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WEIGHTED L^p-BOUNDEDNESS FOR HIGHER ORDER COMMUTATORS OF OSCILLATORY SINGULAR INTEGRALS

Dedicated to Professor Satoru Igari on his sixtieth birthday

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Abstract. In this paper the authors study the weighted L^p-boundedness for higher order commutators of a class of oscillatory singular integrals with rough kernel. The main result in this paper gives a necessary and sufficient condition so that this higher order commutator is bounded on the weighted L^p space with certain weight.

1. Introduction.

Let us consider the oscillatory singular integral defined by

$$Tf(x)=\text{p.v.}\int_{\boldsymbol{R}^n} e^{iP(x,y)}K(x-y)f(y)dy\ ,$$

where $P(x, y)$ is a real polynomial on $\boldsymbol{R}^n\times\boldsymbol{R}^n$, and $K(x)=h(|x|)\Omega(x/|x|)|x|^{-n}$ with $h(r)\in\text{BV}(\boldsymbol{R}_+)$, where $\text{BV}(\boldsymbol{R}_+)$ denotes the class of functions of bounded variation on $\boldsymbol{R}_+$. In 1987, Ricci and Stein [7] proved that T is bounded on $L^p(\boldsymbol{R}^n)$, $1<p<\infty$, if $K\in C^1(\boldsymbol{R}^n\setminus 0)$ and $h\equiv 1$. In 1992, Lu and Zhang [6] improved the result of Ricci and Stein and showed that T is bounded on $L^p(\boldsymbol{R}^n)$, $1<p<\infty$, provided $\Omega\in L^q(S^{n-1})$, $1<q\le\infty$ and $\int_{S^{n-1}}\Omega(x')d\sigma(x')=0$, where S^{n-1} denotes the unit sphere in $\boldsymbol{R}^n$. Moreover, the authors of [6] gave a necessary and sufficient condition so that T is bounded on $L^p(\boldsymbol{R}^n)$. Recently, the above result in [6] was extended by Jiang and Lu [5] to the case of $\Omega\in L\log^+L(S^{n-1})$. The purpose of this paper is to study the weighted L^p-boundedness for higher order commutators formed by T and a function in $\text{BMO}(\boldsymbol{R}_+)$. If we restrict ourselves to the case where $P(x, y)$ is a nontrivial polynomial, then we shall get a criterion on weighted L^p-boundedness for the higher order commutators mentioned above.

Let us first give some definitions.

DEFINITION 1. Let $b(r)\in L_{\text{loc}}(\boldsymbol{R}_+)$. We say $b(r)\in\text{BMO}(\boldsymbol{R}_+)$, if

$$\|b\|_{*,+}=\sup_{I\subset\boldsymbol{R}_+}\frac{1}{|I|}\int_I|b(r)-b_I|dr<\infty\ ,$$

where $b_I=|I|^{-1}\int_I b(r)dr$.

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DEFINITION 2. Suppose that $\omega(r)\geq 0$ and $\omega\in L_{\mathrm{loc}}(\boldsymbol{R}_+)$. For $1<p<\infty$, we say $\omega\in A_p(\boldsymbol{R}_+)$, if there is a $C>0$ such that for any $I\subset\boldsymbol{R}_+$,

$$\left(\frac{1}{|I|}\int_I\omega(r)dr\right)\left(\frac{1}{|I|}\int_I\omega(r)^{-1/(p-1)}dr\right)^{p-1}\leq C<\infty\,.$$

Moreover, if there is a $C>0$ such that

$$\omega^*(r)\leq C\omega(r)\quad\text{a.e. } r\in\boldsymbol{R}_+\,,$$

then we say $\omega\in A_1(\boldsymbol{R}_+)$, where ω^* denotes the Hardy-Littlewood maximal function of ω defined by

$$\omega^*(t)=\sup_{t\in I\subset\boldsymbol{R}_+}\frac{1}{|I|}\int_I\omega(r)dr\,.$$

DEFINITION 3. For $1<p<\infty$, we denote

$$\widetilde{A_p}(\boldsymbol{R}_+)=\{\omega:\omega\geq 0,\,\omega\in L_{\mathrm{loc}}(\boldsymbol{R}_+)\text{ and }\omega^2\in A_p(\boldsymbol{R}_+)\}\,.$$

Now, we may formulate our results as follows:

THEOREM 1. *Let* $1<p<\infty$, $\Omega\in L\log^+L(S^{n-1})$, *homogeneous of degree zero*, $\int_{S^{n-1}}\Omega(x')d\sigma(x')=0$, $h(|x|)\in\mathrm{BV}(\boldsymbol{R}_+)$, $b(x)=b(|x|)\in\mathrm{BMO}(\boldsymbol{R}_+)$, *and* $\omega(x)=\omega(|x|)\in\widetilde{A_p}(\boldsymbol{R}_+)$. *If the operator*

$$\bar{T}f(x)=\mathrm{p.v.}\int\frac{\Omega(x-y)}{|x-y|^n}h(|x-y|)f(y)dy$$

is bounded on $L^p(\omega)$, *then for any* $m\in\boldsymbol{Z}_+$ *and any real polynomial* $P(x,y)$ *on* $\boldsymbol{R}^n\times\boldsymbol{R}^n$, *the higher order commutator*

$$T_b^mf(x)=\mathrm{p.v.}\int e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^n}h(|x-y|)[b(x)-b(y)]^mf(y)dy$$

is also bounded on $L^p(\omega)$.

The following theorem is the main result of this paper.

THEOREM 2. *Let* $1<p<\infty$. *If* Ω, h, b, m and ω *are as in Theorem* 1, *then the following three statements are equivalent*:

(i) *If* $P(x,y)$ *is a nontrivial polynomial* (*i.e.*, $P(x,y)$ *does not take the form of* $P_1(x)+P_2(y)$ (*see* [6])), *then* T_b^m *is bounded on* $L^p(\omega)$.

(ii) *If a nontrivial polynomial* $P(x,y)$ *satisfies*

$$P(x,y)=P(x-t,y-t)+R_1(x,t)+R_2(y,t)\,,\qquad t\in\boldsymbol{R}^n\,,\tag{1.1}$$

where R_1 *and* R_2 *are real polynomials, then* T_b^m *is bounded on* $L^p(\omega)$.

(iii) *The truncated operator*

$$\bar{T}^m_{b,0}f(x) = \text{p.v.}\int_{|x-y|<1} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|)[b(x)-b(y)]^m f(y)dy$$

is bounded on $L^p(\omega)$.

In proving Theorems 1 and 2, the operator M_Ω defined by

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |\Omega(x-y)f(y)|dy ,$$

a variant of the Hardy-Littlewood maximal function associated with $\Omega \in L^1(S^{n-1})$, shall play a key role.

In 1993, Duoandikoetxea [3] gave a weighted result for M_Ω:

THEOREM A. *Let* $1<p<\infty$ *and* $\omega(x)=\omega(|x|)=v_1(|x|)v_2(|x|)^{1-p}$, *where either* $v_i \in A_1(\boldsymbol{R}_+)$ *and is decreasing or* $v_i^2 \in A_1(\boldsymbol{R}_+)$, $i=1, 2$. *Then* M_Ω *is bounded on* $L^p(\omega)$ *and*

$$\|M_\Omega f\|_{p,\omega} \le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{p,\omega} .$$

In proving Theorem 2, we shall use the following weighted L^p-boundedness of $M^m_{\Omega,b}$, a maximal operator related to higher order commutators, defined by

$$M^m_{\Omega,b}f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |\Omega(x-y)||b(x)-b(y)|^m|f(y)|dy ,$$

where $b \in \text{BMO}(\boldsymbol{R}_+)$.

THEOREM 3. *Let* $1<p<\infty$ *and* $\Omega \in L^1(S^{n-1})$, *homogeneous of degree zero,* $b(x)=b(|x|) \in \text{BMO}(\boldsymbol{R}_+)$, $m \in \boldsymbol{Z}_+$ *and* $\omega(x)=\omega(|x|) \in \tilde{A}_p(\boldsymbol{R}_+)$. *Then* $M^m_{\Omega,b}$ *is bounded on* $L^p(\omega)$ *and*

$$\|M^m_{\Omega,b}f\|_{p,\omega} \le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{p,\omega} .$$

2. Some results on $\tilde{A}_p(\boldsymbol{R}_+)$.

LEMMA 1. *If* $1<p<\infty$, *then the weights in* $\tilde{A}_p(\boldsymbol{R}_+)$ *have the following properties*:

(i) $\tilde{A}_p(\boldsymbol{R}_+) \subset A_p(\boldsymbol{R}_+)$.

(ii) *For any* $\omega(r) \in \tilde{A}_p(\boldsymbol{R}_+)$, *there are weights* v_1, v_2 *such that* $\omega = v_1 \cdot v_2^{1-p}$ *and* v_1^2, $v_2^2 \in A_1(\boldsymbol{R}_+)$.

(iii) *For any* $\omega(r) \in \tilde{A}_p(\boldsymbol{R}_+)$, *there exists an* $\varepsilon>0$ *so that* $\omega^{1+\varepsilon} \in \tilde{A}_p(\boldsymbol{R}_+)$.

(iv) *For any* $\omega(r) \in \tilde{A}_p(\boldsymbol{R}_+)$, *there exists an* $\varepsilon>0$ *so that* $p-\varepsilon>1$ *and* $\omega \in \tilde{A}_{p-\varepsilon}(\boldsymbol{R}_+)$.

The above facts can be easily deduced from the definition of $\tilde{A}_p(\boldsymbol{R}_+)$ and corresponding properties of $A_p(\boldsymbol{R}_+)$. We omit the details here.

REMARK 1. By (ii) in Lemma 1 and Theorem A, we see that if $\omega \in \tilde{A}_p(\boldsymbol{R}_+)$, $1<p<\infty$, then M_Ω is bounded on $L^p(\omega)$ and

$$\|M_\Omega f\|_{p,\omega} \le C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{p,\omega}.$$

LEMMA 2. *Let* $1<p<\infty$. *If* $b(r)\in \mathrm{BMO}(\boldsymbol{R}_+)$, *then there is a* $\lambda>0$ *such that* $e^{\lambda b(r)}\in \widetilde{A}_p(\boldsymbol{R}_+)$.

PROOF. From the John-Nirenberg inequality for BMO and the reverse Hölder inequality for the weights in $A_p(\boldsymbol{R}_+)$, it follows that there is a $\lambda_0>0$ such that $e^{\lambda_0 b(r)}\in A_p(\boldsymbol{R}_+)$ (see, e.g., [4]). Now we take $\lambda=\lambda_0/2$. Then $(e^{\lambda b})^2\in A_p(\boldsymbol{R}_+)$, i.e., $e^{\lambda b}\in \widetilde{A}_p(\boldsymbol{R}_+)$ by the definition of $\widetilde{A}_p(\boldsymbol{R}_+)$.

LEMMA 3. *For* $1<p<\infty$ *and* $\lambda>0$, *there exists an* $\eta=\eta(\lambda,p)>0$ *such that if* $b(r)\in \mathrm{BMO}(\boldsymbol{R}_+)$ *and* $\|b\|_{*,+}<\eta$, *then* $e^{\lambda b(r)}\in \widetilde{A}_p(\boldsymbol{R}_+)$.

PROOF. If we take $\eta_0=\min\{c/\lambda, c(p-1)/\lambda\}$, where c is the absolute constant in the John-Nirenberg inequality, then when $\|b\|_{*,+}<\eta_0$ we have $e^{\lambda b(r)}\in A_p(\boldsymbol{R}_+)$ (see [4]). Now we let $\eta=\eta_0/2$. Obviously, if $\|b\|_{*,+}<\eta$, i.e., $\|2b\|_{*,+}<\eta_0$, then

$$e^{2\lambda b(r)}\in A_p(\boldsymbol{R}_+).$$

By the definition of $\widetilde{A}_p(\boldsymbol{R}_+)$, we have $e^{\lambda b(r)}\in \widetilde{A}_p(\boldsymbol{R}_+)$.

3. **Proof of Theorem 3.** Let us first give the proof of Theorem 3 by induction on m. By Theorem A, we see that Theorem 3 holds for $m=0$. Now we assume that the conclusion of Theorem 3 holds for $m-1$, and prove the conclusion for m. Since $\omega\in \widetilde{A}_p(\boldsymbol{R}_+)$, we can choose an $\varepsilon>0$ so that $\omega^{1+\varepsilon}\in \widetilde{A}_p$ by Lemma 1. Then by the assumption of induction, $M^{m-1}_{\Omega,b}$ is bounded on $L^p(\omega^{1+\varepsilon})$ and

$$\|M^{m-1}_{\Omega,b}\varphi\|_{p,\omega^{1+\varepsilon}} \le C_1\|\Omega\|_{L^1(S^{n-1})}\|\varphi\|_{p,\omega^{1+\varepsilon}}, \qquad \text{for} \quad \varphi\in L^p(\omega^{1+\varepsilon}). \tag{3.1}$$

On the other hand, by taking $\lambda=p(1+\varepsilon)/\varepsilon$ and Lemma 3, we see that there exists an $\eta>0$ such that

$$e^{pb(1+\varepsilon)/\varepsilon}\in \widetilde{A}_p(\boldsymbol{R}_+), \qquad \text{if} \quad \|b\|_{*,+}<\eta.$$

Since $b\in\mathrm{BMO}$ implies that $tb\in\mathrm{BMO}$ for $|t|\le 1$ with a smaller BMO norm, we have

$$e^{tpb(1+\varepsilon)/\varepsilon}\in \widetilde{A}_p(\boldsymbol{R}_+), \qquad \text{for} \quad |t|\le 1. \tag{3.2}$$

Without loss of generality we may assume that $\|b\|_{*,+}<\eta$. Indeed, otherwise we take $0<\delta_0<\eta$ and set $a(x)=\delta_0 b(x)/\|b\|_{*,+}$. Thus, $\|a\|_{*,+}=\delta_0<\eta$ and

$$M^m_{\Omega,b}f(x)=\left(\frac{\|b\|_{*,+}}{\delta_0}\right)^m M^m_{\Omega,a}f(x).$$

Therefore, it suffices to consider $M^m_{\Omega,a}$. By the assumption of induction and (3.2), we see that for any $\theta\in[0,2\pi]$ and $\varphi\in L^p(e^{pb(1+\varepsilon)\cos\theta/\varepsilon})$, (3.3) $$\|M^{m-1}_{\Omega,b}\varphi\|_{p,e^{pb(1+\varepsilon)\cos\theta/\varepsilon}}\le C_2\|\Omega\|_{L^1(S^{n-1})}\|\varphi\|_{p,e^{pb(1+\varepsilon)\cos\theta/\varepsilon}},$$

where C_2 depends on p, b and ω, but not on θ. Applying the Stein-Weiss interpolation theorem (see [8] or [2]) between (3.1) and (3.3), we obtain that for any $\theta\in[0,2\pi]$ and $\varphi\in L^p(\omega e^{pb\cos\theta})$,

(3.4) $$\|M^{m-1}_{\Omega,b}\varphi\|_{p,\omega e^{pb\cos\theta}}\le C\|\Omega\|_{L^1(S^{n-1})}\|\varphi\|_{p,\omega e^{pb\cos\theta}},$$

where $C=\max\{C_1, C_2\}$ and depends only on p, b and ω, but not on θ. In the following, we shall use the equality

(3.5) $$b(x)-b(y)=\frac{1}{2\pi}\int_0^{2\pi}e^{e^{i\theta}[b(x)-b(y)]}e^{-i\theta}d\theta.$$

In fact, let $g(z)=e^{z[b(x)-b(y)]}$, $z\in \boldsymbol{C}$. Then by the analyticity of $g(z)$ on $\boldsymbol{C}$ and the Cauchy integration formula, we have

$$b(x)-b(y)=g'(0)=\frac{1}{2\pi i}\int_{|z|=1}\frac{g(z)}{|z|^2}dz.$$

And this is just (3.5). Moreover, if we denote $g_\theta(x)=f(x)e^{-b(x)\cos\theta}$ for any $\theta\in[0,2\pi]$, then it follows from $f\in L^p(\omega)$ that

(3.6) $$g_\theta\in L^p(\omega e^{pb\cos\theta})\quad\text{and}\quad \|g_\theta\|_{p,\omega e^{pb\cos\theta}}=\|f\|_{p,\omega}.$$

Hence, by (3.5) and (3.6), we have

$$\begin{aligned}M^m_{\Omega,b}f(x)&=\sup_{r>0}\frac{1}{r^n}\int_{|x-y|<r}|\Omega(x-y)||b(x)-b(y)|^m|f(y)|dy\\&=\sup_{r>0}\frac{1}{r^n}\int_{|x-y|<r}|\Omega(x-y)||b(x)-b(y)|^{m-1}\\&\quad\cdot\left|\frac{1}{2\pi}\int_0^{2\pi}e^{e^{i\theta}[b(x)-b(y)]}e^{-i\theta}d\theta\right||f(y)|dy\\&\le\frac{1}{2\pi}\int_0^{2\pi}\sup_{r>0}\frac{1}{r^n}\int_{|x-y|<r}|\Omega(x-y)||b(x)-b(y)|^{m-1}\\&\quad\cdot|f(y)e^{-b(y)\cos\theta}|dye^{b(x)\cos\theta}d\theta\\&=\frac{1}{2\pi}\int_0^{2\pi}M^{m-1}_{\Omega,b}(g_\theta)(x)\cdot e^{b(x)\cos\theta}d\theta.\end{aligned}$$

Using Minkowski's inequality, (3.4), (3.6) and the above, we get

$$\|M^m_{\Omega,b}f\|_{p,\omega}\le\left(\int_{\boldsymbol{R}^n}\left|\frac{1}{2\pi}\int_0^{2\pi}M^{m-1}_{\Omega,b}(g_\theta)(x)e^{b(x)\cos\theta}d\theta\right|^p\omega(x)dx\right)^{1/p}$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi}\left(\int_{\boldsymbol{R}^n}\left[M_{\Omega,b}^{m-1}(g_\theta)(x)\right]^p \omega(x)e^{pb(x)\cos\theta}dx\right)^{1/p}d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\|M_{\Omega,b}^{m-1}(g_\theta)\|_{p,\omega e^{pb\cos\theta}}d\theta$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi}C\cdot\|\Omega\|_{L^1(S^{n-1})}\|g_\theta\|_{p,\omega e^{pb\cos\theta}}d\theta$$

$$=C\|\Omega\|_{L^1(S^{n-1})}\|f\|_{p,\omega}\,.$$

This finishes the proof of Theorem 3.

4. Some lemmas. Before proving Theorems 1 and 2, we give some lemmas. Let

$$Gf(x)=\text{p.v.}\int_{\boldsymbol{R}^n}K(x,y)f(y)dy$$

and

$$G_0f(x)=\text{p.v.}\int_{|x-y|<1}K(x,y)f(y)dy\,.$$

LEMMA 4. *Let $1\leq p<\infty$ and $\omega(x)\geq 0$. If G_0 is bounded on $L^p(\omega)$, then the inequality*

$$\left(\int_{|x-t|<\varepsilon}|G_0f(x)|^p\omega(x)dx\right)^{1/p}\leq C_\varepsilon\left(\int_{|y-t|<1+\varepsilon}|f(y)|^p\omega(y)dy\right)^{1/p} \tag{4.1}$$

holds for any $\varepsilon>0$, where C_ε is independent of t and f. Conversely, if (4.1) *holds for certain $\varepsilon>0$, then G_0 is bounded on $L^p(\omega)$.*

See [5] for the proof.

LEMMA 5. *Let $1<p<\infty$, $b(x)=b(|x|)\in \mathrm{BMO}(\boldsymbol{R}_+)$, $m\in \boldsymbol{Z}_+$, $\omega(x)=\omega(|x|)\in \widetilde{A_p}(\boldsymbol{R}_+)$. If $\Omega\in L^1(S^{n-1})$, homogeneous of degree zero, $|K(x,y)|\leq C|\Omega(x-y)||b(x)-b(y)|^m|x-y|^{-n}$, and G is bounded on $L^p(\omega)$, then so is G_0.*

PROOF. By Lemma 4, it will suffice to prove

$$\int_{|x-t|<1/4}|G_0f(x)|^p\omega(x)dx\leq C\int_{|y-t|<5/4}|f(y)|^p\omega(y)dy\,,\qquad t\in\boldsymbol{R}^n\,.$$

Now, we split f into three parts $f=f_1+f_2+f_3$ for given t, where

$$f_1(y)=f(y)\chi_{\{|y-t|<1/2\}}(y)\,,$$

$$f_2(y)=f(y)\chi_{\{1/2\leq|y-t|<5/4\}}(y)\,,$$

and

$$f_3(y)=f(y)\chi_{\{|y-t|\geq 5/4\}}(y)\,.$$

Note that $|x-t|<1/4$ and $|y-t|<1/2$ imply $|x-y|<1$. Thus, we have

$$G_0 f_1(x)=G f_1(x)\,,\qquad |x-t|<1/4\,.$$

Since G is bounded on $L^p(\omega)$, we get

$$\begin{aligned}\int_{|x-t|<1/4}|G_0 f_1(x)|^p\omega(x)dx&=\int_{|x-t|<1/4}|Gf_1(x)|^p\omega(x)dx\\&\leq C\int_{|y-t|<1/2}|f(y)|^p\omega(y)dy\,.\end{aligned}$$

By the assumption on $K(x,y)$, we have

$$\begin{aligned}|G_0 f_2(x)|&\leq\int_{1/4<|x-y|<3/2}\frac{C|\Omega(x-y)|}{|x-y|^n}|b(x)-b(y)|^m|f_2(y)|dy\\&\leq CM^m_{\Omega,b}f_2(x)\,.\end{aligned}$$

Thus, it follows from Theorem 3 and the above that

$$\left(\int_{|x-t|<1/4}|G_0 f_2(x)|^p\omega(x)dx\right)^{1/p}\leq C\left(\int_{|y-t|<5/4}|f(y)|^p\omega(y)dy\right)^{1/p}.$$

Finally, we notice that $|x-t|<1/4$ and $|y-t|>5/4$ imply $|x-y|>1$. Thus, $G_0 f_3(x)=0$ if $|x-t|<1/4$. This completes the proof of Lemma 5.

LEMMA 6. *Let* $1<p<\infty$, $h(r)\in L^\infty(\boldsymbol{R}_+)$ *and* $\omega(x)=\omega(|x|)\in\widetilde{A}_p(\boldsymbol{R}_+)$. *If* $\Omega\in L^1(S^{n-1})$, *homogeneous of degree zero and* $\bar{T}$, *defined in Theorem* 1, *is bounded on* $L^p(\omega)$, *then for any* $b(x)=b(|x|)\in\mathrm{BMO}(\boldsymbol{R}_+)$, $m\in\boldsymbol{Z}_+$ *and any real polynomial* $P(x,y)$, *the operator*

$$T^m_{b,0}f(x)=\mathrm{p.v.}\int_{|x-y|<1}e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^n}h(|x-y|)[b(x)-b(y)]^m f(y)dy$$

is bounded on $L^p(\omega)$.

PROOF. By Lemma 5 for $m=0$, we see that the truncated operator of $\bar{T}$ defined by

$$\bar{T}_0 f(x)=\mathrm{p.v.}\int_{|x-y|<1}\frac{\Omega(x-y)}{|x-y|^n}h(|x-y|)f(y)dy$$

is bounded on $L^p(\omega)$. By Theorem A (or Remark 1) and Lemma 3 in [5], we see that Lemma 6 holds for $m=0$. On the other hand, it follows from Lemmas 1–3 that the results on commutators of linear operators given in [1] also hold if we use $\mathrm{BMO}(\boldsymbol{R}_+)$ and $\widetilde{A}_p(\boldsymbol{R}_+)$ instead of $\mathrm{BMO}(\boldsymbol{R}^n)$ and $A_p(\boldsymbol{R}^n)$ respectively. Thus, we see that the m-th commutator of $\bar{T}$ and $b\in\mathrm{BMO}(\boldsymbol{R}_+)$, defined by

$$\bar{T}_b^m(x)=\text{p.v.}\int\frac{\Omega(x-y)}{|x-y|^n}h(|x-y|)[b(x)-b(y)]^m f(y)dy\,,$$

is bounded on $L^p(\omega)$. Using Lemma 5 again, we get that the truncated operator of $\bar{T}_b^m$, i.e.

$$\bar{T}_{b,0}^m f(x)=\text{p.v.}\int_{|x-y|<1}\frac{\Omega(x-y)}{|x-y|^n}h(|x-y|)[b(x)-b(y)]^m f(y)dy\,,$$

is also bounded on $L^p(\omega)$. By the above result, Theorem 3 and Lemma 4, and using the method proving Lemma 3 in [5], one can prove that the truncated operator $T_{b,0}^m$ is bounded on $L^p(\omega)$. We omit the details.

LEMMA 7. *Let* $1<p<\infty$. *If* Ω, h, m *and* ω *are as in Theorem* 1, *then the operator*

$$T_{b,\infty}^m f(x)=\int_{|x-y|\geq 1}e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^n}h(|x-y|)[b(x)-b(y)]^m f(y)dy$$

is bounded on $L^p(\omega)$ *for any real nontrivial polynomial* $P(x,y)$ *on* $\boldsymbol{R}^n\times\boldsymbol{R}^n$.

PROOF. We split $T_{b,\infty}^m$ as follows:

$$\begin{aligned}T_{b,\infty}^m f(x)&=\int_{|x-y|\geq 1}e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^n}h(|x-y|)[b(x)-b(y)]^m f(y)dy\\&=\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\int_{2^{j-1}\leq|x-y|<2^j}e^{iP(x,y)}\frac{\Omega_k(x-y)}{|x-y|^n}h(|x-y|)[b(x)-b(y)]^m f(y)dy\\&:=\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}T_{b,j,k}^m f(x)\,,\end{aligned}$$

where

$$\Omega_k(x')=\Omega(x')\chi_{E_k}(x')\,,$$
$$E_0=\{x'\in S^{n-1}:|\Omega(x')|<1\}\,,$$

and

$$E_k=\{x'\in S^{n-1}:2^{k-1}\leq|\Omega(x')|<2^k\}\,,\qquad k\in\boldsymbol{N}\,.$$

Now, if we can prove the following two inequalities:

$$\|T_{b,j,k}^m f\|_{p,\omega}\leq C2^{-\delta j}\|\Omega_k\|_{L^\infty(S^{n-1})}\|f\|_{p,\omega}\tag{4.2}$$

and

$$\|T_{b,j,k}^m f\|_{p,\omega}\leq C\|\Omega_k\|_{L^1(S^{n-1})}\|f\|_{p,\omega}\,,\tag{4.3}$$

where $j=1,2,\ldots$, $k=0,1,\ldots$, and $\delta>0$ is independent of k, f and Ω, then we shall deduce the conclusion of Lemma 7. Indeed, we choose a positive integer $M>1/\delta$. Then

$$\begin{aligned}\|T^m_{b,\infty}f\|_{p,\omega}&\le\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\|T^m_{b,j,k}f\|_{p,\omega}\\&=\sum_{j=1}^{\infty}\|T^m_{b,j,0}f\|_{p,\omega}+\sum_{k=1}^{\infty}\sum_{1\le j\le Mk}\|T^m_{b,j,k}f\|_{p,\omega}+\sum_{k=1}^{\infty}\sum_{j>Mk}\|T^m_{b,j,k}f\|_{p,\omega}\\&:=I_1+I_2+I_3\,.\end{aligned}$$

Using (4.2), we get

$$I_1\le\sum_{j=1}^{\infty}C2^{-\delta j}\|\Omega_0\|_{L^\infty(S^{n-1})}\|f\|_{p,\omega}\le C\|f\|_{p,\omega}$$

and

$$\begin{aligned}I_3&\le C\sum_{k=1}^{\infty}\sum_{j>Mk}2^{-\delta j}\|\Omega_k\|_{L^\infty(S^{n-1})}\|f\|_{p,\omega}\\&\le C\sum_{k=1}^{\infty}\sum_{j>Mk}2^{-\delta j}\cdot 2^k\|f\|_{p,\omega}\le C\|f\|_{p,\omega}\sum_{k=1}^{\infty}2^{-(M\delta-1)k}\le C\|f\|_{p,\omega}\,.\end{aligned}$$

By (4.3), we have

$$\begin{aligned}I_2&\le C\sum_{k=1}^{\infty}\sum_{1\le j\le Mk}\|\Omega_k\|_{L^1(S^{n-1})}\|f\|_{p,\omega}\\&\le C\|f\|_{p,\omega}\sum_{k=1}^{\infty}k2^k\cdot|E_k|\le C\|f\|_{p,\omega}\|\Omega\|_{L\log^+L(S^{n-1})}\le C\|f\|_{p,\omega}\,.\end{aligned}$$

Thus, we obtain

$$\|T^m_{b,\infty}\|_{p,\omega}\le C\|f\|_{p,\omega}\,.$$

This confirms the above assertion. It remains to prove (4.2) and (4.3). Let us first prove (4.3). Since

$$\begin{aligned}|T^m_{b,j,k}f(x)|&\le\|h\|_{L^\infty(\boldsymbol{R}^n)}\int_{2^{j-1}\le|x-y|<2^j}\frac{|\Omega_k(x-y)|}{|x-y|^n}|b(x)-b(y)|^m|f(y)|dy\\&\le C\cdot M^m_{\Omega_k,b}f(x)\,,\end{aligned}$$

we get

$$\|T^m_{b,j,k}f\|_{p,\omega}\le C\|M^m_{\Omega_k,b}f\|_{p,\omega}\le C\|\Omega_k\|_{L^1(S^{n-1})}\|f\|_{p,\omega}$$

by Theorem 3. This proves (4.3).

Let us now turn to the proof of (4.2). The proof is completed by induction on m. Since $h\in\mathrm{BV}(\boldsymbol{R}_+)$ and $P(x,y)$ is a real nontrivial polynomial, by a method similar to that in [6], we can prove that there exists an $\eta>0$ such that

(4.4) $$\|T^0_{b,j,k}f\|_p \le C2^{-\eta j}\|\Omega_k\|_{L^\infty(S^{n-1})}\|f\|_p,$$

where C is independent of j and k. On the other hand, we have

$$|T^0_{b,j,k}f(x)| \le \int_{2^{j-1}\le|x-y|<2^j} \frac{|\Omega_k(x-y)||h(|x-y|)|}{|x-y|^n}|f(y)|dy \le C\cdot M_{\Omega_k}f(x).$$

From (iii) in Lemma 1 and Remark 1, it follows that

$$\|T^0_{b,j,k}f\|_{p,\omega^{1+\varepsilon}} \le C\cdot\|M_{\Omega_k}f\|_{p,\omega^{1+\varepsilon}} \le C\|\Omega_k\|_{L^1(S^{n-1})}\|f\|_{p,\omega^{1+\varepsilon}} \le C\|\Omega_k\|_{L^\infty(S^{n-1})}\|f\|_{p,\omega^{1+\varepsilon}}.$$

Combining the above with (4.4), and using the Stein-Weiss theorem of interpolation with change of measure [8], we get

$$\|T^0_{b,j,k}f\|_{p,\omega} \le C2^{-\eta_1 j}\|\Omega_k\|_{L^\infty(S^{n-1})}\|f\|_{p,\omega},$$

where $\eta_1>0$ is independent of j, k, f and Ω. This shows that (4.2) holds for $m=0$. We now assume that (4.2) holds for $m-1$, i.e., for any $\varphi\in L^p(\omega)$ with $\omega\in\widetilde{A}_p(\boldsymbol{R}_+)$ we have

(4.5) $$\|T^{m-1}_{b,j,k}\varphi\|_{p,\omega} \le C2^{-\eta_{m-1}j}\|\Omega_k\|_{L^\infty(S^{n-1})}\|\varphi\|_{p,\omega}.$$

By $\omega\in\widetilde{A}_p(\boldsymbol{R}_+)$ and Lemma 1, there exists an $\varepsilon>0$ so that $\omega^{1+\varepsilon}\in\widetilde{A}_p(\boldsymbol{R}_+)$. Therefore, for any $\varphi\in L^p(\omega^{1+\varepsilon})$,

(4.6) $$\|T^{m-1}_{b,j,k}\varphi\|_{p,\omega^{1+\varepsilon}} \le C2^{-\eta'_{m-1}j}\|\Omega_k\|_{L^\infty(S^{n-1})}\|\varphi\|_{p,\omega^{1+\varepsilon}}.$$

Repeating the proof of Theorem 3, we can obtain the following results: For any $\theta\in[0,2\pi]$ and $\varphi\in L^p(e^{pb(1+\varepsilon)\cos\theta/\varepsilon})$,

(4.7) $$\|T^{m-1}_{b,j,k}\varphi\|_{p,e^{pb(1+\varepsilon)\cos\theta/\varepsilon}} \le C2^{-\eta''_{m-1}j}\|\Omega_k\|_{L^\infty(S^{n-1})}\|\varphi\|_{p,e^{pb(1+\varepsilon)\cos\theta/\varepsilon}},$$

where C and η''_{m-1} depend on p, b and ω, but not on j, k and θ. By interpolating with change measure between (4.6) and (4.7), we see that for any $\theta\in[0,2\pi]$ and $\varphi\in L^p(\omega e^{pb\cos\theta})$,

(4.8) $$\|T^{m-1}_{b,j,k}\varphi\|_{p,\omega e^{pb\cos\theta}} \le C2^{-\delta j}\|\Omega_k\|_{L^\infty(S^{n-1})}\|\varphi\|_{p,\omega e^{pb\cos\theta}},$$

where C and $\delta>0$ are independent of j, k and θ. Moreover, if we let $g_\theta(x)=f(x)e^{-b(x)e^{i\theta}}$, then it is easy to check that for any $\theta\in[0,2\pi]$,

(4.9) $$g_\theta\in L^p(\omega e^{pb\cos\theta}) \quad\text{and}\quad \|g_\theta\|_{p,\omega e^{pb\cos\theta}}=\|f\|_{p,\omega}.$$

For simplicity, we denote

$$K_{m-1}(x,y)=e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^n}h(|x-y|)[b(x)-b(y)]^{m-1}\chi_{[2^{j-1}\le|x-y|<2^j]}(x-y).$$

Thus, by (3.5) and the above notation, we have

$$\begin{aligned}T^m_{b,j,k}f(x)&=\int_{R^n}K_{m-1}(x,y)[b(x)-b(y)]f(y)dy\\&=\int_{R^n}K_{m-1}(x,y)\left(\frac{1}{2\pi}\int_0^{2\pi}e^{e^{i\theta}[b(x)-b(y)]}e^{-i\theta}d\theta\right)f(y)dy\\&=\frac{1}{2\pi}\int_0^{2\pi}\int_{R^n}K_{m-1}(x,y)f(y)e^{-b(y)e^{i\theta}}dy\cdot e^{b(x)e^{i\theta}}\cdot e^{-i\theta}d\theta\\&=\frac{1}{2\pi}\int_0^{2\pi}T^{m-1}_{b,j,k}(g_\theta)(x)\cdot e^{b(x)e^{i\theta}}\cdot e^{-i\theta}d\theta\,.\end{aligned}$$

Hence, by Minkowski's inequality, we get

$$\begin{aligned}\|T^m_{b,j,k}f\|_{p,\omega}&\le\frac{1}{2\pi}\int_0^{2\pi}\left(\int_{R^n}\left|T^{m-1}_{b,j,k}(g_\theta)(x)e^{b(x)e^{i\theta}}\right|^p\omega(x)dx\right)^{1/p}d\theta\\&=\frac{1}{2\pi}\int_0^{2\pi}\|T^{m-1}_{b,j,k}(g_\theta)\|_{p,\omega e^{pb\cos\theta}}d\theta\,.\end{aligned}$$

From (4.8) and (4.9), it follows that

$$\begin{aligned}\|T^m_{b,j,k}f\|_{p,\omega}&\le\frac{1}{2\pi}\int_0^{2\pi}C2^{-\delta j}\|\Omega_k\|_{L^\infty(S^{n-1})}\|g_\theta\|_{p,\omega e^{pb\cos\theta}}d\theta\\&=C2^{-\delta j}\|\Omega_k\|_{L^\infty(S^{n-1})}\|f\|_{p,\omega}\,.\end{aligned}$$

Thus, we proved (4.2) for m and the proof of Lemma 7 is completed.

5. Proofs of Theorems 1 and 2. Theorem 1 can be directly deduced from Lemmas 6 and 7. Let us now give the proof of Theorem 2.

(i)$\Rightarrow$(ii). This step is obvious.

(ii)$\Rightarrow$(iii). Set

$$\begin{aligned}T^m_bf(x)&=\text{p.v.}\int_{|x-y|<1}e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^n}h(|x-y|)[b(x)-b(y)]^mf(y)dy\\&\quad+\int_{|x-y|\ge1}e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^n}h(|x-y|)[b(x)-b(y)]^mf(y)dy\\&:=T^m_{b,0}f(x)+T^m_{b,\infty}f(x)\,.\end{aligned}$$

From Lemma 7, it follows that $T^m_{b,\infty}$ is bounded on $L^p(\omega)$. So $T^m_{b,0}$ is a bounded operator on $L^p(\omega)$. We take a $t\in R^n$. For $|x-t|<1$, we have

$$T^m_{b,0}f(x)=T^m_{b,0}[f(\cdot)\chi_{B(t,2)}(\cdot)](x)\,.$$

Thus, by Lemma 4, we have

$$(5.1)\qquad \left(\int_{|x-t|<1} |T^m_{b,0}f(x)|^p\omega(x)dx\right)^{1/p} \le C\left(\int_{|y-t|<2} |f(y)|^p\omega(y)dy\right)^{1/p},$$

where C is independent of t and f. By (1.1), we write

$$\tilde{T}^m_{b,0}f(x)=e^{-iR_1(x,t)}\text{p.v.}\int_{|x-y|<1} e^{iP(x,y)}K_m(x,y)f(y)e^{-iP(x-t,y-t)}e^{-iR_2(y,t)}dy$$

for $t\in \boldsymbol{R}^n$, where

$$K_m(x,y)=\frac{\Omega(x-y)}{|x-y|^n}h(|x-y|)[b(x)-b(y)]^m .$$

Express $e^{-iP(x-t,y-t)}$ into the Taylor series:

$$e^{-iP(x-t,y-t)}=\sum_{k=0}^{\infty}\frac{(-i)^k}{k!}[P(x-t,y-t)]^k=\sum_{k=0}^{\infty}\frac{(-i)^k}{k!}\left[\sum_{\alpha,\beta}a_{\alpha,\beta}(x-t)^\alpha(y-t)^\beta\right]^k$$
$$=\sum_{k=0}^{\infty}\frac{(-i)^k}{k!}\sum_{\mu,\nu}b_{\mu,\nu}(x-t)^\mu(y-t)^\nu .$$

Thus, we have

$$\left(\int_{|x-t|<1}|\tilde{T}^m_{b,0}f(x)|^p\omega(x)dx\right)^{1/p}\le\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\mu,\nu}|b_{\mu,\nu}|\cdot\left(\int_{|x-t|<1}\left|(x-t)^\mu\right.\right.$$
$$\left.\left.\cdot\int_{|x-y|<1}\exp(iP(x,y))K_m(x,y)f(y)\exp(-iR_2(y,t))(y-t)^\nu dy\right|^p\omega(x)dx\right)^{1/p}$$
$$\le\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\mu,\nu}|b_{\mu,\nu}||\xi^\mu|\left(\int_{|x-t|<1}\left|T^m_{b,0}[\exp(-iR_2(\cdot,t))f(\cdot)(\cdot-t)^\nu](x)\right|^p\omega(x)dx\right)^{1/p},$$

where $\xi=(1,1,\dots,1)$. By (5.1), we obtain

$$\left(\int_{|x-t|<1}|\tilde{T}^m_{b,0}f(x)|^p\omega(x)dx\right)^{1/p}$$
$$\le C\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\mu,\nu}|b_{\mu,\nu}||\xi^\mu|\left(\int_{|y-t|<2}|f(y)(y-t)^\nu|^p\omega(y)dy\right)^{1/p}$$
$$\le C\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\mu,\nu}|b_{\mu,\nu}||\xi^\mu\cdot\eta^\nu|\left(\int_{|y-t|<2}|f(y)|^p\omega(y)dy\right)^{1/p}$$
$$\le C\sum_{k=0}^{\infty}\frac{1}{k!}\left[\sum_{\alpha,\beta}|a_{\alpha,\beta}||\xi^\alpha\cdot\eta^\beta|^k\right]\left(\int_{|y-t|<2}|f(y)|^p\omega(y)dy\right)^{1/p}$$

$$= C \exp\left(\sum_{\alpha,\beta} |a_{\alpha,\beta}| |\xi^\alpha \cdot \eta^\beta|\right)\left(\int_{|y-t|<2} |f(y)|^p \omega(y)dy\right)^{1/p}$$
$$\leq C\left(\int_{|y-t|<2} |f(y)|^p \omega(y)dy\right)^{1/p},$$

where $\eta = (2, 2, \ldots, 2)$. By Lemma 4, we see that the above implies (iii).

(iii) $\Rightarrow$ (i). This step is just a direct result of Lemmas 6 and 7. This completes the proof of Theorem 2.

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