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# WEIGHTED FRIEDRICHS INEQUALITIES IN AMALGAMS

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## 0. INTRODUCTION

**0.1.** A Lebesgue measurable function $f$ defined on $(0,\infty)$ is said to belong to the *weighted amalgam of $L^p$ and $\ell^{\bar p}$*, with weight $w$, if

$$\|f\|_{w,p,\bar p} = \left\{ \sum_{n=0}^{\infty} \left[ \int_n^{n+1} w(x)|f(x)|^p \,\mathrm{d}x \right]^{\bar p/p} \right\}^{1/\bar p} < \infty \tag{0.1}$$

with $1 < p,\ \bar p < \infty$. We shall write

$$f \in L^p(\ell^{\bar p}, w). \tag{0.2}$$

In Carton-Lebrun, Heinig and Hofmann [1], the *Hardy inequality in amalgams* is studied,

$$\|Hf\|_{u,q,\bar q} \leqslant C\|f\|_{v,p,\bar p} \tag{0.3}$$

with $u$, $v$ weight functions and $H$ the Hardy operator

$$(Hf)(x) = \int_0^x f(t)\,\mathrm{d}t \quad \Big(\text{or} \quad \int_x^\infty f(t)\,\mathrm{d}t\Big).$$

The inequality (0.3) can be rewritten in the "differential" form

$$\|F\|_{u,q,\bar q} \leqslant C\|F'\|_{v,p,\bar p} \tag{0.4}$$

where $F(0) = 0$ (or $F(\infty) = 0$).

The aim of this note is twofold:

(i) to extend inequality (0.4) to higher order derivatives,

$$\|F\|_{u,q,\bar q} \leqslant C\|F^{(k)}\|_{v,p,\bar p},\ k>1, \tag{0.5}$$

with appropriate "boundary conditions" on $F$;

(ii) to extend inequality (0.4) to functions of several variables, i.e., to derive for some functions $f=f(x)$, $x\in\mathbf{R}^N$, an inequality of the form

$$\|f\|_{u,q,\bar q} \leqslant C\|\nabla f\|_{v,p,\bar p} \tag{0.6}$$

where now

$$\|g\|_{w,p,\bar p} = \left\{\sum_{n=0}^{\infty}\left[\int\limits_{n<|x|<n+1} w(x)|g(x)|^p\,\mathrm{d}x\right]^{\bar p/p}\right\}^{1/\bar p}. \tag{0.7}$$

Let us point out that inequality (0.5) generalizes to amalgams the "classical" higher order inequality investigated in Kufner and Heinig [2] while (0.6) generalizes to amalgams an analogous result of Sinnamon [3]. The inequalities (0.5) and (0.6) will be called *Friedrichs inequalities in (weighted) amalgams.*

We start with two theorems from [1] which will be substantially used:

**0.2. Theorem.** *Suppose $u$, $v$ are weight functions and $f\geqslant 0$. Let $1<q,\ \bar q<\infty$, $1<p\leqslant q$, $1<\bar p\leqslant \bar q$. Then there is a constant $C>0$ such that*

$$\left\{\sum_{n=0}^{\infty}\left[\int_n^{n+1} u(x)\left(\int_0^x f(t)\,\mathrm{d}t\right)^q\mathrm{d}x\right]^{\bar q/q}\right\}^{1/\bar q} \leqslant C\left\{\sum_{n=0}^{\infty}\left[\int_n^{n+1} v(x)f^p(x)\,\mathrm{d}x\right]^{\bar p/p}\right\}^{1/\bar p} \tag{0.8}$$

*if and only if*

$$\sup_{m\in\mathbf{N}}\left\{\sum_{n=m}^{\infty}\left(\int_n^{n+1} u(x)\,\mathrm{d}x\right)^{\bar q/q}\right\}^{1/\bar q}\left\{\sum_{n=0}^{m}\left(\int_n^{n+1} v^{1-p'}(x)\,\mathrm{d}x\right)^{\bar p'/p'}\right\}^{1/\bar p'} \equiv C_1<\infty \tag{0.9}$$

*and*

$$\sup_{m\in\mathbf{N}}\ \sup_{m<s<m+1}\left(\int_s^{m+1} u(x)\,\mathrm{d}x\right)^{1/q}\left(\int_m^s v^{1-p'}(x)\,\mathrm{d}x\right)^{1/p'} \equiv C_2<\infty. \tag{0.10}$$

**0.3. Theorem.** *Suppose $u$, $v$ are weight functions and $f \geqslant 0$. Let $1 < q, \bar{q} < \infty$, $1 < p \leqslant q$, $1 < \bar{p} \leqslant \bar{q}$. Then there is a constant $C > 0$ such that*

$$\left\{\sum_{n=0}^{\infty}\left[\int_n^{n+1} u(x)\left(\int_x^{\infty} f(t)\,\mathrm{d}t\right)^q \mathrm{d}x\right]^{\bar{q}/q}\right\}^{1/\bar{q}}$$

$$\leqslant C\left\{\sum_{n=0}^{\infty}\left[\int_n^{n+1} v(x) f^p(x)\,\mathrm{d}x\right]^{\bar{p}/p}\right\}^{1/\bar{p}} \tag{0.11}$$

*if and only if*

$$\sup_{m\in\mathbf{N}}\left\{\sum_{n=0}^{m}\left(\int_n^{n+1} u(x)\,\mathrm{d}x\right)^{\bar{q}/q}\right\}^{1/\bar{q}}\left\{\sum_{n=m}^{\infty}\left(\int_n^{n+1} v^{1-p'}(x)\,\mathrm{d}x\right)^{\bar{p}'/p'}\right\}^{1/\bar{p}'} \equiv C_3 < \infty \tag{0.12}$$

*and*

$$\sup_{m\in\mathbf{N}}\ \sup_{m<s<m+1}\left(\int_m^{s} u(x)\,\mathrm{d}x\right)^{1/q}\left(\int_s^{m+1} v^{1-p'}(x)\,\mathrm{d}x\right)^{1/p'} \equiv C_4 < \infty. \tag{0.13}$$

**0.4.** Notice that for $p > 0$, $p \neq 1$,

$$p' = \frac{p}{p-1}. \tag{0.14}$$

In the sequel, we will frequently use the formulas

$$p = \frac{p'}{p'-1}, \quad \frac{p}{p'} = p - 1 = \frac{1}{p'-1}, \quad p(p'-1) = p' \quad \text{etc.} \tag{0.15}$$

and also the inequality

$$\left(\sum_n a_n^r\right)^{1/r} \leqslant \sum_n a_n, \quad \text{i.e.,} \quad \sum_n a_n^r \leqslant \left(\sum_n a_n\right)^r, \tag{0.16}$$

which holds for $r > 1$, $a_n \geqslant 0$.

**0.5.** Remarks. (i) Theorems 0.2 and 0.3 have been formulated only for $p$, $q$, $\bar{p}$, $\bar{q}$ such that

$$1 < p \leqslant q < \infty, \ 1 < \bar{p} \leqslant \bar{q} < \infty. \tag{0.17}$$

In [1] also the case

$$1 < q < p < \infty,\ 1 < \bar{q} < \bar{p} < \infty \tag{0.18}$$

is investigated and it is shown that then the following two conditions are *sufficient* for (0.8) to hold:

$$\left(\sum_{k=0}^{\infty}\left[\sum_{n=k}^{\infty}\left(\int_n^{n+1} u(x)\,\mathrm{d}x\right)^{\bar{q}/q}\right]^{\bar{r}/\bar{q}}\left[\sum_{n=0}^{k}\left(\int_n^{n+1} v^{1-p'}(x)\,\mathrm{d}x\right)^{\bar{p}'/p'}\right]^{\bar{r}/\bar{q}'} \times\left(\int_k^{k+1} v^{1-p'}(x)\,\mathrm{d}x\right)^{\bar{p}'/p'}\right)^{1/\bar{r}} \equiv C_5 < \infty \tag{0.19}$$

and

$$\sum_{n=0}^{\infty}\left[\int_n^{n+1}\left(\int_s^{n+1} u(x)\,\mathrm{d}x\right)^{r/q}\left(\int_n^{s} v^{1-p'}(x)\,\mathrm{d}x\right)^{r/q'} v^{1-p'}(s)\,\mathrm{d}s\right]^{1/r} \equiv C_6 < \infty, \tag{0.20}$$

where

$$\frac{1}{r} = \frac{1}{q} - \frac{1}{p}, \qquad \frac{1}{\bar{r}} = \frac{1}{\bar{q}} - \frac{1}{\bar{p}}.$$

(ii) It is quite evident how to formulate *sufficient* conditions for (0.11) to hold when (0.18) takes places.

(iii) Besides (0.17) and (0.18), there are also other possibilities as far as concerns the mutual relations between the parameters $p$, $q$, $\bar{p}$, $\bar{q}$. Let us mention the case

$$1 < q < p < \infty, \qquad 1 < \bar{p} \leqslant \bar{q} < \infty, \tag{0.21}$$

which will be needed in the sequel. We will formulate the corresponding result and, for completeness, give here its proof, which in fact follows the ideas of the proof of Theorems 0.2 and 0.3 in [1].

**0.6. Theorem.** *Suppose $u$, $v$ are weight functions and $f \geqslant 0$. Let $p$, $q$, $\bar{p}$, $\bar{q}$ satisfy* (0.21). *Then there is a constant $C > 0$ such that* (0.8) *holds if and only if the conditions* (0.9) *and*

$$\sup_{n\in\mathbb{N}}\left[\int_n^{n+1}\left(\int_s^{n+1} u(x)\,\mathrm{d}x\right)^{r/q}\left(\int_n^{s} v^{1-p'}(x)\,\mathrm{d}x\right)^{r/q'} v^{1-p'}(s)\,\mathrm{d}s\right]^{1/r} \equiv C_7 < \infty \tag{0.22}$$

*are fulfilled where*

$$\frac{1}{r} = \frac{1}{q} - \frac{1}{p}. \tag{0.23}$$

P r o o f. (A) *Sufficiency of conditions* (0.9) *and* (0.22). Denote

$$S_1 = \left\{ \sum_{n=0}^{\infty} \left[ \int_n^{n+1} u(x) \left( \int_0^n f(t)\,dt \right)^q dx \right]^{\bar{q}/q} \right\}^{1/\bar{q}}, \tag{0.24}$$

$$S_2 = \left\{ \sum_{n=0}^{\infty} \left[ \int_n^{n+1} u(x) \left( \int_n^x f(t)\,dt \right)^q dx \right]^{\bar{q}/q} \right\}^{1/\bar{q}}. \tag{0.25}$$

Using the fact that $\int_0^x = \int_0^n + \int_n^x$ with $x \in (n, n+1)$, we can estimate the left hand side in (0.8) by

$$c_1(S_1 + S_2)$$

where $c_1 = 2^{1/\bar{q}}$ if $\bar{q} < q$ and $c_1 = 1$ if $\bar{q} \geqslant q$ (we apply Minkowski's inequality twice). To estimate $S_1$ we rewrite $S_1$ as

$$\begin{aligned} S_1 &= \left\{ \sum_{n=0}^{\infty} \left[ \int_n^{n+1} u(x) \left( \sum_{l=0}^{n-1} \int_l^{l+1} f(t)\,dt \right)^q dx \right]^{\bar{q}/q} \right\}^{1/\bar{q}} \\ &= \left\{ \sum_{n=0}^{\infty} \left( \int_n^{n+1} u(x)\,dx \right)^{\bar{q}/q} \left( \sum_{l=0}^{n-1} \int_l^{l+1} f(t)\,dt \right)^{\bar{q}} \right\}^{1/\bar{q}} = \\ &= \left\{ \sum_{n=0}^{\infty} U_n \left( \sum_{l=0}^{n-1} a_l \right)^{\bar{q}} \right\}^{1/\bar{q}}, \end{aligned} \tag{0.26}$$

where

$$U_n = \left( \int_n^{n+1} u(x)\,dx \right)^{\bar{q}/q}, \quad a_l = \int_l^{l+1} f(t)\,dt. \tag{0.27}$$

The discrete Hardy inequality yields

$$\left\{ \sum_{n=0}^{\infty} U_n \left( \sum_{l=0}^{n} a_l \right)^{\bar{q}} \right\}^{1/\bar{q}} \leqslant c_2 \left\{ \sum_{n=0}^{\infty} V_n a_n^{\bar{p}} \right\}^{1/\bar{p}} \tag{0.28}$$

for $1 < \bar{p} \leqslant \bar{q} < \infty$ provided

$$(0.29) \qquad \sup_{n\in\mathbf{N}} \Big(\sum_{m=n}^{\infty} U_m\Big)^{1/\bar{q}} \Big(\sum_{n=0}^{n} V_m^{1-\bar{p}'}\Big)^{1/\bar{p}'} \equiv c_3 < \infty$$

(see Andersen and Heinig [4], Theorem 4.1). Using the second formula in (0.27) and Hölder's inequality, we obtain that

$$\sum_{n=0}^{\infty} V_n a_n^{\bar{p}} = \sum_{n=0}^{\infty} V_n \Big(\int_n^{n+1} f(t)\,\mathrm{d}t\Big)^{\bar{p}} = \sum_{n=0}^{\infty} V_n \Big(\int_n^{n+1} v^{1/p}(t) f(t) v^{-1/p}(t)\,\mathrm{d}t\Big)^{\bar{p}}$$

$$(0.30) \qquad \leqslant \sum_{n=0}^{\infty} V_n \Big(\int_n^{n+1} v(t) f^p(t)\,\mathrm{d}t\Big)^{\bar{p}/p} \Big(\int_n^{n+1} v^{1-p'}(t)\,\mathrm{d}t\Big)^{\bar{p}/p'}.$$

If we take

$$(0.31) \qquad V_n = \Big(\int_n^{n+1} v^{1-p'}(t)\,\mathrm{d}t\Big)^{-\bar{p}/p'},$$

then the formulas (0.26), (0.28) and (0.30) imply

$$(0.32) \qquad S_1 \leqslant c_2 \Big\{ \sum_{n=0}^{\infty} \Big(\int_n^{n+1} v(t) f^p(t)\,\mathrm{d}t\Big)^{\bar{p}/p} \Big\}^{1/\bar{p}};$$

moreover, due to (0.27) and (0.31), condition (0.29) is exactly the condition (0.9).

To estimate $S_2$, we use the "classical" Hardy inequality

$$(0.33) \qquad \Big[\int_n^{n+1} u(x) \Big(\int_n^{x} f(t)\,\mathrm{d}t\Big)^{q} \mathrm{d}x\Big]^{1/q} \leqslant c_{3,n} \Big[\int_n^{n+1} v(x) f^p(x)\,\mathrm{d}x\Big]^{1/p}$$

which holds for $1 < q < p < \infty$ and $n = 0, 1, \ldots$ provided

$$(0.34) \qquad \Big[\int_n^{n+1} \Big(\int_s^{n+1} u(x)\,\mathrm{d}x\Big)^{r/q} \Big(\int_n^{s} v^{1-p'}(x)\,\mathrm{d}x\Big)^{r/q'} v^{1-p'}(s)\,\mathrm{d}s\Big]^{1/r} = c_{4,n} < \infty$$

with $1/r = 1/q - 1/p$ (see, e.g., Opic and Kufner [5], Theorem 1.15). Condition (0.22) implies that (0.34) holds and that $c_{3,n}$ in (0.33) can be choosen independent on $n$, $c_{3,n} = c_3$. Using (0.16) with $r = \bar{q}/\bar{p} \geqslant 1$, we obtain from (0.25) and (0.33) that

$$S_2 \leqslant c_3\Big\{\sum_{n=0}^{\infty}\Big[\int_n^{n+1} v(x)f^p(x)\,\mathrm{d}x\Big]^{\bar{q}/p}\Big\}^{1/\bar{q}} = c_3\Big\{\sum_{n=0}^{\infty}\Big[\int_n^{n+1} v(x)f^p(x)\,\mathrm{d}x\Big]^{\bar{p}/p\cdot\bar{q}/\bar{p}}\Big\}^{\bar{p}/\bar{q}\cdot 1/\bar{p}}$$

$$\leqslant c_3\Big\{\sum_{n=0}^{\infty}\Big[\int_n^{n+1} v(x)f^p(x)\,\mathrm{d}x\Big]^{\bar{p}/p}\Big\}^{1/\bar{p}}.$$

This formula together with (0.32) yields (0.8).

(B) *Necessity of conditions* (0.9) *and* (0.22). Suppose that (0.8) holds. Since for $f \geqslant 0$ and $x \in (n, n+1)$

$$\int_0^x f(t)\,\mathrm{d}t = \sum_{l=0}^{n-1}\int_l^{l+1} f(t)\,\mathrm{d}t + \int_n^x f(t)\,\mathrm{d}t \geqslant \sum_{l=0}^{n-1}\int_l^{l+1} f(t)\,\mathrm{d}t = \Delta_n \tag{0.35}$$

the left hand side of (0.8) is not smaller than

$$\Big\{\sum_{n=0}^{\infty}\Delta_n^{\bar{q}}\Big(\int_n^{n+1} u(x)\,\mathrm{d}x\Big)^{\bar{q}/q}\Big\}^{1/\bar{q}} \geqslant \Big\{\sum_{n=m}^{\infty}\Delta_n^{\bar{q}}\Big(\int_n^{n+1} u(x)\,\mathrm{d}x\Big)^{\bar{q}/q}\Big\}^{1/\bar{q}} \tag{0.36}$$

for every fixed $m \in \mathbf{N}$. Let us fix $m$ and choose $f \geqslant 0$ such that

$$f(x) = 0 \quad \text{for } x \geqslant m. \tag{0.37}$$

Then it is $\Delta_n = \Delta_m$ for $n \geqslant m$ and the right hand side of (0.36) is not smaller than

$$\Delta_m\Big\{\sum_{n=m}^{\infty}\Big(\int_n^{n+1} u(x)\,\mathrm{d}x\Big)^{\bar{q}/q}\Big\}^{1/\bar{q}}. \tag{0.38}$$

Moreover, taking

$$f(x) = a_l v^{1-p'}(x) \quad \text{for } x \in (l, l+1),\ l = 0, 1, \ldots, m-1, \tag{0.39}$$

where $a_l$ are arbitrary non-negative real numbers, then

$$\Delta_m = \sum_{l=0}^{m-1}\int_l^{l+1} a_l v^{1-p'}(x)\,\mathrm{d}x$$

$$= \sum_{l=0}^{m-1}\Big[a_l^p\int_l^{l+1} v^{1-p'}(x)\,\mathrm{d}x\Big]^{1/p}\Big[\int_l^{l+1} v^{1-p'}(x)\,\mathrm{d}x\Big]^{1/p'} = \sum_{l=0}^{m-1}\alpha_l\beta_l$$

where

$$(0.40)\qquad \begin{aligned} \alpha_l &= \left[a_l^p \int_l^{l+1} v^{1-p'}(x)\,\mathrm{d}x\right]^{1/p}, \\ \beta_l &= \left[\int_l^{l+1} v^{1-p'}(x)\,\mathrm{d}x\right]^{1/p'}, \qquad l = 0, 1, \ldots, m-1. \end{aligned}$$

If we denote

$$(0.41)\qquad K_m = \left\{\sum_{n=m}^{\infty}\left(\int_n^{n+1} u(x)\,\mathrm{d}x\right)^{\bar{q}/q}\right\}^{1/\bar{q}},$$

then we obtain that (0.38) is equal to

$$(0.42)\qquad K_m \sum_{l=0}^{m-1} \alpha_l \beta_l .$$

The right hand side in (0.8) attains for $f$ defined in (0.37) and (0.39) the form

$$C\left\{\sum_{n=0}^{m-1}\left(\int_n^{n+1} v(x)\left[a_n v^{1-p'}(x)\right]^p \mathrm{d}x\right)^{\bar{p}/p}\right\}^{1/\bar{p}}$$

$$= C\left\{\sum_{n=0}^{m-1}\left[a_n^p \int_n^{n+1} v^{1-p'}(x)\,\mathrm{d}x\right]^{\bar{p}/p}\right\}^{1/\bar{p}} = C\left(\sum_{n=0}^{m-1} \alpha_n^{\bar{p}}\right)^{1/\bar{p}}$$

with $\alpha_n$ from (0.40), and from (0.8), it follows due to (0.42) that

$$\sum_{l=0}^{m-1} \alpha_l \beta_l \leqslant \frac{C}{K_m}\left(\sum_{l=0}^{m-1} \alpha_l^{\bar{p}}\right)^{1/\bar{p}}.$$

But the vector $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1})$ is arbitrary, since the numbers $a_l$ have been chosen arbitrarily, and consequently the vector $(\beta_0, \beta_1, \ldots, \beta_{m-1})$ belongs to $\ell^{\bar{p}'}$ and, moreover,

$$\left(\sum_{l=0}^{m-1} \beta_l^{\bar{p}'}\right)^{1/\bar{p}'} \leqslant \frac{C}{K_m}, \quad \text{i.e.} \quad K_m\left(\sum_{l=0}^{m-1} \beta_l^{\bar{p}'}\right)^{1/\bar{p}'} \leqslant C.$$

But this last inequality implies (0.9) in view of the notation from (0.41) and (0.40). Consequently, the condition (0.9) is necessary.

Now, take

$$f(x)=\begin{cases} g(x) \text{ for } x\in(m,m+1),\\ 0 \quad \text{otherwise}\end{cases}$$

in (0.8) with $m\in\mathbb{N}$ arbitrary but fixed. Then (0.8) implies that

$$\begin{aligned}
C\left\{\left[\int_m^{m+1} v(x)g^p(x)\,\mathrm{d}x\right]^{\tilde p/p}\right\}^{1/\tilde p} &= C\left[\int_m^{m+1} v(x)g^p(x)\,\mathrm{d}x\right]^{1/p}\\
&\geqslant \left\{\sum_{n=0}^{\infty}\left[\int_n^{n+1} u(x)\Big(\int_0^x f(t)\,\mathrm{d}t\Big)^q\mathrm{d}x\right]^{\tilde q/q}\right\}^{q/\tilde q}\\
&\geqslant \left[\int_m^{m+1} u(x)\Big(\int_0^x f(t)\,\mathrm{d}t\Big)^q\mathrm{d}x\right]^{1/q} = \left[\int_m^{m+1} u(x)\Big(\int_m^x g(t)\,\mathrm{d}t\Big)^q\mathrm{d}x\right]^{1/q},
\end{aligned}$$

that is

$$\left(\int_m^{m+1} u(x)\Big(\int_m^x g(t)\,\mathrm{d}t\Big)^q\mathrm{d}x\right)^{1/q} \leqslant C\left[\int_m^{m+1} v(x)g^p(x)\,\mathrm{d}x\right]^{1/p}.$$

But this is the Hardy inequality on $(m,m+1)$ and its validity implies that (0.34) holds with $c_{4,m}\leqslant C$. Since $m$ was arbitrary, we have shown that also condition (0.22) is necessary. □

## 1. The case of higher order derivatives

**1.1.** For a fixed integer $k\geqslant 2$, write

$$k=k_1+k_2,\ k_i\in\mathbb{N}, \tag{1.1}$$

and denote by $AC_{1,2}^{(k-1)}$ the set of all functions $F$ defined on $(0,\infty)$ whose $(k-1)$-st derivative is absolutely continuous and which satisfy the "boundary conditions"

$$\begin{aligned} F(0)=F'(0)=\ldots=F^{(k_1-1)}(0)=0,\\ F^{(k_1)}(\infty)=F^{(k_1+1)}(\infty)=\ldots=F^{(k-1)}(\infty)=0.\end{aligned} \tag{1.2}$$

Consequently, we have $k_1$ conditions on the left end of the interval $(0,\infty)$ and $k_2$ conditions on the right end.

**1.2. Theorem.** *Suppose $u$, $v$ are weight functions and $F \in AC^{(k-1)}_{1,2}$. Let $1 < q$, $\bar{q} < \infty$, $1 < p \leqslant q$, $1 < \bar{p} \leqslant \bar{q}$. Then the Friedrichs inequality in amalgams,*

$$\left\{ \sum_{n=0}^{\infty} \left[ \int_n^{n+1} u(x) |F(x)|^q \, dx \right]^{\bar{q}/q} \right\}^{1/\bar{q}} \leqslant C \left\{ \sum_{n=0}^{\infty} \left[ \int_n^{n+1} v(x) |F^{(k)}(x)|^p \, dx \right]^{\bar{p}/p} \right\}^{1/\bar{p}} \tag{1.3}$$

*holds with a constant $C$ independent of $F$ if and only if the following four conditions are fulfilled:*

$$\sup_{m \in \mathbb{N}} \left\{ \sum_{n=m+1}^{\infty} \left( \int_n^{n+1} u(x) x^{(k_1-1)q} \, dx \right)^{\bar{q}/q} \right\}^{1/\bar{q}} \times \left\{ \sum_{n=0}^{\infty} \left( \int_n^{n+1} v^{1-p'}(x) x^{k_2 p'} \, dx \right)^{\bar{p}'/p'} \right\}^{1/\bar{p}'} < \infty, \tag{1.4}$$

$$\sup_{m \in \mathbb{N}} \left\{ \sum_{n=0}^{m} \left( \int_n^{n+1} u(x) x^{k_1 q} \, dx \right)^{\bar{q}/q} \right\}^{1/\bar{q}} \times \left\{ \sum_{n=m+1}^{\infty} \left( \int_n^{n+1} v^{1-p'}(x) x^{(k_2-1)p'} \, dx \right)^{\bar{p}'/p'} \right\}^{1/\bar{p}'} < \infty, \tag{1.5}$$

$$\sup_{m \in \mathbb{N}} \sup_{m<s<m+1} \left( \int_s^{m+1} u(x) x^{(k_1-1)q} \, dx \right)^{1/q} \left( \int_m^{s} v^{1-p'}(x) x^{k_2 p'} \, dx \right)^{1/p'} < \infty, \tag{1.6}$$

$$\sup_{m \in \mathbb{N}} \sup_{m<s<m+1} \left( \int_m^{s} u(x) x^{k_1 q} \, dx \right)^{1/q} \left( \int_s^{m+1} v^{1-p'}(x) x^{(k_2-1)p'} \, dx \right)^{1/p'} < \infty. \tag{1.7}$$

Proof. According to [2, Theorem 1], $F \in AC^{(k-1)}_{1,2}$ can be written in the form

$$F(x) = \int_0^x K_1(x,t) f(t) \, dt + \int_x^{\infty} K_2(x,t) f(t) \, dt \equiv (Tf)(x) \tag{1.8}$$

where

$$F^{(k)}(x) = f(x). \tag{1.9}$$

Consequently, instead of the inequality (1.3) [which in fact is the inequality (0.5)] we can investigate the inequality

$$\|Tf\|_{u,q,\bar q} \leqslant C\|f\|_{v,p,\bar p}. \tag{1.10}$$

We have

$$K_1(x,t) \approx x^{k_1-1}t^{k_2} \quad \text{for} \quad 0 < t < x < \infty \tag{1.11}$$

and

$$K_2(x,t) \approx x^{k_1}t^{k_2-1} \quad \text{for} \quad 0 < x < t < \infty. \tag{1.12}$$

Here $A(x,t) \approx B(x,t)$ means that there are two positive constants $c_1$, $c_2$ such that $c_1A(x,t) \leqslant B(x,t) \leqslant c_2A(x,t)$ for $x$, $t$ from the domain of definition of $A$, $B$ (see again [2]).

Denote

$$(J_1f)(x) = \int_0^x x^{k_1-1}t^{k_2}f(t)\,\mathrm{d}t = x^{k_1-1}\int_0^x t^{k_2}f(t)\,\mathrm{d}t, \tag{1.13}$$

$$(J_2f)(x) = \int_x^\infty x^{k_1}t^{k_2-1}f(t)\,\mathrm{d}t = x^{k_1}\int_x^\infty t^{k_2-1}f(t)\,\mathrm{d}t. \tag{1.14}$$

According to Theorem 0.2, the inequality

$$\left\{\sum_{n=0}^{\infty}\left[\int_n^{n+1} u(x)(J_1f)^q(x)\,\mathrm{d}x\right]^{\bar q/q}\right\}^{1/\bar q} \leqslant C\left\{\sum_{n=0}^{\infty}\left[\int_n^{n+1} v(x)f^p(x)\,\mathrm{d}x\right]^{\bar p/p}\right\}^{1/\bar p} \tag{1.15}$$

holds if and only if conditions (1.4) and (1.6) are satisfied. Indeed: Due to (1.13), the left and right hand sides of (1.15) can be rewritten as

$$\left\{\sum_{n=0}^{\infty}\left[\int_n^{n+1} u(x)x^{(k_1-1)q}\left(\int_0^x t^{k_2}f(t)\mathrm{d}t\right)^q \mathrm{d}x\right]^{\bar q/q}\right\}^{1/\bar q}$$

and

$$\left\{\sum_{n=0}^{\infty}\left[\int_{n}^{n+1} v(x)x^{-k_2 p}\left(x^{k_2}f(x)\right)^{p}\,\mathrm{d}x\right]^{\tilde{p}/p}\right\}^{1/\tilde{p}},$$

respectively, and the inequality (1.15) is nothing but inequality (0.8) with $f(x)$ replaced by $\tilde{f}(x) = x^{k_2}f(x)$, $u(x)$ replaced by $\tilde{u}(x) = u(x)x^{(k_1-1)q}$ and $v(x)$ replaced by $\tilde{v}(x) = v(x)x^{-k_2 p}$. Now conditions (1.4) and (1.6) are the conditions (0.9), (0.10) for the weights $\tilde{u}$, $\tilde{v}$ instead of $u$, $v$.

Analogously, we can show that, according to Theorem 0.3, the inequality

$$\text{(1.16)}\quad \left\{\sum_{n=0}^{\infty}\left[\int_{n}^{n+1} u(x)(J_2 f)^{q}(x)\,\mathrm{d}x\right]^{\tilde{q}/q}\right\}^{1/\tilde{q}} \leqslant C\left\{\sum_{n=0}^{\infty}\left[\int_{n}^{n+1} v(x)f^{p}(x)\,\mathrm{d}x\right]^{\tilde{p}/p}\right\}^{1/\tilde{p}}$$

holds if and only if conditions (1.5) and (1.7) are satisfied. Indeed, due to (1.14), the inequality (1.16) is nothing but inequality (0.11) with $f(x)$, $u(x)$, $v(x)$ replaced by $f^*(x) = x^{k_2-1}f(x)$, $u^*(x) = u(x)x^{k_1 q}$, $v^*(x) = v(x)x^{-(k_2-1)p}$, respectively, and conditions (1.5), (1.7) are the conditions (0.12), (0.13) for $u^*$, $v^*$ instead of $u$, $v$.

Due to (1.8), (1.13), (1.14),

$$\|F\|_{u,q,\tilde{q}} = \|Tf\|_{u,q,\tilde{q}} \leqslant c_3\left(\|J_1 f\|_{u,q,\tilde{q}} + \|J_2 f\|_{u,q,\tilde{q}}\right)$$

and (1.3) follows from (1.15) and (1.16) due to (1.9). Consequently, the conditions (1.4)–(1.7) are *sufficient* for (1.3) to hold.

But these conditions are also *necessary*. Indeed: Since the operators $J_1$, $J_2$ from (1.13), (1.14) are positive, we have

$$(J_i f)(x) \leqslant (J_1 f)(x) + (J_2 f)(x) \leqslant c_4(Tf)(x) = c_4 F(x),\ i = 1, 2,$$

due to (1.11), (1.12). Consequently, the validity of (1.3) implies the validity of both (1.15), (1.16) which again implies that (1.4), (1.6) and (1.5), (1.7) are satisfied, respectively. □

## 2. The multidimensional case

**2.1.** Following Sinnamon [3] we introduce, for $x \in \mathbf{R}^N$, the operators $P$, $Q$ defined by

$$(Pf)(x) = \int_0^1 f(xt)\frac{\mathrm{d}t}{t}, \tag{2.1}$$

$$(Qf)(x) = \int_1^\infty f(xt)\frac{\mathrm{d}t}{t}. \tag{2.2}$$

Denoting further

$$(Rf)(x) = x \cdot \nabla f(x) = \sum_{i=1}^N x_i \frac{\partial f(x)}{\partial x_i} \tag{2.3}$$

we have

$$f = P(Rf) \quad \text{provided} \quad f(0) = 0 \tag{2.4}$$

and

$$f = Q(-Rf) \quad \text{provided} \quad f(\infty) = 0 \tag{2.5}$$

[$f(\infty) = 0$ means that $\lim_{t\to\infty} f(xt) = 0$ for every $x \in \mathbf{R}^N$]. Indeed:

$$\begin{aligned} P(Rf)(x) &= \int_0^1 (Rf)(xt)\frac{\mathrm{d}t}{t} = \int_0^1 xt \cdot \nabla f(xt)\frac{\mathrm{d}t}{t} = \int_0^1 x \cdot \nabla f(xt)\,\mathrm{d}t \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(xt)\,\mathrm{d}t = f(xt)\big|_0^1 = f(x) - f(0) = f(x), \end{aligned}$$

and analogously for $Q(-Rf)$.

**2.2.** We will deal with *radial* weights only. This means that for $x = \sigma t$, where $\sigma$, $t$ are the polar (spherical) coordinates, $\sigma \in \Sigma$ (= the unit $N$-sphere) and $t > 0$, it is

$$u(x) = u(\sigma t) = u(t), \ v(x) = v(\sigma t) = v(t).$$

**2.3.** Remark. We will consider the *N-dimensional Friedrichs inequality* in weighted amalgams in the form

$$\begin{aligned}\Big\{\sum_{k=0}^{\infty}\Big[\int\limits_{k<|x|<k+1} u(x)|f(x)|^q\,\mathrm{d}x\Big]^{\bar q/q}\Big\}^{1/\bar q}\\ \leqslant C\Big\{\sum_{k=0}^{\infty}\Big[\int\limits_{k<|x|<k+1} v(x)|x\cdot\nabla f(x)|^p\,\mathrm{d}x\Big]^{\bar p/p}\Big\}^{1/\bar p}.\end{aligned}\tag{2.6}$$

This is a generalization of the "classical" $N$-dimensional Hardy inequality

$$\Big(\int\limits_{\mathbf{R}^N} u(x)|f(x)|^q\,\mathrm{d}x\Big)^{1/q}\leqslant C\Big(\int\limits_{\mathbf{R}^N} v(x)|x\cdot\nabla f(x)|^p\,\mathrm{d}x\Big)^{1/p}\tag{2.7}$$

investigated in [3]: We obtain (2.7) from (2.6), taking $\bar q = q$ and $\bar p = p$. But in [3, Theorem 3.4] it is shown that (2.7) cannot hold if $p < q$, and consequently, we cannot expect that (2.6) will hold if we assume that $1 < p < q$, $1 < \bar p \leqslant \bar q$.

Therefore, in the next two theorems we will assume that $1 < q$, $\bar q < \infty$ and

$$1 < p = q,\ 1 < \bar p \leqslant \bar q.\tag{2.8}$$

**2.4. Theorem.** *Suppose $u$, $v$ are radial weights on $\mathbf{R}^N$, $f$ differentiable on $\mathbf{R}^N$, $f(0) = 0$. Let $p$, $\bar p$, $q$, $\bar q$ satisfy (2.8). Then the inequality (2.6) holds with a constant $C > 0$ independent of $f$ if and only if the following two conditions are satisfied:*

$$\begin{aligned}\sup_{m\in\mathbf{N}}\Big\{\sum_{k=0}^{m}\Big[\int\limits_{k<|x|<k+1} u(x)\,\mathrm{d}x\Big]^{\bar q/q}\Big\}^{1/\bar q}\\ \times\Big\{\sum_{k=m}^{\infty}\Big[\int\limits_{k<|x|<k+1} v^{1-p'}(x)|x|^{-Np'}\,\mathrm{d}x\Big]^{\bar p'/p'}\Big\}^{1/\bar p'}<\infty,\end{aligned}\tag{2.9}$$

$$\begin{aligned}\sup_{n\in\mathbf{N}}\sup_{k-1<s<k}\Big(\int\limits_{s<|x|<k} u(x)\,\mathrm{d}x\Big)^{1/q}\\ \times\Big(\int\limits_{k-1<|x|<s} v^{1-p'}(x)|x|^{-Np'}\,\mathrm{d}x\Big)^{1/p'}<\infty.\end{aligned}\tag{2.10}$$

Proof. (i) *Sufficiency.* Using (2.4) and (2.1), we can rewrite the left side of (2.6) as

$$\left\{\sum_{k=0}^{\infty}\left[\int\limits_{k<|x|<k+1}|P(Rf)(x)|^{q}u(x)\,\mathrm{d}x\right]^{\bar{q}/q}\right\}^{1/\bar{q}}$$

$$=\left\{\sum_{k=0}^{\infty}\left[\int\limits_{k<|x|<k+1}\left|\int_{0}^{1}(Rf)(xt)\frac{\mathrm{d}t}{t}\right|^{q}u(x)\,\mathrm{d}x\right]^{\bar{q}/q}\right\}^{1/\bar{q}}.$$

Using polar coordinates, $x=\sigma y$, and then the substitution $yt=s$, this is

$$\left\{\sum_{k=0}^{\infty}\left[\int_{\Sigma}\int_{k}^{k+1}\left|\int_{0}^{1}(Rf)(\sigma yt)\frac{\mathrm{d}t}{t}\right|^{q}u(\sigma y)y^{N-1}\,dy\,\mathrm{d}\sigma\right]^{\bar{q}/q}\right\}^{1/\bar{q}}$$

$$=\left\{\sum_{k=0}^{\infty}\left[\int_{k}^{k+1}u(y)y^{N-1}\int_{\Sigma}\left|\int_{0}^{y}(Rf)(\sigma s)\frac{\mathrm{d}s}{s}\right|^{q}\mathrm{d}\sigma\,dy\right]^{\bar{q}/q}\right\}^{1/\bar{q}}$$

[note that since the weight is radial, $u(\sigma y)=u(y)$]. Since

$$\int_{0}^{y}=\int_{0}^{k}+\int_{k}^{y}=\sum_{l=0}^{k-1}\int_{l}^{l+1}+\int_{k}^{y}\quad\text{for}\quad y\in(k,k+1),$$

we have that the left side of (2.6) can be estimated from above by

$$c_1(S_1+S_2)\tag{2.11}$$

where

$$S_1=\left\{\sum_{k=0}^{\infty}\left[\int_{k}^{k+1}u(y)y^{N-1}\int_{\Sigma}\left|\sum_{l=0}^{k-1}\int_{l}^{l+1}(Rf)(\sigma s)\frac{\mathrm{d}s}{s}\right|^{q}\mathrm{d}\sigma\,dy\right]^{\bar{q}/q}\right\}^{1/\bar{q}}$$

$$=\left\{\sum_{k=0}^{\infty}\left[\int_{k}^{k+1}u(y)y^{N-1}\,dy\right]^{\bar{q}/q}\left[\int_{\Sigma}\left|\sum_{l=0}^{k-1}\int_{l}^{l+1}(Rf)(\sigma s)\frac{\mathrm{d}s}{s}\right|^{q}\mathrm{d}\sigma\right]^{\bar{q}/q}\right\}^{1/\bar{q}},\tag{2.12}$$

$$S_2=\left\{\sum_{k=0}^{\infty}\left[\int_{\Sigma}\left(\int_{k}^{k+1}u(y)y^{N-1}\left|\int_{y}^{k}(Rf)(\sigma s)\frac{\mathrm{d}s}{s}\right|^{q}dy\right)\mathrm{d}\sigma\right]^{\bar{q}/q}\right\}^{1/\bar{q}}.\tag{2.13}$$

In (2.11), it is $c_1=1$ if $\bar{q}\geqslant q$ (we used Minkowski's inequality twice) and $c_1=2^{1/\bar{q}}$ if $\bar{q}<q$.

To estimate $S_1$, denote

$$(2.14)\qquad U_k = \Big(\int_k^{k+1} u(y) y^{N-1}\, dy\Big)^{\bar q/q} = c_2\Big(\int_{k<|x|<k+1} u(x)\, \mathrm{d}x\Big)^{\bar q/q},$$

$$(2.15)\qquad A_l = \Big(\int_\Sigma \Big|\int_l^{l+1} (Rf)(\sigma s)\frac{\mathrm{d}s}{s}\Big|^q \mathrm{d}\sigma\Big)^{1/q}.$$

The Minkowski inequality yields

$$\Big(\int_\Sigma \Big|\sum_{l=0}^{k-1}\int_l^{l+1} (Rf)(\sigma s)\frac{\mathrm{d}s}{s}\Big|^q \mathrm{d}\sigma\Big)^{1/q} \leqslant \sum_{l=0}^{k-1}\Big(\int_\Sigma \Big|\int_l^{l+1} (Rf)(\sigma s)\frac{\mathrm{d}s}{s}\Big|^q\Big)^{1/q} = \sum_{l=0}^{k-1} A_l$$

and consequently

$$(2.16)\qquad S_1 \leqslant \Big\{\sum_{k=0}^{\infty} U_k \Big(\sum_{l=0}^{k-1} A_l\Big)^{\bar q}\Big\}^{1/\bar q}.$$

Using the discrete Hardy inequality (see [4, Theorem 4.1]) we can estimate the right side of (2.16) by

$$(2.17)\qquad c_3\Big\{\sum_{k=0}^{\infty} V_k A_k^{\bar p}\Big\}^{1/\bar p}$$

provided the (necessary and sufficient) condition

$$(2.18)\qquad \sup_{m\in\mathbf{N}} \Big(\sum_{k=m}^{\infty} U_k\Big)^{1/\bar q}\Big(\sum_{k=0}^{m} V_k^{1-\bar p'}\Big)^{1/\bar p'} < \infty$$

is fulfilled.

Let us choose an appropriate $V_k$. Using the integral Minkowski inequality and then the Hölder inequality we obtain from (2.15) that

$$\begin{aligned} A_k &\leqslant \int_k^{k+1} \Big[\int_\Sigma |(Rf)(\sigma s)|^q \,\mathrm{d}\sigma\Big]^{1/q} \frac{\mathrm{d}s}{s} \\ (2.19)\qquad &= \int_k^{k+1} v^{1/q}(s) s^{(N-1)/q}\Big[\int_\Sigma |(Rf)(\sigma s)|^q \,\mathrm{d}\sigma\Big]^{1/q} v^{-1/q}(s) s^{-(N-1)/q-1}\mathrm{d}s \\ &\leqslant \Big(\int_k^{k+1} v(s) s^{N-1}\int_\Sigma |(Rf)(\sigma s)|^q \,\mathrm{d}\sigma\, \mathrm{d}s\Big)^{1/q}\Big(\int_k^{k+1} v^{1-q'}(s) s^{N-1-Nq'}\,\mathrm{d}s\Big)^{1/q'} \end{aligned}$$

and if we denote

$$
(2.20)\qquad
\begin{aligned}
V_k &= \Bigg( \int\limits_k^{k+1} v^{1-q'}(s) s^{N-1-Nq'} \, \mathrm{d}s \Bigg)^{-\bar{p}/q'} \\
&= c_4 \Bigg( \int\limits_{k<|x|<k+1} v^{1-q'}(x) |x|^{-Nq'} \, \mathrm{d}x \Bigg)^{-\bar{p}/q'}
\end{aligned}
$$

we obtain in view of (2.19) and (2.3) that

$$
(2.21)\qquad
\begin{aligned}
V_k A_k^{\bar{p}} &\leqslant c_5 \Bigg[ \int\limits_{\Sigma} \Bigg( \int\limits_k^{k+1} |(Rf)(\sigma s)|^q v(s) s^{N-1} \, \mathrm{d}s \Bigg) \mathrm{d}\sigma \Bigg]^{\bar{p}/q} \\
&= c_5 \Bigg( \int\limits_{k<|x|<k+1} v(x) |x \cdot \nabla f(x)|^q \, \mathrm{d}x \Bigg)^{\bar{p}/q}.
\end{aligned}
$$

This inequality together with (2.16) and (2.17) implies that

$$
(2.22)\qquad
\begin{aligned}
S_1 &\leqslant c_6 \Bigg\{ \sum_{k=0}^{\infty} \Bigg[ \int\limits_{k<|x|<k+1} v(x) |x \cdot \nabla f(x)|^q \, \mathrm{d}x \Bigg]^{\bar{p}/q} \Bigg\}^{1/\bar{p}} \\
&= c_6 \Bigg\{ \sum_{k=0}^{\infty} \Bigg[ \int\limits_{k<|x|<k+1} v(x) |x \cdot \nabla f(x)|^p \, \mathrm{d}x \Bigg]^{\bar{p}/p} \Bigg\}^{1/\bar{p}}
\end{aligned}
$$

[note that $q = p$ due to (2.8)]. Moreover, the condition (2.18) is, in view of (2.14) and (2.20), exactly the condition (2.9) (again we use the fact that $q = p$).

To estimate $S_2$, we use the Hardy inequality for the function $G(y) = (Rf)(\sigma y)/y$. It reads (see, e.g. Opic and Kufner [5, Theorem 1.14])

$$
(2.23)\qquad
\begin{aligned}
\int\limits_k^{k+1} u(y) y^{N-1} \Bigg| \int\limits_y^{k+1} G(s) \, \mathrm{d}s \Bigg|^q dy &\leqslant c_7 \Bigg( \int\limits_k^{k+1} v(y) y^{N-1+p} |G(y)|^p \, dy \Bigg)^{q/p} \\
&= c_7 \Bigg( \int\limits_k^{k+1} v(y) y^{N-1} |(Rf)(\sigma y)|^p \, dy \Bigg)^{q/p}
\end{aligned}
$$

provided the necessary and sufficient condition

$$
(2.24)\qquad \sup_{k<s<k+1} \Bigg( \int\limits_s^{k+1} u(y) y^{N-1} \, dy \Bigg)^{1/q} \Bigg( \int\limits_k^{s} v^{1-p'}(y) y^{N-1-Np'} \, dy \Bigg)^{1/p'} < \infty
$$

is fulfilled (this condition concerns the case $q \geqslant p$; we in fact consider $q = p$).

Using (2.23) in (2.13), we obtain (with $p = q$)

$$S_2 \leqslant c_8\left\{\sum_{k=0}^{\infty}\left[\int_{\Sigma}\left(\int_k^{k+1} v(y)y^{N-1}|(Rf)(\sigma y)|^p\,dy\right)\mathrm{d}\sigma\right]^{\bar{q}/p}\right\}^{1/\bar{q}}$$

$$(2.25)\qquad = c_8\left\{\sum_{k=0}^{\infty}\left[\int_{k<|x|<k+1} v(x)|x\cdot\nabla f(x)|^p\,\mathrm{d}x\right]^{\bar{p}/p\cdot\bar{q}/\bar{p}}\right\}^{\bar{p}/\bar{q}\cdot 1/\bar{p}}$$

$$\leqslant c_8\left\{\sum_{k=0}^{\infty}\left[\int_{k<|x|<k+1} v(x)|x\cdot\nabla f(x)|^p\,\mathrm{d}x\right]^{\bar{p}/p}\right\}^{1/\bar{p}}$$

[we used (0.16) with $r = \bar{q}/\bar{p} \geqslant 1$]. But (2.25) together with (2.22) leads to (2.6) (with $p = q$). Moreover, the condition (2.24) is fulfilled since, in view of (2.14) and (2.20), it follows from the condition (2.10).

(ii) *Necessity.* Take

$$(2.26)\qquad f(x) = \int_0^{|x|} g(t)\,\mathrm{d}t.$$

Then $f(0) = 0$ and $x\cdot\nabla f(x) = g(|x|)|x|$. Using polar coordinates, $x = \sigma t$, and taking into account that the weights $u$, $v$ are radial, we can rewrite (2.6) in the form

$$\left\{\sum_{k=0}^{\infty}\left[\int_k^{k+1} u(t)t^{N-1}\left|\int_0^t g(s)\,\mathrm{d}s\right|^q \mathrm{d}t\right]^{\bar{q}/q}\right\}^{1/\bar{q}}$$

$$(2.27)\qquad \leqslant C\left\{\sum_{k=0}^{\infty}\left[\int_k^{k+1} v(t)t^{N-1+p}|g(t)|^p\,\mathrm{d}t\right]^{\bar{p}/p}\right\}^{1/\bar{p}}.$$

But this is the inequality (0.8) from Theorem 0.2, with $f$ replaced by $g$, $u(t)$ by $u(t)t^{N-1}$ and $v(t)$ by $v(t)t^{N-1+p}$. The necessary and sufficient conditions (0.9), (0.10) then take the form

$$\sup_{m\in\mathbf{N}}\Big\{\sum_{k=m}^{\infty}\Big(\int_k^{k+1} u(t)t^{N-1}\,dt\Big)^{\bar q/q}\Big\}^{1/\bar q}$$

$$\times\Big\{\sum_{k=0}^{m}\Big(\int_k^{k+1} v^{1-p'}(t)t^{N-1-Np'}\,dt\Big)^{\bar p/p'}\Big\}^{1/\bar p'}<\infty, \tag{2.28}$$

$$\sup_{m\in\mathbf{N}}\ \sup_{m<s<m+1}\Big(\int_s^{m+1} u(t)t^{N-1}\,dt\Big)^{1/q}\Big(\int_m^{s} v^{1-p'}(t)t^{N-1-Np'}\,dt\Big)^{1/p'}<\infty, \tag{2.29}$$

and in view of (2.14) and (2.20), (2.28) and (2.29) are the conditions (2.9), (2.10). □

Completely analogously, using only the operator $Q$ from (2.2) and formula (2.5) instead of the operator $P$ from (2.1) and formula (2.4), and taking $f(x)=-\int_{|x|}^{\infty} g(t)\,dt$ instead of (2.26), we can prove the following assertion, quoting Theorem 0.3.

**2.5. Theorem.** *Suppose $u$, $v$ are radial weights on $\mathbf{R}^N$, $f$ differentiable on $\mathbf{R}^N$, $f(\infty)=0$. Let $p$, $\bar p$, $q$, $\bar q$ satisfy (2.8). Then the inequality (2.6) holds with a constant $C>0$ independent of $f$ if and only if the following two conditions are satisfied:*

$$\sup_{m\in\mathbf{N}}\Big\{\sum_{k=0}^{m}\Big[\int_{k<|x|<k+1} u(x)\,dx\Big]^{\bar q/q}\Big\}^{1/\bar q}$$

$$\times\Big\{\sum_{k=m}^{\infty}\Big[\int_{k<|x|<k+1} v^{1-p'}(x)|x|^{-Np'}\,dx\Big]^{\bar p'/p'}\Big\}^{1/\bar p'}<\infty, \tag{2.30}$$

$$\sup_{k\in\mathbf{N}}\ \sup_{k-1<s<k}\Big(\int_{k-1<|x|<s} u(x)\,dx\Big)^{1/q}\Big(\int_{s<|x|<k} v^{1-p'}(x)|x|^{-Np'}\,dx\Big)^{1/p'}<\infty. \tag{2.31}$$

**2.6.** Remark. Up to now, we considered, in this section, only the case (2.8), i.e., $p=q$, $\bar p\leqslant\bar q$. Now, let us suppose that

$$1<q<p<\infty,\ 1<\bar p\leqslant\bar q<\infty \tag{2.32}$$

and investigate inequality (2.6), again for radial weights $u$, $v$ and for $f$ satisfying

$$f(0)=0.$$

We can proceed as in the proof of Theorem 2.4 up to formula (2.18). Instead of formula (2.19) we have now

$$A_k \leqslant \int_k^{k+1} \left[ \int_\Sigma |(Rf)(\sigma s)|^q \, \mathrm{d}\sigma \right]^{1/q} \frac{\mathrm{d}s}{s} \leqslant c_1 \int_k^{k+1} \left[ \int_\Sigma |(Rf)(\sigma s)|^p \, \mathrm{d}\sigma \right]^{1/p} \frac{\mathrm{d}s}{s}$$

$$= c_1 \int_k^{k+1} v^{1/p}(s) s^{(N-1)/p} \left[ \int_\Sigma |(Rf)(\sigma s)|^p \mathrm{d}\sigma \right]^{1/p} v^{-1/p}(s) s^{-(N-1)/p-1} \, \mathrm{d}s$$

$$\leqslant c_1 \left( \int_k^{k+1} v(s) s^{N-1} \left[ \int_\Sigma |(Rf)(\sigma s)|^p \, \sigma s \right] \mathrm{d}s \right)^{1/p} \left( \int_k^{k+1} v^{1-p'}(s) s^{N-1-Np'} \, \mathrm{d}s \right)^{1/p'};$$

here, we used Minkowski's inequality, then Hölder's inequality for the surface integral with parameter $p/q > 1$, and then again Hölder's inequality for the integral over $s \in (k, k+1)$ with parameter $p$.

Now we proceed again as in the proof of Theorem 2.4 and obtain (2.20), (2.21) and (2.22), of course with $p$, $p'$ instead of $q$, $p$. Among other changes we have

$$V_k = c_2 \left( \int_{k<|x|<k+1} v^{1-p'}(x) |x|^{-Np'} \, \mathrm{d}x \right)$$

and the condition (2.18) is exactly the condition (2.9).

To estimate $S_2$, we arrive again at (2.23), but now we have $q < p$, and consequently, the necessary and sufficient condition reads

$$\left\{ \int_k^{k+1} \left( \int_s^{k+1} u(y) y^{N-1} \, dy \right)^{r/q} \left( \int_k^s v^{1-p'}(y) y^{N-1-Np'} \, dy \right)^{r/q'} \right.$$

$$\left. \times\, v^{1-p'}(s) s^{N-1-Np'} \, \mathrm{d}s \right\}^{1/r} < \infty \tag{2.33}$$

with $1/r = 1/q - 1/p$ (see, e.g., [5, Theorem 1.15]), note that condition (2.33) replaces condition (2.24). Using again (2.23) in (2.13), but now with $q < p$, we obtain

$$S_2 \leqslant c_3 \left\{ \sum_{k=0}^\infty \left[ \int_\Sigma \left( \int_k^{k+1} v(y) y^{N-1} |(Rf)(\sigma s)|^p \, dy \right)^{q/p} \mathrm{d}\sigma \right]^{\bar{q}/q} \right\}^{1/\bar{q}}. \tag{2.34}$$

The Hölder inequality, used for the surface integral with parameter $p/q > 1$, yields

$$
\begin{aligned}
S_2 &\leqslant c_4\left\{\sum_{k=0}^{\infty}\left[\int_{\Sigma}\int_{k}^{k+1} v(y)y^{N-1}\left|(Rf)(\sigma y)\right|^p\,dy\,\mathrm{d}\sigma\right]^{q/p\cdot\bar q/q}\right\}^{1/\bar q}\\
&= c_4\left\{\sum_{k=0}^{\infty}\left[\int_{k<|x|<k+1} v(x)\left|x\cdot\nabla f(x)\right|^p\,\mathrm{d}x\right]^{\bar p/p\cdot\bar q/\bar p}\right\}^{\bar p/\bar q\cdot 1/\bar p}\\
&\leqslant c_4\left\{\sum_{k=0}^{\infty}\left[\int_{k<|x|<k+1} v(x)\left|x\cdot\nabla f(x)\right|^p\,\mathrm{d}x\right]^{\bar p/p}\right\}^{1/\bar p}
\end{aligned}
\tag{2.35}
$$

where we used (0.16) with $r = \bar q/\bar p \geqslant 1$. Formulas (2.22) (with $p$ instead of $q$) and (2.35) lead to (2.6).

Consequently we have proved the sufficiency part of the following assertion.

**2.7. Theorem.** *Suppose $u$, $v$ are radial weights on $\mathbf{R}^N$, $f$ differentiable on $\mathbf{R}^N$, $f(0) = 0$. Let $p$, $\bar p$, $q$, $\bar q$ satisfy (2.32). Then the inequality (2.6) holds with a constant $C > 0$ independent of $f$ if and only if the conditions (2.9) and*

$$
\sup_{k\in\mathbf{N}}\left\{\int_{k<|x|<k+1}\left(\int_{|x|<|z|<k+1} u(z)\,\mathrm{d}z\right)^{r/q}\left(\int_{k<|z|<|x|} v^{1-p'}(z)|z|^{-Np'}\,\mathrm{d}z\right)^{r/q'} \times v^{1-p'}(x)|x|^{-Np'}\,\mathrm{d}x\right\}^{1/r} \equiv C_1^* < \infty
\tag{2.36}
$$

*are satisfied, where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.*

Proof. The sufficiency of (2.9) and (2.36) was shown in Remark 2.6 since (2.36) implies (2.33).

To prove necessity, we again choose $f$ from (2.26) and obtain (2.27), and then we use Theorem 0.6 similarly as we used Theorem 0.2 in the proof of Theorem 2.4. □

The theorem analogous to Theorem 2.7 for $f$ such that $f(\infty) = 0$ has the following form:

**2.8. Theorem.** *Suppose $u$, $v$ are radial weights on $\mathbf{R}^N$, $f$ differentiable on $\mathbf{R}^N$, $f(\infty) = 0$. Let $p$, $\bar p$, $q$, $\bar q$ satisfy (2.32). Then the inequality (2.6) holds with a constant $C > 0$ independent of $f$ if and only if the conditions (2.30) and*

$$
\sup_{k\in\mathbf{N}}\left\{\int_{k<|x|<k+1}\left(\int_{k<|z|<|x|} u(z)\,\mathrm{d}z\right)^{r/q}\left(\int_{|x|<|z|<k+1} v^{1-p'}(z)|z|^{-Np'}\,\mathrm{d}z\right)^{r/q'} \times v^{1-p'}(x)|x|^{-Np'}\,\mathrm{d}x\right\}^{1/r} \equiv C_2^* < \infty
\tag{2.37}
$$

*are satisfied, where* $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

## 3. Remarks and examples

**3.1.** According to Remark 0.5 (i), the theorems from Section 1 and Section 2 can be extended to the case

$$1 < q < p < \infty, \quad 1 < \bar{q} < \bar{p} < \infty$$

and at least sufficient conditions for the validity of the corresponding weighted Hardy (Friedrichs) inequalities in amalgams can be derived. The formulation of the corresponding results is left to the reader.

**3.2.** Let us consider a special case of parameters $p$, $\bar{p}$, $q$, $\bar{q}$, namely such that

$$q < \bar{q} \quad \text{and} \quad p > \bar{p}. \tag{3.1}$$

The inequality (0.16), used for $r = \bar{q}/q$, leads to the estimate

$$\begin{aligned}\left\{\sum_{n=0}^{\infty}\left[\int_n^{n+1} u(x)|f(x)|^q \,\mathrm{d}x\right]^{\bar{q}/q}\right\}^{1/\bar{q}} &\leqslant \left(\sum_{n=0}^{\infty}\int_n^{n+1} u(x)|f(x)|^q \,\mathrm{d}x\right)^{1/q} \\ &= \left(\int_0^{\infty} u(x)|f(x)|^q \,\mathrm{d}x\right)^{1/q}.\end{aligned} \tag{3.2}$$

Supposing that

$$1 < p \leqslant q \quad \text{and} \quad f(0) = 0, \tag{3.3}$$

we can estimate the last integral by the "classical" Hardy inequality

$$\left(\int_0^{\infty} u(x)|f(x)|^q \,\mathrm{d}x\right)^{1/q} \leqslant C\left(\int_0^{\infty} v(x)|f'(x)|^p \,\mathrm{d}x\right)^{1/p} \tag{3.4}$$

provided the (necessary and sufficient) condition

$$\sup_{0<s<\infty}\left(\int_s^{\infty} u(x)\,\mathrm{d}x\right)^{1/q}\left(\int_0^{s} v^{1-p'}(x)\,\mathrm{d}x\right)^{1/p'} \equiv C_1 < \infty \tag{3.5}$$

holds. Using again inequality (0.16), now with $r = p/\bar{p}$, we obtain that

$$\Big(\int_0^\infty v(x)|f'(x)|^p\,\mathrm{d}x\Big)^{1/p} = \Big\{\sum_{n=0}^\infty \int_n^{n+1} v(x)|f'(x)|^p\,\mathrm{d}x\Big\}^{1/p}$$

$$= \Big\{\sum_{n=0}^\infty \Big(\Big[\int_n^{n+1} v(x)|f'(x)|^p\,\mathrm{d}x\Big]^{\bar{p}/p}\Big)^{p/\bar{p}}\Big\}^{1/p}$$

$$\leqslant \Big\{\sum_{n=0}^\infty \Big[\int_n^{n+1} v(x)|f'(x)|^p\,\mathrm{d}x\Big]^{\bar{p}/p}\Big\}^{1/\bar{p}}. \tag{3.6}$$

From (3.2), (3.4) and (3.6) we obtain the Hardy inequality in amalgams,

$$\|f\|_{u,q,\bar{q}} \leqslant C\|f'\|_{v,p,\bar{p}} \tag{3.7}$$

which was derived under the *unique* condition (3.5), supposing, of course, that (3.1) and (3.3) hold.

Condition (3.5) does not depend on $\bar{p}$, $\bar{q}$. Since conditions (3.1) and (3.3) imply

$$1 < \bar{p} \leqslant \bar{q}, \tag{3.8}$$

we have from Theorem 0.2 that the two conditions (0.9) and (0.10) are necessary and sufficient for (3.7) to hold.

Thus, we can derive (3.7) in two ways: either from Theorem 0.2 under the necessary and sufficient conditions, or via the "classical" Hardy inequality (3.4) under condition (3.5). The following example shows that condition (3.5), being *sufficient* for (3.7) to hold, is *not necessary*.

**3.3.** Example. Take

$$\begin{aligned} u(x) &= \frac{1}{n+1} \quad \text{for } x \in (n, n+1), \quad n = 0, 1, 2, \ldots, \\ v(x) &\equiv 1 \quad \text{for } x \in (0, \infty). \end{aligned} \tag{3.9}$$

Taking $s = m \in \mathbb{N}$ in (3.5), we have that

$$\Big(\int_m^\infty u(x)\,\mathrm{d}x\Big)^{1/q}\Big(\int_0^m v^{1-p'}(x)\,\mathrm{d}x\Big)^{1/p'} = \Big(\sum_{n=m}^\infty \int_n^{n+1} \frac{1}{n+1}\,\mathrm{d}x\Big)^{1/q} m^{1/p'}$$

$$= m^{1/p'}\Big(\sum_{n=m}^\infty \frac{1}{n+1}\Big)^{1/q} = \infty;$$

consequently, *condition* (3.5) *is not fulfilled.*

Further, we have that

$$\Big(\int_{m+s}^{m+1} u(x)\,\mathrm{d}x\Big)^{1/q}\Big(\int_{m}^{m+s} v^{1-p'}(x)\,\mathrm{d}x\Big)^{1/p'} = \Big[\frac{1}{m+1}(1-s)\Big]^{1/q} s^{1/p'} \leqslant C$$

for every $s \in (0,1)$ and for $m = 0, 1, 2, \ldots$, so that *condition* (0.10) *is satisfied.*

Finally

$$\Big\{\sum_{n=m}^{\infty}\Big(\int_{n}^{n+1} u(x)\,\mathrm{d}x\Big)^{\bar{q}/q}\Big\}^{1/\bar{q}}\Big\{\sum_{n=0}^{m}\Big(\int_{n}^{n+1} v^{1-p'}(x)\,\mathrm{d}x\Big)^{\bar{p}'/p'}\Big\}^{1/\bar{p}'}$$

$$= \Big\{\sum_{n=m}^{\infty}\Big(\frac{1}{n+1}\Big)^{\bar{q}/q}\Big\}^{1/\bar{q}}\{m\}^{1/\bar{p}'} \leqslant Cm^{(-\bar{q}/q+1)/\bar{q}+1/\bar{p}'}$$

$$= Cm^{-1/q+1/\bar{q}+1/\bar{p}'}, \quad m \in \mathbf{N}$$

and this will be finite if

$$(3.10) \qquad -\frac{1}{q}+\frac{1}{\bar{q}}+\frac{1}{\bar{p}'} \leqslant 0.$$

Then *also condition* (0.9) *will be satisfied.*

But (3.10) is fulfilled if we take, e.g.

$$q = \frac{11}{2}, \ \bar{q} = 11, \ \bar{p} = \frac{11}{10}$$

and also conditions (3.1), (3.3) and (3.8) are satisfied.

This example shows that the consideration of weighted amalgam norms allows to use a broader class of weight functions $u$, $v$ (e.g., that from (3.9)) than the application of the weighted $L^p$-norm.

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