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WEIGHTED INEQUALITIES FOR ONE-SIDED MAXIMAL FUNCTIONS

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ABSTRACT. Let M_g^+ be the maximal operator defined by

$$M_g^+ f(x) = \sup_{h>0} \left(\int_x^{x+h} |f(t)| g(t)\, dt \right) \left(\int_x^{x+h} g(t)\, dt \right)^{-1},$$

where g is a positive locally integrable function on $\mathbf{R}$. We characterize the pairs of nonnegative functions (u, v) for which M_g^+ applies $L^p(v)$ in $L^p(u)$ or in weak-$L^p(u)$. Our results generalize Sawyer's (case $g = 1$) but our proofs are different and we do not use Hardy's inequalities, which makes the proofs of the inequalities self-contained.

1. INTRODUCTION

In this paper we will study the operator M_g^+ acting on measurable real functions on $\mathbf{R}$ defined by

$$(1.1) \qquad M_g^+ f(x) = \sup_{h>0} \int_x^{x+h} |f(t)| g(t)\, dt \left(\int_x^{x+h} g(t)\, dt \right)^{-1},$$

where g is a locally integrable and positive function. If $g = 1$ we obtain the one-sided Hardy-Littlewood maximal operator which has been studied by Sawyer [7].

We will characterize the pairs of weights (u, v) such that M_g^+ is of weak and strong type (p, p) with respect to the measures $v\,dx$ and $u\,dx$. Our results include Sawyer's as particular cases, but with different proofs. The proof of the theorem about the weak type (p, p) $(p > 1)$ is adapted from [1]. On the other hand, the proof of the theorem about the strong type (p, p) is simpler than the corresponding one in [7] (our proof follows the pattern of the proof in [6]) and besides we do not use Hardy's inequalities which makes the proofs of the inequalities self-contained. We also include the weak type (1,1) that is not

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studied in [7]. Finally we give several results about the good weights for M_g^+ such as relations with Muckenhoupt's classes, factorization, and extrapolation.

2. Notation and main results

Throughout this paper, g will be a positive locally integrable function and C a positive constant not necessarily the same at each occurrence. If $p > 1$, then its conjugate exponent will be denoted by p', and for a Lebesgue measurable set A, χ_A will be its characteristic function and $|A|$ its measure.

We will say that a pair of nonnegative functions (u, v) satisfies condition $A_p^+(g)$, $p > 1$, if there exists a constant $C > 0$ such that for every y, x, b with $y \le x \le b$,

$$\int_y^x u \left(\int_x^b g^{p'}\sigma\right)^{p-1} \le C\left(\int_y^b g\right)^p, \tag{2.1}$$

where $\sigma = v^{-1/p-1}$ (as usual, we consider $0 \cdot \infty = 0$).

Condition $A_1^+(g)$ is given by

$$M_g^-(g^{-1}u) \le Cg^{-1}v \quad \text{a.e.}, \tag{2.2}$$

where M_g^- is the left maximal operator defined in the obvious way.

A pair of nonnegative functions (u, v) satisfies condition $S_p^+(g)$, $p > 1$, if there exists a constant $C > 0$ such that for every interval $I = (a, b)$ with $\int_{(-\infty, a)} u > 0$,

$$\int_a^b (M_g^+(\chi_I g^{1/p-1}\sigma))^p u \le C\int_a^b g^{p'}\sigma < \infty. \tag{2.3}$$

Our main results are the following three theorems.

Theorem 1. *M_g^+ is of weak type (p, p), $p > 1$, with respect to the measures $v\,dx$ and $u\,dx$ if and only if (u, v) satisfies $A_p^+(g)$.*

Theorem 2. *M_g^+ is of strong type (p, p), $p \ge 1$, from $L^p(v)$ to $L^p(u)$ if and only if (u, v) satisfies $S_p^+(g)$.*

Theorem 3. *If $u = v$ and $p > 1$, $A_p^+(g)$ and $S_p^+(g)$ are equivalent conditions, that is, the weak type (p, p) is equivalent to the strong type (p, p).*

3. Proof of Theorem 1 for $p = 1$

We will need two lemmas:

Lemma 1. *Let w be a positive increasing function defined on $I = [a, b]$ (i.e., $s \le t$ implies $w(s) \le w(t)$). Let f be a positive function on I. Suppose for some positive number λ*

$$\int_t^b gf \ge \lambda \int_t^b g \quad \text{for every } t \in I.$$

Then $\lambda \int_a^b gw \le \int_a^b gfw$.

Proof of Lemma 1. Let $B > 1$. Let

$$E = \left\{ t \in [a, b] : \lambda \int_a^b gw \right.$$
$$\left. \leq \lambda \int_a^t gw + Bw(t) \left[\lambda \int_t^b g - \int_t^b fg \right] + B \int_t^b fgw \right\}.$$

Let $\tau = \inf E$ (E is nonempty). We claim that $\tau = a$. If $a < \tau$, let $\eta \in (a, \tau)$ such that $Bw(\eta) \geq \operatorname{ess\,sup}\{w(t) : a < t \leq \tau\}$. We will prove that $\eta \in E$, which will contradict that $\tau = \inf E$. Since $\tau \in E$ we have

$$\lambda \int_a^b gw \leq \lambda \int_a^\tau gw + Bw(\tau) \left[\lambda \int_\tau^b g - \int_\tau^b fg \right] + B \int_\tau^b fgw.$$

Now, the fact that w is increasing, the assumptions of the lemma and $\eta < \tau$ give

$$\lambda \int_a^b gw \leq \lambda \int_a^\eta gw + \lambda \int_\eta^\tau gw + Bw(\eta) \left[\lambda \int_\tau^b g - \int_\tau^b fg \right]$$
$$+ B \int_\eta^b fgw - B \int_\eta^\tau fgw.$$

If we use again that w is increasing and the election of η, we obtain

$$\lambda \int_a^b gw \leq \lambda \int_a^\eta gw + Bw(\eta) \left[\lambda \int_\tau^b g - \int_\tau^b fg \right]$$
$$+ B \int_\eta^b fgw + Bw(\eta) \left[\lambda \int_\eta^\tau g - \int_\eta^\tau fg \right]$$
$$= \lambda \int_a^\eta gw + Bw(\eta) \left[\lambda \int_\eta^b g - \int_\eta^b fg \right] + B \int_\eta^b fgw.$$

This means that $\eta \in E$, a contradiction. Hence, $\tau = a$ and then $a \in E$, that is

$$\lambda \int_a^b gw \leq Bw(a) \left[\lambda \int_a^b g - \int_a^b fg \right] + B \int_a^b fgw.$$

Since the expression in brackets is nonpositive, we obtain $\lambda \int_a^b gw \leq B \int_a^b fgw$. Letting B tend to 1, we have the result.

Lemma 2. *If* (u, v) *satisfies* $A_1^+(g)$ *and* $[a, b]$ *is an interval, then there exists an increasing function* w *on* $[a, b]$ *such that*

(i) $w(s) \leq Cg^{-1}(s)v(s)$ *a.e.* $s \in [a, b]$.

(ii) $\int_a^b u \leq \int_a^b gw$.

Proof of Lemma 2. Let $G(y) = M_g^-(g^{-1}u\chi_{[a,b]})(y)$. The function G is lower semicontinuous and finite a.e. by $A_1^+(g)$.

Let $w(x) = \min_{x\le y\le b} G(y)$. It is obvious that w is increasing and verifies (i). To see (ii), let $0 < B < 1$ and

$$A = \left\{ t \in [a, b] \colon \int_y^b gw \ge B \int_y^b u \ \text{ for every } y \in [t, b] \right\}.$$

It is clear that A is a closed interval $[\tau, b]$. We will prove that $\tau = a$.

Suppose $\tau > a$. Since G is lower semicontinuous, there exists $\delta > 0$ such that $G(x) \ge BG(\tau)$ if $x \in [\tau - \delta, \tau)$. For such an x,

$$\begin{aligned} w(x) &= \min_{x\le y\le b} G(y) = \min\left\{ \min_{x\le y\le \tau} G(y), \min_{\tau\le y\le b} G(y) \right\} \\ &\ge \min\{BG(\tau), w(\tau)\} \ge Bw(\tau). \end{aligned}$$

By the definition of w, there exists γ with $\tau \le \gamma \le b$ such that $w(\tau) = G(\gamma)$. For every $x \in [\tau, \gamma]$, $w(x) = w(\tau) = G(\gamma)$, and for every $x \in [\tau - \delta, \tau)$, $w(x) \ge Bw(\tau) = BG(\gamma)$. Therefore, if $x \in [\tau - \delta, \gamma]$ then $w(x) \ge BG(\gamma)$. Hence

$$\int_x^\gamma gw \ge BG(\gamma) \int_x^\gamma g \ge B \int_x^\gamma u \quad \text{for every } x \in [\tau - \delta, \gamma].$$

This means that $\tau - \delta \in A$, which contradicts that τ is the infimum of A. Therefore $\tau = a$ and then $\int_a^b gw \ge B \int_a^b u$. Letting B tend to 1 the proof is finished.

Now, it is easy to prove that $A_1^+(g)$ is sufficient for the weak (1,1) inequality. Let f be a positive function with support bounded from above, and let $\lambda, N > 0$. Let $O_{\lambda, N} = (-N, \infty) \cap \{x \colon M_g^+ f(x) > \lambda\}$. $O_{\lambda, N}$ is a bounded open set and therefore there exists a sequence of maximal pairwise disjoint finite intervals $\{(a_j, b_j)\}$ such that $O_{\lambda, N} = \bigcup (a_j, b_j)$ and $\int_x^{b_j} fg \ge \lambda \int_x^{b_j} g$ for every $x \in (a_j, b_j)$. For each j, by Lemma 2, there exists an increasing function w_j on $[a_j, b_j]$ such that

$$w_j(t) \le Cg^{-1}(t)v(t) \quad \text{a.e. } t \in [a_j, b_j] \tag{3.1}$$

and

$$\int_{a_j}^{b_j} u \le \int_{a_j}^{b_j} gw_j. \tag{3.2}$$

If we apply Lemma 1 to each w_j, we obtain

$$\lambda \int_{a_j}^{b_j} gw_j \le \int_{a_j}^{b_j} gfw_j. \tag{3.3}$$

Now (3.2), (3.3), and (3.1) give

$$\begin{aligned} \int_{O_{\lambda, N}} u &= \sum_j \int_{a_j}^{b_j} u \le \sum_j \int_{a_j}^{b_j} gw_j \\ &\le \lambda^{-1} \sum_j \int_{a_j}^{b_j} gfw_j \le C\lambda^{-1} \sum_j \int_{a_j}^{b_j} fv = C\lambda^{-1} \int_{O_{\lambda, N}} fv. \end{aligned}$$

Letting N tend to infinity we obtain $\int_{\{x\,;\,M_g^+f(x)>\lambda\}} u \le C\lambda^{-1}\int_{-\infty}^{+\infty} fv$.

Conversely, let us suppose that M_g^+ is of weak type (1,1) with respect to the measures $v\,dx$ and $u\,dx$. For every natural number N we consider the set $E_N = \{x\colon g^{-1}(x)v(x) \le N\}$ and the function $v_N = v\chi_{E_N}$. Let F_N and H_N be the Lebesgue sets of $g^{-1}v_N$ and χ_{E_N} respectively. It is clear that if $F = \bigcap_N F_N \cap H_N$ then $|\mathbf{R} - F| = 0$. Let x be in F, and let δ, $\varepsilon > 0$ such that $\int_{x-\delta}^{x+\varepsilon} g \le 2\int_{x-\delta}^{x} g$. Now consider N with $g^{-1}(x)v(x) \le N$. If $f_N = g^{-1}\chi_{E_N\cap(x\,,\,x+\varepsilon)}$ and $y \in (x-\delta\,,\,x)$ then

$$M_g^+ f_N(y) \ge \int_x^{x+\varepsilon} \chi_{E_N}\left(2\int_{x-\delta}^{x} g\right)^{-1}.$$

Therefore, by the weak type inequality,

$$\int_{x-\delta}^{x} u \le 2C\left(\int_{x-\delta}^{x} g\right)\left(\int_x^{x+\varepsilon} g^{-1}v_N\right)\left(\int_x^{x+\varepsilon}\chi_{E_N}\right)^{-1}.$$

If we let ε tend to zero and then N to infinity we get

$$\int_{x-\delta}^{x} u \le 2C\left(\int_{x-\delta}^{x} g\right)\left(g^{-1}v\right)(x).$$

Since δ is an arbitrary positive number we obtain $M_g^- u(x) \le 2C(g^{-1}v)(x)$ for all x in F and thus for almost every x in $\mathbf{R}$.

4. Proof of Theorem 1 for $p > 1$

Suppose that $(u\,,\,v)$ satisfies $A_p^+(g)$ and $\int_{(-\infty\,,\,b)} g = \infty$ for every b in $\mathbf{R}$. Then if $y \le x \le b$

$$\left(\int_y^x u\right)\left(\int_x^b g^{p'}\sigma\right)^{p-1} \le C\left(\int_y^b g\right)^p, \tag{4.1}$$

with C independent of x, y, and b. Let $\alpha > 0$. Multiplying both sides of (4.1) by $g(y)(\int_y^b g)^{-p-\alpha-1}$ and integrating with respect to y on $(-\infty\,,\,x)$ we get

$$\begin{aligned}\left(\int_x^b g^{p'}\sigma\right)^{p-1}&\int_{-\infty}^{x} g(y)\left(\int_y^x u\right)\left(\int_y^b g\right)^{-p-\alpha-1} dy\\ &\le C\int_{-\infty}^{x} g(y)\left(\int_y^b g\right)^{-\alpha-1} dy\end{aligned} \tag{4.2}$$

for every x. Computing the right-hand side of (4.2), we obtain

(4.3)

$$\left(\int_x^b g^{p'}\sigma\right)^{p-1}\int_{-\infty}^{x} g(y)\left(\int_y^x u\right)\left(\int_y^b g\right)^{-p-\alpha-1} dy \le C\alpha^{-1}\left(\int_x^b g\right)^{-\alpha}.$$

Besides

$$
\begin{aligned}
&\int_{-\infty}^{x} g(y)\left(\int_{y}^{x} u(t)\,dt\right)\left(\int_{y}^{b} g\right)^{-p-\alpha-1} dy \\
(4.4)\qquad &= \int_{-\infty}^{x} u(t)\left(\int_{-\infty}^{t} g(y)\left(\int_{y}^{b} g\right)^{-p-\alpha-1} dy\right) dt \\
&= \int_{-\infty}^{x} u(t)(p+\alpha)^{-1}\left(\int_{t}^{b} g\right)^{-p-\alpha} dt.
\end{aligned}
$$

(4.3) and (4.4) give

$$
(4.5)\quad \left(\int_{x}^{b} g^{p'}\sigma\right)^{p-1}\int_{-\infty}^{x} u(t)\left(\int_{t}^{b} g\right)^{-p-\alpha} dt \le C(p+\alpha)\alpha^{-1}\left(\int_{x}^{b} g\right)^{-\alpha}.
$$

It is interesting to note that (4.5) holds even if $g^{p'}\sigma$ is not locally integrable, since $A_p^+(g)$ implies that if the integral of $g^{p'}\sigma$ on $[x_1, x_2]$ is infinite then $u(t) = 0$ for a.e. $t < x_1$.

Let f be a positive function with support bounded from above. For $\lambda > 0$ and natural N, let $O_{\lambda, N} = \{x\colon M_g^+ f(x) > \lambda\} \cap (-N, \infty)$. Let (a, b) be a connected component of O_λ. We have

$$
(4.6)\qquad \lambda\int_{x}^{b} g \le \int_{x}^{b} fg \quad \text{for every } x \text{ in } (a, b).
$$

Let $A = \{x \in [a, b]\colon \int_x^b g^{p'}\sigma = \infty\}$. If $A \ne \varnothing$, let $x_0 = \sup A$; if $A = \varnothing$, let $x_0 = a$. Then $\int_x^b g^{p'}\sigma < \infty$ for every $x > x_0$ and it follows from $A_p^+(g)$ that $u(x) = 0$ a.e. in $[a, x_0]$. Thus $\int_a^b u = \int_{x_0}^b u$.

Let H and h be the functions defined on (a, b) by

$$
H(x) = \int_{a}^{x} u(t)\left(\int_{t}^{b} g\right)^{-p-\alpha} dt \quad \text{and} \quad h(x) = \left(\int_{x}^{b} g^{p'}\sigma\right)^{1/p'}.
$$

It is clear that $H(x) = 0$ and $h(x) = \infty$ if $x < x_0$ and from (4.5) we get

$$
(4.7)\qquad (h(x))^{p} H(x) \le C(p+\alpha)\alpha^{-1}\left(\int_{x}^{b} g\right)^{-\alpha} \quad \text{for every } x \in [a, b].
$$

On the other hand we have

$$
(4.8)\qquad \int_{a}^{b} u(x)\,dx = \int_{x_0}^{b} H'(x)\left(\int_{x}^{b} g\right)^{p+\alpha} dx.
$$

Integration by parts in the right-hand side of (4.8) gives

$$\begin{aligned}\int_a^b u(x)\,dx &= (p+\alpha)\int_{x_0}^b H(x)g(x)\left(\int_x^b g\right)^{p+\alpha-1} dx \\ &\le C(p+\alpha)^2\alpha^{-1}\int_{x_0}^b h^{-p}(x)g(x)\left(\int_x^b g\right)^{p-1} dx,\end{aligned} \tag{4.9}$$

by (4.7). Again integration by part gives

$$\begin{aligned}\int_a^b u \le C(p+\alpha)^2\alpha^{-1}&\left(p^{-1}h^{-p}(x_0)\left(\int_{x_0}^b g\right)^p\right. \\ &\left.-\int_{x_0}^b h'(x)h^{-p-1}(x)\left(\int_x^b g\right)^p dx\right).\end{aligned} \tag{4.10}$$

On the other hand, if we raise both sides of (4.6) to the pth power and apply Hölder's inequality with exponents p and p', after introducing suitable factors, we obtain

$$\begin{aligned}\lambda^p &\le \left(\int_x^b g\right)^{-p}\left(\int_x^b gf\right)^p \\ &\le \left(\int_x^b g\right)^{-p}\left(\int_x^b f^p hv\right)\left(\int_x^b g^{p'}\sigma h^{-p'/p}\right)^{p/p'}.\end{aligned} \tag{4.11}$$

Computing the last integral of the above inequality gives us

$$\begin{aligned}\int_x^b g^{p'}\sigma h^{-p'/p} &= \int_x^b g^{p'}(t)\sigma(t)\left(\int_t^b g^{p'}\sigma\right)^{-1/p} dt \\ &= p'\left(\int_x^b g^{p'}\sigma\right)^{1/p'} = p'h(x).\end{aligned} \tag{4.12}$$

Then (4.11) becomes

$$\left(\int_x^b g\right)^p \le \lambda^{-p}p'^{p-1}\left(\int_x^b f^p hv\right)h^{p-1}(x). \tag{4.13}$$

If we set $x = x_0$ in (4.13), we obtain an inequality that allows us to majorize the first addend of the right-hand side in (4.10), i.e.,

$$h^{-p}(x_0)\left(\int_{x_0}^b g\right)^p \le \lambda^{-p}p'^{p-1}\left(\int_{x_0}^b f^p hv\right)h^{-1}(x_0). \tag{4.14}$$

To majorize the second addend we will use (4.13), the positivity of $-h'$, and integration by parts:

$$(4.15)\qquad \begin{aligned} &-\int_{x_0}^{b} h^{-p-1}(x)h'(x)\left(\int_x^b g\right)^p dx \\ &\le -\lambda^{-p}p'^{p-1}\int_{x_0}^{b} h^{-2}(x)h'(x)\left(\int_x^b f^p hv\right) dx \\ &= \lambda^{-p}p'^{p-1}\left(-h^{-1}(x_0)\int_{x_0}^{b} f^p hv + \int_{x_0}^{b} f^p v\right). \end{aligned}$$

Finally, (4.14), (4.15), and (4.10) give

$$(4.16)\quad \int_a^b u \le C\lambda^{-p}(p+\alpha)^2 p'^{p-1}\alpha^{-1}\int_{x_0}^{b} f^p v \le C\lambda^{-p}(p+\alpha)^2 p'^{p-1}\alpha^{-1}\int_a^b f^p v.$$

We have proved that

$$\int_{O_{\lambda,N}} u \le C\lambda^{-p}\int_{O_{\lambda,N}} f^p v.$$

Letting N tend to infinity we obtain the weak inequality.

Everything we have just done is based on the assumption $\int_{(-\infty,b)} g = \infty$ for every b. If it is not true we define $g_n = g$ if $x \ge -n$ and $g_n = \max\{g, 1\}$ if $x < -n$, for every natural n. Every g_n verifies $\int_{-\infty}^{b} g_n = \infty$ for every b, and since $(u, v) \in A_p^+(g)$ we have that $(u, g^{-p}g_n^p v)$ satisfies $A_p^+(g_n)$. Then, by what we have already shown

$$(4.17)\qquad \int_{\{x\,:\, M_{g_n}^+ f(x)>\lambda\}} u \le C\lambda^{-p}\int_{-\infty}^{+\infty} |f|^p g_n^p g^{-p} v$$

for every f, where C depends only on the constant of the $A_p^+(g)$ condition. Now if we apply (4.17) to the functions $f\chi_{(-n,\infty)}$ we have

$$\int_{\{x\ge -n\,:\, M_g^+ f(x)>\lambda\}} u \le C\lambda^{-p}\int_{-n}^{+\infty} |f|^p v.$$

Letting n tend to infinity we obtain the weak type inequality.

Conversely, suppose that M_g^+ is of weak type (p, p) with respect to the measures $v\,dx$ and $u\,dx$. Let x, y, b be given with $x \le y \le b$. For every natural n, let $h_n = g^{p'}\sigma\chi_{\{x\,:\, g^{p'}\sigma(x)<n\}}$. $\{h_n\}$ is an increasing sequence with limit $g^{p'}\sigma$. Let $f = \chi_{(y,b)}(g^{-1}v)^{-1/p-1}\chi_{\{x\,:\, g^{p'}\sigma<n\}}$ and $B_n = \int_y^b h_n(\int_x^b g)^{-1}$. If $z \in [x, y]$ we have $M_g^+ f(z) \ge B_n$. Then, the weak type inequality gives

$$\int_x^y u \le CB_n^{-p}\int_y^b h_n,$$

or equivalently

$$\left(\int_x^y u\right)\left(\int_y^b h_n\right)^{p-1} \le C\left(\int_x^b g\right)^p.$$

Now the $A_p^+(g)$ condition follows from the monotone convergence theorem.

5. Proof of Theorem 2

The necessity of $S_p^+(g)$ for the two-norm inequality is trivial. For the converse it will suffice to prove the strong type inequality for bounded positive f in $L^p(v)$ with support bounded from above.

Let N be a positive integer. For $k > 0$ let

$$O_k = \{x \in \mathbf{R}\colon M_g^+ f(x) > 2^k\} \cap (-N, +\infty).$$

Each O_k is an open set and, therefore, there exists a sequence $\{I_{jk}\}_j$ of open pairwise disjoint intervals with $O_k = \bigcup_j I_{jk}$ and such that

$$\int_x^{b_{jk}} gf \geq 2^k \int_x^{b_{jk}} g \quad \text{for every } x \in I_{jk} = (a_{jk}, b_{jk}). \tag{5.1}$$

It is clear that $\sup_{j,k} |I_{jk}| < \infty$. For every j and k let $A_{jk} = \{x \in I_{jk}\colon \int_x^{b_{jk}} g^{p'} \sigma = \infty\}$. If $A_{jk} \neq \varnothing$, let $x_{jk} = \sup A$; if $A_{jk} = \varnothing$, let $x_{jk} = a_{jk}$. It is clear that $\int_x^{b_{jk}} g^{p'} \sigma < \infty$ if $x > x_{jk}$ and $u = 0$ a.e. x in (a_{jk}, x_{jk}) by $S_p^+(g)$. For every j and k let

$$E_{jk} = I_{jk} \cap \{x\colon M_g^+ f(x) \leq 2^{k+1}\} \quad \text{and} \quad F_{jk} = (x_{jk}, b_{jk}) \cap E_{jk}.$$

The sets E_{jk} are pairwise disjoint and for every k

$$\bigcup_j E_{jk} = \{x\colon 2^k < M_g^+ f(x) \leq 2^{k+1}\} \cap (-N, \infty). \tag{5.2}$$

Then

$$\begin{aligned}\int_{-N}^{+\infty} (M_g^+ f)^p u &= \sum_k \int_{(-N, +\infty)\cap\{x:\ 2^k < M_g^+ f(x) \leq 2^{k+1}\}} (M_g^+ f)^p u \\ &= \sum_{k,j} \int_{E_{jk}} (M_g^+ f)^p u = \sum_{k,j} \int_{F_{jk}} (M_g^+ f)^p u.\end{aligned}$$

By the definition of F_{jk} and by (5.1) we have that the last term is smaller than or equal to

$$2^p \sum_{k,j} \int_{F_{jk}} u(x) \left(\int_x^{b_{jk}} fg\right)^p \left(\int_x^{b_{jk}} g\right)^{-p} dx.$$

Therefore

$$\begin{aligned}\int_{-N}^{+\infty} (M_g^+ f)^p u &\leq 2^p \sum_{k,j} \int_{F_{jk}} u(x) \left(\int_x^{b_{jk}} fg\right)^p \left(\int_x^{b_{jk}} g^{p'} \sigma\right)^{-p} \\ &\quad \times \left(\int_x^{b_{jk}} g^{p'} \sigma\right)^p \left(\int_x^{b_{jk}} g\right)^{-p} dx.\end{aligned} \tag{5.3}$$

Let $X = \mathbf{Z} \times \mathbf{Z} \times \mathbf{R}$ and let ω be the product measure $\nu \times \nu \times m$ where ν is the counting measure on $\mathbf{Z}$ and m is the Lebesgue measure on $\mathbf{R}$. Let φ be the real function defined on X by

$$\varphi(j, k, x) = \chi_{F_{jk}}(x) u(x) \left(\int_x^{b_{jk}} g^{p'} \sigma\right)^p \left(\int_x^{b_{jk}} g\right)^{-p}$$

and let T be the linear operator

$$Th(j, k, x) = \int_x^{b_{jk}} h g^{p'} \sigma \left(\int_x^{b_{jk}} g^{p'} \sigma\right)^{-1}.$$

With these notations inequality (5.3) can be written in the following way:

$$\int_{-N}^{+\infty} (M_g^+ f)^p u \leq 2^p \int_X [T(f(g^{-1}v)^{1/p-1})]^p \varphi \, d\omega. \tag{5.4}$$

If we prove that the operator T is bounded from $L^p(g^{p'} \sigma \, dx)$ to $L^p(X, \varphi d\omega)$, we will get

$$\int_{-N}^{+\infty} (M_g^+ f)^p u \leq C 2^p \int_{-\infty}^{+\infty} (f(g^{-1}v)^{1/p-1})^p g^{p'} \sigma = C 2^p \int_{-\infty}^{+\infty} f^p v,$$

and, letting N tend to infinity, the proof will be finished.

To prove the boundedness of T we observe that it is obviously bounded in L^∞ and by Marcinkiewickz's interpolation theorem it will be enough to prove the weak type (1,1), i.e., $\int_{\{(j,k,x)\in X \,:\, Th(j,k,x)>\lambda\}} \varphi d\omega \leq C\lambda^{-1} \int_{-\infty}^{+\infty} h g^{p'} \sigma$ with C the constant of condition (2.3).

Let $A_{jk}(\lambda) = F_{jk} \cap \{x : Th(j, k, x) > \lambda\}$. The sets A_{jk} are pairwise disjoint. For each pair j, k, let $s_{jk}(\lambda) = \inf A_{jk}(\lambda)$ and $J_{jk} = J_{jk}(\lambda) = [s_{jk}(\lambda), b_{jk})$. If we pick up two of these intervals J_{jk} and J_{lm}, then they are either disjoint or one of them is contained in the other. Also it is clear that each J_{jk} verifies

$$\int_{J_{jk}} h g^{p'} \sigma \geq \lambda \int_{J_{jk}} g^{p'} \sigma. \tag{5.5}$$

Let $\{J_i\}$ be the maximal elements of the family $\{J_{jk}\}$. These maximal elements exist since the intervals J_{jk} have uniformly bounded lengths. Also the intervals J_i verify (5.5). Then

$$\begin{aligned}\int_{\{(j,k,x)\,:\,Th(j,k,x)>\lambda\}} \varphi(j,k,x)\,d\omega &= \sum_{k,j}\int_{A_{jk}(\lambda)} u(x)\left(\int_x^{b_{jk}} g^{p'}\sigma\right)^p\left(\int_x^{b_{jk}} g\right)^{-p} dx\\ &\le \sum_i \sum_{\{(k,j)\,:\,J_i\supset J_{jk}\}} \int_{A_{jk}(\lambda)} u(x)\left(\int_x^{b_{jk}} g^{p'}\sigma\right)^p\left(\int_x^{b_{jk}} g\right)^{-p} dx\\ &\le \sum_i \int_{J_i} (M_g^+(\chi_{J_i} g^{1/p-1}\sigma)(x))^p u(x)\,dx\\ &\le C\sum_i \int_{J_i} g^{p'}(x)\sigma(x)\,dx \quad \text{by (2.3)},\end{aligned}$$

and by (5.5) the last term is smaller than or equal to

$$C\lambda^{-1}\sum_i \int_{J_i} h(x) g^{p'}(x)\sigma(x)\,dx \le C\lambda^{-1}\int_{-\infty}^{+\infty} h(x) g^{p'}(x)\sigma(x)\,dx.$$

This proves the weak (1,1) inequality for T and hence the proof of Theorem 2 is finished.

6. Proof of Theorem 3

Suppose that $u \in A_p^+(g)$. Let $I = (a, b)$ be an interval such that $\int_{-\infty}^a u > 0$. This implies that $\int_a^b g^{p'}\sigma < \infty$ where $\sigma = u^{-1/p-1}$. Let $x \in I$ then there exist $h > 0$ with $x + h \in I$ such that

$$\tag{6.1} \frac{3}{4} M_g^+(\chi_I g^{1/p-1}\sigma)(x) \le \int_x^{x+h} g^{p'}\sigma\left(\int_x^{x+h} g\right)^{-1}.$$

For this h there exists t with $0 < t < h$ such that $2\int_x^{x+t} g = \int_x^{x+h} g$. This t verifies

$$\tag{6.2} \int_x^{x+t} g^{p'}\sigma\left(\int_x^{x+t} g\right)^{-1} \le M_g^+(\chi_I g^{1/p-1}\sigma)(x).$$

(6.1) and (6.2) give

$$\tag{6.3} M_g^+(\chi_I g^{1/p-1}\sigma)(x) \le 4\int_{x+t}^{x+h} g^{p'}\sigma\left(\int_x^{x+h} g\right)^{-1}.$$

On the other hand, condition $A_p^+(g)$ for u gives

$$\tag{6.4} \int_{x+t}^{x+h} g^{p'}\sigma\left(\int_x^{x+h} g\right)^{-1} \le C\left(\int_x^{x+h} g\right)^{p'-1}\left(\int_x^{x+t} u\right)^{1-p'}.$$

Now (6.3) together with (6.4) gives

$$\begin{aligned}(6.5)\qquad M_g^+(\chi_I g^{1/p-1}\sigma)(x) &\le C\left(\int_x^{x+h} g\right)^{p'-1}\left(\int_x^{x+t} u\right)^{1-p'}\\ &\le C(M_u^+(\chi_I g u^{-1})(x))^{p'-1}.\end{aligned}$$

Raising to p and multiplying by $u(x)$, we get

$$(6.6)\qquad (M_g^+(\chi_I g^{1/p-1}\sigma)(x))^p u(x) \le C(M_u^+(\chi_I g u^{-1})(x))^{p'} u(x).$$

But M_u^+ is bounded in $L^{p'}(u)$, because $u \in A_r^+(u)$ for every $r > 1$. Then

$$\int_I (M_g^+(\chi_I g^{1/p-1}\sigma)(x))^p u(x)\,dx \le C\int_I g^{p'}\sigma.$$

Therefore u satisfies $S_p^+(g)$.

The fact that $S_p^+(g)$ implies $A_p^+(g)$ is a consequence of Theorems 1 and 2, not only for a weight, but for pairs of weights. However, we are going to give a direct proof. Suppose that the pair (u, v) satisfies $S_p^+(g)$. Let a, b, and c be real numbers with $a \le b \le c$. If the integral of $g^{p'}\sigma$ on $[b, c]$ is equal to infinity, then, by $S_p^+(g)$, $u(x) = 0$ a.e. x in $[a, b]$ and the inequality

$$\int_a^b u\left(\int_b^c g^{p'}\sigma\right)^{p-1} \le C\left(\int_a^c g\right)^p$$

is trivially satisfied.

Now suppose that the integral of $g^{p'}\sigma$ over $[b, c]$ is finite. We define a possibly finite decreasing sequence by $x_0 = b$ and x_{k+1} the real number such that

$$2^{k+1}\int_b^c g^{p'}\sigma = \int_{x_{k+1}}^c g^{p'}\sigma$$

if

$$\int_{-\infty}^{x_k} g^{p'}\sigma \ge 2^k\int_b^c g^{p'}\sigma$$

otherwise the sequence finishes in x_k.

Suppose first that the sequence is finite and x_r is its last term. If $r = 0$, then

$$\begin{aligned}\int_a^b u(x)\left(\int_x^c g\right)^{-p}\left(\int_b^c g^{p'}\sigma\right)^p dx &\le \int_a^b u(x)(M_g^+(\chi_{(a,b)} g^{1/p-1}\sigma)(x))^p\,dx\\ &\le C\int_a^c g^{p'}\sigma \le 2C\int_b^c g^{p'}\sigma.\end{aligned}$$

This implies trivially $A_p^+(g)$.

If $r > 0$, let $a' < x_r$ and $a' < a$. Then,

$$\begin{aligned}
&\int_a^b u(x)\left(\int_x^c g\right)^{-p}\left(\int_b^c g^{p'}\sigma\right)^p dx \\
&\quad\le 2^{-rp}\int_{a'}^{x_r} u(x)\left(\int_x^c g\right)^{-p}\left(\int_{x_r}^c g^{p'}\sigma\right)^p dx \\
&\qquad+\sum_{k=0}^{r-1} 2^{-kp}\int_{x_{k+1}}^{x_k} u(x)\left(\int_x^c g\right)^{-p}\left(\int_{x_k}^c g^{p'}\sigma\right)^p dx \\
&\quad\le 2^{-rp}\int_{a'}^{x_r} u(x)(M_g^+(\chi_{(a',c)}g^{1/p-1}\sigma)(x))^p\,dx \\
&\qquad+\sum_{k=0}^{r-1} 2^{-kp}\int_{x_{k+1}}^{x_k} (M_g^+(\chi_{(x_{k+1},c)}g^{1/p-1}\sigma)(x))^p u(x)\,dx \\
&\quad\le C\left(\sum_{k=0}^{r} 2^{-kp+k+1}\right)\int_b^c g^{p'}\sigma \le C\int_b^c g^{p'}\sigma.
\end{aligned}$$

Finally, suppose that the sequence is infinite and let $d = \lim x_k$. If d is finite, then $u = 0$ a.e. in $(-\infty, d)$ by $S_p^+(g)$. So, whether d is finite or not we have

$$\int_a^b u(x)\left(\int_x^c g\right)^{-p}\left(\int_b^c g^{p'}\sigma\right)^p dx \le \int_d^b u(x)\left(\int_x^c g\right)^{-p}\left(\int_b^c g^{p'}\sigma\right)^p dx.$$

Using the reasoning above with sum from 0 to ∞ completes the proof.

7. Further results

(A) Relations with Muckenhoupt's $A_p(g)$ classes and Sawyer's $S_p(g)$ classes. Consider the weighted two-sided Hardy-Littlewood maximal operator defined by

$$M_g f(x) = \sup_{h,s>0}\left(\int_{x-s}^{x+h}|f|g\right)\left(\int_{x-s}^{x+h} g\right)^{-1}.$$

It is clear that the following relation holds:

$$\frac{1}{2}(M_g^+ + M_g^-) \le M_g \le M_g^+ + M_g^-. \tag{7.1}$$

We have the following results for M_g (see e.g. [5, 6]):

(i) Let $1 \le p < \infty$. M_g is of weak type (p, p) with respect to the measures $v\,dx$ and $u\,dx$ if and only if the pair (u, v) satisfies $A_p(g)$, i.e.

$A_p(g)$: There exists $C > 0$ such that

$$\left(\int_a^b u\right)\left(\int_a^b g^{p'}\sigma\right)^{p-1} \le C\left(\int_a^b g\right)^p$$

for every interval (a, b) and $p > 1$.

$A_1(g)$: There exists $C > 0$ such that $M_g(g^{-1}u) \le Cg^{-1}v$ a.e.

(ii) Let $1 < p < \infty$. M_g is of strong type (p, p) with respect to the measures vdx and udx if and only if the pair (u, v) satisfies $S_p(g)$, i.e.,

$S_p(g)$: There exists $C > 0$ such that for every interval (a, b)

$$\int_a^b |M_g(\chi_{(a,b)}g^{1/p-1}\sigma)|^p u \le C\int_a^b g^{p'}\sigma < \infty.$$

Of course, if $u = v$ then $A_p(g)$ and $S_p(g)$ are equivalent conditions.

It follows from these results, our theorems, and (7.1) that $A_p(g) = A_p^+(g) \cap A_p^-(g)$ $(1 \le p < \infty)$ and $S_p(g) = S_p^+(g) \cap S_p^-(g)$ $(1 < p < \infty)$. We will now give direct proofs of these equalities and so results (i) and (ii) will be consequences of the results in this paper.

Theorem 4. (a) $A_p(g) = A_p^+(g) \cap A_p^-(g)$ $(1 \le p < \infty)$.

(b) $S_p(g) = S_p^+(g) \cap S_p^-(g)$ $(1 < p < \infty)$.

Proof of Theorem 4. (a) For $p = 1$ the equality is trivial by (7.1). Let $1 < p$. Since it is clear that $A_p^+(g) \cap A_p^-(g) \supset A_p(g)$ we only have to prove $A_p(g) \supset A_p^+(g) \cap A_p^-(g)$. Let (u, v) be in $A_p^+(g) \cap A_p^-(g)$, let a and c be real numbers with $a \le c$, let N be a natural number, define $G_N(x) = (g^{p'}\sigma)(x)$ if $(g^{p'}\sigma)(x) \le N$ and $G_N(x) = 0$ otherwise. There exists h such that

$$\int_a^c G_N = 2\int_a^h G_N = 2\int_h^c G_N.$$

Then

$$\int_a^c u\left(\int_a^c G_N\right)^{p-1} = 2^{p-1}\int_a^h u\left(\int_h^c G_N\right)^{p-1} + 2^{p-1}\int_h^c u\left(\int_a^h G_N\right)^{p-1}$$
$$\le 2^p C\left(\int_a^c g\right)^p \quad \text{by } A_p^+(g) \text{ and } A_p^-(g).$$

Letting N tend to infinity we get $A_p(g)$.

Finally, (b) follows clearly from (7.1).

(B) Factorization. We will give here a result that generalizes the theorem of Coifman, Jones and Rubio de Francia [2] (see also [4]). As consequences, we will obtain the factorization of $A_p^+(g)$ and $A_p^-(g)$ weights.

Theorem 5. *Let F and G be two sublinear operators acting on measurable functions of a measure space $(X, \mathfrak{M}, \mu)$. For $p > 1$ let $W_p = \{w\colon F$ is bounded in $L^p(wd\mu)\}$ and $U_p = \{u\colon G$ is bounded in $L^p(ud\mu)\}$. Let g be a positive function, and let $W_1 = \{w\colon G(g^{-1}w) \le Cg^{-1}w$ a.e.$\}$ and $U_1 = \{u\colon F(g^{-1}u) \le Cg^{-1}u$ a.e.$\}$. Then $g^{p-1}W_1U_1^{1-p} \supset W_p \cap g^pU_p^{1-p}$, i.e., if $w \in W_p$ and $g^{p'}w^{-1/p-1} \in U_p$ then there exist $w_0 \in W_1$ and $u_0 \in U_1$ such that $w = g^{p-1}w_0u_0^{1-p}$.*

If $F = G$ and $g = 1$ we obtain the above-mentioned result of Coifman, Jones, and Rubio de Francia.

Proof. This proof follows the proof of Theorem 5.2 in [4], with the obvious changes. Suppose $1 < p \le 2$. Let $w \in W_p \cap g^p U_p^{1-p}$. We have to find v such that

(i) $vw \in W_1$, i.e., $G(g^{-1}vw) \le Cg^{-1}vw$ a.e.,

(ii) $gv^{1/p-1} \in U_1$, i.e., $F(v^{1/p-1}) \le Cv^{1/p-1}$ a.e.

Let us define an operator S by $S(u) = |G(g^{-1}uw)|w^{-1}g + (F(|u|^{1/p-1}))^{p-1}$. The operator S is positive, sublinear, and bounded on $L^{p'}(w)$. So, S verifies the conditions of Lemma 5.1 in [4], and it ensures the existence of such a v. Then $w_0 = vw$ and $u_0 = gv^{1/p-1}$.

Corollary 1. *$w \in A_p^+(g)$ if and only if $w = g^{p-1}w_0w_1^{1-p}$ with $w_0 \in A_1^+(g)$ and $w_1 \in A_1^-(g)$.*

Proof. If in Theorem 5, we take $F = M_g^+$ and $G = M_g^-$, the classes of good weights are, respectively $W_p = A_p^+(g)$ and $U_p = A_p^-(g)$. Then Theorem 5 assures

$$g^{p-1}A_1^+(g)(A_1^-(g))^{1-p} \supset A_p^+(g) \cap g^p(A_p^-(g))^{1-p}.$$

But $A_p^+(g) \cap g^p(A_p^-(g))^{1-p} = A_p^+(g)$, and this proves the factorization of a weight in $A_p^+(g)$.

Conversely, take $w_0 \in A_1^+(g)$ and $w_1 \in A_1^-(g)$, and let $w = g^{p-1}w_0w_1^{1-p}$. If $a \le b \le c$,

$$\begin{aligned}
&\int_a^b w\left(\int_b^c g^{p'}w^{-1/p-1}\right)^{p-1}\\
&\quad = \int_a^b w_0(g^{-1}w_1)^{1-p}\left(\int_b^c w_1(g^{-1}w_0)^{1-p'}\right)^{p-1}\\
&\quad \le C\left(\int_a^b w_0(x)\left(\int_x^{x+h} g\right)^{p-1}\left(\int_x^{x+h} w_1\right)^{1-p} dx\right)\\
&\quad\quad \times\left(\int_b^c w_1(x)\left(\int_{x-s}^x g\right)^{p'-1}\left(\int_{x-s}^x w_0\right)^{1-p'} dx\right)^{p-1}
\end{aligned}$$

for every $h, s > 0$ by condition $A_1^+(g)$ for w_0 and $A_1^-(g)$ for w_1. In particular, if $h = c - x$ and $s = x - a$ we obtain

$$\begin{aligned}
\int_a^b w\left(\int_b^c g^{p'} w^{-1/p-1}\right)^{p-1} \\
&\leq C\left(\int_a^c g\right)^p \left(\int_a^b w_0(x)\left(\int_x^c w_1\right)^{1-p} dx\right) \\
&\times \left(\int_b^c w_1(x)\left(\int_a^x w_0\right)^{1-p'} dx\right)^{p-1} \\
&\leq C\left(\int_a^c g\right)^p \left(\int_b^c w_1\right)^{1-p}\left(\int_a^b w_0\right)\left(\int_b^c w_1\right)^{p-1}\left(\int_a^b w_0\right)^{(1-p')(p-1)} \\
&= C\left(\int_a^c g\right)^p .
\end{aligned}$$

(C).

Theorem 6. *If w is in $A_1^+(g)$ then there exists $\delta > 0$ such that*

$$\int_a^b g^{-\delta} w^{1+\delta}\left(\int_a^b g\right)^{-1} \leq C_\delta \int_a^b w\left(\int_a^b g\right)^{-\delta} g^{-\delta}(b) w^\delta(b)$$

for every a and a.e. b. For this δ, $g^{-\delta}w^{1+\delta}$ is in $A_1^+(g)$.

Proof. Let a and b be real numbers with $a < b$ and with b verifying

$$M_g^-(g^{-1}w)(b) \leq C(g^{-1}w)(b).$$

Let $O_\lambda = \{x : M_g^-(g^{-1}w\chi_{(a,b)})(x) > \lambda\}$ be open. Then there exists a sequence of pairwise disjoint open intervals $I_j = (a_j, b_j)$ such that $O_\lambda = \bigcup I_j$ with

$$\int_{a_j}^x w\chi_{(a,b)}\left(\int_{a_j}^x g\right)^{-1} > \lambda \quad \text{for every } x \in (a_j, b_j) \tag{7.2}$$

and with

$$\int_{a_j}^{b_j} w\chi_{(a,b)}\left(\int_{a_j}^{b_j} g\right)^{-1} = \lambda \quad \text{for every } j. \tag{7.3}$$

It is clear that each a_j is bigger than a. Then, if $\lambda > C(g^{-1}w)(b)$, where C is the $A_1^+(g)$ constant of w, each I_j verifies either $(a, b) \supset I_j$ or $I_j \cap (a, b) = \varnothing$, since if I_j is not contained in (a, b) and $I_j \cap (a, b) \neq \varnothing$, then $b \in I_j$ and therefore

$$\int_{a_j}^b w\chi_{(a,b)}\left(\int_{a_j}^b g\right)^{-1} > \lambda > C(g^{-1}w)(b)$$

which goes against the election of b.

By Lebesgue's differentiation theorem we have that $\{x \in (a, b): (g^{-1}w)(x) > \lambda\}$ is contained in O_λ. This relation, (7.3), and condition $A_1^+(g)$ for w imply

$$\int_{\{x\in(a,b)\,:\,(g^{-1}w)(x)>\lambda\}} w \le \lambda \sum_{\{j\,:\,(a,b)\supset I_j\}} \int_{I_j} g \le \lambda \int_{\{x\in(a,b)\,:\,C(g^{-1}w)(x)>\lambda\}} g. \tag{7.4}$$

Let $\delta > 0$. Multiplying the last inequalities by $\lambda^{\delta-1}$ and then integrating with respect to λ from $C(g^{-1}w)(b)$ to $+\infty$ we get

$$\begin{aligned}&\int_{C(g^{-1}w)(b)}^{+\infty} \lambda^{\delta-1}\left(\int_{\{x\in(a,b)\,:\,(g^{-1}w)(x)>\lambda\}} w(x)\,dx\right) d\lambda \\ &\quad \le C^{\delta+1}(1+\delta)^{-1}\int_a^b (g^{-1}w)^{1+\delta}(x)g(x)\,dx.\end{aligned} \tag{7.5}$$

On the other hand, the first item of (7.5) is equal to

$$\begin{aligned}&\int_a^b \left(\int_{C(g^{-1}w)(b)}^{(g^{-1}w)(x)} \lambda^{\delta-1}\,d\lambda\right) w(x)\,dx \\ &\quad = \delta^{-1}\int_a^b g^{-\delta}w^{1+\delta} - C^\delta\delta^{-1}(g^{-1}w)^\delta(b)\int_a^b w.\end{aligned} \tag{7.6}$$

(7.5) together with (7.6) gives

$$(\delta^{-1} - C^{1+\delta}(1+\delta)^{-1})\int_a^b g^{-\delta}w^{1+\delta} \le C^\delta\delta^{-1}(g^{-1}w)^\delta(b)\int_a^b w. \tag{7.7}$$

Choosing δ such that $\delta^{-1} - C^{1+\delta}(1+\delta)^{-1} > 0$, we obtain the result.

Corollary 2. *Let $1 < p < \infty$. If w is in $A_p^+(g)$ then there exists $\varepsilon > 0$ such that $p - \varepsilon > 1$ and w is in $A_{p-\varepsilon}^+(g)$.*

Proof. Let $w \in A_p^+(g)$. By factorization, there exist w_0 in $A_1^+(g)$ and w_1 in $A_1^-(g)$ such that $w = g^{p-1}w_0w_1^{1-p}$. By Theorem 6 there exist $\delta > 0$ such that $g^{-\delta}w_1^{1+\delta} \in A_1^+(g)$. Then

$$w = g^{p-1}w_0w_1^{1-p} = g^{p-\varepsilon-1}w_0(g^{-\delta}w_1^{1+\delta})^{1-(p-\varepsilon)} \quad \text{with } \varepsilon = \delta(p-1)(1+\delta)^{-1}$$

and the result follows from Corollary 1.

Corollary 3. *If $w \in A_1^+(g)$ then there exists γ with $0 < \gamma < 1$, a function k with k and k^{-1} in L^∞, and a function f such that $w = kg(M_g^- f)^\gamma$.*

Proof. By Theorem 6, there exists $\delta > 0$ such that $M_g^-(g^{-1}w)^{1+\delta})^{1/(1+\delta)} \le Cg^{-1}w$ a.e. On the other hand, Lebesgue's differentiation theorem gives $g^{-1}w \le (M_g^-(g^{-1}w)^{1+\delta})^{1/(1+\delta)}$. Let $k(x) = g^{-1}(x)w(x)(M_g^-(g^{-1}w)^{1+\delta}(x))^{-1/(1+\delta)}$. Then $C^{-1} \le k \le 1$ and $w = gk(M_g^- f)^\gamma$ where $\gamma = (1+\delta)^{-1}$ and $f = (g^{-1}w)^{1+\delta}$.

(D) Extrapolation. We can also state the following theorem.

Theorem 7. *Let* T *be a sublinear operator acting on measurable functions on* $\mathbf{R}$. *Suppose that for a certain* p_0, $1 \le p_0 < \infty$, *and for every* w *in* $A^+_{p_0}(g)$, T *is of weak type* (p_0, p_0) *with respect to the measure* $w\,dx$. *Then for every* p *with* $1 < p < \infty$ *and every* w *in* $A^+_p(g)$, T *is bounded on* $L^p(w\,dx)$.

The proof follows that of [3] with the obvious changes, which are essentially the definition of G in Lemma 1 in [3] (now $G = (gM^-_g(g^{-1}h^{1/t}w)w^{-1})^t$) and the fact that w is in $A^+_p(g)$ if and only if $g^{p'}\sigma$ is in $A^-_{p'}(g)$.

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