39}$$

It is obvious that, in view of (3), the validity of inequality (38) is equivalent to the simultaneous validity of inequality (1) and the following inequalities:

$$\left(\int_a^b u(x)\left(\int_a^x w(t)\, dt\right)^{q/r} \cdot \left(\int_x^b K(s,x) f(s)\, ds\right)^q dx\right)^{1/q} \le \widetilde{C_1}\left(\int_a^b v(t) \cdot f^p(t)\, dt\right)^{1/p}, \tag{40}$$

$$\left(\int_a^b u(x)\left(\int_a^x K^r(x,t) w(t)\, dt\right)^{q/r} \cdot \left(\int_x^b f(s)\, ds\right)^q dx\right)^{1/q} \le \widetilde{C_2}\left(\int_a^b v(t) \cdot f^p(t)\, dt\right)^{1/p}. \tag{41}$$

Inequality (40) can be treated by Theorem A, while inequality (41) is the standard Hardy-type inequality. This means that if we combine Theorem 2 and the known results on Hardy-type inequalities, we can characterize (38) for the case $1 < p \le q < \infty$ and $1 < r < \infty$. Similar splitting can be done for inequality (39). In [14] inequalities (38) and (39) are completely characterized for all relations between p, q, and r, where $1 \le p < \infty$, $0 < q < \infty$, and $0 < r \le \infty$. The characterization method in [14] is not based on the integral splitting. Thus, due to the splitting, our main inequalities (1) and (2) allow characterizing inequalities (38) and (39). However, inversely, inequalities (38) and (39) do not help to characterize inequalities (1) and (2).

Let us also notice that when $K(\cdot,\cdot) = 1$ inequalities (38) and (39) were considered in [15–17].

3. Applications

(1) Let a function $g : I \to R$ have generalized derivatives up to nth order; $n > 1$. Let $0 \le k \le n-1$. Now we consider the inequality

$$\left\| \left\| R_{n-k}\left(\cdot,\cdot,g^{(k)}\right)\right\|_{r,w}\right\|_{q,u} \le C\left\|g^{(n)}\right\|_{p,v}, \tag{42}$$

where the inside norm $\|R_{n-k}(\cdot,\cdot,g^{(k)})\|_{r,w}$ is taken with respect to the second argument of the function R_{n-k} and the function $R_{n-k}(t,x,g^{(k)})$ is the $(n-k)$th remainder of Taylor's formula of $g^{(k)}$; that is,

$$R_{n-k}\left(t,x,g^{(k)}\right) = g^{(k)}(t) - \sum_{i=0}^{n-k-1} \frac{g^{(k+i)}(x)(t-x)^i}{i!}. \tag{43}$$

In the case $k = 0$ we have

$$R_n(t,x,g) = g(t) - \sum_{i=0}^{n-1} \frac{g^{(i)}(x)(t-x)^i}{i!}. \tag{44}$$

Moreover, $\|\cdot\|_{p,v}$ stands for

$$\|f\|_{p,v} = \left(\int_a^b v(x)\left|f(x)\right|^p dx\right)^{1/p}. \tag{45}$$

By integration by parts it is easy to see that for $x > t$ we have

$$R_{n-k}\left(t,x,g^{(k)}\right) = \frac{(-1)^{n-k}}{(n-k-1)!}\int_t^x (s-t)^{n-k-1} g^{(n)}(s)\, ds := G_k^{-}\left(t,x,g^{(n)}\right). \tag{46}$$

Similarly, for $x < t$ we get

$$R_{n-k}\left(t,x,g^{(k)}\right) = \frac{1}{(n-k-1)!}\int_x^t (t-s)^{n-k-1} g^{(n)}(s)\, ds := G_k^{+}\left(t,x,g^{(n)}\right). \tag{47}$$

Therefore, inequality (42) holds if and only if the following inequalities simultaneously hold:

$$\left(\int_a^b u(x)\left(\int_a^x \left|G_k^-\left(t,x,g^{(n)}\right)\right|^r w(t)\,dt\right)^{q/r} dx\right)^{1/q} \le C\left(\int_a^b v(x)\left|g^{(n)}(x)\right|^p dx\right)^{1/p},$$
$$\left(\int_a^b u(x)\left(\int_x^b \left|G_k^+\left(t,x,g^{(n)}\right)\right|^r w(t)\,dt\right)^{q/r} dx\right)^{1/q} \le C\left(\int_a^b v(x)\left|g^{(n)}(x)\right|^p dx\right)^{1/p}. \quad (48)$$

Thus, if we denote $E_k^\pm := E^\pm$ and $F_k^\pm := F^\pm$ when $K(s,t) = (s-t)^{n-k-1}$, from Theorems 2, 3, 5, and 6 we have the following.

Theorem 9. *Let $0 \le k \le n-1$. Let $1 < p \le q < \infty$ and $1 < r < \infty$. Inequality (42) holds if and only if $E_k = \max\{E_k^-, E_k^+\} < \infty$. Moreover, $E_k \approx C$, where C is the best constant in (42).*

Theorem 10. *Let $0 \le k \le n-1$. Let $0 < q < p < \infty$, $p > 1$, and $1 < r < \infty$. Inequality (42) holds if $F_k = \max\{F_k^-, F_k^+\} < \infty$. Moreover, $C \ll F_k$, where C is the best constant in (42).*

Remark 11. If inequality (42) holds, then the inequalities

$$\left(\int_a^b u(x)\left(\int_a^b \left|g^{(n-1)}(x) - g^{(n-1)}(t)\right|^r w(t)\,dt\right)^{q/r} dx\right)^{1/q} \le C\left(\int_a^b v(x)\left|g^{(n)}(x)\right|^p dx\right)^{1/p},$$
$$\left(\int_a^b u(x)\left(\int_a^b \left|g^{(k)}(x) - g^{(k)}(t)\right|^r w(t)\,dt\right)^{q/r} dx\right)^{1/q} \le C\left(\left(\int_a^b v(x)\left|g^{(n)}(x)\right|^p dx\right)^{1/p} + \sum_{i=k+1}^{n-1}\left(\int_a^b u(x)\left(\int_a^b \left|g^{(i)}(t)(x-t)^{i-k}\right|^r w(t)\,dt\right)^{q/r} dx\right)^{1/q}\right), \quad 0 \le k < n-1, \quad (49)$$

also hold.

(2) In this part of the paper we investigate the inequality

$$\left(\int_a^b u(x)\left(\int_a^b \left|g^{(k)}(x) - g^{(k)}(t)\right|^r w(t)\,dt\right)^{q/r} dx\right)^{1/q} \le C\left(\int_a^b v(x)\left|g^{(n)}(x)\right|^p dx\right)^{1/p} \quad (50)$$

for $0 \le k < n-1$ with the conditions

$$\lim_{t\to a^+} g^{(i)}(t) = g^{(i)}(a) = 0, \quad \lim_{t\to b^-} g^{(i)}(t) = g^{(i)}(b) = 0 \quad \text{for } i = k, k+1, \ldots, n-1. \quad (51)$$

When $r = q$ inequality (50) turns to the inequality

$$\left(\int_a^b \int_a^b \left|g^{(k)}(x) - g^{(k)}(t)\right|^q u(x)\,w(t)\,dt\,dx\right)^{1/q} \le C\left(\int_a^b v(x)\left|g^{(n)}(x)\right|^p dx\right)^{1/p}. \quad (52)$$

Characterization of inequality (52) with conditions (51) is associated with open problem 2 in book [13, page 297].

First, we consider the inequality

$$\left(\int_a^b u(x)\left(\int_a^x \left|g^{(k)}(x) - g^{(k)}(t)\right|^r w(t)\,dt\right)^{q/r} dx\right)^{1/q} \le C\left(\int_a^b v(x)\left|g^{(n)}(x)\right|^p dx\right)^{1/p} \quad (53)$$

for $0 \le k < n-1$ with the conditions

$$\lim_{t\to b^-} g^{(i)}(t) = g^{(i)}(b) = 0 \quad \text{for } i = k, k+1, \ldots, n-1. \quad (54)$$

Let $g^{(n)}(t) = f(t)$; then we have

$$g^{(k)}(t) = \frac{(-1)^{n-k}}{(n-k-1)!}\int_t^b (s-t)^{n-k-1} f(s)\,ds, \quad k = 0, 1, \ldots, n-1. \quad (55)$$

Therefore, for $x > t$,

$$g^{(k)}(t) - g^{(k)}(x) = \frac{(-1)^{n-k}}{(n-k-1)!}\left[\int_t^b (s-x)^{n-k-1} f(s)\,ds - \int_x^b (s-t)^{n-k-1} f(s)\,ds\right]$$

$$= \frac{(-1)^{n-k}}{(n-k-1)!} \left[\int_t^x (s-t)^{n-k-1} f(s)\, ds + \int_x^b \left[(s-t)^{n-k-1} - (s-x)^{n-k-1} \right] f(s)\, ds \right]. \quad (56)$$

In the case $n > 2$ and $k = n-2$ we have

$$g^{(n-2)}(t) - g^{(n-2)}(x) = \int_t^x (s-t) f(s)\, ds + (x-t) \int_x^b f(s)\, ds. \quad (57)$$

In the case $n > 2$ and $0 \le k < n-2$ for $s \ge x \ge t$ we use the following relation:

$$(s-t)^{n-k-1} - (s-x)^{n-k-1} \approx (x-t) \left[(s-x)^{n-k-2} + (x-t)^{n-k-2} \right]. \quad (58)$$

From (56), (57), and (58) we have

$$g^{(k)}(x) - g^{(k)}(t) \approx \int_t^x (s-t)^{n-k-1} f(s)\, ds + (x-t)^{n-k-1} \int_x^b f(s)\, ds + \gamma(k,n)(x-t) \int_x^b (s-t)^{n-k-2} f(s)\, ds, \quad (59)$$

where $\gamma(k,n) = 0$ when $k = n-2$ and $\gamma(k,n) = 1$ when $0 \le k < n-2$.

From (59) we obtain

$$\left(\int_a^x \left| g^{(k)}(x) - g^{(k)}(t) \right|^r w(t)\, dt \right)^{1/r} \approx \left(\int_a^x \left| \int_t^x (s-t)^{n-k-1} f(s)\, ds \right|^r w(t)\, dt \right)^{1/r} + \left(\int_a^x (x-t)^{r(n-k-1)} w(t)\, dt \right)^{1/r} \left| \int_x^b f(s)\, ds \right| + \gamma(k,n) \left(\int_a^x (x-t)^r w(t)\, dt \right)^{1/r} \cdot \left| \int_x^b (s-t)^{n-k-2} f(s)\, ds \right|. \quad (60)$$

Consequently, the validity of inequality (53) is equivalent to the simultaneous validity of the following inequalities:

$$\left(\int_a^b u(x) \cdot \left(\int_a^x \left(\int_t^x (s-t)^{n-k-1} f(s)\, ds \right)^r w(t)\, dt \right)^{q/r} dx \right)^{1/q} \le C_1 \left(\int_a^b v(x) f^p(x)\, dx \right)^{1/p}, \quad (61)$$

$$\left(\int_a^b W_m^-(x) \left(\int_x^b (s-x)^{m-k} f(s)\, ds \right)^q dx \right)^{1/q} \le C_2 \left(\int_a^b v(x) f^p(x)\, dx \right)^{1/p} \quad (62)$$

for $m = k$ and $m = n-2$, where $W_k^-(x) = u(x)(\int_a^x (x-t)^{r(n-k-1)} w(t) dt)^{1/r}$, $0 \le k < n-1$.

Inequality (61) can be characterized by Theorems 2 and 5 when $K(s,t) = (s-t)^{n-k-1}$.

Inequality (62) can be characterized by Theorem A if we denote $A_{q,m}^-(a,b) := A^-(a,b)$ and $B_{q,m}^-(a,b) := B^-(a,b)$, where we replace $K(s,t)$ by $(s-t)^{m-k}$, w by W_m^-, and r by q. Then by Theorem A inequality (62) is valid if and only if $A_{q,m}^-(a,b) < \infty$ for $1 < p \le q < \infty$ and $B_{q,m}^-(a,b) < \infty$ for $1 < q < p < \infty$.

Thus, the following hold.

Proposition 12. *Let $n \ge 2$ and $0 \le k < n-1$. Let $1 < p \le q < \infty$ and $1 < r < \infty$. Suppose that g satisfies condition (54). Then inequality (53) holds if and only if $D_k^- = \max\{\gamma(k,n) A_{q,n-2}^-(a,b), A_{q,k}^-(a,b), E_k^-\} < \infty$. Moreover, $D_k^- \approx C$, where C is the best constant in (53).*

Proposition 13. *Let $n \ge 2$ and $0 \le k < n-1$. Let $1 < q < p < \infty$ and $1 < r < \infty$. Suppose that g satisfies condition (54). Then inequality (53) holds if $M_k^- = \max\{\gamma(k,n) B_{q,n-2}^-(a,b), B_{q,k}^-(a,b), F_k^-\} < \infty$. Moreover, $M_k^- \approx C$, where C is the best constant in (53).*

A similar result can be written for the inequality

$$\left(\int_a^b u(x) \left(\int_x^b \left| g^{(k)}(t) - g^{(k)}(x) \right|^r w(t)\, dt \right)^{q/r} dx \right)^{1/q} \le C \left(\int_a^b v(x) \left| g^{(n)}(x) \right|^p dx \right)^{1/p} \quad (63)$$

with the conditions

$$\lim_{t \to a^+} g^{(i)}(t) = g^{(i)}(a) = 0 \quad \text{for } i = k, k+1, \ldots, n-1. \quad (64)$$

Here we need the following notations: $A_{q,m}^+(a,b) := A^+(a,b)$ and $B_{q,m}^+(a,b) := B^+(a,b)$, where we replace $K(s,t)$ by

$(s-t)^{m-k}$, w by W_m^+ for $m = k$ and $m = n-2$, where $W_k^+(x) = u(x)(\int_x^b (t-x)^{r(n-k-1)} w(t)dt)^{1/r}$, $0 \le k < n-1$, and r by q.

Proposition 14. *Let $n \ge 2$ and $0 \le k < n-1$. Let $1 < p \le q < \infty$ and $1 < r < \infty$. Suppose that g satisfies condition (64). Then inequality (63) holds if and only if $D_k^+ = \max\{\gamma(k,n)A_{q,n-2}^+(a,b), A_{q,k}^+(a,b), E_k^+\} < \infty$. Moreover, $D_k^+ \approx C$, where C is the best constant in (63).*

Proposition 15. *Let $n \ge 2$ and $0 \le k < n-1$. Let $1 < q < p < \infty$ and $1 < r < \infty$. Suppose that g satisfies condition (64). Then inequality (63) holds if $M_k^+ = \max\{\gamma(k,n)B_{q,n-2}^+(a,b), B_{q,k}^+(a,b), F_k^+\} < \infty$. Moreover, $M_k^+ \approx C$, where C is the best constant in (63).*

The validity of inequality (50) with conditions (51) is equivalent to the simultaneous validity of inequalities (53) with (54) and (63) with (64). Therefore, from Propositions 12, 13, 14, and 15 we have the following.

Theorem 16. *Let $0 \le k < n-1$. Let $1 < p \le q < \infty$ and $1 < r < \infty$. Suppose that g satisfies conditions (51). Then inequality (50) holds if and only if $D_k = \max\{D_k^-, D_k^+\} < \infty$. Moreover, $D_k \approx C$, where C is the best constant in (50).*

Theorem 17. *Let $0 \le k < n-1$. Let $1 < q < p < \infty$ and $1 < r < \infty$. Suppose that g satisfies conditions (51). Then inequality (50) holds if $M_k = \max\{M_k^-, M_k^+\} < \infty$. Moreover, $M_k \approx C$, where C is the best constant in (50).*

Competing Interests

The authors declare that they have no competing interests.

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