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Weighted inequalities for multilinear fractional integral operators

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Abstract

A weighted theory for multilinear fractional integral operators and maximal functions is presented. Sufficient conditions for the two weight inequalities of these operators are found, including "power and logarithmic bumps" and an A_∞ condition. For one weight inequalities a necessary and sufficient condition is then obtained as a consequence of the two weight inequalities. As an application, Poincaré and Sobolev inequalities adapted to the multilinear setting are presented.

1. Introduction

As it is well-known, Muckenhoupt [15] characterized the weights w, for which the Hardy-Littlewood maximal operator, M, is bounded on $L^p(w)$ for $1 < p < \infty$. He showed that M is bounded on $L^p(w)$ if and only if w belongs to the class A_p, i.e.,

$$[w]_{A_p} = \sup_Q \Big(\frac{1}{|Q|}\int_Q w\ dx\Big)\Big(\frac{1}{|Q|}\int_Q w^{1-p'}\ dx\Big)^{p-1} < \infty.$$

Sawyer [21] characterized the two weight inequality, showing that $M : L^p(v) \to L^p(u)$ if and only if the pair (u, v) satisfies the testing condition

$$[u, v]_{S_p} = \sup_Q \frac{\int_Q M(\chi_Q\sigma)^p u\ dx}{\sigma(Q)} < \infty$$

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where $\sigma = v^{1-p'}$. The Hardy-Littlewood maximal operator and the weighted estimates it satisfies play a very important role in harmonic analysis. In particular M is intimately related to the study of singular integral operators.

Also of importance in harmonic analysis is the study of fractional type operators and associated maximal functions. Recall the definition of the fractional integral operator or Riesz potential,

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n\,.$$

and the related maximal function,

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)|\, dy, \quad 0 \le \alpha < n\,,$$

where the supremum is taken over all cubes Q containing x. Note that the case $\alpha = 0$ corresponds to the Hardy-Littlewood maximal operator. A nice exposition about the properties of the operators particularly I_α can be found in the books by Stein [24] and Grafakos [8].

Weighted estimates for I_α have been studied as well. In Muckenhoupt and Wheeden [16], characterized the one weight strong type inequality,

$$\Big(\int_{\mathbb{R}^n} (I_\alpha f w)^q \, dx \Big)^{1/q} \le C \Big(\int_{\mathbb{R}^n} (fw)^p \, dx \Big)^{1/p}, \tag{1.1}$$

where, $f \ge 0$, $1 < p < n/\alpha$ and q is defined by $1/q = 1/p - \alpha/n$. They showed that (1.1) holds if and only if $w \in A_{p,q}$ i.e.,

$$[w]_{A_{p,q}} = \sup_Q \Big(\frac{1}{|Q|} \int_Q w^q \, dx \Big)^{1/q} \Big(\frac{1}{|Q|} \int_Q w^{-p'} \, dx \Big)^{1/p'} < \infty\,.$$

These estimates are of interest on their own and they also have relevance to partial differential equations and quantum mechanics. We refer the reader to Sawyer and Wheeden [23], Kerman and Sawyer [13] for further information and applications.

A characterization for the two weight inequality for I_α was given by Sawyer [22]. Further results for these operators and more general potential operators were obtained by Pérez [18, 19]. Of particular interest are the results in [19] where the "power bump" condition due to Neugebauer [17] is extended using Banach function spaces. See Cruz-Uribe, Martell, and Pérez, [4], for further references and historical remarks.

Multilinear maximal functions appear naturally in connection with multilinear Calderón-Zygmund theory. Lerner, Ombrosi, Pérez, Torres, and Trujillo-González [14] developed a weighted theory for the multi(sub)linear maximal function

$$\mathcal{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{i=1}^m \frac{1}{|Q|} \int_Q |f_i(y_i)| \, dy_i$$

where $\vec{f} = (f_1, \ldots, f_m)$. They showed that for $1 < p_1, \ldots, p_m < \infty$, and p given by $1/p = 1/p_1 + \cdots + 1/p_m$,

$$\|\mathcal{M}\vec{f}\|_{L^p(\nu_{\vec{w}})} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}$$

where $\nu_w = \prod_{i=1}^m w_i^{p/p_i}$ if and only if $\vec{w} \in A_{\vec{P}}$ i.e.,

$$[\vec{w}]_{A_{\vec{P}}} = \sup_Q \Big(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}\, dx\Big)^{1/p} \prod_{i=1}^m \Big(\frac{1}{|Q|}\int_Q w_i^{1-p_i'}\, dx\Big)^{1/p_i'} < \infty\,.$$

This lead to a development of a multilinear weighted theory for multilinear Calderón-Zygmund operators and other operators which answered questions posed in earlier works on the subject by Grafakos and Torres [10, 11], and Pérez and Torres [20].

Motivated by the work in [14] we consider here the multilinear fractional case. Multilinear fractional integral operators were studied by Grafakos [7], Kenig and Stein [12], Grafakos and Kalton [9]. These works present multilinear generalizations of the bilinear operator,

$$B_\alpha(f,g)(x) = \int_{\mathbb{R}^n} \frac{f(x+t)g(x-t)}{|t|^{n-\alpha}} dt, \qquad 0 < \alpha < n\,.$$

They showed that B_α maps $L^{p_1} \times L^{p_2}$ into L^q where $1/q = 1/p_1 + 1/p_2 - \alpha/n$. As a tool to understand B_α, the operators

$$\mathcal{I}_\alpha \vec{f}(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1| + \cdots + |x-y_m|)^{mn-\alpha}} d\vec{y} \qquad 0 < \alpha < nm$$

were studied as well. We examine the one and two weight theory for these last operators and the corresponding multi(sub)linear fractional maximal operators,

$$\mathcal{M}_\alpha \vec{f}(x) = \sup_{Q \ni x} \prod_{i=1}^m \frac{1}{|Q|^{1-\alpha/nm}} \int_Q |f_i(y_i)|\, dy_i.$$

We obtain much of the multilinear counter part of the linear results in [19]. The extension to the multilinear setting, however, is not immediate and new ideas are required. We also prove the multilinear analog of the result in [16] In particular we show that the inequality,

$$\Big(\int_{\mathbb{R}^n} (|\mathcal{I}_\alpha \vec{f}|(\Pi_{i=1}^m w_i))^q\, dx\Big)^{1/q} \le C \prod_{i=1}^m \Big(\int_{\mathbb{R}^n} (|f_i| w_i)^{p_i}\, dx\Big)^{1/p_i}$$

if and only if $\vec{w} = (w_1, \ldots, w_m)$ satisfies the $A_{\vec{P},q}$ condition,

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q (\Pi_{i=1}^m w_i)^q\Big)^{1/q} \prod_{i=1}^m \Big(\frac{1}{|Q|}\int_Q {w_i}^{-p_i'}\Big)^{1/p_i'} < \infty\,.$$

The general organization of the paper is as follows. Section 2 contains some preliminary definitions and the statements of the two weight results. Many of these are corollaries of our main results Theorem 2.2 and Theorem 2.8. Section 3 contains the statements of the one weight results. The proof of the two weight theorems are in Section 4 while the proof of the one weight theorems are in Section 5. In Section 6 we present a version of Theorems 2.2 and 2.8 in the more general context of Banach function spaces and give examples. Finally, we present in Section 7 some applications of the theory, including Poincaré and Sobolev inequalities for products of functions.

The work [14] has spurred several efforts by other authors in the study of multilinear fractional integrals and maximal operators. After the research presented here was completed we were informed of recent, simultaneous, and independent results by Xi Chen and Qingying Xue [2] and Alberto de la Torre [5]. In the one weight case, Chen and Xue have investigated sufficient conditions for boundedness of multilinear fractional operators, arriving to the same one we have. De la Torre has also found the same condition we found. He has a different proof for the sufficiency of the condition for the multilinear fractional maximal operator. We not only prove sufficiency but also the necessity of the condition and our methods of proof are different from those in [2] and [5].

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2. Two weight results

We recall some standard notation. Throughout this paper we consider cubes, usually denoted Q, with sides parallel to the axes. The side length of a cube Q is denoted $\ell(Q)$, and aQ, $a>0$, denotes the cube concentric with Q and side length $a\ell(Q)$. The set of dyadic cubes in $\mathbb{R}^d$, denoted $\mathcal{D}(\mathbb{R}^d)$ or simply $\mathcal{D}$ when the dimension is evident, is the set of all half open cubes of the form $Q_{m,k}=2^k(m+[0,1)^d)$ where $k\in\mathbb{Z}$ and $m\in\mathbb{Z}^d$. For convinience if $Q\in\mathcal{D}$ we will say $\ell(Q)=2^k$ for some $k\in\mathbb{Z}$. In this article a weight is simply a non-negative measurable function w.

Given a measurable set E, $w(E)$ will denoted the weighted measure of E, i.e., $w(E)=\int_E w\,dx$. Occasionally we will use the notation $T:X\to Y$ to mean T is a bounded operator from X to Y, i.e. $\|Tx\|_Y\leq C\|x\|_X$ for all x in X. The multilinear version, $T:X_1\times\cdots\times X_m\to Y$ means, $\|T(x_1,\ldots,x_m)\|_Y\leq C\Pi_i\|x_i\|_{X_i}$ for all $(x_1,\ldots,x_m)$ in $X_1\times\cdots\times X_m$. For each $1\leq q\leq\infty$, q' will denote the dual exponent of q, i.e., $q'=q/(q-1)$ with the usual modifications $1'=\infty$ and $\infty'=1$. Finally, given a set of m exponents $1\leq p_1,\ldots,p_m\leq\infty$, $\vec{P}$ will denote the m-tuple, $(p_1,\ldots,p_m)$, and p will often denote the number defined by the Hölder relationship

$$\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}.$$

DEFINITION 2.1 Let α be a number such that $0<\alpha<mn$ and $\vec{f}=(f_1,\ldots,f_m)$ be a collection of m functions on $\mathbb{R}^n$. We define the multilinear fractional integral as

$$\mathcal{I}_\alpha\vec{f}(x)=\int_{(\mathbb{R}^n)^m}\frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}}d\vec{y}\,,$$

where the integral is convergent if $\vec{f}\in\mathcal{S}\times\cdots\times\mathcal{S}$.

Our main result about $\mathcal{I}_\alpha$ is the following.

Theorem 2.2

Suppose that $0<\alpha<nm$, $1<p_1,\ldots,p_m<\infty$ and q is a number that satisfies $1/m<p\leq q<\infty$. Suppose that one of the following two conditions holds.

i) *$q > 1$ and $(u, \vec{v})$ are weights that satisfy*

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q-1/p} \Big(\frac{1}{|Q|} \int_Q u^{qr} \, dx \Big)^{1/qr} \prod_{i=1}^m \Big(\frac{1}{|Q|} \int_Q {v_i}^{-p_i' r} \, dx \Big)^{1/p_i' r} < \infty \tag{2.1}$$

for some $r > 1$.
ii) *$q \le 1$ and $(u, \vec{v})$ are weights that satisfy*

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q-1/p} \Big(\frac{1}{|Q|} \int_Q u^{q} \, dx \Big)^{1/q} \prod_{i=1}^m \Big(\frac{1}{|Q|} \int_Q {v_i}^{-p_i' r} \, dx \Big)^{1/p_i' r} < \infty \tag{2.2}$$

for some $r > 1$.
Then the inequality,

$$\Big(\int_{\mathbb{R}^n} (|\mathcal{I}_\alpha \vec{f}| u)^q \, dx \Big)^{1/q} \le C \prod_{i=1}^m \Big(\int_{\mathbb{R}^n} (|f_i| v_i)^{p_i} \, dx \Big)^{1/p_i}$$

holds for all $\vec{f} \in L^{p_1}(v_1^{p_1}) \times \cdots \times L^{p_m}(v_m^{p_m})$.

Remark 2.3 We notice that conditions (2.1) and (2.2) are two sided and one sided conditions respectively. For $q > 1$ we must require a stronger norm i.e., a "power bump" for u^q and the ${v_i}^{-p_i'}$'s. However for $q \le 1$ we only need to power bump the $v_i^{-p_i'}$ weights. This is similar to the condition for the operator $\mathcal{M}_\alpha$ below.

Corollary 2.4

Suppose that $0 < \alpha < nm$, $1 < p_1, \ldots, p_m < \infty$ and q is such that $1/m < p \le q < \infty$. Further suppose that $u, v_1, \ldots, v_m$ are weights with $u^q, {v_1}^{-p_1'}, \ldots, {v_m}^{-p_m'} \in A_\infty$, that satisfy,

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q-1/p} \Big(\frac{1}{|Q|} \int_Q u^{q} \, dx \Big)^{1/q} \prod_{i=1}^m \Big(\frac{1}{|Q|} \int_Q {v_i}^{-p_i'} \, dx \Big)^{1/p_i'} < \infty.$$

Then,

$$\Big(\int_{\mathbb{R}^n} (|\mathcal{I}_\alpha \vec{f}| u)^q \, dx \Big)^{1/q} \le C \prod_{i=1}^m \Big(\int_{\mathbb{R}^n} (|f_i| v_i)^{p_i} \, dx \Big)^{1/p_i}$$

holds for all $\vec{f} \in L^{p_1}(v_1^{p_1}) \times \cdots \times L^{p_m}(v_m^{p_m})$.

The above corollary is a direct consequence of the fact that A_∞ weights satisfy the reverse Hölder condition. We give the proof of Theorem 2.2 in full detail in Section 4. We stated the above results and proofs in the L^p context for clarity in the presentation and because it is what is needed for the one weight theory. There is however a better result that we obtain using Banach function spaces. In particular we have the following.

Theorem 2.5

Suppose that $0 < \alpha < nm$, $1 < p_1, \ldots, p_m < \infty$ and q is a number that satisfies $1/m < p \le q < \infty$. Suppose that one of the following two conditions holds.

i) $q > 1$ *and* $(u, \vec{v})$ *are weights that satisfy*

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q-1/p} \|u\|_{L^q(\log L)^{q-1+\delta}(Q,dx/|Q|)} \times \prod_{i=1}^m \|v_i{}^{-1}\|_{L^{p'_i}(\log L)^{p'_i-1+\delta}(Q,dx/|Q|)} < \infty \tag{2.3}$$

for some $\delta > 0$.
ii) $q \leq 1$ *and* $(u, \vec{v})$ *are weights that satisfy*

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q-1/p} \Big(\frac{1}{|Q|}\int_Q u^q\ dx\Big)^{1/q} \prod_{i=1}^m \|v_i{}^{-1}\|_{L^{p'_i}(\log L)^{p'_i-1+\delta}(Q,dx/|Q|)} < \infty \tag{2.4}$$

for some $\delta > 0$.
Then the inequality,

$$\Big(\int_{\mathbb{R}^n} (|\mathcal{I}_\alpha \vec{f}|u)^q\ dx\Big)^{1/q} \leq C \prod_{i=1}^m \Big(\int_{\mathbb{R}^n} (|f_i|v_i)^{p_i}\ dx\Big)^{1/p_i}$$

holds for all $\vec{f} \in L^{p_1}(v_1^{p_1}) \times \cdots \times L^{p_m}(v_m^{p_m})$.

We describe in Section 6 how to modify the proof of Theorem 2.2 so that it applies to the more abstract setting of Banach function spaces. Also see Section 6 for pertinent definitions in the above theorem.

We now turn our attention to the fractional multi(sub)linear maximal operater.

DEFINITION 2.6 For $0 \leq \alpha < nm$ and $\vec{f} = (f_1, \ldots, f_m) \in L^1_{\rm loc} \times \cdots \times L^1_{\rm loc}$, we define the multi(sub)linear maximal operator $\mathcal{M}_\alpha$ by

$$\mathcal{M}_\alpha \vec{f}(x) = \sup_{Q \ni x} \prod_{i=1}^m \frac{\ell(Q)^{\alpha/m}}{|Q|} \int_Q |f_i(y_i)|\ dy_i.$$

We will refer to this as the multilinear maximal function. Notice that the case $\alpha = 0$ corresponds to the multi(sub)linear maximal function $\mathcal{M}$ studied in [14]. In [14] it is shown that, for $1 \leq p_1, \ldots, p_m < \infty$ the weak inequality

$$\|\mathcal{M}\vec{f}\|_{L^{p,\infty}(u)} \leq C \prod_{i=1}^m \|f_i\|_{L^{p_i}(v_i)}$$

holds if and only if $(u, \vec{v})$ satisfies the condition,

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q u\ dx\Big)^{1/p} \prod_{i=1}^m \Big(\frac{1}{|Q|}\int_Q v_i{}^{1-p'_i}\ dx\Big)^{1/p'_i} < \infty. \tag{2.5}$$

When $p_j = 1$, $(\frac{1}{|Q|}\int_Q v_j{}^{1-p'_j}\ dx)^{1/p'_j}$ is understood as $(\inf_Q v_j)^{-1}$. There is a corresponding weak characterization for $\mathcal{M}_\alpha$.

Theorem 2.7

Suppose that $0 \le \alpha < nm$, $1 \le p_1, \dots, p_m < \infty$ *and* q *is a number satisfying* $1/m < p \le q < \infty$. *Then the inequality,*

$$\|\mathcal{M}_\alpha \vec{f}\|_{L^{q,\infty}(u)} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(v_i)}$$

holds if and only if the weights $(u, \vec{v})$ *satisfy,*

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q-1/p} \Big(\frac{1}{|Q|}\int_Q u\ dx\Big)^{1/q} \prod_{i=1}^m \Big(\frac{1}{|Q|}\int_Q {v_i}^{1-p_i'}\ dx\Big)^{1/p_i'} < \infty.$$

Here $\big(\frac{1}{|Q|}\int_Q {v_j}^{1-p_j'}\ dx\big)^{1/p_j'}$ *is understood as* $(\inf_Q v_j)^{-1}$ *when* $p_j = 1$.

We now state the main theorem for $\mathcal{M}_\alpha$. For the next results, we preform the normalization $u \mapsto u^q$ and $v_i \mapsto v_i^{p_i}$. This simplifies matters for the extensions to Banach function spaces in Section 6 and makes computations in the one weight case easier.

Theorem 2.8

Suppose $0 \le \alpha < nm$, $1 < p_1, \dots, p_m < \infty$, *and* q *is a number such that* $1/m < p \le q < \infty$. *If* $(u, \vec{v})$ *are weights that satisfy*

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q-1/p} \Big(\frac{1}{|Q|}\int_Q u^q\ dx\Big)^{1/q} \prod_{i=1}^m \Big(\frac{1}{|Q|}\int_Q v_i^{-rp_i'}\ dx\Big)^{1/rp_i'} < \infty, \tag{2.6}$$

for some $r > 1$, *then*

$$\Big(\int_{\mathbb{R}^n} (\mathcal{M}_\alpha \vec{f} u)^q\ dx\Big)^{1/q} \le C \prod_{i=1}^m \Big(\int_{\mathbb{R}^n} (|f_i| v_i)^{p_i}\ dx\Big)^{1/p_i}$$

holds for all $\vec{f} \in L^{p_1}(v_1^{p_1}) \times \cdots \times L^{p_m}(v_m^{p_m})$.

Remark 2.9 As before this is a one-sided condition and the previous remarks can be adapted to the multilinear fractional maximal function, i.e., we may assume that $v_i^{-p_i'}$ are A_∞ weights and the same conclusion holds.

Once again we obtain a better result using Banach function spaces.

Theorem 2.10

Suppose $0 \le \alpha < nm$, $1 < p_1, \dots, p_m < \infty$, *and* q *is a number such that* $1/m < p \le q < \infty$. *If* $(u, \vec{v})$ *are weights that satisfy*

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q-1/p} \Big(\frac{1}{|Q|}\int_Q u^q\ dx\Big)^{1/q} \prod_{i=1}^m \|{v_i}^{-1}\|_{L^{p_i'}(\log L)^{p_i'-1+\delta}(Q, dx/|Q|)} < \infty \tag{2.7}$$

for some $\delta > 0$, *then*

$$\Big(\int_{\mathbb{R}^n} (\mathcal{M}_\alpha \vec{f} u)^q\ dx\Big)^{1/q} \le C \prod_{i=1}^m \Big(\int_{\mathbb{R}^n} (|f_i| v_i)^{p_i}\ dx\Big)^{1/p_i}$$

holds for all $\vec{f} \in L^{p_1}(v_1^{p_1}) \times \cdots \times L^{p_m}(v_m^{p_m})$.

Remark 2.11 For $\alpha = 0$ and $p = q$, Theorems 2.8 and 2.10 give new results for the operator, $\mathcal{M}$, studied in [14]. As already mentioned, in [14] necessary and sufficient conditions for two weight weak bounds of $\mathcal{M}$ were found, while Theorems 2.8 and 2.10 give sufficient conditions for the strong boundedness. We note, however, that the two weight conditions in [14] are not sufficient for the strong boundedness of $\mathcal{M}$. See Remark 7.3 in Section 7.

3. One weight theory

We now turn our attention to the multi-linear one vector weight case. As in the linear case we also have that $\mathcal{M}_\alpha$ is a smaller operator that $\mathcal{I}_\alpha$, more specifically, $\mathcal{M}_\alpha \vec{f} \leq C\mathcal{I}_\alpha \vec{f}$ for $f_i \geq 0$. However we also have the reverse inequality in norm. We obtain the following theorem relating $\mathcal{I}_\alpha$ and $\mathcal{M}_\alpha$ as an application of the extrapolation theorem of Cruz-Uribe, Pérez, and Martell [3].

Theorem 3.1

Suppose that $0 < \alpha < mn$, then for every $w \in A_\infty$ and all $0 < q < \infty$, we have,

$$\int_{\mathbb{R}^n} |\mathcal{I}_\alpha \vec{f}(x)|^q w(x)\, dx \leq C \int_{\mathbb{R}^n} \mathcal{M}_\alpha \vec{f}(x)^q w(x)\, dx$$

for all functions $\vec{f}$ with f_i bounded with compact support.

If we assume, say ${v_i}^{-p_i'} \in A_\infty$, then the two weight characterization for the fractional maximal function becomes,

$$\Big(\int_{\mathbb{R}^n} (\mathcal{M}_\alpha(\vec{f})u)^q\, dx\Big)^{1/q} \leq C \prod_{i=1}^m \Big(\int_{\mathbb{R}^n} (|f_i|v_i)^{p_i}\, dx\Big)^{1/p_i}$$

if and only if

$$\sup_Q |Q|^{\alpha/n+1/q-1/p} \Big(\frac{1}{|Q|}\int_Q u^q\Big)^{1/q} \prod_{i=1}^m \Big(\frac{1}{|Q|}\int_Q {v_i}^{-p_i'}\Big)^{1/p_i'} < \infty. \tag{3.1}$$

Notice that when we have the Sobolev relationship,

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n},$$

we obtain from (3.1)

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q u^q\Big)^{1/q} \prod_{i=1}^m \Big(\frac{1}{|Q|}\int_Q {v_i}^{-p_i'}\Big)^{1/p_i'} < \infty.$$

In this situation, the Lesbegue differentiation theorem gives then

$$u \leq C \prod_{i=1}^m v_i.$$

With this motivation we define a one weight condition as follows.

DEFINITION 3.2 Let $1 < p_1, \ldots, p_m < \infty$ and q be a number $1/m < p \le q < \infty$. We say that a vector of weights $\vec{w} = (w_1, \ldots, w_m)$ is in the class $A_{\vec{P},q}$, or that it satisfies the $A_{\vec{P},q}$ condition, if

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q (\Pi_{i=1}^m w_i)^q\Big)^{1/q} \prod_{i=1}^m \Big(\frac{1}{|Q|}\int_Q {w_i}^{-p_i'}\Big)^{1/p_i'} < \infty.$$

Remark 3.3 If $p_i \le q_i$ and $1/q = 1/q_1 + \cdots + 1/q_m$ then by Hölder's inequality we have,

$$\begin{aligned} &\Big(\frac{1}{|Q|}\int_Q (\Pi_i w_i)^q\Big)^{1/q} \prod_{i=1}^m \Big(\frac{1}{|Q|}\int_Q {w_i}^{-p_i'}\Big)^{1/p_i'} \\ &\le \prod_{i=1}^m \Big(\frac{1}{|Q|}\int_Q {w_i}^{q_i}\Big)^{1/q_i} \Big(\frac{1}{|Q|}\int_Q {w_i}^{-p_i'}\Big)^{1/p_i'}, \end{aligned}$$

and hence

$$\bigcup_{q_1,\ldots,q_m} \prod_{i=1}^m A_{p_i,q_i} \subseteq A_{\vec{P},q}, \tag{3.2}$$

where the union is over all $q_i \ge p_i$ that satisfy $1/q = 1/q_1 + \cdots + 1/q_m$. We will show that this containment is strict. See Remark 7.5.

We examine the properties of the weights in the class $A_{\vec{P},q}$. We have the follow theorem which is a variant of Theorem 3.6 in [14].

Theorem 3.4

Suppose, $1 < p_1, \ldots, p_m < \infty$, *and* $\vec{w} \in A_{\vec{P},q}$, *then*

$$(\Pi_{i=1}^m w_i)^q \in A_{mq} \quad \text{and} \quad w_i^{-p_i'} \in A_{mp_i'}.$$

We now state the main theorem for these weights. In the one weight situation we obtain necessary and sufficent conditions for the boundedness of $\mathcal{I}_\alpha$ and $\mathcal{M}_\alpha$.

Theorem 3.5

Suppose that $0 < \alpha < nm$ *and* $1 < p_1, \ldots, p_m < \infty$ *are exponents with* $1/m < p < n/\alpha$ *and* q *is the exponent defined by* $1/q = 1/p - \alpha/n$. *Then the inequality*

$$\Big(\int_{\mathbb{R}^n} (|\mathcal{I}_\alpha \vec{f}|(\Pi_i w_i))^q \, dx\Big)^{1/q} \le C \prod_{i=1}^m \Big(\int_{\mathbb{R}^n} (|f_i| w_i)^{p_i} \, dx\Big)^{1/p_i}$$

holds for every $\vec{f} \in L^{p_1}(w_1^{p_1}) \times \cdots \times L^{p_m}(w_m^{p_m})$ *if and only if* w *satisfies the* $A_{\vec{P},q}$ *condition.*

In light of Theorem 3.4 and Corollary 2.4 the sufficiency of the $A_{\vec{P},q}$ condition follows from the two weight case with $u = \Pi_i w_i$ and $v_i = w_i$. The necessity of the $A_{\vec{P},q}$ condition follows from Theorem 2.7 and the fact that $\mathcal{I}_\alpha$ is a bigger operator than $\mathcal{M}_\alpha$.

Theorem 3.6

Suppose that $0 < \alpha < nm$ and $1 < p_1, \dots, p_m < \infty$ are exponents with $1/m < p < n/\alpha$ and q is the exponent defined by $1/q = 1/p - \alpha/n$. Then the inequality

$$\Big(\int_{\mathbb{R}^n} (\mathcal{M}_\alpha(\vec{f})(\Pi_i w_i))^q\, dx\Big)^{1/q} \leq C \prod_{i=1}^m \Big(\int_{\mathbb{R}^n} (|f_i| w_i)^{p_i}\, dx\Big)^{1/p_i}$$

holds for every $\vec{f} \in L^{p_1}(w_1^{p_1}) \times \cdots \times L^{p_m}(w_m^{p_m})$ if and only if $\vec{w}$ satisfies the $A_{\vec{P},q}$ condition.

Once again the sufficiency of the $A_{\vec{P},q}$ condition follows from the two weight case Theorem 2.8 and the necessity follows from the weak characterization in Theorem 2.8. We do note, however, that Theorem 3.6 combined with Theorems 3.1 and 3.4 gives a different proof of the sufficiency of the $A_{\vec{P},q}$ condition in Theorem 3.5. When $\alpha = 0$ (so $p = q$) we recover the result from [14].

4. Proof of the two weight theorems

Proof of Theorem 2.2 We first treat the case $q \geq 1$. We wish to show that

$$\Big(\int_{\mathbb{R}^n} (|\mathcal{I}_\alpha \vec{f}(x)| u(x))^q\, dx\Big)^{1/q} \leq C \prod_{i=1}^m \Big(\int_{\mathbb{R}^n} (|f_i(x)| v_i(x))^{p_i}\, dx\Big)^{1/p_i}.$$

Equivalently, since $\mathcal{I}_\alpha$ is a positive operator, it is enough to show that

$$\int_{\mathbb{R}^n} \mathcal{I}_\alpha \vec{f}(x) u(x) g(x)\, dx \leq C \Big(\int_{\mathbb{R}^n} g(x)^{q'}\, dx\Big)^{1/q'} \prod_{i=1}^m \Big(\int_{\mathbb{R}^n} (|f_i(x)| v_i(x))^{p_i}\, dx\Big)^{1/p_i}$$

for all $g \in L^{q'}(\mathbb{R}^n)$, with $g \geq 0$, and all $f_i \geq 0$, bounded with compact support. We apply a discretization technique similar to that used in [18] for the operator $\mathcal{I}_\alpha$.

For a fixed $x \in \mathbb{R}^n$ and $l \in \mathbb{Z}$ there is a unique dyadic cube of side length 2^l that contains x. Hence we have

$$\begin{aligned}
\mathcal{I}_\alpha \vec{f}(x) &= \sum_{\nu \in \mathbb{Z}} \int_{2^{\nu-1} < \sum_i |x-y_i| \leq 2^\nu} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1| + \cdots + |x-y_m|)^{nm-\alpha}} d\vec{y} \\
&= \sum_{\nu \in \mathbb{Z}} \sum_{\substack{Q \in \mathcal{D}(\mathbb{R}^n) \\ \ell(Q) = 2^\nu}} \chi_Q(x) \int_{\ell(Q)/2 < \sum_i |x-y_i| \leq \ell(Q)} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1| + \cdots + |x-y_m|)^{nm-\alpha}} d\vec{y} \\
&\leq C \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \frac{\ell(Q)^\alpha}{|Q|^m} \int_{\sum_i |x-y_i| \leq \ell(Q)} f_1(y_1)\cdots f_m(y_m)\, d\vec{y}\, \chi_Q(x) \\
&\leq C \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \frac{\ell(Q)^\alpha}{|Q|^m} \int_{\sup_i |x-y_i| \leq \ell(Q)} f_1(y_1)\cdots f_m(y_m)\, d\vec{y}\, \chi_Q(x). \\
&\leq C \sum_{Q \in \mathcal{D}} \frac{\ell(Q)^\alpha}{|3Q|^m} \int_{(3Q)^m} f_1(y_1)\cdots f_m(y_m)\, d\vec{y}\, \chi_Q(x).
\end{aligned}$$

Let g be a non-negative function in $L^{q'}(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} \mathcal{I}_\alpha \vec{f}(x)u(x)g(x)\, dx \le C \sum_{Q\in\mathcal{D}} \frac{\ell(Q)^\alpha}{|3Q|^m} \int_{(3Q)^m} f_1(y_1)\cdots f_m(y_m)\, d\vec{y} \int_Q w(x)g(x)\, dx.$$

Further, define

$$\mathcal{M}_{3\mathcal{D}}\vec{h}(x) = \sup_{x\in Q\in\mathcal{D}} \prod_{i=1}^m \frac{1}{|3Q|}\int_{3Q} |h_i(y_i)|\, dy_i,$$

to be the maximal function with the basis of triples of dyadic cubes. Notice that $\mathcal{M}_{3\mathcal{D}}\vec{f} \le \mathcal{M}\vec{f}$. Let $\|\mathcal{M}\|$ be the constant from the $L^1\times\cdots\times L^1 \to L^{1/m,\infty}$ inequality for $\mathcal{M}$, $a > 6^n\|\mathcal{M}\|$ and

$$D_k = \left\{x\in\mathbb{R}^d : \mathcal{M}_{3\mathcal{D}}\vec{f}(x) > a^k\right\}.$$

If D_k is non-empty we can find a dyadic cube Q with $x\in Q$ and

$$\frac{1}{|3Q|^m}\int_{(3Q)^m} f_1(y_1)\cdots f_m(y_m)\, d\vec{y} > a^k.$$

Since f_i is bounded with compact support we can find a dyadic cube that satisfies this condition and is maximal with respect to inclusion. Thus, we get $D_k = \bigcup_j Q_{k,j}$ where, for each k the cubes $Q_{k,j}$ are maximal, disjoint, dyadic cubes that satisfy

$$a^k < \frac{1}{|3Q_{k,j}|^m}\int_{(3Q_{k,j})^m} f_1(y_1)\cdots f_m(y_m) d\vec{y} \le 2^{nm}a^k.$$

Fix $Q_{k,j}$, we compute the part of $Q_{k,j}$ covered by D_{k+1}. We have,

$$Q_{k,j}\cap D_{k+1} = \left\{x\in Q_{k,j} : \mathcal{M}_{3\mathcal{D}}\vec{f}(x) > a^{k+1}\right\}.$$

Since $x\in Q_{k,j}$ the supremum in

$$\mathcal{M}_{3\mathcal{D}}\vec{f}(x) = \sup_{x\in P\in\mathcal{D}} \frac{1}{|3P|^m}\int_{(3P)^m} f_1(y_1)\cdots f_m(y_m)\, d\vec{y} > a^{k+1}.$$

is taken over all dyadic cubes that contain $Q_{k,j}$ or are contained in $Q_{k,j}$. But the maximality of $Q_{k,j}$ implies

$$\frac{1}{|3P|^m}\int_{(3P)^m} f_1(y_1)\cdots f_m(y_m)\, d\vec{y} \le a^k \quad \forall P \supsetneq Q_{k,j}.$$

It now follows that if $x\in Q_{k,j}$ and $\mathcal{M}_{3\mathcal{D}}\vec{f}(x) > a^{k+1}$, then $\mathcal{M}_{3\mathcal{D}}(f_1\chi_{3Q_{k,j}},\ldots,f_m\chi_{3Q_{k,j}})(x) > a^{k+1}$. We have,

$$\begin{aligned}
|Q_{k,j}\cap D_{k+1}| &= \left|\{x\in Q_{k,j} : \mathcal{M}_{3\mathcal{D}}\vec{f}(x) > a^{k+1}\}\right| \\
&\le \left|\{x\in Q_{k,j} : \mathcal{M}_{3\mathcal{D}}(f_1\chi_{3Q_{k,j}},\ldots,f_m\chi_{3Q_{k,j}})(x) > a^{k+1}\}\right| \\
&\le \left|\{x\in\mathbb{R}^n : \mathcal{M}(f_1\chi_{3Q_{k,j}},\ldots,f_m\chi_{3Q_{k,j}})(x) > a^{k+1}\}\right| \\
&\le \Big(\frac{\|\mathcal{M}\|}{a^{k+1}}\prod_{i=1}^m\int_{3Q_{k,j}} f_i(y_i)\, dy_i\Big)^{1/m} \\
&\le \Big(\frac{\|\mathcal{M}\|}{a^{k+1}}\frac{1}{|3Q_{k,j}|^m}\int_{(3Q_{k,j})^m} f_1(y_1)\cdots f_m(y_m)\, d\vec{y}\Big)^{1/m}|3Q_{k,j}| \\
&\le \frac{6^n\|\mathcal{M}\|^{1/m}}{a^{1/m}}|Q_{k,j}|.
\end{aligned}$$

Thus,

$$|Q_{k,j} \cap D_{k+1}| \le \beta |Q_{k,j}|$$

for some $0 < \beta < 1$. If $E_{k,j} = Q_{k,j} \backslash D_{k+1}$ then $\{E_{k,j}\}_{k,j}$ is a disjoint family of sets that satisfy

$$|Q_{k,j}| \le C|E_{k,j}|$$

for some $C > 0$. Let,

$$\mathcal{C}^k = \Big\{ Q \in \mathcal{D} : a^k < \frac{1}{|3Q|^m} \int_{(3Q)^m} f_1(y_1) \cdots f_m(y_m) d\vec{y} \le a^{k+1} \Big\}.$$

Then,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathcal{I}_\alpha \vec{f}(x) u(x) g(x)\ dx \\
&\qquad \le C \sum_{Q\in\mathcal{D}} \frac{\ell(Q)^\alpha}{|3Q|^m} \int_{(3Q)^m} f_1(y_1) \cdots f_m(y_m) d\vec{y} \int_Q u(x) g(x)\ dx \\
&\qquad \le C \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{C}^k} \frac{\ell(Q)^\alpha}{|3Q|^m} \int_{(3Q)^m} f_1(y_1) \cdots f_m(y_m) d\vec{y} \int_Q u(x) g(x)\ dx \\
&\qquad \le C \sum_{k\in\mathbb{Z}} a^{k+1} \sum_{j\in\mathbb{Z}} \sum_{\substack{Q\in\mathcal{C}^k \\ Q\subset Q_{k,j}}} \ell(Q)^\alpha \int_Q u(x) g(x)\ dx \\
&\qquad \le Ca \sum_{k\in\mathbb{Z}} a^k \sum_{j\in\mathbb{Z}} \ell(Q_{k,j})^\alpha \int_{Q_{k,j}} u(x) g(x)\ dx. \\
&\qquad \le C \sum_{k,j} \ell(3Q_{k,j})^\alpha \prod_{i=1}^m \frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} f_i(y_i) v_i(y) v_i^{-1}(y_i) dy_i \\
&\qquad\quad \times \frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} u(x) g(x)\ dx |Q_{k,j}|.
\end{aligned}$$

Using now Hölder's inequality repeatedly and replacing $Q_{k,j}$ with the disjoint $E_{k,j}$ we have,

$$\begin{aligned}
&\le C \sum_{k,j} \ell(3Q_{k,j})^\alpha \prod_{i=1}^m \Big(\frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} (f_i v_i)^{(rp_i')'}\ dx \Big)^{1/(rp_i')'} \Big(\frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} v_i^{-rp_i'}\ dx \Big)^{1/rp_i'} \\
&\qquad \times \Big(\frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} u^{qr}\ dx \Big)^{1/qr} \Big(\frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} g^{(qr)'}\ dx \Big)^{1/(qr)'} |Q_{k,j}| \\
&\quad \le CK \sum_{k,j} \Big(\frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} g^{(qr)'}\ dx \Big)^{1/(qr)'} \\
&\qquad \times \prod_{i=1}^m \Big(\frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} (f_i v_i)^{(rp_i')'}\ dx \Big)^{1/(rp_i')'} |E_{k,j}|^{1/p+1/q'} \\
&\quad \le CK \Bigg(\sum_{k,j} \Big(\frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} g^{(qr)'}\ dx \Big)^{p'/(qr)'} |E_{k,j}|^{p'/q'} \Bigg)^{1/p'}
\end{aligned} \tag{4.1}$$

$$
\begin{aligned}
&\times \bigg(\sum_{k,j} \prod_{i=1}^{m} \Big(\frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} (f_i v_i)^{(rp_i')'}\, dx \Big)^{p/(rp_i')'} |E_{k,j}| \bigg)^{1/p} \\
&\le CK \bigg(\sum_{k,j} \Big(\frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} g^{(qr)'}\, dx \Big)^{q'/(qr)'} |E_{k,j}| \bigg)^{1/q'} \\
&\times \prod_{i=1}^{m} \Big(\sum_{k,j} \bigg(\frac{1}{|3Q_{k,j}|} \int_{3Q_{k,j}} (f_i v_i)^{(rp_i')'}\, dx \Big)^{p_i/(rp_i')'} |E_{k,j}| \bigg)^{1/p_i} \\
&\le CK \Big(\sum_{k,j} \int_{E_{k,j}} M_{(rq)'}(g)(x)^{q'}\, dx \Big)^{1/q'} \prod_{i=1}^{m} \Big(\sum_{k,j} \int_{E_{k,j}} M_{(rp_i')'}(f_i v_i)(x)^{p_i}\, dx \Big)^{1/p_i} \\
&\le CK \Big(\int_{\mathbb{R}^n} M_{(rq)'}(g)(x)^{q'}\, dx \Big)^{1/q'} \prod_{i=1}^{m} \Big(\int_{\mathbb{R}^n} M_{(rp_i')'}(f_i v_i)(x)^{p_i}\, dx \Big)^{1/p_i}
\end{aligned}
$$

where K is the constant in (2.1) and M_s is the operator $M_s f = M(|f|^s)^{1/s}$. Notice that since $r > 1$ we have, have

$$
\begin{cases}
M_{(rq)'} : L^{q'}(\mathbb{R}^n) \to L^{q'}(\mathbb{R}^n) \\
M_{(rp_1')'} : L^{p_1}(\mathbb{R}^n) \to L^{p_1}(\mathbb{R}^n) \\
\quad\vdots \\
M_{(rp_m')'} : L^{p_m}(\mathbb{R}^n) \to L^{p_m}(\mathbb{R}^n)
\end{cases} . \tag{4.2}
$$

Hence using these boundedness properties above we obtain

$$
\int_{\mathbb{R}^n} \mathcal{I}_\alpha \vec{f}(x) u(x) g(x)\, dx \le C \left(\int_{\mathbb{R}^n} g(x)^{q'}\, dx \right)^{1/q'} \prod_{i=1}^{m} \left(\int_{\mathbb{R}^n} (|f_i(x)| v_i(x))^{p_i}\, dx \right)^{1/p_i}
$$

which concludes the case $q > 1$.

Now suppose $1/m < p \le q \le 1$, then we work directly with the norm $\|\mathcal{I}_\alpha \vec{f} w\|_{L^q}$. Using the same discritization technique as above, and $q \le 1$ we obtain

$$
\mathcal{I}_\alpha \vec{f}(x)^q \le C \sum_{Q \in \mathcal{D}} \Big(\frac{\ell(Q)^\alpha}{|3Q|^m} \int_{(3Q)^m} f_1(y_1) \cdots f_m(y_m)\, d\vec{y} \Big)^q \chi_Q(x).
$$

Multiplying by u^q and integrating,

$$
\begin{aligned}
&\Big(\int_{\mathbb{R}^n} (\mathcal{I}_\alpha \vec{f}(x) u(x))^q\, dx \Big)^{1/q} \\
&\qquad \le \left(C \sum_{Q \in \mathcal{D}} \Big(\frac{\ell(Q)^\alpha}{|3Q|^m} \int_{(3Q)^m} f_1(y_1) \cdots f_m(y_m)\, d\vec{y} \Big)^q \int_Q u(x)^q\, dx \right)^{1/q}.
\end{aligned}
$$

Performing the same decomposition as above we obtain $\{Q_{k,j}\}_{k,j}$ and construct $\{E_{k,j}\}$

satisfying the same properties. Thus,

$$\Big(C\sum_{Q\in\mathcal{D}}\Big(\frac{\ell(Q)^\alpha}{|3Q|^m}\int_{(3Q)^m} f_1(y_1)\cdots f_m(y_m)\ d\vec{y}\Big)^q\int_Q u(x)^q\ dx\Big)^{1/q}$$
$$\le C\Big(C\sum_{k\in\mathbb{Z}} a^{(k+1)q}\sum_{j\in\mathbb{Z}}\sum_{\substack{Q\in C^k\\ Q\subset Q_{k,j}}}\ell(Q)^{\alpha q}\int_Q u(x)^q\ dx\Big)^{1/q}$$
$$\le C\Bigg(\sum_{k,j}\frac{\ell(3Q_{k,j})^{\alpha q}}{|3Q_{k,j}|}\int_{3Q_{k,j}} u(x)^q\ dx\prod_{i=1}^m\Big(\frac{1}{|3Q_{k,j}|}\int_{3Q_{k,j}} f_i(y_i)v_i(y_i)v_i^{-1}(y_i)dy_i\Big)^q|Q_{k,j}|\Bigg)^{1/q}. \tag{4.3}$$

Using Hölder's inequality and condition (2.2),

$$\le CK\sum_{k,j}\Bigg(\prod_{i=1}^m\Big(\frac{1}{|3Q_{k,j}|}\int_{3Q_{k.j}}(f_iv_i)^{(rp_i')'}\ dx\Big)^{q/(rp_i')'}|Q_{k,j}|^{q/p}\Bigg)^{1/q}.$$

Using $p\le q$, replacing the $Q_{k,j}$'s with $E_{k,j}$'s and multilinear Hölder's inequality again we have,

$$\le CK\sum_{k,j}\Bigg(\prod_{i=1}^m\Big(\frac{1}{|3Q_{k,j}|}\int_{3Q_{k.j}}(f_iv_i)^{(rp_i')'}\ dx\Big)^{p/(rp_i')'}|E_{k,j}|\Bigg)^{1/p}$$
$$\le CK\prod_{i=1}^m\Bigg(\sum_{k,j}\Big(\frac{1}{|3Q_{k,j}|}\int_{3Q_{k,j}}(f_iv_i)^{(rp_i')'}\ dx\Big)^{p_i/(rp_i')'}|E_{k,j}|\Bigg)^{1/p_i}$$
$$\le CK\prod_{i=1}^m\Big(\int_{\mathbb{R}^n}M_{(rp_i')'}(f_iv_i)^{p_i}\ dx\Big)^{1/p_i}$$
$$\le CK\prod_{i=1}^m\Big(\int_{\mathbb{R}^n}(f_i(y_i)v_i(y_i))^{p_i}\ dy_i\Big)^{1/p_i}.$$

This concludes the proof of Theorem 2.2. □

Remark 4.1 A close examination of the above proof yeilds that the operator norm denoted $\|\mathcal{I}_\alpha\|$ has the dependence,

$$\|\mathcal{I}_\alpha\|\le CK$$

where C is a dimensional constant and K is the constant from (2.1).

Proof of Theorem 2.7 The proof is similar to that of the weak inequality given in [14]. We only present the case where $p_1,\dots,p_m>1$ as a the case when some $p_j=1$ is a minor modification of the linear case. Suppose that $\mathcal{M}_\alpha$ is weakly bounded i.e.

$$u(\{x\in\mathbb{R}^n:\mathcal{M}_\alpha\vec{f}(x)>\lambda\})\le\Big(\frac{C}{\lambda}\prod_{i=1}^m\|f_i\|_{L^{p_i}(v_i)}\Big)^q$$

for all $\lambda > 0$. Let $f_i \geq 0$ and fix a cube Q with $\Pi_i |Q|^{\alpha/nm-1} \int_Q f_i > 0$. Notice that for $x \in Q$ we have

$$\prod_{i=1}^m \frac{\ell(Q)^{\alpha/m}}{|Q|} \int_Q f_i \leq \mathcal{M}_\alpha(f_1\chi_Q, \ldots, f_m\chi_Q)(x).$$

Hence, if $\lambda < \Pi_i |Q|^{\alpha/nm-1} \int_Q f_i \leq \mathcal{M}_\alpha(f_1\chi_Q, \ldots, f_m\chi_Q)(x)$ we have

$$Q \subset \{x \in \mathbb{R}^n : \mathcal{M}_\alpha(f_1\chi_Q, \ldots, f_m\chi_Q)(x) > \lambda\}.$$

Thus,

$$u(Q) \leq u(\{x \in \mathbb{R}^n : \mathcal{M}_\alpha(f_1\chi_Q, \ldots, f_m\chi_Q)(x) > \lambda\}) \leq \left(\frac{C}{\lambda} \prod_{i=1}^m \Big(\int_Q {f_i}^{p_i} v_i\Big)^{1/p_i}\right)^q.$$

Since this holds for all $\lambda < \Pi_i |Q|^{\alpha/nm-1} |Q| \int_Q f_i$ it follows that

$$|Q|^{\alpha/n-m} u(Q)^{1/q} \prod_{i=1}^m \Big(\int_Q f_i\Big) \leq C \prod_{i=1}^m \Big(\int_Q {f_i}^{p_i} v_i\Big)^{1/p_i}.$$

If we set $f_i = {v_i}^{1-p_i'}$ we get

$$|Q|^{\alpha/n-m} u(Q)^{1/q} \prod_{i=1}^m \Big(\int_Q {v_i}^{1-p_i'}\Big) \leq C \prod_{i=1}^m \Big(\int_Q {v_i}^{1-p_i'}\Big)^{1/p_i}$$

which gives the $A_{\vec{P},q}$ condition. Conversely, suppose that $(u, \vec{v}) \in A_{\vec{P},q}$ and assume for the moment that for all $1 \leq i \leq m$ $\|f_i\|_{L^{p_i}(v_i)} = 1$. We will also use the centered fractional multilinear maximal function $\mathcal{M}_\alpha^c$ where the supremum is taken over all cubes centered at x. Clearly $\mathcal{M}_\alpha \approx \mathcal{M}_\alpha^c$.

Given x fix a cube, Q, centered at x. Then Hölder's inequality yeilds

$$\begin{aligned}
\prod_{i=1}^m \frac{\ell(Q)^{\alpha/m}}{|Q|} \int_Q |f_i| &= |Q|^{\alpha/n-m} \prod_{i=1}^m \int_Q |f_i| {v_i}^{1/p_i} {v_i}^{-1/p_i} \\
&\leq |Q|^{\alpha/n-m} \prod_{i=1}^m \Big(\int_Q |f_i|^{p_i} v_i\Big)^{1/p_i} \Big(\int_Q {v_i}^{1-p_i'}\Big)^{1/p_i'} \\
&= |Q|^{\alpha/n-m} u(Q)^{1/q} \prod_{i=1}^m \Big(\int_Q {v_i}^{1-p_i'}\Big)^{1/p_i'} u(Q)^{-1/q} \prod_{i=1}^m \Big(\int_Q |f_i|^{p_i} v_i\Big)^{1/p_i} \\
&\leq C u(Q)^{-1/q} \prod_{i=1}^m \Big(\int_Q |f_i|^{p_i} v_i\Big)^{1/p_i}.
\end{aligned}$$

Now, since we are assuming that $\|f_i\|_{L^{p_i}(v_i)} = 1$, we have

$$\prod_{i=1}^m \Big(\int_Q |f_i|^{p_i} v_i\Big)^{1/p_i} \leq 1.$$

Moreover, since $p/q \le 1$ we have

$$\begin{aligned}
\prod_{i=1}^m \frac{\ell(Q)^{\alpha/m}}{|Q|}\int_Q |f_i| &\le C\frac{1}{u(Q)^{1/q}}\prod_{i=1}^m\Big(\int_Q |f_i|^{p_i}v_i\Big)^{1/p_i}\\
&\le C\frac{1}{u(Q)^{1/q}}\bigg(\prod_{i=1}^m\Big(\int_Q |f_i|^{p_i}v_i\Big)^{1/p_i}\bigg)^{p/q}\\
&= C\bigg(\frac{1}{u(Q)^{1/p}}\prod_{i=1}^m\Big(\int_Q |f_i|^{p_i}v_i\Big)^{1/p_i}\bigg)^{p/q}\\
&= C\bigg(\prod_{i=1}^m\Big(\frac{1}{u(Q)}\int_Q |f_i|^{p_i}v_iu^{-1}u\Big)^{1/p_i}\bigg)^{p/q}\\
&\le C\Big(\prod_{i=1}^m M_u^c(|f_i|^{p_i}v_i/u)(x)^{1/p_i}\Big)^{p/q}.
\end{aligned}$$

Hence,

$$\mathcal{M}_\alpha^c\vec{f}(x) \le C\Big(\prod_{i=1}^m M_u^c(|f_i|^{p_i}v_i/u)(x)^{1/p_i}\Big)^{p/q}.$$

Using a weak-type Hölder's inequality we have,

$$\begin{aligned}
\|\mathcal{M}_\alpha^c\vec{f}\|_{L^{q,\infty}(u)} &\le C\|(\Pi_i M_u^c(|f_i|^{p_i}v_i/u)^{1/p_i})^{p/q}\|_{L^{q,\infty}(u)}\\
&= C\|\Pi_i M_u^c(|f_i|^{p_i}v_i/u)^{1/p_i}\|_{L^{p,\infty}(u)}^{p/q}\\
&\le C\Big(\prod_{i=1}^m \|M_u^c(|f_i|^{p_i}v_i/u)\|_{L^{p_i,\infty}(u)}\Big)^{p/q}\\
&= C\Big(\prod_{i=1}^m \|M_u^c(|f_i|^{p_i}v_i/u)\|_{L^{1,\infty}(u)}^{1/p_i}\Big)^{p/q}\\
&\le C\Big(\prod_{i=1}^m \||f_i|^{p_i}v_i/u\|_{L^1(u)}^{1/p_i}\Big)^{p/q}\\
&= C\Big(\prod_{i=1}^m \|f_i\|_{L^{p_i}(v_i)}\Big)^{p/q} = C\,.
\end{aligned}$$

For general f_i the result follows if we replace $f_i \to f_i/\|f_i\|_{L^{p_i}(v_i)}$. □

Proof of Theorem 2.8 We first prove the boundedness for the dyadic version,

$$\mathcal{M}_\alpha^d\vec{f}(x) = \sup_{Q\in\mathcal{D}:x\in Q}\prod_{i=1}^m \frac{\ell(Q)^\alpha}{|Q|}\int_Q |f_i(y_i)|dy_i.$$

Let a be a constant satisfying $a > 2^{nm}$ and let

$$D_k = \{x\in\mathbb{R}^n : \mathcal{M}_\alpha^d f(x) > a^k\}\,.$$

If D_k is non-empty then we can write $D_k = \bigcup_j Q_{k,j}$ where each $Q_{k,j}$ is a maximal dyadic cube satisfying

$$a^k < \prod_{i=1}^m |Q_{k,j}|^{\alpha/(nm)-1}\int_{Q_{k,j}} f_i(y_i)dy_i < 2^{mn-\alpha}a^k \le 2^{mn}a^k.$$

Also, each $D_{k+1} \subseteq D_k$ and each $Q_{k+1,l}$ is contained in $Q_{k,j}$ for some j by properties of dyadic cubes and we have

$$|Q_{k,j} \cap D_{k+1}| \leq \frac{2^n}{a^{1/m}} |Q_{k,j}|.$$

Hence the sets $E_{k,j} = Q_{k,j} \backslash (Q_{k,j} \cap D_{k+1})$ are disjoint and satisfy

$$|Q_{k,j}| < \beta |E_{k,j}|$$

for some $\beta > 1$. Thus, we have

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^n} (\mathcal{M}_\alpha^d \vec{f}(x) u(x))^q \, dx \Big)^{1/q} \\
&= \Big(\sum_k \int_{D_k \backslash D_{k+1}} (\mathcal{M}_\alpha^d \vec{f}(x) u(x))^q \, dx \Big)^{1/q} \\
&\leq \Big(\sum_k a^{(k+1)q} \int_{D_k} u^q(x) \, dx \Big)^{1/q} \\
&\leq a \Big(\sum_{k,j} a^{kq} \int_{Q_{k,j}} u^q(x) \, dx \Big)^{1/q} \\
&\leq a \Bigg(\sum_{k,j} \Big(\prod_{i=1}^m \frac{\ell(Q_{k,j})^{\alpha/m}}{|Q_{k,j}|} \int_{Q_{k,j}} f_i(y_i) v_i(y_i) v_i(y_i)^{-1} dy_i \Big)^q \int_{Q_{k,j}} u^q(x) \, dx \Bigg)^{1/q}.
\end{aligned}$$

This equation is the same as (4.3) in the proof of Theorem 2.2 and the dyadic version of the theorem follows. The non-dyadic version follows from the inequality

$$\mathcal{M}_\alpha^k \vec{f}(x)^q \leq \frac{C_{\alpha,n}}{|B_k|} \int_{B_k} (\tau_{-t} \circ \mathcal{M}_\alpha^d \circ \vec{\tau}_t)(\vec{f})(x)^q \, dt \tag{4.4}$$

for all $x \in \mathbb{R}^n$ and $f_i \geq 0$. Where $B_k = [-2^{k+2}, 2^{k+2}]^n$, $\mathcal{M}_\alpha^k \vec{f}$ is the maximal function with the supremum taken over cubes of side length less than 2^k, $\tau_t g(x) = g(x-t)$, $\vec{\tau}_t \vec{f} = (\tau_t f_1, \ldots, \tau_t f_m)$. The inequality (4.4) holds for all $0 < q < \infty$, and a proof for the linear case can be found in [6, p. 431] and the multilinear case is a slight modification. From (4.4) it follows that

$$\|\mathcal{M}_\alpha \vec{f} u\|_{L^q} \leq \sup_t \|\tau_t \circ \mathcal{M}_\alpha^d \circ \vec{\tau}_t \vec{f} u\|_{L^q}.$$

Noe that if $(u, \vec{v})$ satisfy condition (2.6), then $(\tau_t u, \vec{\tau}_t \vec{v})$ satisfy the condition (2.6) independent of t. By the dyadic case we have,

$$\begin{aligned}
\|(\tau_{-t} \circ \mathcal{M}_\alpha^d \circ \vec{\tau}_t) \vec{f} u\|_{L^q} &= \|(\mathcal{M}_\alpha^d \circ \tau_t) \vec{f} \tau_t u\|_{L^q} \\
&\leq C \prod_{i=1}^m \|\tau_t f_i \tau_t v_i\|_{L^{p_i}} = C \prod_{i=1}^m \|f_i\|_{L^{p_i}(v_i)},
\end{aligned}$$

where the constant C is independent of t. It now follows that,

$$\|\mathcal{M}_\alpha \vec{f} u\|_{L^q} \leq C \sup_t \|\tau_t \circ \mathcal{M}_\alpha^d \circ \vec{\tau}_t \vec{f} u\|_{L^q} \leq C \prod_{i=1}^m \|f_i v_i\|_{L^{p_i}}.$$

□

5. Proof of the one weight theorems

Proof of Theorem 3.1 In light of the extrapolation theorem in [3] we just need to show that the result holds for $q=1$ and all $w\in A_\infty$. Using the same decomposition as in Theorem 2.2 with $g=1$ we have,

$$\int_{\mathbb{R}^n}\mathcal{I}_\alpha\vec{f}w\;dx\leq c\sum_{k,j}\prod_{i=1}^m\frac{\ell(3Q_{k,j})^\alpha}{|3Q_{k,j}|}\int_{3Q_{k,j}}f_i(y_i)dy_iw(Q_{k,j}).$$

Since $w\in A_\infty$ and $|Q_{k,j}|\leq C|E_{k,j}|$ we have,

$$w(Q_{k,j})\leq Cw(E_{k,j}).$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}^n}\mathcal{I}_\alpha\vec{f}w\;dx&\leq C\sum_{k,j}\prod_{i=1}^m\frac{\ell(3Q_{k,j})^\alpha}{|3Q_{k,j}|}\int_{3Q_{k,j}}f_i(y_i)dy_iw(E_{k,j})\\&\leq C\sum_{k,j}\int_{E_{k,j}}\mathcal{M}_\alpha\vec{f}w\;dx\\&\leq\int_{\mathbb{R}^n}\mathcal{M}_\alpha\vec{f}w\;dx.\end{aligned}$$

□

Proof of Theorem 3.4 We use techniques similar to those in [14]. Since $p\leq q$, if we let $q_i=qp_i/p$ then, $q_i\geq p_i$ and $1/q=1/q_1+\cdots+1/q_m$. Further, we have

$$\frac{1}{q_1'}+\cdots+\frac{1}{q_m'}=m-\frac{1}{q},$$

and hence Hölders inequality with $r_i=(m-1/q)q_i'$ can be applied to get

$$\begin{aligned}&\Big(\frac{1}{|Q|}\int_Q(\Pi_iw_i)^q\Big)^{1/mq}\Big(\frac{1}{|Q|}\int_Q(\Pi_iw_i)^{-q/(mq-1)}\Big)^{(mq-1)/mq}\\&\leq\Big(\frac{1}{|Q|}\int_Q(\Pi_iw_i)^q\Big)^{1/mq}\bigg(\prod_i\Big(\frac{1}{|Q|}\int_Q w_i^{-q_i'}\Big)^{1/q_i'}\bigg)^{1/m}.\end{aligned}$$

We now use Hölder's with $p_i'/q_i'>1$ to get

$$\leq\quad\bigg(\Big(\frac{1}{|Q|}\int_Q(\Pi_iw_i)^q\Big)^{1/q}\prod_i\Big(\frac{1}{|Q|}\int_Q w_i^{-p_i'}\Big)^{1/p_i'}\bigg)^{1/m}.$$

This shows that $\Pi_iw_i{}^q\in A_{mq}$. Now to show that $w_i^{-p_i'}\in A_{mp_i'}$, for this fix $1\leq i\leq m$, then the $A_{mp_i'}$ condition is,

$$\sup_Q\Big(\frac{1}{|Q|}\int_Q w_i^{-p_i'}\Big)^{1/mp_i'}\Big(\frac{1}{|Q|}\int_Q w_i^{p_i'/(mp_i'-1)}\Big)^{(mp_i'-1)/mp_i'}<\infty.$$

If we set

$$r_i = p\Big(m - 1 + \frac{1}{p_i}\Big) = p\Big(m - \frac{1}{p_i'}\Big) \quad \text{and} \quad r_j = \frac{p_j}{p_j - 1}\frac{r_i}{p} = \frac{p_j'}{p} r_i \quad 1 \le j \ne i \le m.$$

Then notice $1 < r_j < \infty$ and

$$\sum_{j=1}^{m} \frac{1}{r_j} = \frac{1}{r_j}\Big(1 + \sum_{1\le j\ne i\le m} \frac{p}{p_j'}\Big) = 1.$$

Further

$$\begin{aligned}\Big(\frac{1}{|Q|}\int_Q {w_i}^{p_i'/(mp_i'-1)}\Big)^{(mp_i'-1)/mp_i'} &= \Big(\frac{1}{|Q|}\int_Q {w_i}^{p/r_i}\Big)^{r_i/mp} \\ &= \Big(\frac{1}{|Q|}\int_Q (\Pi_{j=1}^m w_j)^{p/r_i}\Pi_{j\ne i} w_j^{-p/r_i}\Big)^{r_i/mp}.\end{aligned}$$

We use Hölder's inequality with exponents $r_1, \ldots, r_m$ to get

$$\begin{aligned}&\Big(\frac{1}{|Q|}\int_Q (\Pi_j w_j)^{p/r_i}\Pi_{j\ne i} w_j^{-p/r_i}\Big)^{r_i/mp} \\ &\le \frac{1}{|Q|^{r_i/mp}}\Big[\Big(\int_Q (\Pi_j w_j)^p\Big)^{1/r_i}\prod_{j\ne i}\Big(\int_Q {w_j}^{-p_j'}\Big)^{1/r_j}\Big)^{r_i/mp} \\ &= \frac{1}{|Q|^{r_i/mp}}\Big(\Big(\int_Q (\Pi_j w_j)^p\Big)^{1/p}\prod_{j\ne i}\Big(\int_Q {w_j}^{-p_j'}\Big)^{1/p_j'}\Big)^{1/m} \\ &= \Big(\Big(\frac{1}{|Q|}\int_Q (\Pi_j w_j)^p\Big)^{1/p}\prod_{j\ne i}\Big(\frac{1}{|Q|}\int_Q {w_j}^{-p_j'}\Big)^{1/p_j'}\Big)^{1/m} \\ &\le \Big(\Big(\frac{1}{|Q|}\int_Q (\Pi_j w_j)^q\Big)^{1/q}\prod_{j\ne i}\Big(\frac{1}{|Q|}\int_Q {w_j}^{-p_j'}\Big)^{1/p_j'}\Big)^{1/m}.\end{aligned}$$

The second to last inequality follows since

$$\frac{r_i}{p} = m - \frac{1}{p_i'} = \frac{1}{p} + \sum_{1\le j\ne i\le m}\frac{1}{p_j'},$$

and the last inequality follows from Hölder's with q/p. Thus we arrive at the inequality,

$$\begin{aligned}&\Big(\frac{1}{|Q|}\int_Q w_i^{-p_i'}\Big)^{1/mp_i'}\Big(\frac{1}{|Q|}\int_Q {w_i}^{p_i'/(mp_i'-1)}\Big)^{(mp_i'-1)/mp_i'} \\ &\quad\le \Big(\Big(\frac{1}{|Q|}\int_Q (\Pi_j w_j)^q\Big)^{1/q}\prod_{j=1}^{m}\Big(\frac{1}{|Q|}\int_Q {w_j}^{-p_j'}\Big)^{1/p_j'}\Big)^{1/m}.\end{aligned}$$

This shows that $w_i^{-p_i'} \in A_{mp_i'}$. □

6. Banach function spaces

Suppose X is a Banach function space over $\mathbb{R}^n$ with respect to Lebesgue measure. We refer the reader to [1] for a detailed account of Banach functions spaces. X has an associate Banach function space X' for which the generalized Hölder inequality,

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \le \|f\|_X \|g\|_{X'},$$

holds. Examples of Banach function spaces are the Lebesgue L^p spaces, Lorentz spaces, and Orlicz spaces which we shall describe next. The Orlicz space $L^B = L^B(\mathbb{R}^n)$ is defined by a Young function B (see [19]) with and consists of all measurable functions f such that

$$\int_{\mathbb{R}^n} B\Big(\frac{|f(y)|}{\lambda}\Big)\,dy < \infty$$

for some $\lambda > 0$. The space is then equipped with a norm given by

$$\|f\|_B = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} B\Big(\frac{|f(y)|}{\lambda}\Big)\,dy \le 1\right\}.$$

As in [19], for a function $f \in X$ and a cube $Q \subset \mathbb{R}^n$ we define the X average of f over Q to be

$$\|f\|_{X,Q} = \|\delta_{\ell(Q)}(f\chi_Q)\|_X,$$

where for $a > 0$, $\delta_a f(x) = f(ax)$. Observe that if $X = L^r$ then

$$\|f\|_{X,Q} = \Big(\frac{1}{|Q|}\int_Q |f|^r\,dx\Big)^{1/r}$$

and if $X = L^B$ then,

$$\|f\|_{B,Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q B\Big(\frac{|f(y)|}{\lambda}\Big)\,dy \le 1\right\}.$$

We define the maximal operator associated to the Banach function space X to be

$$M_X f(x) = \sup_{Q \ni x} \|f\|_{X,Q}.$$

When X is the Orlicz space L^B we denote M_X by M_B. Notice that if M is the Hardy-Littlewood maximal operator, then $M_{L^1} = M$ and $M_{L^r} f(x) = M_r f = M(f^r)^{1/r}$. If $Y_1, \ldots, Y_m$ are Banach function spaces we define the multi(sub)linear maximal function to be

$$\mathcal{M}_{\vec{Y}}\vec{f}(x) = \sup_{Q \ni x} \prod_{i=1}^m \|f_i\|_{Y_i,Q}.$$

Notice that $\mathcal{M}_{\vec{Y}}\vec{f}(x) \le \prod_{i=1}^m M_{Y_i} f_i(x)$. Hence if $1 \le p_1, \ldots, p_m \le \infty$ and $M_{Y_i} : L^{p_i} \to L^{p_i}$ then by Hölder's inequality

$$\mathcal{M}_{\vec{Y}} : L^{p_1} \times \cdots \times L^{p_m} \to L^p.$$

We have have the following generalized version of Theorem 2.2.

Theorem 6.1

Suppose $0 < \alpha < nm$, $1 < p_1, \dots, p_m < \infty$, with $1/p = 1/p_1 + \cdots + 1/p_m$, and $Y_1, \dots, Y_m$ are Banach function spaces over $\mathbb{R}^n$ such that

$$\mathcal{M}_{\vec{Y}'} : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \tag{6.1}$$

where $\mathcal{M}_{\vec{Y}'}$ is the multilinear maximal function associated to $Y_1', \dots, Y_m'$. Let q be an exponent satisfying $1/m < p \le q < \infty$. Suppose that one of the following two conditions holds.

i) $q > 1$, *X is a Banach function space that satisfing*

$$M_{X'} : L^{q'}(\mathbb{R}^n) \to L^{q'}(\mathbb{R}^n), \tag{6.2}$$

and $(u, \vec{v})$ are weights that satisfy

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q-1/p} \|u\|_{X,Q} \prod_{i=1}^m \|{v_i}^{-1}\|_{Y_i,Q} < \infty.$$

ii) $q \le 1$ *and $(u, \vec{v})$ are weights that satisfy*

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q-1/p} \Big(\frac{1}{|Q|} \int_Q u^q \, dx \Big)^{1/q} \prod_{i=1}^m \|{v_i}^{-1}\|_{Y_i,Q} < \infty.$$

Then the inequality

$$\Big(\int_{\mathbb{R}^n} (|\mathcal{I}_\alpha \vec{f}| u)^q \, dx \Big)^{1/q} \le C \prod_{i=1}^m \Big(\int_{\mathbb{R}^n} (|f_i| v_i)^{p_i} \, dx \Big)^{1/p_i}$$

holds for all $\vec{f} \in L^{p_1}(v_1^{p_1}) \times \cdots \times L^{p_m}(v_m^{p_m})$.

Remark 6.2 Theorem 2.2 is the specific case of Theorem 6.1 when $X = L^{qr}$ and $Y_i = L^{rp_i'}$ for some $r > 1$. In this case the boundedness of the maximal functions in (6.1) and (6.2) are automatic.

The proof of Theorem 6.1 is very similar to the proof Theorem 2.2. The main ingredients are the generalized Hölder inequality which is used in place of equation (4.1) and the assumed boundedness of the maximal functions $\mathcal{M}_{\vec{Y}'}$ in (6.1) and $M_{X'}$ in (6.2) are used in place of (4.2). For the Orlicz spaces, L^B, the boundedness of the corresponding maximal functions M_B has been developed by Pérez [19]. He showed that

$$M_B : L^s \to L^s$$

if and only if there exists $c > 0$ such that

$$\int_c^\infty \frac{B(t)}{t^s} \frac{dt}{t} < \infty.$$

Thus we have the following theorem.

Theorem 6.3

Suppose $0<\alpha<nm$, $1<p_1,\dots,p_m<\infty$, with $1/p=1/p_1+\cdots+1/p_m$, q is an exponent with $1/m<p\le q<\infty$ and $\Psi,\Phi_1,\dots,\Phi_m$ are Young functions that satisfy

$$\int_c^\infty \frac{\Psi(t)}{t^q}\frac{dt}{t}<\infty \tag{6.3}$$

and

$$\int_c^\infty \frac{\Phi_i(t)}{t^{p_i'}}\frac{dt}{t}<\infty, \qquad i=1,\dots,m \tag{6.4}$$

for some $c>0$. Let q be an exponent satisfying $1/m<p\le q<\infty$ and assume that

$$\sup_Q \ell(Q)^\alpha|Q|^{1/q-1/p}\|u\|_{\Psi,Q}\prod_{i=1}^m\|{v_i}^{-1}\|_{\Phi_i,Q}<\infty.$$

Then the inequality

$$\Big(\int_{\mathbb{R}^n}(|\mathcal{I}_\alpha\vec{f}|u)^q\ dx\Big)^{1/q}\le C\prod_{i=1}^m\Big(\int_{\mathbb{R}^n}(|f_i|v_i)^{p_i}\ dx\Big)^{1/p_i}$$

holds for all $\vec{f}\in L^{p_1}(v_1^{p_1})\times\cdots\times L^{p_m}(v_m^{p_m})$.

We notice that the functions $\Psi(t)=t^q(\log(1+t))^{q-1+\delta}$ and $\Phi_i=t^{p_i'}(\log(1+t))^{p_i'-1+\delta}$ satisfy (6.3) and (6.4) respectively if $\delta>0$. From here we obtain Theorem 2.5. Similary we extend Theorem 2.8.

Theorem 6.4

Suppose $0\le\alpha<nm$, $1<p_1,\dots,p_m<\infty$, with $1/p=1/p_1+\cdots+1/p_m$, and q is an exponent satisfying $1/m<p\le q<\infty$, and $Y_1,\dots,Y_m$ are are translation invariant Banach function spaces with

$$\mathcal{M}_{\vec{Y}'}:L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)\to L^p(\mathbb{R}^n).$$

If $(u,\vec{v})$ are weights that satisfy

$$\sup_Q \ell(Q)^\alpha|Q|^{1/q-1/p}\Big(\frac{1}{|Q|}\int_Q u^q\ dx\Big)^{1/q}\prod_{i=1}^m\|{v_i}^{-1}\|_{Y_i,Q}<\infty$$

then the inequality

$$\Big(\int_{\mathbb{R}^n}(\mathcal{M}_\alpha\vec{f}u)^q\ dx\Big)^{1/q}\le C\prod_{i=1}^m\Big(\int_{\mathbb{R}^n}(|f_i|v_i)^{p_i}\ dx\Big)^{1/p_i}$$

holds for all $\vec{f}\in L^{p_1}(v_1^{p_1})\times\cdots\times L^{p_m}(v_m^{p_m})$.

Remark 6.5 Once again the proof Theorem 6.4 is almost identical to the proof of Theorem 2.8 using the generalized Hölder inequality and the boundedness of $\mathcal{M}_{\vec{Y}'}$ in the right places. The translation invariance is used to pass from the dyadic version via equation (4.4). Theorem 2.8 is also a particular case of Theorem 6.4 where $Y_i=L^{rp_i'}$.

In the context of Orlicz space we have the following theorem for $\mathcal{M}_\alpha$.

Theorem 6.6

Suppose $0 < \alpha < nm$, $1 < p_1, \ldots, p_m < \infty$, *with* $1/p = 1/p_1 + \cdots + 1/p_m$, q *is an exponent with* $1/m < p \leq q < \infty$ *and* $\Phi_1, \ldots, \Phi_m$ *are Young functions that satisfy*

$$\int_c^\infty \frac{\Phi_i(t)}{t^{p_i'}} \frac{dt}{t} < \infty, \qquad i = 1, \ldots, m \tag{6.5}$$

for some $c > 0$. *Let* q *be an exponent satisfying* $1/m < p \leq q < \infty$.

$$\sup_Q \ell(Q)^\alpha |Q|^{1/q-1/p} \Big(\frac{1}{|Q|} \int_Q u^q \; dx \Big)^{1/q} \prod_{i=1}^m \|{v_i}^{-1}\|_{\Phi_i, Q} < \infty.$$

Then the inequality

$$\Big(\int_{\mathbb{R}^n} (|\mathcal{M}_\alpha \vec{f}| u)^q \; dx \Big)^{1/q} \leq C \prod_{i=1}^m \Big(\int_{\mathbb{R}^n} (|f_i| v_i)^{p_i} \; dx \Big)^{1/p_i}$$

holds for all $\vec{f} \in L^{p_1}(v_1^{p_1}) \times \cdots \times L^{p_m}(v_m^{p_m})$.

Again setting $\Phi_i(t) = t^{p_i'}(\log(1+t))^{p_i'-1+\delta}$ for some $\delta > 0$ we obtain Theorem 2.10.

7. Applications and examples

With the multi-linear fractional integral operator we have some Poincaré and Sobolev type inequalities for products of functions. We do the estimates with two functions but the interested reader may generalize these inequalites to m functions.

Theorem 7.1

Suppose that $1 < r, s < \infty$ *with* $1/p = 1/r + 1/s$ *and* $1/2 < p < n$. *If* $1/q = 1/p - 1/n$ *and* $(u, v) \in A_{(r,s),q}$ *with* $\nu = uv$, *then there exists a constant* $C > 0$ *such that*

$$\|fg\nu\|_{L^q} \leq C(\|(\nabla f)u\|_{L^r} \|gv\|_{L^s} + \|fu\|_{L^r} \|(\nabla g)v\|_{L^s})$$

for all $f, g \in C_c^\infty(\mathbb{R}^n)$.

Proof. Given $y_1, y_2 \in \mathbb{R}^n$ Denote $\vec{y} \in \mathbb{R}^{2n}$ by $\vec{y} = (y_1, y_2) = (y_{1,1}, \ldots, y_{1,n}, y_{2,1}, \ldots, y_{2,n})$. and $\nabla_{2n} = (\partial_{1,1}, \cdots, \partial_{1,n}, \partial_{2,1}, \ldots, \partial_{2,n})$ be the gradient in $\mathbb{R}^{2n}$. Since f, g have compact support and are smooth we have, see Stein [24, p. 125] we have

$$\begin{aligned} |f(x)g(x)| &\leq C \int_{\mathbb{R}^{2n}} \frac{|\nabla_{2n} f(x-y_1) g(x-y_2)|}{|\vec{y}|^{2n-1}} d\vec{y} \\ &\leq C(\mathcal{I}_1(|\nabla f|, |g|)(x) + \mathcal{I}_1(|f|, |\nabla g|)(x)), \end{aligned}$$

where ∇f and ∇g are the gradients of f and g in $\mathbb{R}^n$. It now follows that,

$$\begin{aligned} \|fg\nu\|_{L^q} &\leq C(\|\mathcal{I}_1(|\nabla f|, |g|)\nu\|_{L^q} + \|\mathcal{I}_1(|f|, |\nabla g|)\nu\|_{L^q}) \\ &\leq C(\|(\nabla f)u\|_{L^r} \|gv\|_{L^s} + \|fu\|_{L^r} \|(\nabla g)v\|_{L^s}). \end{aligned}$$

□

For $x \in \mathbb{R}^n$ let $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial {x_i}^2}$ denote the Laplacian operator in $\mathbb{R}^n$ and for $\vec{x} = (x_1, x_2) \in \mathbb{R}^{2n}$ let $\Delta_{2n} = \sum_{i=1}^2 \sum_{j=1}^n \frac{\partial^2}{\partial x_{i,j}^2}$ denote the Laplacian operator in $\mathbb{R}^{2n}$.

Theorem 7.2

Suppose that $n > 1$ and $1 < r, s < \infty$ with $1/p = 1/r + 1/s$ and $1/2 < p < n/2$. Then if $1/q = 1/p - 2/n$ and $(u, v) \in A_{(r,s),q}$ with $\nu = uv$ there exists a constant $C > 0$ such that

$$\|fg\nu\|_{L^q} \leq C(\|(\Delta f)u\|_{L^r}\|gv\|_{L^s} + \|f\|_{L^r}\|(\Delta g)v\|_{L^s})$$

for all $f, g \in C_c^\infty(\mathbb{R}^n)$.

Proof. Since $f, g \in C_c^\infty(\mathbb{R}^{2n})$ we can write

$$f(x)g(x) = C\int_{\mathbb{R}^{2n}} \frac{\Delta_{2n} f(y_1)g(y_2)}{(|x-y_1|^2 + |x-y_2|^2)^{(2n-2)/2}} d\vec{y}.$$

Notice we are restricting the integral to a set of measure zero, however this is ligitimate since it is an absolutely convergent integral. Then we have

$$\begin{aligned} f(x)g(x) &= C\int_{\mathbb{R}^{2n}} \frac{\Delta_{2n} f(y_1)g(y_2)}{(|x-y_1|^2 + |x-y_2|^2)^{(2n-2)/2}} d\vec{y} \\ &= C\int_{\mathbb{R}^{2n}} \frac{g(y_2)\Delta f(y_1) + f(y_1)\Delta g(y_2)}{(|x-y_1|^2 + |x-y_2|^2)^{(2n-2)/2}} d\vec{y}. \end{aligned}$$

Thus we have

$$|f(x)g(x)| \leq C\mathcal{I}_2(|\Delta f|, |g|)(x) + C\mathcal{I}_2(|f|, |\Delta g|)(x).$$

Using the boundedness of $\mathcal{I}_2$ we obtain,

$$\begin{aligned} \|fg\nu\|_{L^q} &\leq C(\|\mathcal{I}_2(|\Delta f|, |g|)\nu\|_{L^q} + \|\mathcal{I}_2(|f|, |\Delta g|)\nu\|_{L^q)} \\ &\leq C(\|(\Delta f)u\|_{L^r}\|gv\|_{L^s} + \|fu\|_{L^r}\|(\Delta g)v\|_{L^s}). \end{aligned}$$

□

Remark 7.3 Condition (2.5) is not sufficient for the strong boundedness of $\mathcal{M}$.

This uses an arguement similar to the linear case. We require the following lemma which us a multilinear version of the fact $w \in A_p$ implies $w^{1-p'} \in A_{p'}$ in the linear situation.

Lemma 7.4

Given $1 < p_1, \ldots, p_m < \infty$ and $(u, v_1, \ldots, v_m)$ that satisfy (2.5). If we set $\vec{P}' = ({p_1}', \ldots, {p_m}')$, and $q = p/(pm-1)$ so

$$\frac{1}{q} = m - \frac{1}{p} = \frac{1}{{p_1}'} + \cdots + \frac{1}{{p_m}'}.$$

Then the weights

$$(v_1^{-q/p_1} \cdots v_m^{-q/p_m}, u^{1-{p_1}'}, \ldots, u^{1-{p_m}'})$$

satisfy (2.5) with respect to $\vec{P}'$.

Proof. The condition for $(v_1^{-q/p_1}\cdots v_m^{-q/p_m}, u^{1-p_1{}'},\ldots,u^{1-p_m{}'})$ is

$$\sup_Q\Big(\frac{1}{|Q|}\int_Q v_1^{-q/p_1}\cdots v_m^{-q/p_m}\,dx\Big)^{1/q}\prod_{i=1}^m\Big(\frac{1}{|Q|}\int_Q (u^{1-p_i'})^{1-p_i}\,dx\Big)^{1/p_i'}<\infty$$

Using Hölder's inequality,

$$\Big(\frac{1}{|Q|}\int_Q v_1^{-q/p_1}\cdots v_m^{-q/p_m}\,dx\Big)^{1/q}\prod_{i=1}^m\Big(\frac{1}{|Q|}\int_Q (u^{1-p_i'})^{1-p_i}\,dx\Big)^{1/p_i'}$$
$$\le\prod_{i=1}^m\Big(\frac{1}{|Q|}\int_Q v_i^{-p_i'/p_i}\,dx\Big)^{1/p_i'}\Big(\frac{1}{|Q|}\int_Q u\,dx\Big)^{1/p_i'}.$$

Thus the lemma follows from this inequality. □

Now notice that $(u, Mu,\ldots,Mu)$ satisfy condition 2.5. By the lemma with $q=p/(mp-1)$ we have $(Mu^{-\frac{q}{p}}, u^{1-p_1{}'},\ldots,u^{1-p_m{}'})$ satisfying condition 2.5 with respect to $\vec{P}'$. If condition 2.5 were sufficient then we would have

$$\Big(\int_{\mathbb{R}^n}\mathcal{M}(f_1,\ldots,f_m)^q Mu^{-\frac{q}{p}}\,dx\Big)^{1/q}\le C\prod_{i=1}^m\Big(\int_{\mathbb{R}^n}|f_i|^{p_i'}u^{1-p_i{}'}\,dx\Big)^{1/p_i{}'}.$$

But setting $f_1=\cdots=f_m=u$, and using the fact that $\mathcal{M}(u,\ldots,u)=(Mu)^m$ we have

$$\Big(\int_{\mathbb{R}^n}Mu\,dx\Big)\le C\Big(\int_{\mathbb{R}^n}u\,dx\Big),$$

which is a clear contradiction.

Remark 7.5 In general we have strict containment in (3.2), i.e.

$$\bigcup_{q_1,\ldots,q_m}\prod_{i=1}^m A_{p_i,q_i}\subsetneq A_{\vec{P},q}.$$

Take for example, $n=1$, $m=2$, $p_1=p_2=2$, and $q=3/2$. We use a similar example to the one given in [14] let

$$w_1(x)=\begin{cases}|x-1|^{-1/2} & x\in[0,2]\\ 1 & \text{otherwise}\end{cases}$$

and $w_2(x)=|x|^{-1/2}$. Then $(w_1w_2)^q$ is in A_1 and $\inf_Q(w_1w_2)^q\sim(\inf_Q w_1^q)(\inf_Q w_2^q)$ but for any power $r\ge 2$ $w_i^r\notin L^1_{\text{loc}}$ and hence cannot be in $A_{r,2}$ for any such r.

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