Weighted Geometric Means of Positive Operators

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ABSTRACT. A weighted version of the geometric mean of k (≥ 3) positive invertible operators is given. For operators $A_1, ..., A_k$ and for nonnegative numbers $\alpha_1, ..., \alpha_k$ such that $\sum_{i=1}^{k} \alpha_i = 1$, we define weighted geometric means of two types, the first type by a direct construction through symmetrization procedure, and the second type by an indirect construction through the non-weighted (or uniformly weighted) geometric mean. Both of them reduce to $A_1^{\alpha_1} \cdots A_k^{\alpha_k}$ if $A_1, ..., A_k$ commute with each other. The first type does not have the property of permutation invariance, but satisfies a weaker one with respect to permutation invariance. The second type has the property of permutation invariance. We also show a reverse inequality for the arithmetic-geometric mean inequality of the weighted version.

1. Introduction

In [14] and [9], a new definition of the geometric mean of k (≥ 3) (bounded linear) positive invertible operators (or positive definite matrices) was introduced, borrowing the technique due to Ando-Li-Mathias [2]. In succession of those results, we give a weighted version of geometric means of k positive invertible operators, which extends the weighted mean

$$A \#_\alpha B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$$

of two positive invertible operators A and B for $\alpha \in [0, 1]$.

Let Ω be the positive cone of all (positive invertible) operators acting on a Hilbert space H or all $n \times n$ positive definite matrices, that is, operators for $\dim H = n$. We assume that Ω is provided with the Thompson metric ([3], [4]) defined by

$$d(A, B) = \| \log A^{-1/2} B A^{-1/2} \| \quad \text{for } A, B \in \Omega.$$

Then Ω is complete with respect to the metric and the corresponding metric topology agrees with the relative norm topology. As the most fundamental inequality related to this metric, between weighted geometric means of two operators, the following one holds ([3]):

$$(1.1) \qquad d(A_1 \#_\alpha A_2, B_1 \#_\alpha B_2) \le (1-\alpha)d(A_1, B_1) + \alpha d(A_2, B_2)$$
$$\text{for } A_1, A_2, B_1, B_2 \in \Omega \text{ and } \alpha \in (0,1).$$

A definition of a weighted geometric mean G of k operators $A_1, ..., A_k \in \Omega$ corresponding to a k-weight $w = (\alpha_1, ..., \alpha_k)$, that is, an ordered k-tuple of nonnegative numbers α_i with $\sum_{i=1}^{k} \alpha_i = 1$ is (cf. [15], [1], [5])

$$(1.2) \qquad G = g(w; A_1, ..., A_k) = A_1 \#_{x_1} \cdots \#_{x_{k-1}} A_k$$
$$:= \overbrace{(\cdots (A_1 \#_{x_1} A_2) \#_{x_2} \cdots)}^{k-2} \#_{x_{k-2}} A_{k-1}) \#_{x_{k-1}} A_k,$$

where $x_i = \frac{\alpha_{i+1}}{\sum_{j=1}^{i+1} \alpha_j}$ $\left(\frac{0}{0} = 0\right)$ for $i = 1, ..., k-1$. Then, of course, G reduces to $A_1^{\alpha_1} \cdots A_k^{\alpha_k}$ if all A_i commute with each other. But G does not have always the property of permutation-invariance (PW3, stated below), one of desirable properties as an operator mean. So we have to ask another reasonable definition of a weighted geometric mean. In [14], [9], a usual (non-weighted) geometric mean of k operators was defined such that the mean has ten properties P1-P10 postulated by Ando-Li-Mathias [2]. Let $G = G(w; A_1, ..., A_k) = G(\alpha_1, ..., \alpha_k; A_1, ..., A_k)$ be a weighted geometric mean of k operators $A_1, ..., A_k$ for any weight $w = (\alpha_1, ..., \alpha_k)$. Then, parallel to the properties P1-P10 (in [2]), we want to request the following ten properties for G to be a reasonable weighted mean:

PW1 Consistency with scalars. If all A_i commute then $G = A_1^{\alpha_1} \cdots A_k^{\alpha_k}$.

PW2 Joint homogeneity. $G(w; a_1 A_1, ..., a_k A_k) = a_1^{\alpha_1} \cdots a_k^{\alpha_k} G$.

PW3 Permutation-invariance. For any permutation $\pi \in \mathbf{S(k)}$, *the symmetric group of k letters*

$$G = G(\pi(w; A_1, ..., A_k)) = G(\pi(w); \pi(A_1, ..., A_k)).$$

PW4 Monotonicity. The map $(A_1, ..., A_k) \mapsto G(w; A_1, A_2, \ldots, A_k)$ is monotone, i.e., if $A_i \ge B_i$ for $i = 1, \ldots, k$, then $G(w; A_1, A_2, \ldots, A_k) \ge G(w; B_1, B_2, \ldots, B_k)$.

PW5 Continuity from above. If $\{A_1^{(n)}\}, \{A_2^{(n)}\}, \ldots, \{A_k^{(n)}\}$ are monotone decreasing sequences converging to $A_1, A_2, \ldots, A_k$, respectively, then $\{G(w; A_1^{(n)}, A_2^{(n)}, \ldots, A_k^{(n)})\}$ converges to $G(w; A_1, A_2, \ldots, A_k)$.

PW6 Congruence invariance. For any invertible operator S,

$$G(w; S^*A_1S, S^*A_2S, \ldots, S^*A_kS) = S^*G(w; A_1, A_2, \ldots, A_k)S.$$

PW7 Joint concavity. The map $(A_1, A_2, \ldots, A_k) \mapsto G(w; A_1, A_2, \ldots, A_k)$ is jointly concave: $G(w; \lambda A_1 + (1-\lambda)A_1', \ldots, \lambda A_k + (1-\lambda)A_k')$
$$\geq \lambda G(w; A_1, A_2, \ldots, A_k) + (1-\lambda)G(w; A_1', A_2', \ldots, A_k') \quad (\lambda \in [0,1]).$$

PW8 Self-duality. $G(w; A_1, A_2, \ldots, A_k)^* = G(w; A_1, A_2, \ldots, A_k)$. The dual $G(w; A_1, A_2, \ldots, A_k)^*$ is defined by

$$G(w; A_1, A_2, \ldots, A_k)^* = G(w; A_1^{-1}, A_2^{-1}, \ldots, A_k^{-1})^{-1}.$$

PW9 (*In case $A_1, A_2, \ldots, A_k$ are matrices.*) Determinant identity.

$$\det G(w; A_1, A_2, \ldots, A_k) = (\det A_1)^{\alpha_1} \cdots (\det A_k)^{\alpha_k}.$$

PW10 The arithmetic-geometric-harmonic mean inequaility.

$$\sum_{i=1}^{k} \alpha_i A_i \geq G \geq \left(\sum_{i=1}^{k} \alpha_i A_i^{-1}\right)^{-1}.$$

In this paper, we define weighted geometric means of k operators of two types, the first one directly obtained by using symmetrization procedure, and the second one indirectly by using non-weighted geometric means. The second type of the weighted geometric means has all the ten properties above, but the first type does not have PW3, though it satisfies a weaker condition related to PW3. Kantorovich type reverse inequality for weighted geometric means are also given.

2. Weighted geometric means of k operators by direct construction; first type

A sequence $\{A_n\}$ of operators converges to A with respect to strong operator topology (denoted by $A_n \to^s A$) if $\| (A_n - A)x \| \to 0$ as $n \to \infty$ for all vectors x ($\in H$). A norm-convergent sequence is also convergent with respect to strong operator topology, so that a sequence in Ω is convergent with respect to strong operator topology if it is convergent with respect to the Thompson metric.

Now we want to prepare a useful lemma related to strong convergence of operators. Recall that we, in (1.2), defined a successive composition of two-operator weighted geometric means:

$$g(\alpha_1, \ldots, \alpha_k; A_1, \ldots, A_k) = A_1 \#_{x_1} \cdots \#_{x_{k-1}} A_k \quad \left(x_i = \frac{\alpha_{i+1}}{\sum_{j=1}^{i+1} \alpha_j}\right).$$

Related to this mean we have:

Lemma 2.1. *Let* $\{X_1^{(r)}\}, ..., \{X_k^{(r)}\}$ *be arbitrary* k (≥ 2) *sequences of positive operators* ($\in \Omega$) *such that* $0 < mI \leq X_1^{(r)}, ..., X_k^{(r)} \leq MI$ *for some scalars* m *and* M, *and let* $(\alpha_1, ..., \alpha_k)$ *be a* k-*weight of positive numbers. If*

$$\delta^{(r)} := \sum_{i=1}^{k} \alpha_i X_i^{(r)} - g(\alpha_1, ..., \alpha_k; X_1^{(r)}, ..., X_k^{(r)}) \rightarrow^s 0 \ as \ r \rightarrow \infty,$$

then $X_i^{(r)} - X_j^{(r)} \rightarrow^s 0$ *for* i, j ($i \neq j$) $= 1, ..., k$.

Proof. To show the lemma by induction, first consider for $k = 2$. Let $Y_r = X_1^{(r)}, Z_r = X_2^{(r)}$, and $1 - h = \alpha_1, h = \alpha_2$. Note that for any $t \geq 0$,

(2.1) $(1 - h) + ht - t^h \geq \min\{h, 1 - h\}(1 - t^{\frac{1}{2}})^2.$

In this inequality, replacing t by $Y_r^{-\frac{1}{2}} Z_r Y_r^{-\frac{1}{2}}$ and multiplying $Y_r^{\frac{1}{2}}$ from both sides, we can obtain

$$\delta^{(r)} = (1 - h)Y_r + hZ_r - Y_r \#_h Z_r \geq \min\{h, 1 - h\}Y_r^{\frac{1}{2}}\{I - (Y_r^{-\frac{1}{2}} Z_r Y_r^{-\frac{1}{2}})^{\frac{1}{2}}\}^2 Y_r^{\frac{1}{2}}.$$

Hence, if $\delta^{(r)} \rightarrow^s 0$ then, putting $W_r = (Y_r^{-\frac{1}{2}} Z_r Y_r^{-\frac{1}{2}})^{\frac{1}{2}}$, we have $Y_r^{\frac{1}{2}}(I - W_r)^2 Y_r^{\frac{1}{2}} \rightarrow^s 0$, so that also $(I - W_r)Y_r^{\frac{1}{2}} \rightarrow^s 0$. Hence we have, using boundedness assumption,

$$Y_r - Z_r = Y_r^{\frac{1}{2}}(I - W_r^2)Y_r^{\frac{1}{2}} = Y_r^{\frac{1}{2}}(I + W_r)(I - W_r)Y_r^{\frac{1}{2}} \rightarrow^s 0.$$

Now we assume that the lemma is true for $(k - 1)$ ($k \geq 3$), and further assume that

(2.2) $\delta^{(r)} := \sum_{i=1}^{k} \alpha_i X_i^{(r)} - g(\alpha_1, ..., \alpha_k; X_1^{(r)}, ..., X_k^{(r)})) \rightarrow^s 0$ as $r \rightarrow \infty.$

Then, take a basic inequality:

(2.3) $g(\alpha_1, ..., \alpha_k; X_1^{(r)}, ..., X_k^{(r)}) \leq (1 - \alpha_k)g(\alpha_1', ..., \alpha_{k-1}'; X_1^{(r)}, ..., X_{k-1}^{(r)}) + \alpha_k X_k^{(r)},$

$$\alpha_j' = \frac{a_j}{1 - \alpha_k} \quad (j = 1, ..., k - 1).$$

From (2.2) and (2.3),

$$\delta^{(r)} \geq \sum_{i=1}^{k-1} \alpha_i X_i^{(r)} - (1 - \alpha_k)g(\alpha_1', ..., \alpha_{k-1}'; X_1^{(r)}, ..., X_{k-1}^{(r)})$$

$$= (1 - \alpha_k) \cdot \left\{ \sum_{i=1}^{k-1} \alpha_i' X_i^{(r)} - g(\alpha_1', ..., \alpha_{k-1}'; X_1^{(r)}, ..., X_{k-1}^{(r)}) \right\} \ (\geq 0)$$

and $\delta^{(r)} \to^s 0$, so that by the assumption on $(k-1)$, we obtain that for all $i, j \leq k-1, i \neq j$, $X_i^{(r)} - X_j^{(r)} \to^s 0$. Further, note that

$$\epsilon^{(r)} := (1-\alpha_k)X_1^{(r)} + \alpha_k X_k^{(r)} - X_1^{(r)} \#_{\alpha_k} X_k^{(r)}$$

$$= \sum_{i=1}^{k-1} \alpha_i \cdot X_1^{(r)} + \alpha_k X_k^{(r)} - g(\alpha_1, ..., \alpha_k; \overbrace{X_1^{(r)}, ..., X_1^{(r)}}^{k-1}, X_k^{(r)})$$

$$\left. \begin{array}{c} = \displaystyle\sum_{i=1}^{k} \alpha_i X_i^{(r)} - g(\alpha_1, ..., \alpha_k; X_1^{(r)}, ..., X_k^{(r)}) \\[2em] + \displaystyle\sum_{i=1}^{k-1} \alpha_i (X_1^{(r)} - X_i^{(r)}) \\[2em] + g(\alpha_1, ..., \alpha_k; X_1^{(r)}, ..., X_k^{(r)}) - g(\alpha_1, ..., \alpha_k; \overbrace{X_1^{(r)}, ..., X_1^{(r)}}^{k-1}, X_k^{(r)}). \end{array} \right\}$$

Hence $\epsilon^{(r)} \to^s 0$, because we have, for the first term of the third identity in above,

$$\sum_{i=1}^{k} \alpha_i X_i^{(r)} - g(\alpha_1, ..., \alpha_k; X_1^{(r)}, ..., X_k^{(r)}) = \delta^{(r)} \to^s 0$$

by assumption, for the second one $\sum_{i=1}^{k-1} \alpha_i(X_1^{(r)} - X_i^{(r)}) \to^s 0$ clearly, and for the last one, by continuity of $g(\alpha_1, ..., \alpha_k; X_1, ..., X_k)$ with respect to $(X_1, ..., X_k)$, it tends to 0, or, by using (1.1) successively,

$$d\left(g(\alpha_1, ..., \alpha_k; X_1^{(r)}, ..., X_k^{(r)}), g(\alpha_1, ..., \alpha_k; \overbrace{X_1^{(r)}, ..., X_1^{(r)}}^{k-1}, X_k^{(r)}) \right)$$

$$\leq \alpha_2 d(X_2^{(r)}, X_1^{(r)}) + \cdots + \alpha_{k-1} d(X_{k-1}^{(r)}, X_1^{(r)}) \to^s 0.$$

Hence by the lemma for the case $k = 2$, we have $X_1^{(r)} - X_k^{(r)} \to^s 0$, which implies the desired conclusion. $\qquad\square$

Theorem 2.2. *Let $A_1, \ldots, A_k$ be k operators in Ω. For a k-weight $w = (\alpha_1, \ldots, \alpha_k)$ of positive numbers and a subset $S = \{\pi_1, \ldots, \pi_k\}$ in $\mathbf{S(k)}$, we define k sequences $\{A_1^{(r)}\}, \ldots, \{A_k^{(r)}\}$ as follows:*

$$(2.4) \qquad \left\{ \begin{array}{l} A_i^{(1)} = A_i \quad i = 1, \ldots, k, \quad and \ for \ r \geq 1, \ i = 1, \ldots, k, \\ A_i^{(r+1)} = g(\pi_i(w; A_1, \ldots, A_k)) = g(\pi_i(\alpha_1, \ldots, \alpha_k \; ; A_1, \ldots, A_k)) \\ \qquad (= g(\alpha_{\pi_i(1)}, \ldots, \alpha_{\pi_i(k)} \; ; A_{\pi_i(1)}, \ldots, A_{\pi_i(k)})). \end{array} \right.$$

Then the above k sequences converge with respect to strong operator topology and have a common limit (denoted by)

$$G_S = G_S(w; A_1, \ldots, A_k) = G_S(\alpha_1, \ldots, \alpha_k \; ; A_1, \ldots, A_k).$$

The constructed mean G_S has the following properties:

(i) *G_S has the properties PW1-PW10 except PW3.*

(ii) *If $\sigma \in S$ satisfies*

(2.5) $$(\sigma\pi_1, \ldots, \sigma\pi_k) = (\pi_{\sigma(1)}, \ldots, \pi_{\sigma(k)}),$$

then G_S is permutation-invariant with respect to σ.

Proof. First assume that $mI \leq A_i \leq MI$ $(i = 1, ..., k)$. Then easily we see $mI \leq A_i^{(r)} \leq MI$. Note that

$$A_i^{(r+1)} \leq (1 - \alpha_{\pi_i(k)})g(\alpha'_{\pi_i(1)}, \ldots, \alpha'_{\pi_i(k-1)} \; ; A_{\pi_i(1)}^{(r)}, \ldots, A_{\pi_i(k-1)}^{(r)}) + \alpha_{\pi_i(k)}A_{\pi_i(k)}^{(r)}$$

$$\leq \alpha_1 A_1^{(r)} + \cdots + \alpha_k A_k^{(r)} \quad (i = 1, \ldots, k).$$

Here $\alpha'_{\pi_i(j)} = \dfrac{\alpha_{\pi_i(j)}}{1 - \alpha_{\pi_i(k)}}$. If we write

$$C_i^{(r)} = g(\alpha'_{\pi_i(1)}, \ldots, \alpha'_{\pi_i(k-1)} \; ; A_{\pi_i(1)}^{(r)}, \ldots, A_{\pi_i(k-1)}^{(r)}) \quad \text{and} \quad D^{(s)} = \sum_{j=1}^{k} \alpha_j A_j^{(s)},$$

then from the above inequalities

$$D^{(r+1)} = \alpha_1 A_1^{(r+1)} + \cdots + \alpha_k A_k^{(r+1)}$$

$$\leq \alpha_1 \Big\{ (1 - \alpha_{\pi_1(k)})C_1^{(r)} + \alpha_{\pi_1(k)}A_{\pi_1(k)}^{(r)} \Big\} + \cdots + \alpha_k \Big\{ (1 - \alpha_{\pi_k(k)})C_k^{(r)} + \alpha_{\pi_k(k)}A_{\pi_k(k)}^{(r)} \Big\}$$

$$\leq \alpha_1 D^{(r)} + \cdots + \alpha_k D^{(r)} = D^{(r)}.$$

Hence $\{D^{(r)}\}$ is a decreasing sequence, so that it converges with respect to strong operator topology. (We shall denote the limit by G_S.) If we put

$$E^{(r)} = \alpha_1 \Big\{ (1 - \alpha_{\pi_1(k)})C_1^{(r)} + \alpha_{\pi_1(k)}A_{\pi_1(k)}^{(r)} \Big\} + \cdots + \alpha_k \Big\{ (1 - \alpha_{\pi_k(k)})C_k^{(r)} + \alpha_{\pi_k(k)}A_{\pi_k(k)}^{(r)} \Big\},$$

then $D^{(r)} - E^{(r)} \to^s 0$ as $r \to \infty$, and

$$D^{(r)} - E^{(r)} = \sum_{i=1}^{k} \alpha_i I_i^{(r)},$$

where

$$(2.6) \quad I_i^{(r)} = D^{(r)} - (1 - \alpha_{\pi_i(k)})C_i^{(r)} - \alpha_{\pi_i(k)}A_{\pi_i(k)}^{(r)}$$

$$= \sum_{\ell=1}^{k-1} \alpha_{\pi_i(\ell)}A_{\pi_i(\ell)}^{(r)} - (1 - \alpha_{\pi_i(k)})C_i^{(r)}$$

$$= (1 - \alpha_{\pi_i(k)})\left\{\sum_{\ell=1}^{k-1} \alpha'_{\pi_i(\ell)}A_{\pi_i(\ell)}^{(r)} - g\left(\alpha'_{\pi_i(1)}, ..., \alpha'_{\pi_i(k-1)}; A_{\pi_i(1)}^{(r)}, ..., A_{\pi_i(k-1)}^{(r)}\right)\right\}.$$

Then since $I_i^{(r)} \geq 0$ (for all $i = 1, ..., k$), we see (from $D^{(r)} - E^{(r)} \to^s 0$) that $I_i^{(r)} \to^s 0$. Hence by Lemma 2.1 we have

$$A_{\pi_i(\ell)}^{(r)} - A_{\pi_i(\ell')}^{(r)} \to^s 0 \text{ for } \ell, \ \ell' \leq k - 1,$$

or

$$A_{\pi_i(\ell)}^{(r)} - A_{\pi_i(1)}^{(r)} \to^s 0 \text{ for } \ell \leq k - 1.$$

Now note that

$$(2.7) \qquad C_i^{(r)} - A_{\pi_i(1)}^{(r)}$$

$$= g(\alpha'_{\pi_i(1)}, \ldots, \alpha'_{\pi_i(k-1)}; A_{\pi_i(1)}^{(r)}, \ldots, A_{\pi_i(k-1)}^{(r)})$$

$$- g(\alpha'_{\pi_i(1)}, \ldots, \alpha'_{\pi_i(k-1)}; \overbrace{A_{\pi_i(1)}^{(r)}, \ldots, A_{\pi_i(1)}^{(r)}}^{k-1}) \to^s 0$$

by continuity of $g(\alpha'_{\pi_i(1)}, \ldots, \alpha'_{\pi_i(k-1)}; A_1, \ldots, A_{k-1})$. Further, note that $D^{(r-1)} \geq A_j^{(r)}$ (or $D^{(r-1)} \geq A_{\pi_i(j)}^{(r)}$) for $r \geq 2$ $(j = 1, ..., k)$, and

$$\epsilon^{(r)} := (1 - \alpha_{\pi_i(k)})(D^{(r-1)} - A_{\pi_i(1)}^{(r)}) + \alpha_{\pi_i(k)}(D^{(r-1)} - A_{\pi_i(k)}^{(r)})$$

$$= D^{(r-1)} - \left\{(1 - \alpha_{\pi_i(k)})A_{\pi_i(1)}^{(r)} + \alpha_{\pi_i(k)}A_{\pi_i(k)}^{(r)}\right\}$$

$$= D^{(r-1)} - \left\{(1 - \alpha_{\pi_i(k)})(A_{\pi_i(1)}^{(r)} - C_i^{(r)}) + (1 - \alpha_{\pi_i(k)})C_i^{(r)} + \alpha_{\pi_i(k)}A_{\pi_i(k)}^{(r)}\right\}$$

$$= (D^{(r-1)} - D^{(r)}) + \left\{D^{(r)} - (1 - \alpha_{\pi_i(k)})C_i^{(r)} - \alpha_{\pi_i(k)}A_{\pi_i(k)}^{(r)}\right\} - (1 - \alpha_{\pi_i(k)})(A_{\pi_i(1)}^{(r)} - C_i^{(r)})$$

$$= (D^{(r-1)} - D^{(r)}) + I^{(r)} - (1 - \alpha_{\pi_i(k)})(A_{\pi_i(1)}^{(r)} - C_i^{(r)}).$$

Then since $D^{(r-1)} - D^{(r)} \to^s 0$, $I^{(r)} \to^s 0$ and $A_{\pi_i(1)}^{(r)} - C_i^{(r)} \to^s 0$, we see $\epsilon^{(r)} \to^s 0$, so that $D^{(r-1)} - A_{\pi_i(1)}^{(r)}, \ D^{(r-1)} - A_{\pi_i(k)}^{(r)} \to^s 0$. Hence

$$A_{\pi_i(1)}^{(r)}, \ A_{\pi_i(k)}^{(r)} \to^s \lim_{r \to \infty} D^{(r)} = G_S,$$

which is desired.   □

For the facts (i)-(ii), (i) can be shown by induction without difficulty. So it
suffices to show (ii). Let $S = \{\pi_1, \ldots, \pi_k\}$ be a subset of $\mathbf{S(k)}$, and let σ be an
element in S. Put

$$(2.8) \quad (\beta_1, \ldots, \beta_k) = \sigma(\alpha_1, \ldots, \alpha_k) = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)}), \text{ i.e., } \beta_i = \alpha_{\sigma(i)}, \text{ and}$$
$$(B_1, \ldots, B_k) = \sigma(A_1, \ldots, A_k) = (A_{\sigma(1)}, \ldots, A_{\sigma(k)}), \text{ i.e., } B_i = A_{\sigma(i)}.$$

We then define sequences $\{B_1^{(r)}\}, \ldots, \{B_k^{(r)}\}$, similarly as, $\{A_1^{(r)}\}, \ldots, \{A_k^{(r)}\}$ by
(2.4), that is,

$$B_i^{(1)} = B_i \quad (i = 1, \ldots, k), \text{ and for } r \geq 1,$$
$$B_i^{(r+1)} = g(\pi_i(\beta_1, \ldots, \beta_k; B_1^{(r)}, \ldots, B_k^{(r)}))$$
$$= g(\beta_{\pi_i(1)}, \ldots, \beta_{\pi_i(k)}; B_{\pi_i(1)}^{(r)}, \ldots, B_{\pi_i(k)}^{(r)}).$$

We show, by induction on r, that

$$(2.9) \quad\quad\quad\quad B_i^{(r)} = A_{\sigma(i)}^{(r)} \text{ for } i = 1, \ldots, k, \text{ and for } r \geq 1,$$

which implies that all sequences $\{B_i^{(r)}\}$, as a whole, coincide with those of $\{A_i^{(r)}\}$,
so that G_S is invariant under σ. Now for (2.9), it is clear for $r = 1$. So assume that
(2.9) holds (for r). Then

$$B_i^{(r+1)} = g(\pi_i(\beta_1, \ldots, \beta_k ; B_1^{(r)}, \ldots, B_k^{(r)}))$$
$$= g(\beta_{\pi_i(1)}, \ldots, \beta_{\pi_i(k)} ; B_{\pi_i(1)}^{(r)}, \ldots, B_{\pi_i(k)}^{(r)})$$
$$= g(\alpha_{\sigma(\pi_i(1))}, \ldots, \alpha_{\sigma(\pi_i(k))} ; A_{\sigma(\pi_i(1))}^{(r)}, \ldots, A_{\sigma(\pi_i(k))}^{(r)}) \quad (\text{by } (2.8))$$
$$= g(\alpha_{(\sigma\pi_i)(1)}, \ldots, \alpha_{(\sigma\pi_i)(k)} ; A_{(\sigma\pi_i)(1)}^{(r)}, \ldots, A_{(\sigma\pi_i)(k)}^{(r)})$$
$$= g(\sigma\pi_i(\alpha_1, \ldots, \alpha_k ; A_1^{(r)}, \ldots, A_k^{(r)}))$$
$$= g(\pi_{\sigma(i)}(\alpha_1, \ldots, \alpha_k ; A_1^{(r)}, \ldots, A_k^{(r)})) \quad (\text{by } (2.5))$$
$$= A_{\sigma(i)}^{(r+1)}.$$

Remark 2.3-1. Related to the assumption (2.5) in Theorem 2.2, we notice, by
definition,
$$\sigma(\pi_1, \ldots, \pi_k) = (\pi_{\sigma(1)}, \ldots, \pi_{\sigma(k)}),$$
so that the left side usually does not coincide with $(\sigma\pi_1, \ldots, \sigma\pi_k)$, which is multipli-
cation by σ to $(\pi_1, \ldots, \pi_k)$ from the left side. Hence (2.5) does not hold in general.

Remark 2.3-2. The assumption (2.5) in Theorem 2.2 needs the relation $S =$

$\sigma \cdot S \ (= \{\sigma\pi_1, \ldots, \sigma\pi_k\})$. In this case *the order of σ must be a factor of k, so that if k is a prime (and if $\sigma \neq id$) then S is the cyclic group (with order k) generated by σ*. The authors are very grateful to the referee for communicating those facts.

Example 2.4-1. Let $S = \{\pi_1, \pi_2, \pi_3, \pi_4\} \subset \mathbf{S(4)}$, with

$$(\pi_1, \pi_2, \pi_3, \pi_4) = (id, (12)(34), \tau, (12)(34)\tau),$$

where τ is an element in $\mathbf{S(4)}$. If $\sigma = \pi_2$, then
$$(\sigma\pi_1, \sigma\pi_2, \sigma\pi_3, \sigma\pi_4) = (\pi_2, \pi_1, \pi_4, \pi_3),$$
and
$$(\pi_{\sigma(1)}, \pi_{\sigma(2)}, \pi_{\sigma(3)}, \pi_{\sigma(4)}) = (\pi_2, \pi_1, \pi_4, \pi_3).$$
Hence by Theorem 2.2 (ii), G_S is permutation-invariant with respect to σ.

Example 2.4-2. Let $p = (12 \cdots k) \in \mathbf{S(k)}$ be a cyclic permutation of k letters, and let $S = \{\pi_1, \ldots, \pi_k\}$ with $\pi_i = p^{i-1}$. If $\sigma = p^j$, then

$$(\sigma\pi_1, \ldots, \sigma\pi_k) = (p^j, \ldots, p^{k+j-1}) = (\pi_{1+j}, \ldots, \pi_{k+j}).$$

Here $k + \ell > k$ is identified with ℓ. For $(\pi_{\sigma(1)}, \ldots, \pi_{\sigma(k)})$, since

$$\sigma = (12 \cdots k)^j = \left(\begin{array}{cccc} 1 & 2 & \cdots & k \\ 1+j & 2+j & \cdots & k+j \end{array} \right),$$

we see $\sigma(1) = 1 + j, \ldots, \sigma(k) = k + j$, so that

$$(\pi_{\sigma(1)}, \ldots, \pi_{\sigma(k)}) = (\pi_{1+j}, \ldots, \pi_{k+j}).$$

Hence G_S is permutation-invariant with respect to σ.

Example 2.4-3. Let

$$A_1 = \left[\begin{array}{cc} 5 & 2 \\ 2 & 1 \end{array} \right], \ A_2 = \left[\begin{array}{cc} 3 & \sqrt{2} \\ \sqrt{2} & 1 \end{array} \right], A_3 = \left[\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array} \right], \ A_4 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right],$$

$$w = (\alpha_1, \alpha_2, \alpha_3, \alpha_4), \quad \alpha_1 = \frac{1}{12}, \ \alpha_2 = \frac{1}{6}, \ \alpha_3 = \frac{1}{4}, \ \alpha_4 = \frac{1}{2},$$

$$S_1 = \{\pi_1, \pi_2, \pi_3, \pi_4\}, \quad \pi_1 = id, \ \pi_2 = (1234), \ \pi_3 = (1234)^2, \ \pi_4 = (1234)^3$$

and
$$S_2 = \{\pi_1, \pi_2, \pi_3, \pi_4\}, \quad \pi_1 = (23), \ \pi_2 = (34), \ \pi_3 = (243), \ \pi_4 = (123).$$

Then by numerical computation we have, (discarded less than 10^{-6},)

$$G_{S_1} = G_{S_1}(w; A_1, A_2, A_3, A_4)$$
$$= \left[\begin{array}{cc} 1.246\ 916 & 0.485\ 219 \\ 0.485\ 219 & 0.990\ 793 \end{array} \right] (= A_1^{(r)} = A_2^{(r)} = A_3^{(r)} = A_4^{(r)} \text{ for } r \geq 4),$$

$$G_{S_2} = G_{S_2}(w; A_1, A_2, A_3, A_4)$$

$$= \begin{bmatrix} 1.236\ 306 & 0.463\ 749 \\ 0.463\ 749 & 0.982\ 817 \end{bmatrix} \ (= A_1^{(r)} = A_2^{(r)} = A_3^{(r)} = A_4^{(r)} \text{ for } r \geq 3).$$

In the above, G_{S_1} is permutation-invariant with respect to (all $\sigma \in$) S_1, but G_{S_2} is not so with respect to any $\sigma \in S_2$: Say, for $\sigma = (23)$,

$$G_{S_2}(\sigma(w; A_1, A_2, A_3, A_4))$$
$$= G_{S_2}(\alpha_1,\ \alpha_3,\ \alpha_2,\ \alpha_4; A_1, A_3, A_2, A_4)$$
$$(= G_{S_2}(\beta_1,\ \beta_2,\ \beta_3,\ \beta_4; B_1, B_2, B_3, B_4))$$
$$= \begin{bmatrix} 1.241\ 734 & 0.457\ 126 \\ 0.457\ 126 & 0.973\ 609 \end{bmatrix} \ (= B_1^{(r)} = B_2^{(r)} = B_3^{(r)} = B_4^{(r)} \text{ for } r \geq 4).$$

In Theorem 2.2, we took a subset with k elements in $\mathbf{S(k)}$ to construct a weighted geometric mean of k operators. We could take a subset with ℓ ($\ell \leq k$ or $\ell \geq k$) elements:

Theorem 2.5. *Let* $A_1, A_2, \cdots, A_k$ *be* k *operators in* Ω, $w = (\alpha_1, ..., \alpha_k)$ *be a* k-*weight of nonnegative numbers, and let* $S = \{\pi_1, ..., \pi_\ell\}$ *be a subset of* $\mathbf{S(k)}$. *Define*

$$B_i = g(\pi_i(w; A_1, ..., A_k)), \quad i = 1, ..., \ell.$$

Further, let $v = (\beta_1, ..., \beta_\ell)$ *be an* ℓ-*weight of nonnegative numbers and let* $T = \{\sigma_1, ..., \sigma_\ell\}$ *be a subset of* $\mathbf{S(\ell)}$. *Then, define*

$$\tilde{G} = G_{S,T}(w; v; A_1, ..., A_k) = G_T(v; B_1, ..., B_\ell).$$

Then $\tilde{G}$ *is a weighted geometric mean which has the properties PW1-PW10 except PW3. If* $\ell = k, \beta_i = \alpha_i$ $(i = 1, ..., k)$ *and* $T = S$, *then* $\tilde{G} = G_S(w; A_1, ..., A_k)$.

3. Weighted geometric means of k operators by indirect construction; second type

Recall that non-weighted (or uniformly weighted) geometric means of k operators were defined in [14], [9] (cf. [2]) by induction on k, as follows:

(1) First for $k = 2$, define $G(A_1, A_2) = G_\#(A_1, A_2) = A_1 \# A_2$ (the usual geometric mean) for two operators A_1 and A_2. Then $G(A_1, A_2)$ satisfies all properties P1-P10 (for $k = 2$).

(2) To define geometric means for k (≥ 3) operators, we assume that for $(k-1)$ operators $A_1, ..., A_{k-1}$ we have obtained a geometric mean $G(A_1, ..., A_{k-1})$ possessing all properties P1-P10.

(3) Then we can define a geometric mean $G_\lambda(A_1, ..., A_k)$ of k operators $A_1, ..., A_k$ with a parameter $\lambda \in (0, 1]$ as the common limit (with respect to the Thompson metric) of the sequences $\{A_i^{(r)}\}_{r=1}^\infty$ $(i = 1, ..., k)$ defined by

$$(3.1) \qquad A_i^{(1)} = A_i \ \text{ and } \ A_i^{(r+1)} = G((A_j^{(r)})_{j \neq i}) \#_\lambda A_i^{(r)} \text{ for } r \geq 1.$$

Here

$$G((A_j^{(r)})_{j \neq i}) = G(A_1^{(r)}, ..., A_{i-1}^{(r)}, A_{i+1}^{(r)}, ..., A_k^{(r)}).$$

The operator $G_\lambda(A_1, ..., A_k)$ then again has all the ten properties P1-P10 as desired.

We remark that the above definition (3.1) of the sequences $\{A_i^{(r)}\}_{r=1}^\infty$ is different from that in [9]. However, clearly the common limit is the same as in [9] if the parameter λ is replaced by $1 - \lambda$.

For the convenience sake, we write $G \in \mathbf{G(k)}$ if G *is a geometric mean of k operators in Ω with the properties* P1-P10. We define [9]

$$G_{\#,\lambda_1,...,\lambda_k} = G_{\#,\lambda_1,...,\lambda_k}(A_1, ..., A_{k+2})$$

by

$$G_{\#,\lambda_1,...,\lambda_k} = G_{(\#,\lambda_1,...,\lambda_{k-1}),\lambda_k}.$$

Then $G_{\#,\lambda_1,...,\lambda_k} \in \mathbf{G(k+2)}$ is a geometric mean with parameters $\lambda_1, ..., \lambda_k$ ($k \geq 1$).

Now we define weighted geometric means for k operators through the non-weighted geometric means by induction on k. First for two operators $A_1, A_2 \in \Omega$ and for any 2-weight $w = (\alpha_1, \alpha_2)$ ($\alpha_2 = 1 - \alpha_1$), we put

$$G(w; A_1, A_2) = G(\alpha_1, \alpha_2; A_1, A_2) = A_1 \#_{\alpha_2} A_2.$$

Then it is easy to see that this mean has all ten properties PW1-PW10. In particular, as an basic fact $A_2 \#_{\alpha_1} A_1 = A_1 \#_{1-\alpha_1} A_2$, we see

$$G(\alpha_2, \alpha_1; A_2, A_1) = G(\alpha_1, \alpha_2; A_1, A_2),$$

which implies that $G(w; A_1, A_2)$ has permutation-invariance, PW3.

To define the weighted geometric mean for k (≥ 3) operators, we assume that $G = G(v; X_1, ..., X_{k-1})$ is a weighted geometric mean of $(k-1)$ operators in Ω for any $(k-1)$-weight v. Then for given k operators $A_1, ..., A_k$, and for a k-weight $w = (\alpha_1, ..., \alpha_k)$ (without loss of generality, assuming $0 < \alpha_1, ..., \alpha_k < 1$), define, for $i = 1, ..., k$,

$$B_i = G(w_i'; A_1, ..., A_{i-1}, A_{i+1}, ..., A_k) \#_{\alpha_i} A_i,$$

where $w_i' = \left(\dfrac{\alpha_1}{1 - \alpha_i}, ..., \dfrac{\alpha_{i-1}}{1 - \alpha_i}, \dfrac{\alpha_{i+1}}{1 - \alpha_i}, ..., \dfrac{\alpha_k}{1 - \alpha_i} \right)$. Now choose a (non-weighted) geometric mean $\Gamma \in \mathbf{G(k)}$, and define

$$\tilde{G} = G_\Gamma(w; A_1, ..., A_k) := \Gamma(B_1, ..., B_k).$$

Then it is not difficult to see that $\tilde{G}$ is a desired weighted geometric mean of k operators, which has all the ten properties PW1-PW10. Further, if $w = (\frac{1}{k}, ..., \frac{1}{k})$ then $\tilde{G} = \Gamma(B_1, ..., B_k)$ by definition, so that $\tilde{G}$ is an extension of the original geometric mean. Rewriting the above facts as a result, we have

Theorem 3.1. *For any k (≥ 2) operators $A_1, ..., A_k \in \Omega$ and for any k-weight $w = (\alpha_1, ..., \alpha_k)$, we can define a weighted geometric mean $G_\Gamma(w; A_1, ..., A_k)$ which has the ten properties PW1-PW10 for k operators in Ω, as an extension of a non-weighted geometric mean $\Gamma \in \mathbf{G}(\mathbf{k})$.*

Example 3.2. To compare weighted geometric means of the two types, let

$$A_1 = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$w = (\alpha_1, \alpha_2, \alpha_3), \quad \alpha_1 = \frac{1}{2}, \ \alpha_2 = \frac{1}{3}, \ \alpha_3 = \frac{1}{6}.$$

First choose $\Gamma = G_{\#, \frac{1}{3}} \in \mathbf{G}(3)$. Then

$$G_\Gamma(w; A_1, A_2, A_3) = \Gamma(B_1, B_2, B_3) = G_{\#, \frac{1}{3}}(B_1, B_2, B_3).$$

Here B_1, B_2, B_3 are defined as follows:

$$B_1 = G\left(\frac{\alpha_2}{1-\alpha_1}, \frac{\alpha_3}{1-\alpha_1}; A_2, A_3\right) \#_{\alpha_1} A_1 = (A_2 \#_{\frac{1}{3}} A_3) \#_{\frac{1}{2}} A_1,$$

$$B_2 = G\left(\frac{\alpha_1}{1-\alpha_2}, \frac{\alpha_3}{1-\alpha_2}; A_1, A_3\right) \#_{\alpha_2} A_2 = (A_1 \#_{\frac{1}{4}} A_3) \#_{\frac{1}{3}} A_2,$$

$$B_3 = G\left(\frac{\alpha_1}{1-\alpha_3}, \frac{\alpha_2}{1-\alpha_3}; A_1, A_2\right) \#_{\alpha_3} A_3 = (A_1 \#_{\frac{2}{5}} A_2) \#_{\frac{1}{6}} A_3.$$

We then, (discarded less than 10^{-6},) have

$$G_\Gamma(w; A_1, A_2, A_3) = \begin{bmatrix} 2.039\ 159 & 0.903\ 343 \\ 0.903\ 343 & 0.890\ 577 \end{bmatrix} (= B_1^{(r)} = B_2^{(r)} = B_3^{(r)} \text{ for } r \geq 20).$$

Next for $S = \{id, (123), (123)^2\} \subset \mathbf{S}(\mathbf{3})$, we have

$$G_S(w; A_1, A_2, A_3) = \begin{bmatrix} 2.037\ 846 & 0.895\ 118 \\ 0.895\ 118 & 0.883\ 892 \end{bmatrix} (= A_1^{(r)} = A_2^{(r)} = A_3^{(r)} \text{ for } r \geq 4).$$

The geometric mean $G_\Gamma(w; A_1, A_2, A_3)$ is of second type and $G_S(w; A_1, A_2, A_3)$ is of first type. The former is permutation-invariant with respect to $\mathbf{S}(\mathbf{3})$, and the latter is so with respect to S.

Remark 3.3. We have constructed the weighted geometric means, starting from the definition (1.2) by successive composition of two-operator weighted geometric means. However, we could begin with another definition by such composition of two-operator weighted geometric means, so that we could construct various weighted geometric means. For example, for four operators $A_1, A_2, A_3, A_4 \in \Omega$ with 4-weight

$w = (\alpha_1, ..., \alpha_4)$ of positive numbers, we can consider the following five types of composition:

$$G^{[1]} = g^{[1]}(w; A_1, A_2, A_3, A_4) = \left((A_1 \#_{\frac{\alpha_2}{\alpha_1+\alpha_2}} A_2) \#_{\frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3}} A_3 \right) \#_{\alpha_4} A_4,$$

$$G^{[2]} = g^{[2]}(w; A_1, A_2, A_3, A_4) = \left(A_1 \#_{\frac{\alpha_2+\alpha_3}{\alpha_1+\alpha_2+\alpha_3}} (A_2 \#_{\frac{\alpha_3}{\alpha_2+\alpha_3}} A_3) \right) \#_{\alpha_4} A_4,$$

$$G^{[3]} = g^{[3]}(w; A_1, A_2, A_3, A_4) = \left(A_1 \#_{\frac{\alpha_2}{\alpha_1+\alpha_2}} A_2 \right) \#_{\alpha_3+\alpha_4} \left(A_3 \#_{\frac{\alpha_4}{\alpha_3+\alpha_4}} A_4 \right),$$

$$G^{[4]} = g^{[4]}(w; A_1, A_2, A_3, A_4) = A_1 \#_{\alpha_2+\alpha_3+\alpha_4} \left((A_2 \#_{\frac{\alpha_3}{\alpha_2+\alpha_3}} A_3) \#_{\frac{\alpha_4}{\alpha_2+\alpha_3+\alpha_4}} A_4 \right),$$

$$G^{[5]} = g^{[5]}(w; A_1, A_2, A_3, A_4) = A_1 \#_{\alpha_2+\alpha_3+\alpha_4} \left(A_2 \#_{\frac{\alpha_3+\alpha_4}{\alpha_2+\alpha_3+\alpha_4}} (A_3 \#_{\frac{\alpha_4}{\alpha_3+\alpha_4}} A_4) \right).$$

Recall that $G^{[1]}$ is the type of (1.2) for $k = 4$. Clearly, each of the above $G^{[i]}$ reduce to $A_1^{\alpha_1} A_2^{\alpha_2} A_3^{\alpha_3} A_4^{\alpha_4}$, if all A_i commute. Consequently, we can choose any of them for the starting point of our definition of the weighted geometric mean with respect to four operators. If we choose, *say*, the type of $G^{[3]}$, then, put

$$\begin{cases} B_1 = g^{[3]}(\alpha_1, \alpha_2, \alpha_3, \alpha_4; A_1, A_2, A_3, A_4), \\ B_2 = g^{[3]}(\alpha_1, \alpha_3, \alpha_2, \alpha_4; A_1, A_3, A_2, A_4), \\ B_3 = g^{[3]}(\alpha_1, \alpha_4, \alpha_2, \alpha_3; A_1, A_4, A_2, A_3). \end{cases}$$

First, say, for $S = \{\pi_1 = id, \pi_2 = (123), \pi_3 = (123)^2\} \subset \mathbf{S}(3)$, and a 3-weight $v = (\beta_1, \beta_2, \beta_3)$, define

$$\begin{cases} B_i^{(1)} = B_i \text{ for } i = 1, 2, 3, \text{ and} \\ B_i^{(r+1)} = g(\pi_i(v; B_1^{(r)}, B_2^{(r)}, B_3^{(r)})) \text{ for } i = 1, 2, 3, \ r \geq 1. \end{cases}$$

Then as the common limit of $\{B_i^{(r)}\}$ $(i = 1, 2, 3)$, we obtain a weighted geometric mean of A_1, A_2, A_3, A_4 related to $G^{[3]}$. Denote by

$$\hat{G} = G_S(w; v; A_1, A_2, A_3, A_4) = \lim_{r \to \infty} B_i^{(r)}.$$

Then $\hat{G}$ is a weighted geometric mean corresponding to one given as the first type in Section 2.

Second, define, for a $\Gamma \in \mathbf{G(3)}$,

$$\tilde{G} = G_\Gamma(w; A_1, A_2, A_3, A_4) = \Gamma(B_1, B_2, B_3).$$

Then we obtain a weighted geometric mean $\tilde{G}$ related to $G^{[3]}$, which corresponds to one given as the second type in Section 3.

4. Reverse inequality

Recently, reverse type inequalities of the arithmetic-geometric, the arithmetic-harmonic, or the arithmetic-geometric-harmonic means for two or more than two operators were presented in [1], [6], [7], [16]. The following result was shown in [7]:

Lemma 4.1([7, Theorem 9]). *Let $A_1, A_2, \ldots, A_k$ be operators such that $0 < mI \leq A_i \leq MI$ for $i = 1, 2, \ldots, k$ for some scalars m and M with $0 < m < M$. (The letter I stands the identity operator.) Then*

$$\frac{A_1 + \cdots + A_k}{k} \leq \frac{(M+m)^2}{4Mm} \left(\frac{A_1^{-1} + \cdots + A_k^{-1}}{n} \right)^{-1}.$$

For the reverse version of the weighted mean inequality, the following result was shown in [14]:

Proposition 4.2([14, Proposition 4.3]). *Let $mI \leq A, B, C \leq MI$ for some scalars m and M with $0 < m < M$. Then for $\alpha, \beta, \gamma \geq 0, \alpha + \beta + \gamma = 1, \Gamma = G_{\#, \frac{2}{3}}$,*

$$\alpha A + \beta B + \gamma C \leq \frac{(M+m)^2}{4Mm} G_\Gamma(\alpha, \beta, \gamma \; ; A, B, C).$$

As an extension of both the above results, we have the following:

Theorem 4.3. *Let $A_1, A_2, \ldots, A_k$ be operators such that $0 < mI \leq A_i \leq MI$ for $i = 1, \ldots, k$ for some scalars m and M with $0 < m < M$. Assume that $w = (\alpha_1, ..., \alpha_k)$ is a k-weight. Then*

$$(4.1) \qquad \sum_{i=1}^{k} \alpha_i A_i \leq \frac{(M+m)^2}{4Mm} G_\Gamma(w; A_1, \ldots, A_k)$$

and

$$(4.2) \qquad \sum_{i=1}^{k} \alpha_i A_i \leq \frac{(M+m)^2}{4Mm} G_S(w; A_1, \ldots, A_k),$$

where Γ is a geometric mean in $\mathbf{G}(k)$ and S is a subset of k elements in $\mathbf{S}(\mathbf{k})$.

Proof. Let $\Phi : B(H) \oplus \cdots \oplus B(H) \mapsto B(H) \oplus \cdots \oplus B(H)$ be a map defined as follows:

$$\Phi(X_1, \ldots, X_k) = \begin{bmatrix} \alpha_1 X_1 + \cdots + \alpha_k X_k & & 0 \\ & \ddots & \\ 0 & & \alpha_1 X_1 + \cdots + \alpha_k X_k \end{bmatrix}.$$

Then Φ is positive linear, and normalized, that is, $\Phi(I^{(k)}) = I^{(k)}$ $(I^{(k)} = (I, \cdots, I).)$ Furthermore, it is not difficult to see that $mI^{(k)} \leq \Phi(A_1, \ldots, A_k) \leq MI^{(k)}$. Hence by a result due to Mond-Pečarić [8, Theorem 1.32], we have

$$\Phi(A_1, \ldots, A_k) \leq \frac{(M+m)^2}{4Mm} \Phi(A_1^{-1}, \ldots, A_k^{-1})^{-1}.$$

Hence

$$\alpha_1 X_1 + \cdots + \alpha_k X_k \leq \frac{(M+m)^2}{4Mm}(\alpha_1 X_1^{-1} + \cdots + \alpha_k X_k^{-1})^{-1}.$$

From this inequality and PW10 for $G = G_\Gamma, G_S$, we obtain the desired inequalities (4.1) and (4.2). $\qquad\square$

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