

WEIGHTED INTEGRALS OF HOLOMORPHIC FUNCTIONS IN THE UNIT POLYDISC

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Let f be a measurable function defined on the unit polydisc U^n in \mathbf{C}^n and let $\omega_j(z_j)$, $j = 1, \dots, n$, be admissible weights on the unit disk U , with distortion functions $\psi_j(z_j)$, $\mathcal{L}_{\omega, N}^{p, q}(U^n) = \{f \mid \|f\|_{\mathcal{L}_{\omega, N}^{p, q}} < \infty\}$, where $\|f\|_{\mathcal{L}_{\omega, N}^{p, q}}^q = \int_{[0,1]^n} M_p^q(f, r) \prod_{j=1}^n \omega_j(r_j) dr_j$, and $\mathcal{A}_{\omega, N}^{p, q}(U^n) = \mathcal{L}_{\omega, N}^{p, q}(U^n) \cap H(U^n)$. We prove the following result: if $p, q \in [1, \infty)$ and for all $j = 1, \dots, n$, $\psi_j(z_j)(\partial f / \partial z_j)(z) \in \mathcal{L}_{\omega, N}^{p, q}$, then $f \in \mathcal{A}_{\omega, N}^{p, q}$ and there is a positive constant $C = C(p, q, \omega_j, n)$ such that $\|f\|_{\mathcal{A}_{\omega, N}^{p, q}} \leq C(|f(0)| + \sum_{j=1}^n \|\psi_j(\partial f / \partial z_j)\|_{\mathcal{L}_{\omega, N}^{p, q}})$.

1. Introduction

Let $U^1 = U$ be the unit disk in the complex plane, $dm(z) = (1/\pi) dr d\theta$ the normalized Lebesgue measure on U , U^n the unit polydisc in complex vector space \mathbf{C}^n and $H(U^n)$ the space of all analytic functions on U^n . For $z, w \in \mathbf{C}^n$ we write $z \cdot w = (z_1 w_1, \dots, z_n w_n)$; $e^{i\theta}$ is an abbreviation for $(e^{i\theta_1}, \dots, e^{i\theta_n})$; $dt = dt_1 \cdots dt_n$; $d\theta = d\theta_1 \cdots d\theta_n$ and r, θ are vectors in \mathbf{C}^n . If we write $0 \leq r < 1$, where $r = (r_1, \dots, r_n)$ it means $0 \leq r_j < 1$ for $j = 1, \dots, n$.

For $f \in H(U^n)$ and $p \in (0, \infty)$ we usually write

$$M_p(f, r) = \left(\frac{1}{(2\pi)^n} \int_{[0, 2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta \right)^{1/p}, \quad \text{for } 0 \leq r < 1 \quad (1.1)$$

for the integral means of f .

Let $\omega(s)$, $0 \leq s < 1$, be a weight function which is positive and integrable on $(0, 1)$. We extend ω on U by setting $\omega(z) = \omega(|z|)$. We may assume that our weights are normalized so that $\int_0^1 \omega(s) ds = 1$.

Let $\mathcal{L}_{\omega}^p = \mathcal{L}_{\omega}^p(U^n)$ denotes the class of all measurable functions defined on U^n such that

$$\|f\|_{\mathcal{L}_{\omega}^p}^p = \int_{U^n} |f(z)|^p \prod_{j=1}^n \omega_j(z_j) dm(z) < \infty, \quad (1.2)$$

where $\omega_j(z_j)$, $j = 1, \dots, n$, are admissible weights on the unit disk U .

The weighted Bergman space \mathcal{A}_ω^p is the intersection of \mathcal{L}_ω^p and $H(U^n)$. For $\omega_j(z_j) = (1 - |z_j|^2)^{\alpha_j}$, $\alpha_j > -1$, $j = 1, \dots, n$, we obtain the classical Bergman space $\mathcal{A}^p(dV_{\bar{\alpha}})$, see [1, page 33].

Let $\mathcal{L}_{\omega,N}^{p,q} = \mathcal{L}_{\omega,N}^{p,q}(U^n)$, $p, q > 0$, denotes the class of all measurable functions defined on U^n such that

$$\|f\|_{\mathcal{L}_{\omega,N}^{p,q}}^q = \int_{[0,1]^n} M_p^q(f,r) \prod_{j=1}^n \omega_j(r_j) dr_j < \infty, \tag{1.3}$$

and $\mathcal{A}_{\omega,N}^{p,q}$ be the intersection of $\mathcal{L}_{\omega,N}^{p,q}$ and $H(U^n)$. When $p = q$ we denote $\mathcal{A}_{\omega,N}^{p,q}$ by $\mathcal{A}_{\omega,N}^p$.

In the case $p = q$, these two norms are equivalent on the space $H(U^n)$, but the later one is more suitable for calculations than the first one. The result is contained in the following lemma.

LEMMA 1.1. *The norms $\|\cdot\|_{\mathcal{L}_\omega^p}$ and $\|\cdot\|_{\mathcal{A}_{\omega,N}^p}$ are equivalent on the space $H(U^n)$.*

Proof. By the polar coordinates it is easy to see that $\|f\|_{\mathcal{A}_{\omega,N}^p} \leq 2^n \|f\|_{\mathcal{L}_\omega^p}$ for every $f \in H(U^n)$, moreover $\|f\|_{\mathcal{L}_\omega^p} \leq 2^n \|f\|_{\mathcal{A}_{\omega,N}^p}$ for every f measurable on U^n .

Now we prove that there is a positive constant C , which is independent of f , such that

$$\|f\|_{\mathcal{A}_{\omega,N}^p} \leq C \|f\|_{\mathcal{L}_\omega^p}, \tag{1.4}$$

for every $f \in H(U^n)$. Without loss of generality we may assume that $n = 2$. Let $f \in H(U^n)$, then

$$\|f\|_{\mathcal{A}_{\omega,N}^p}^p = \int_0^{1/2} \int_0^{1/2} + \int_0^{1/2} \int_{1/2}^1 + \int_{1/2}^1 \int_0^{1/2} + \int_{1/2}^1 \int_{1/2}^1 g(r_1, r_2) dr_1 dr_2, \tag{1.5}$$

where $g(r_1, r_2) = M_p^p(f, r_1, r_2) \omega_1(r_1) \omega_2(r_2)$. Now we estimate these four integrals, which we denote by I_i , $i = 1, 2, 3, 4$.

Since $f \in H(U^2)$ then the function f is analytic in each variable separately on U and consequently $M_p^p(f, r_1, r_2)$ is nondecreasing function in r_1 and r_2 , see, for example, [3]. Let

$$C_{\omega_i} = \int_0^{1/2} \omega_i(r_i) / \int_{1/2}^1 \omega_i(r_i), \quad i = 1, 2. \tag{1.6}$$

Note that C_{ω_i} , $i = 1, 2$, are well defined and finite numbers since ω_i are positive integrable functions on $(0, 1)$.

Using the above mentioned facts and definitions, we have

$$\begin{aligned}
 I_1 &\leq M_p^p(f, 1/2, 1/2) \int_0^{1/2} \int_0^{1/2} \omega_1(r_1)\omega_2(r_2)dr_1 dr_2 \\
 &= C_{\omega_1}C_{\omega_2}M_p^p(f, 1/2, 1/2) \int_{1/2}^1 \int_{1/2}^1 \omega_1(r_1)\omega_2(r_2)dr_1 dr_2 \\
 &\leq C_{\omega_1}C_{\omega_2} \int_{1/2}^1 \int_{1/2}^1 M_p^p(f, r_1, r_2)\omega_1(r_1)\omega_2(r_2)dr_1 dr_2 \\
 &\leq 4C_{\omega_1}C_{\omega_2} \int_{1/2}^1 \int_{1/2}^1 M_p^p(f, r_1, r_2)\omega_1(r_1)\omega_2(r_2)r_1r_2dr_1 dr_2,
 \end{aligned} \tag{1.7}$$

$$\begin{aligned}
 I_2 &\leq \int_{1/2}^1 M_p^p(f, 1/2, r_2) \int_0^{1/2} \omega_1(r_1)dr_1\omega_2(r_2)dr_2 \\
 &= C_{\omega_1} \int_{1/2}^1 \int_{1/2}^1 M_p^p(f, 1/2, r_2)\omega_1(r_1)\omega_2(r_2)dr_1 dr_2 \\
 &\leq C_{\omega_1} \int_{1/2}^1 \int_{1/2}^1 M_p^p(f, r_1, r_2)\omega_1(r_1)\omega_2(r_2)dr_1 dr_2 \\
 &\leq 4C_{\omega_1} \int_{1/2}^1 \int_{1/2}^1 M_p^p(f, r_1, r_2)\omega_1(r_1)\omega_2(r_2)r_1r_2dr_1 dr_2.
 \end{aligned} \tag{1.8}$$

Similarly

$$I_3 \leq 4C_{\omega_2} \int_{1/2}^1 \int_{1/2}^1 M_p^p(f, r_1, r_2)\omega_1(r_1)\omega_2(r_2)r_1r_2dr_1 dr_2. \tag{1.9}$$

Finally, it is clear that

$$I_4 \leq 4 \int_{1/2}^1 \int_{1/2}^1 M_p^p(f, r_1, r_2)\omega_1(r_1)\omega_2(r_2)r_1r_2dr_1 dr_2. \tag{1.10}$$

From (1.7)–(1.10), we obtain

$$\|f\|_{\mathcal{S}_{\omega, N}^p}^p \leq (C_{\omega_1} + 1)(C_{\omega_2} + 1)\|f\|_{\mathcal{S}_{\omega}^p}^p, \tag{1.11}$$

as desired.

Following [8], for a given weight ω we define the function

$$\psi(r) = \psi_{\omega}(r) \stackrel{\text{def}}{=} \frac{1}{\omega(r)} \int_r^1 \omega(u)du, \quad 0 \leq r < 1, \tag{1.12}$$

and we call it the *distortion function* of ω . We put $\psi(z) = \psi(|z|)$ for $z \in U$. □

Definition 1.2 [8]. We say that a weight ω is *admissible* if it satisfies the following conditions:

- (i) there is a positive constant $A = A(\omega)$ such that

$$\omega(r) \geq \frac{A}{1-r} \int_r^1 \omega(u)du, \quad \text{for } 0 \leq r < 1; \tag{1.13}$$

(ii) ω is differentiable and there is a positive constant $B = B(\omega)$ such that

$$\omega'(r) \leq \frac{B}{1-r} \omega(r), \quad \text{for } 0 \leq r < 1; \tag{1.14}$$

(iii) for each sufficiently small positive δ there is a positive constant $C = C(\delta, \omega)$ such that

$$\sup_{0 \leq r < 1} \frac{\omega(r)}{\omega(r + \delta\psi(r))} \leq C. \tag{1.15}$$

Observe that (i) implies $A\psi(r) \leq 1 - r$ thus for sufficiently small positive δ we have $r + \delta\psi(r) < 1$ and the quantity in the denominator of the fraction in (iii) is well defined.

For a list of examples of admissible weights, see [8, pages 660–663]. The following theorem was proved in [8].

THEOREM 1.3. *Suppose $1 \leq p < \infty$ and ω is an admissible weight with distortion function ψ . Then*

$$\int_U |f(z)|^p \omega(z) dm(z) \asymp |f(0)|^p + \int_U |f'(z)|^p \psi(z)^p \omega(z) dm(z), \tag{1.16}$$

for all analytic functions f on the unit disc U , where $dm(z) = r dr d\theta / \pi$ denotes the normalized Lebesgue area measure on U .

The above means that there are finite positive constants C and C' independent of f such that the left- and right-hand sides $L(f)$ and $R(f)$ satisfy

$$CR(f) \leq L(f) \leq C'R(f) \tag{1.17}$$

for all analytic f .

Some generalizations of Theorem 1.3 in many directions can be found in [10, 11]. In [5, 6, 9] was also investigated Bergman space of analytic functions with weights other than classical. Closely related results for the classical weight $\omega(r) = (1 - r)^\alpha$, $\alpha > -1$, are presented in [1, 2, 3, 4, 7, 13].

Using a Bergman type projection $\mathbf{B}_\alpha : \mathcal{L}^p(dV_{\vec{\alpha}}) \rightarrow \mathcal{A}^p(dV_{\vec{\alpha}})$, in [1] the authors proved the following theorem.

THEOREM 1.4. *Let $p \in [1, \infty)$, $\alpha_j > -1$, $j = 1, \dots, n$ and m be a fixed positive integer and let $\mathbf{k} = (k_1, \dots, k_n) \in (\mathbf{Z}_+)^n$. Let f be a holomorphic function defined on the polydisc U^n in \mathbf{C}^n . Then for $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$, $f \in \mathcal{A}^p(dV_{\vec{\alpha}})$ if and only if*

$$\left[\prod_{j=1}^n (1 - |z_j|^2)^{k_j} \right] \frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1} \dots \partial z_n^{k_n}}(z) \in \mathcal{L}^p(dV_{\vec{\alpha}}) \quad \forall |\mathbf{k}| = m. \tag{1.18}$$

Moreover,

$$\|f\|_{\mathcal{A}^p(dV_{\vec{\omega}})} \asymp \left(\sum_{|\mathbf{k}|=0}^{m-1} \left| \frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1} \dots \partial z_n^{k_n}}(0) \right| + \sum_{|\mathbf{k}|=m} \left\| \left[\prod_{j=1}^n (1-|z_j|^2)^{k_j} \right] \frac{\partial^m f}{\partial z_1^{k_1} \dots \partial z_n^{k_n}} \right\|_{\mathcal{L}^p(dV_{\vec{\omega}})} \right). \tag{1.19}$$

In [11] among other things we proved the following theorem which generalizes Theorem 1.4.

THEOREM 1.5. *Let $\mathbf{k} = (k_1, \dots, k_n) \in (\mathbf{Z}_+)^n$, f be a holomorphic function defined on the polydisc U^n in \mathbf{C}^n and $\omega_j(z_j)$, $j = 1, \dots, n$ are admissible weights on the unit disk U , with distortion functions $\psi_j(z_j)$. If $f \in \mathcal{A}_{\vec{\omega}}^p$ and $p > 0$, then*

$$\left[\prod_{j=1}^n \psi_j^{k_j}(z_j) \right] \frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1} \dots \partial z_n^{k_n}}(z) \in \mathcal{L}_{\vec{\omega}}^p. \tag{1.20}$$

Moreover, let m be a fixed positive integer. Then there is a positive constant $C = C(p, \omega_j, m, n)$ such that

$$\|f\|_{\mathcal{A}_{\vec{\omega}}^p} \geq C \left(\sum_{|\mathbf{k}|=0}^{m-1} \left| \frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1} \dots \partial z_n^{k_n}}(0) \right| + \sum_{|\mathbf{k}|=m} \left\| \left[\prod_{j=1}^n \psi_j^{k_j}(z_j) \right] \frac{\partial^m f}{\partial z_1^{k_1} \dots \partial z_n^{k_n}} \right\|_{\mathcal{L}_{\vec{\omega}}^p} \right). \tag{1.21}$$

In the same paper, we formulate and give a sketch of a proof of the following partially converse of Theorem 1.5.

THEOREM 1.6. *Let $f \in H(U^n)$ and $\omega_j(z_j)$, $j = 1, \dots, n$ are admissible weights on the unit disk U , with distortion functions $\psi_j(z_j)$. If $p \in [1, \infty)$ and for all $j = 1, \dots, n$, $\psi_j(z_j)(\partial f/\partial z_j)(z) \in \mathcal{L}_{\vec{\omega}}^p$, then $f \in \mathcal{A}_{\vec{\omega}}^p$ and there is a positive constant $C = C(p, \omega_j, n)$ such that*

$$\|f\|_{\mathcal{A}_{\vec{\omega}}^p} \leq C \left(|f(0)| + \sum_{j=1}^n \left\| \psi_j \frac{\partial f}{\partial z_j} \right\|_{\mathcal{L}_{\vec{\omega}}^p} \right). \tag{1.22}$$

In communication with other specialists in this field it has turned out that the proof is more complicated than we have expected. Hence in this note we will present a clear detailed proof of Theorem 1.6. Also, we prove the following generalization of Theorem 1.6.

THEOREM 1.7. *Let $f \in H(U^n)$ and $\omega_j(z_j)$, $j = 1, \dots, n$ are admissible weights on the unit disk U , with distortion functions $\psi_j(z_j)$. If $p, q \in [1, \infty)$ and for all $j = 1, \dots, n$, $\psi_j(z_j)(\partial f/\partial z_j)(z) \in \mathcal{L}_{\vec{\omega}, N}^{p, q}$, then $f \in \mathcal{A}_{\vec{\omega}, N}^{p, q}$ and there is a positive constant $C = C(p, q, \omega_j, n)$ such that*

$$\|f\|_{\mathcal{A}_{\vec{\omega}, N}^{p, q}} \leq C \left(|f(0)| + \sum_{j=1}^n \left\| \psi_j \frac{\partial f}{\partial z_j} \right\|_{\mathcal{L}_{\vec{\omega}, N}^{p, q}} \right). \tag{1.23}$$

2. An auxiliary results

In this section, we prove an auxiliary result which we use in the proof of the main result.

LEMMA 2.1. *Suppose $1 \leq p < \infty$ and $f \in H(U^n)$. Then*

$$\frac{d}{dt} M_p^p(f, tr) \leq p M_p^{p-1}(f, tr) \sum_{i=1}^n r_i M_p \left(\frac{\partial f}{\partial z_i}, tr \right), \tag{2.1}$$

almost everywhere.

Proof. For $f \equiv 0$ the result is obvious. If $f \not\equiv 0$, at points where f is not zero, we have

$$\begin{aligned} \frac{d}{dt} (|f(tr \cdot e^{i\theta})|^p) &= p |f(tr \cdot e^{i\theta})|^{p-1} \frac{d}{dt} |f(tr \cdot e^{i\theta})| \\ &\leq p |f(tr \cdot e^{i\theta})|^{p-1} \left| \frac{d}{dt} (f(tr \cdot e^{i\theta})) \right| \\ &= p |f(tr \cdot e^{i\theta})|^{p-1} |\langle \nabla f(tr \cdot e^{i\theta}), r \cdot e^{i\theta} \rangle| \\ &\leq p |f(tr \cdot e^{i\theta})|^{p-1} \sum_{i=1}^n r_i \left| \frac{\partial f}{\partial z_i}(tr \cdot e^{i\theta}) \right|. \end{aligned} \tag{2.2}$$

From (2.2) and by the dominated convergence theorem we obtain

$$\frac{d}{dr} M_p^p(f, tr) \leq \frac{p}{(2\pi)^n} \sum_{i=1}^n r_i \int_{[0, 2\pi]^n} |f(tr \cdot e^{i\theta})|^{p-1} \left| \frac{\partial f}{\partial z_i}(tr \cdot e^{i\theta}) \right| d\theta. \tag{2.3}$$

If $p = 1$ the assertion is clear. If $p > 1$, applying in the last integral Hölder’s inequality with exponents $p/(p - 1)$ and p we obtain the result. □

COROLLARY 2.2. *Suppose $p, q \in [1, \infty)$ and $f \in H(U^n)$. Then*

$$\frac{d}{dt} M_p^q(f, tr) \leq q M_p^{q-1}(f, tr) \sum_{i=1}^n r_i M_p \left(\frac{\partial f}{\partial z_i}, tr \right), \tag{2.4}$$

almost everywhere.

Proof. Computing $(d/dt)M_p^q(f, tr)$ and then using Lemma 2.1 we prove the corollary. □

3. Proofs of the theorem

In this section, we prove the main results in this paper.

Proof of Theorem 1.6. Without loss of generality, we may assume that $n = 2$, and $f(0, 0) = 0$. Also we assume that f is not constant and all integrals are finite. In order to avoid some

complicated notations we use $M_p^p(f, r_1 t, r_2 t)$ instead of $M_p^p(r_1 t, r_2 t)$. We have

$$\begin{aligned}
 \|f\|_{\mathcal{A}_{\omega, N}^p}^p &= \int_0^1 \int_0^1 \left(\int_0^1 \frac{d}{dt} M_p^p(r_1 t, r_2 t) dt \right) \omega_1(r_1) \omega_2(r_2) dr_1 dr_2 \\
 &\leq p \int_0^1 \int_0^1 \left(\int_0^1 M_p^{p-1}(r_1 t, r_2 t) \sum_{j=1}^2 M_p \left(\frac{\partial f}{\partial z_j}, r_1 t, r_2 t \right) r_j dt \right) \omega_1(r_1) \omega_2(r_2) dr_1 dr_2 \\
 &\leq p \int_0^1 \int_0^1 \left(\int_0^1 M_p^{p-1}(r_1 t, r_2 t) M_p \left(\frac{\partial f}{\partial z_1}, r_1 t, r_2 t \right) r_1 dt \right) \omega_1(r_1) \omega_2(r_2) dr_1 dr_2 \\
 &\quad + p \int_0^1 \int_0^1 \left(\int_0^1 M_p^{p-1}(r_1 t, r_2 t) M_p \left(\frac{\partial f}{\partial z_2}, r_1 t, r_2 t \right) r_2 dt \right) \omega_1(r_1) \omega_2(r_2) dr_1 dr_2 \\
 &\leq p \int_0^1 \int_0^1 \left(\int_0^{r_1} M_p^{p-1}(s, r_2) M_p \left(\frac{\partial f}{\partial z_1}, s, r_2 \right) ds \right) \omega_1(r_1) \omega_2(r_2) dr_1 dr_2 \\
 &\quad + p \int_0^1 \int_0^1 \left(\int_0^{r_2} M_p^{p-1}(r_1, \tau) M_p \left(\frac{\partial f}{\partial z_2}, r_1, \tau \right) d\tau \right) \omega_1(r_1) \omega_2(r_2) dr_1 dr_2 \\
 &\leq p \int_0^1 \int_0^1 \left(M_p^{p-1}(s, r_2) M_p \left(\frac{\partial f}{\partial z_1}, s, r_2 \right) \int_s^1 \omega_1(r_1) dr_1 \right) \omega_2(r_2) ds dr_2 \\
 &\quad + p \int_0^1 \int_0^1 \left(M_p^{p-1}(r_1, \tau) M_p \left(\frac{\partial f}{\partial z_2}, r_1, \tau \right) \int_\tau^1 \omega_2(r_2) dr_2 \right) \omega_1(r_1) d\tau dr_1 \\
 &= p \int_0^1 \int_0^1 M_p^{p-1}(s, r_2) M_p \left(\frac{\partial f}{\partial z_1}, s, r_2 \right) \psi_1(s) \omega_1(s) \omega_2(r_2) ds dr_2 \\
 &\quad + p \int_0^1 \int_0^1 M_p^{p-1}(r_1, \tau) M_p \left(\frac{\partial f}{\partial z_2}, r_1, \tau \right) \psi_2(\tau) \omega_2(\tau) \omega_1(r_1) d\tau dr_1.
 \end{aligned} \tag{3.1}$$

If $p > 1$, by Hölder inequality, we get

$$\begin{aligned}
 &\int_0^1 \int_0^1 M_p^{p-1}(s, r_2) M_p \left(\frac{\partial f}{\partial z_1}, s, r_2 \right) \psi_1(s) \omega_1(s) \omega_2(r_2) ds dr_2 \\
 &\quad \leq \left(\int_0^1 \int_0^1 M_p^p(s, r_2) \omega_1(s) \omega_2(r_2) ds dr_2 \right)^{(p-1)/p}, \\
 &\left(\int_0^1 \int_0^1 M_p^p \left(\frac{\partial f}{\partial z_1}, s, r_2 \right) \psi_1^p(s) \omega_1(s) \omega_2(r_2) ds dr_2 \right)^{1/p} \\
 &\quad = \|f\|_{\mathcal{A}_{\omega, N}^p}^{p-1} \left(\int_0^1 \int_0^1 M_p^p \left(\frac{\partial f}{\partial z_1}, s, r_2 \right) \psi_1^p(s) \omega_1(s) \omega_2(r_2) ds dr_2 \right)^{1/p}.
 \end{aligned} \tag{3.2}$$

Similarly,

$$\begin{aligned}
 &\int_0^1 \int_0^1 M_p^{p-1}(r_1, \tau) M_p \left(\frac{\partial f}{\partial z_2}, r_1, \tau \right) \psi_2(\tau) \omega_2(\tau) \omega_1(r_1) d\tau dr_1 \\
 &\quad \leq \|f\|_{\mathcal{A}_{\omega, N}^p}^{p-1} \left(\int_0^1 \int_0^1 M_p^p \left(\frac{\partial f}{\partial z_2}, r_1, \tau \right) \psi_2^p(\tau) \omega_2(\tau) \omega_1(r_1) d\tau dr_1 \right)^{1/p}.
 \end{aligned} \tag{3.3}$$

From (3.1)–(3.3) we obtain the result in this case. For $p = 1$ the result it follows from (3.3). If f is constant the result is clear. To remove the restriction of the finiteness of the integrals we consider holomorphic functions $f_\rho(z) = f(\rho z)$, $\rho \in (0, 1)$ and use the Monotone Convergence theorem, when $\rho \rightarrow 1$. \square

Open question 3.1. Does Theorem 1.6 hold in the case $0 < p < 1$?

Remark 3.2. In the case when $\omega_j(z_j)$, $j = 1, \dots, n$, are the classical weights $(1 - |z_j|)^{\alpha_j}$, $j = 1, \dots, n$, a positive answer to Open question 3.1 was given in [12].

Proof of Theorem 1.7. If $f(0, 0) = 0$, is not constant and all integrals are finite, then by Corollary 2.2, as in the proof of Theorem 1.6, we obtain

$$\begin{aligned} \|f\|_{\mathcal{A}_{\omega, N}^{p, q}}^q &\leq \|f\|_{\mathcal{A}_{\omega, N}^{p, q}}^{q-1} \left(\int_0^1 \int_0^1 M_p^q \left(\frac{\partial f}{\partial z_1}, s, r_2 \right) \psi_1^q(s) \omega_1(s) \omega_2(r_2) ds dr_2 \right)^{1/q} \\ &\quad + \|f\|_{\mathcal{A}_{\omega, N}^{p, q}}^{q-1} \left(\int_0^1 \int_0^1 M_p^q \left(\frac{\partial f}{\partial z_2}, r_1, \tau \right) \psi_2^q(\tau) \omega_2(\tau) \omega_1(r_1) d\tau dr_1 \right)^{1/q}, \end{aligned} \quad (3.4)$$

that is,

$$\|f\|_{\mathcal{A}_{\omega, N}^{p, q}} \leq \sum_{j=1}^2 \left\| \psi_j \frac{\partial f}{\partial z_j} \right\|_{\mathcal{L}_{\omega, N}^{p, q}}. \quad (3.5)$$

The rest of the proof is similar to the proof of Theorem 1.6. \square

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