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Weighted L^q-theory for the Stokes resolvent in exterior domains

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1. Introduction.

In an exterior domain $\Omega\subset\boldsymbol{R}^n$, $n\geqq 2$, consider the *generalized Stokes resolvent system*

$$\begin{aligned} \lambda u-\Delta u+\nabla p &= f \quad \text{in } \Omega \\ \operatorname{div} u &= g \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

where $f=(f_1, \cdots, f_n)$ is the prescribed force, g the prescribed divergence and the resolvent parameter $\lambda\in\boldsymbol{C}$ is contained in the sector

$$S_\varepsilon=\{z\in\boldsymbol{C}\,;\, z\neq 0,\ \arg z<\pi-\varepsilon\},\quad 0<\varepsilon<\pi/2.$$

The aim of this paper is to solve (1.1) and estimate the unknown velocity field $u=(u_1, \cdots, u_n)$ and the pressure p in *weighted Sobolev spaces.* Since the domain is unbounded, it is reasonable to solve (1.1) not only in classical Sobolev spaces but to use radially symmetric weights $|x|^\alpha$ or even anisotropic and locally singular weights to describe the behaviour of the solution (u, p) more precisely.

Radial weights are a widely used tool to investigate Poisson's equation or Stokes' equation (with $\lambda=0$) in an exterior domain. The radial symmetry of the problem which is also reflected by the structure of the fundamental solution as well as the Poincaré type inequality $\int|u|^2/|x|^2dx\leqq 4\int|\nabla u|^2dx$ for $u\in C_0^\infty(\boldsymbol{R}^3)$ lead to weights of the form $|x|^\alpha$. From the numerous papers on these topics we mention [**11**, **20**] where a lot of further references can be found. But up to now there seems to be no complete weighted theory of the Stokes resolvent problem which leads via analytic semigroup theory to the investigation of the instationary Stokes and Navier-Stokes equations in weighted spaces. To our knowledge, [**19**] is the only reference for (1.1) in weighted L^q-spaces using the special weights $(1+|x|)^\alpha$ together with the restriction $|\lambda|\geqq\delta>0$; our method is different. For the Stokes resolvent problem in spaces without weights we refer to [**2**, **3**, **9**, **18**, **24**, **25**] when $g=\operatorname{div} u=0$ and to [**5**] when $g\neq 0$. To define the Stokes operator the construction of the Helmholtz projection is needed; see

[7, 14, 17, 25] for results in spaces without weights. For the special weights of the form $(1+|x|)^{\alpha}$ a description of the Helmholtz decomposition has been given in [21] even without the restriction on α below.

In this paper we solve the generalized Stokes resolvent problem (1.1) with $g\neq 0$ and construct the Helmholtz decomposition in weighted L^q-spaces for a large class of weights. A weight function $0\leqq w\in L^1_{\text{loc}}(\boldsymbol{R}^n)$ is said to be of the *Muckenhoupt class* $\mathcal{A}_q$ iff

$$\sup_Q\Big(\frac{1}{|Q|}\int_Q w\,dx\Big)\cdot\Big(\frac{1}{|Q|}\int_Q w^{-1/(q-1)}dx\Big)^{q-1}\leqq C<\infty;$$

here $1<q<\infty$ and $Q\subset\boldsymbol{R}^n$ runs through the set of all bounded cubes of $\boldsymbol{R}^n$ with axes parallel to the coordinate axes, and $|Q|$ denotes the Lebesgue measure of Q. Examples are given by the standard radial weights with fixed $x_0\in\boldsymbol{R}^n$ defined by

$$w(x)=|x-x_0|^{\alpha}\ \text{ or }\ w(x)=(1+|x|)^{\alpha},\quad -n<\alpha<n(q-1);$$

also finite positive sums of these terms multiplied by logarithmic terms

$$\log^{\beta}(2+|x|),\ \log^{\beta}(2+|x-x_0|^{-1}),\quad \beta\in\boldsymbol{R},$$

are allowed. Even the distance $\operatorname{dist}(x, M)^{\alpha}$ of x to a bounded manifold M and anisotropic functions such as

$$(1+|x|)^{\alpha}(1+|x|-x_1)^{\beta},\quad x=(x_1,\cdots,x_n)\in\boldsymbol{R}^n,$$

for certain $\alpha, \beta\in\boldsymbol{R}$ define $\mathcal{A}_q$-weights; see Lemma 2.2, 2.3 and Remark 2.4 below for construction and properties of $\mathcal{A}_q$-weights. The reason to consider weights of class $\mathcal{A}_q$ is the fact that the classical multiplier theorem of Hörmander and Michlin remains true in weighted L^q-spaces on $\boldsymbol{R}^n$ for an $\mathcal{A}_q$-weight.

Given a weight $w\in\mathcal{A}_q$ define the weighted spaces

$$L^q_w(\Omega)=\Big\{u\in L^1_{\text{loc}}(\bar{\Omega});\ \|u\|_{q,w}=\|uw^{1/q}\|_q=\Big(\int_{\Omega}|u|^q w\,dx\Big)^{1/q}<\infty\Big\},$$

$$H^{2,q}_w(\Omega)=\{u\in L^1_{\text{loc}}(\bar{\Omega});\ u,\ \nabla u,\ \nabla^2 u\in L^q_w(\Omega)\}$$

for the velocity, and the homogeneous Sobolev space

$$\dot{H}^{1,q}_w(\Omega)=\{p\in L^1_{\text{loc}}(\bar{\Omega});\ \nabla p\in L^q_w(\Omega)^n\};$$

here $\Omega=\boldsymbol{R}^n$ or $\Omega\subset\boldsymbol{R}^n$ is an exterior domain. Then the main result on the Stokes resolvent problem (with $g=\operatorname{div} u=0$) in $\boldsymbol{R}^n$ reads as follows:

THEOREM 1.1. *Let* $q\in(1,\infty)$, $n\geqq 2$, $w\in\mathcal{A}_q$ *and* $\varepsilon\in(0,\pi/2)$. *Then for every* $\lambda\in S_{\varepsilon}$ *and* $f\in L^q_w(\boldsymbol{R}^n)^n$ *the resolvent problem*

$$\lambda u-\Delta u+\nabla p=f,\quad \operatorname{div} u=0 \quad on\ \boldsymbol{R}^n$$

has a unique solution $(u, p)\in H_w^{2,q}(\boldsymbol{R}^n)^n\times \dot{H}_w^{1,q}(\boldsymbol{R}^n)$, *where* p *is unique only up to an additive constant. Further*

$$\|(\lambda u, \nabla^2 u, \nabla p)\|_{q,w}\leqq c(\varepsilon)\|f\|_{q,w}$$

with a constant $c(\varepsilon)>0$ *independent of* $\lambda\in S_\varepsilon$.

For further details including the case $g=\operatorname{div} u\neq 0$ we refer to Theorem 4.5 below. The crucial part of the proof of this theorem and of the construction of the Helmholtz decomposition of $L_w^q(\boldsymbol{R}^n)^n$ is based on the weighted multiplier theory of Kurtz and Wheeden [**13**].

For an exterior domain $\Omega\subset\boldsymbol{R}^n$ with boundary of class $C^{1,1}$ we need some restrictions on the weights near the boundary of Ω—see Definition 2.5 for the weight class $\mathcal{A}_q(\Omega)$—and also near infinity to get a uniform resolvent estimate for all $\lambda\in S_\varepsilon$. The main results on the Stokes resolvent system with $g=\operatorname{div} u=0$ yielding estimates uniformly in $\lambda\in S_\varepsilon$ are gathered in the following theorem. There $|\cdot|$ denotes the function $x\mapsto|x|$ on $\boldsymbol{R}^n$.

THEOREM 1.2. *Let* $\Omega\subset\boldsymbol{R}^n$, $n\geqq 2$, *be an exterior domain with boundary of class* $C^{1,1}$, *let* $q\in(1,\infty)$, $w\in\mathcal{A}_q(\Omega)$ *and* $\varepsilon\in(0,\pi/2)$. *Then for every* $\lambda\in S_\varepsilon$ *and* $f\in L_w^q(\Omega)^n$ *the resolvent system*

$$\lambda u-\Delta u+\nabla p=f,\quad \operatorname{div} u=0 \quad in\ \Omega,\quad u|_{\partial\Omega}=0$$

has a unique solution $(u, p)\in H_w^{2,q}(\Omega)^n\times\dot{H}_w^{1,q}(\Omega)$. *Additionally assume that* $n\geqq 3$, $x_0\in\boldsymbol{R}^n\backslash\partial\Omega$ *and*

$$w^{s/q}|\cdot-x_0|^{-\gamma s}\in\mathcal{A}_s(\Omega)\quad where\ \gamma=n\Big(\frac{2}{n}+\frac{1}{s}-\frac{1}{q}\Big)\geqq 0\ and\ s\geqq q \tag{1.2}$$

or

$$w^{s/q}|\cdot-x_0|^{\tilde{\gamma}s}\in\mathcal{A}_s(\Omega)\quad where\ \tilde{\gamma}=n\Big(\frac{2}{n}+\frac{1}{q}-\frac{1}{s}\Big)\geqq 0\ and\ s\leqq q. \tag{1.3}$$

Then

$$\|(\lambda u, \nabla^2 u, \nabla p)\|_{q,w}\leqq c(\varepsilon)\|f\|_{q,w}$$

or

$$\|(\lambda u, -\Delta u+\nabla p)\|_{q,w}\leqq c(\varepsilon)\|f\|_{q,w}$$

uniformly in $\lambda\in S_\varepsilon$, *respectively. In particular,* (1.2) *is satisfied for the weight* $w=|\cdot-x_0|^\alpha$ *when*

$$n\geqq 3,\quad 2q-n<\alpha<n(q-1), \tag{1.2'}$$

while (1.3) *is satisfied for this* w *when*

$$n\geqq 3,\quad -n<\alpha<n(q-1)-2q. \tag{1.3'}$$

The same results hold when the term $|\cdot - x_0|$ *is replaced by* $1+|\cdot|$.

The conditions (1.2), (1.3) are not needed if $\lambda \in S_\varepsilon$ is restricted by $|\lambda| \geqq \delta > 0$ and $c(\varepsilon)$ additionally depends on $\delta > 0$, see Theorem 5.5. However, the independence of $c(\varepsilon)$ of δ is very important for applications, see [10]. Theorem 5.5 also treats the case $g = \operatorname{div} u \neq 0$.

To define the Stokes operator we need the Helmholtz decomposition of the weighted space $L_w^q(\Omega)^n$ for an exterior domain $\Omega \subset \boldsymbol{R}^n$; see Corollary 4.4 for the Helmholtz decomposition in $L_w^q(\boldsymbol{R}^n)^n$.

THEOREM 1.3. *Let* $\Omega \subset \boldsymbol{R}^n$ *be an exterior domain with boundary of class* C^1, *let* $q \in (1, \infty)$ *and* $w \in \mathcal{A}_q(\Omega)$.

(i) $L_w^q(\Omega)^n$ *has a unique algebraic and topological decomposition*

$$L_w^q(\Omega)^n = L_{w,\sigma}^q(\Omega) \oplus \nabla \dot{H}_w^{1,q}(\Omega)$$

where $L_{w,\sigma}^q(\Omega)$ *is the closure of* $C_{0,\sigma}^\infty(\Omega) = \{u \in C_0^\infty(\Omega)^n ; \operatorname{div} u = 0\}$ *with respect to the norm* $\|\cdot\|_{q,w}$. *In particular there exists a unique bounded projection operator*

$$P_{q,w} : L_w^q(\Omega)^n \rightarrow L_{w,\sigma}^q(\Omega)$$

with null space $\nabla \dot{H}_w^{1,q}(\Omega) = \{\nabla p ; p \in \dot{H}_w^{1,q}(\Omega)\}$ *and range* $L_{w,\sigma}^q(\Omega)$.

(ii) $(P_{q,w})^* = P_{q',w'}$ *and* $(L_{w,\sigma}^q(\Omega))^* = L_{w',\sigma}^{q'}(\Omega)$ *where* $q' = q/(q-1)$ *and* $w' = w^{-1/(q-1)}$.

(iii) *If* $u \in L_{w_1}^{q_1}(\Omega)^n \cap L_{w_2}^{q_2}(\Omega)^n$ *for* $q_1, q_2 \in (1, \infty)$, $w_1 \in \mathcal{A}_{q_1}(\Omega)$ *and* $w_2 \in \mathcal{A}_{q_2}(\Omega)$, *then* $P_{q_1,w_1} u = P_{q_2,w_2} u$.

Given the Helmholtz projection $P_{q,w}$, the Stokes operator $A_{q,w}$ in $L_{w,\sigma}^q(\Omega)$ for an exterior domain Ω is defined by $A_{q,w} = -P_{q,w}\Delta$ with domain of definition

$$\mathcal{D}(A_{q,w}) = \{u \in H_w^{2,q}(\Omega)^n \cap L_{w,\sigma}^q(\Omega) ; \ u = 0 \text{ on } \partial\Omega\} ;$$

for the entire space $\boldsymbol{R}^n$

$$A_{q,w} = -P_{q,w}\Delta, \quad \mathcal{D}(A_{q,w}) = H_w^{2,q}(\boldsymbol{R}^n)^n \cap L_{w,\sigma}^q(\boldsymbol{R}^n).$$

Now Theorem 1.2 and interpolation theory yield the following result on the resolvent of the Stokes operator $A_{q,w}$, its analytic semigroup $\{e^{-tA_{q,w}} ; t \geqq 0\}$ and its imaginary powers $A_{q,w}^{it}$, $t \in \boldsymbol{R}$.

THEOREM 1.4. *Let* $q \in (1, \infty)$ *and* $w \in \mathcal{A}_q$.

(i) *The resolvent problem*

$$\lambda u + A_{q,w} u = f \qquad u \in \mathcal{D}(A_{q,w}),$$

in $\boldsymbol{R}^n$ *has for every* $f \in L_{w,\sigma}^q(\boldsymbol{R}^n)$ *and* $\lambda \in S_\varepsilon$, $0 < \varepsilon < \pi/2$, *a unique solution* $u \in \mathcal{D}(A_{q,w})$. *This solution satisfies the resolvent estimate*

$$\|(\lambda u,\ A_{q,w}u)\|_{q,w} \leqq c(\varepsilon)\|f\|_{q,w}\,.$$

(ii) *The Stokes operator in* $\boldsymbol{R}^n$ *generates a bounded analytic semigroup* $\{e^{-tA_{q,w}};\ t\geqq 0\}$.

(iii) *The imaginary powers* $\{A_{q,w}^{it};\ t\in\boldsymbol{R}\}$ *define a family of bounded linear operators in* $L^q_{w,\sigma}(\boldsymbol{R}^n)$ *satisfying for every* $\delta>0$ *the estimate*

$$\|A_{q,w}^{it}\| \leqq c(\delta)e^{\delta|t|},\quad t\in\boldsymbol{R}.$$

THEOREM 1.5. *Let* $\Omega\subset\boldsymbol{R}^n$ *be an exterior domain with boundary of class* $C^{1,1}$, *let* $q\in(1,\infty)$ *and* $w\in\mathcal{A}_q(\Omega)$. *Then for every* $\lambda\in S_\varepsilon$, $0<\varepsilon<\pi/2$, *and every* $f\in L^q_{w,\sigma}(\Omega)$ *the resolvent problem*

$$\lambda u+A_{q,w}u=f,\quad u\in\mathcal{D}(A_{q,w}),$$

has a unique solution $u\in\mathcal{D}(A_{q,w})$.

(i) *For every* $\delta>0$ *this solution satisfies the resolvent estimate*

$$\|(\lambda u,\ A_{q,w}u)\|_{q,w} \leqq c(\varepsilon,\delta)\|f\|_{q,w}$$

where $\lambda\in S_\varepsilon$ *restricted by* $|\lambda|\geqq\delta$.

(ii) *Assume for* $n\geqq 3$ *that there exist* $w_0,\ w_1\in\mathcal{A}_q(\Omega)$ *and* $x_0\notin\partial\Omega$ *such that*

$$w_0|\cdot-x_0|^{-2q}\in\mathcal{A}_q(\Omega),\ w_1|\cdot-x_0|^{2q}\in\mathcal{A}_q(\Omega)\ \text{and}\ w=w_0^{1-\theta}w_1^{\theta} \tag{1.4}$$

for some $\theta\in[0,1]$. *Then the constant* $c(\varepsilon,\delta)$ *is independent of* $\delta>0$ *and* $-A_{q,w}$ *generates a bounded analytic semigroup. In particular, condition* (1.4) *is satisfied for* $w=|\cdot-x_0|^\alpha$ *with* $-n<\alpha<n(q-1)$. *The same result holds when the term* $|\cdot-x_0|$ *is replaced by* $1+|\cdot|$.

(iii) $A_{q,w}$ *is a closed operator and* $(A_{q,w})^*=A_{q',w'}$ *where* $q'=q/(q-1)$ *and* $w'=w^{-1/(q-1)}$.

This paper is organized as follows. In Section 2 we introduce—besides some notations—the Muckenhoupt class $\mathcal{A}_q$, $1\leqq q<\infty$, describe the main properties of weight functions and construct several types of weights which are interesting in unbounded domains. Weighted Sobolev spaces are introduced in Section 3 where also the fundamental theory of multiplier operators in weighted spaces is resumed. Further we prove an interpolation theorem and embedding properties which are based on estimates of integral operators on weighted spaces. In Section 4 we consider the generalized Stokes resolvent problem (1.1) and the Helmholtz decomposition in L^q_w for the whole space. The same problems are considered in Section 5 for an exterior domain. Via the localization method the proofs are based on the results of Section 4. This step in which cut-off functions are introduced forces us to consider the generalized resolvent system with nonzero divergence. A further crucial point is the problem to get estimates uniformly in $\lambda \in S_\varepsilon$ requiring the embedding results proved in Section 3.

2. Weight functions of class $\mathcal{A}_q$.

Given a domain $\Omega \subset \boldsymbol{R}^n$ we will use the standard notations $L^q(\Omega)$ with norm $\|\cdot\|_{q,\Omega}$ (or simply $\|\cdot\|_q$, if the domain Ω is known from the context), $L^q_{\text{loc}}(\Omega)$ and $L^q_{\text{loc}}(\bar{\Omega})$ for spaces of measurable functions. Here $u \in L^q_{\text{loc}}(\bar{\Omega})$ iff $u \in L^q_{\text{loc}}(\Omega \cap B)$ for all balls

$$B = B_r(x) = \{y \in \boldsymbol{R}^n ; |y-x|<r\}, \quad r>0,$$

with $\Omega \cap B \neq \emptyset$. Further $H^{1,q}(\Omega)$, $H^{1,q}_0(\Omega)$, $H^{1,q}_{\text{loc}}(\bar{\Omega})$, $H^{2,q}(\Omega)$ etc. will denote standard Sobolev spaces of scalar functions. For vector—or matrix-valued functions $u \in L^q(\Omega)^m$ etc. we also use the symbol $\|\cdot\|_q$ for the L^q-norm; more generally

$$\|(u_1, \cdots, u_m)\|_q = \Big(\sum_{i=1}^m \|u_i\|_q^q\Big)^{1/q}$$

for $u_i \in L^q(\Omega)$ or $u_i \in L^q(\Omega)^n$, $i=1, \cdots, m$. Sometimes we simply write L^q, $H^{1,q}$ etc. if $\Omega = \boldsymbol{R}^n$. Recall the definition of the differential operators ∇, Δ and div; moreover $\nabla^k u$ denotes the set of all kth order partial derivatives $(\partial_1^{k_1} \cdots \partial_n^{k_n} u)$ where $(k_1, \cdots, k_n) \in \{0, \cdots, k\}^n$ runs through the set of all multi-indices of order $k = k_1 + \cdots + k_n$ and where $\partial_i = \partial / \partial x_i$, $1 \leqq i \leqq n$. For $q \in [1, \infty)$ let $q' = q/(q-1)$ with $q' = \infty$ for $q=1$; thus $L^q(\Omega)^* = L^{q'}(\Omega)$. In general X^* denotes the dual space of a Banach space X. Then the dual pairing of X with X^* is denoted by $\langle \cdot, \cdot \rangle$. The symbol $f \sim g$ means that there are positive constants $c_1 \leqq c_2$ with $c_1 f(x) \leqq g(x) \leqq c_2 f(x)$ for all x in the domain of definition of f, g.

Next we introduce weight functions of the Muckenhoupt class $\mathcal{A}_q$, $1 \leqq q < \infty$. Let Q always denote a bounded open cube in $\boldsymbol{R}^n$ lying parallel to the coordinate axes, e.g.,

$$Q_h(x) = (x_1 - h, x_1 + h) \times \cdots \times (x_n - h, x_n + h)$$

for $x = (x_1, \cdots, x_n) \in \boldsymbol{R}^n$ and $h > 0$. For a measurable set $E \subset \boldsymbol{R}^n$ with Lebesgue measure $|E|$ and a measurable function $w \geqq 0$ on $\boldsymbol{R}^n$ let $w(E) = \int_E w(y)dy$ and

$$\fint_E w = \frac{1}{|E|}\int_E w(y)dy = \frac{w(E)}{|E|}.$$

DEFINITION 2.1. (i) For a nonnegative Borel measure μ on $\boldsymbol{R}^n$

$$\mathcal{M}\mu(x) = \sup\Big\{\frac{1}{|Q|}\int_Q d\mu ; Q \subset \boldsymbol{R}^n \text{ cube with } x \in Q\Big\}$$

is the *Hardy-Littlewood maximal operator* of μ. For a measurable function $w \geqq 0$ defining the measure $d\mu(y) = w(y)dy$ we simply write $\mathcal{M}w(x) = \mathcal{M}\mu(x)$.

(ii) A function $0\leqq w\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$, $w\not\equiv 0$, is called a *weight of Muckenhoupt class* $\mathcal{A}_1$, or simply $w\in\mathcal{A}_1$, iff there exists a constant $c>0$ such that

$$\mathcal{M}w(x)\leqq cw(x) \quad \text{a.e. in } \boldsymbol{R}^n. \tag{2.1}$$

(iii) For $q\in(1, \infty)$ a function $0\leqq w\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$, $w\not\equiv 0$, is a *weight of Muckenhoupt class* $\mathcal{A}_q$, or $w\in\mathcal{A}_q$, iff there is a constant $C>0$ such that

$$⨍_Q w\cdot\left(⨍_Q w'\right)^{q-1}\leqq C \quad \text{for all cubes } Q\subset\boldsymbol{R}^n; \tag{2.2}$$

here $w':=w^{-1/(q-1)}=w^{-(q'-1)}=w^{-q'/q}$. The least constant $C=C_q(w)$ in (2.2) is called the $\mathcal{A}_q$-constant of w.

Obviously the $\mathcal{A}_1$-condition (2.1) is equivalent to the condition

$$⨍_Q w\leqq c\operatorname*{ess\,inf}_Q w \quad \text{for all cubes } Q\subset\boldsymbol{R}^n. \tag{2.3}$$

Hölder's inequality implies that $\mathcal{A}_q\subset\mathcal{A}_p$ for all $1\leqq q<p<\infty$. Moreover, since $w\not\equiv 0$ in Definition 2.1 (ii), (iii), for every measurable set E with $|E|>0$

$$0<\operatorname*{ess\,inf}_E w\leqq\operatorname*{ess\,sup}_E w<\infty.$$

Finally for every $q\in(1, \infty)$

$$w\in\mathcal{A}_q \text{ is equivalent to } w'\in\mathcal{A}_{q'}.$$

The following properties are less obvious.

LEMMA 2.2. (i) *Let μ be a nonnegative Borel measure such that $\mathcal{M}\mu(x)<\infty$ a.e.. Then $(\mathcal{M}\mu)^\gamma$ is an $\mathcal{A}_1$-weight for all $\gamma\in[0, 1)$. Conversely for every $\mathcal{A}_1$-weight w there exist constants $h_1>h_0>0$, some function $h:\boldsymbol{R}^n\to[h_0, h_1]$ and a function $0\leqq g\in L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$ such that*

$$w=h(\mathcal{M}g)^\gamma$$

for some $\gamma\in[0, 1)$.

(ii) *A weight w is of class $\mathcal{A}_q$, $1<q<\infty$, iff there are weights $w_1, w_2\in\mathcal{A}_1$ with*

$$w=\frac{w_1}{w_2^{q-1}}.$$

(iii) *For a weight $w\in\mathcal{A}_q$, $1<q<\infty$,*

$$\int_{\boldsymbol{R}^n}(1+|x|)^{-nq}w(x)dx\leqq cw(Q_1)<\infty$$

where $Q_1=Q_1(0)$ and c only depends on the $\mathcal{A}_q$-constant of w.

(iv) *Every* $w\in\mathcal{A}_q$, $1\leqq q<\infty$, *satisfies the* ***reverse Hölder inequality*** RH_δ *with some* $\delta=\delta(w)>1$, *i.e.*,

$$\left(\fint_Q w^\delta\right)^{1/\delta} = c\fint_Q w \quad \textit{for all cubes } Q\subset \boldsymbol{R}^n.$$

(v) *For every* $w\in\mathcal{A}_q$, $1<q<\infty$, *there is an* $\varepsilon=\varepsilon(w)>0$ *such that* $w\in\mathcal{A}_{q-\varepsilon}$ *and* $w^{1+\varepsilon}\in\mathcal{A}_q$.

PROOF. (i) See [**22**, Chapter IX, Proposition 3.3, Theorem 3.4].

(ii) See [**22**, Chapter IX, Proposition 4.3, Theorem 5.5]. Note that the assertion w_1, $w_2^{1-q}\in\mathcal{A}_q$ for given w_1, $w_2\in\mathcal{A}_1$ is trivial due to (2.3).

(iii) See [**22**, Chapter IX, Proposition 4.5].

(iv) See [**22**, Chapter IX, Theorem 3.5, Proposition 4.5].

(v) See [**22**, Proposition 4.5]. Both assertions are easy consequences of (i), (ii). □

By means of Lemma 2.2 it is easy to construct $\mathcal{A}_q$-weights. The following lemma yields further techniques to construct $\mathcal{A}_q$-weights starting from the radial $\mathcal{A}_1$-weights $|\cdot|^\alpha$, $-n<\alpha\leqq 0$, on $\boldsymbol{R}^n$.

LEMMA 2.3. (i) *The minimum, maximum and the sum of a finite set of* $\mathcal{A}_1$*-weights again yield* $\mathcal{A}_1$*-weights.*

(ii) *Let* $\phi:[0,\infty]\to[0,\infty]$ *be concave and nondecreasing. Then* $w\in\mathcal{A}_1$ *yields* $\phi\circ w\in\mathcal{A}_1$.

(iii) *For any two weights* w_1, $w_2\in\mathcal{A}_1$ *and* $\theta\in[0,1]$ *also* $w_1^\theta w_2^{1-\theta}\in\mathcal{A}_1$.

(iv) *For all* $\alpha\in(-n,0]$ *and* $\beta\geqq 0$ *the functions defined by*

$$|x|^\alpha,\ \log^{-\beta}(2+|x|),\ \log^\beta(2+|x|^{-1}),\quad x\in\boldsymbol{R}^n,$$

and products of them are $\mathcal{A}_1$*-weights. The same holds when* $|x|^\alpha$ *is replaced by* $(1+|x|)^\alpha$.

(v) *Let* $1<q<\infty$. *Then for all* $-n<\alpha<n(q-1)$ *and for all* $\beta\in\boldsymbol{R}$ *the functions defined by*

$$|x|^\alpha,\ \log^\beta(2+|x|),\ \log^\beta(2+|x|^{-1}),\quad x\in\boldsymbol{R}^n,$$

and products of them are $\mathcal{A}_q$*-weights. The same holds when* $|x|^\alpha$ *is replaced by* $(1+|x|)^\alpha$.

(vi) *Let* $1\leqq q<\infty$, $k\in\{1,\cdots,n-1\}$ *and let* $M\subset\boldsymbol{R}^n$ *be a* k*-dimensional compact Lipschitzian manifold. Then the function defined by*

$$\mathrm{dist}(x,M)^\alpha,\quad -(n-k)<\alpha<(n-k)(q-1)$$

is an $\mathcal{A}_q$*-weight.*

PROOF. (i) These assertions are trivial by (2.1).

(ii) For all cubes $Q \subset \boldsymbol{R}^n$ and $x \in Q$

$$\fint_Q \phi \circ w \leqq \phi\left(\fint_Q w\right) \leqq \phi(cw(x))$$

due to Jensen's inequality and (2.3). Since the concavity of $\phi \geqq 0$ yields $\phi(2s) \leqq 2\phi(s)$ for all $s \geqq 0$, the monotonicity of ϕ implies that $\phi(cw(x)) \leqq 2^m \phi \circ w(x)$ when $c \leqq 2^m$, $m \in \boldsymbol{N}$.

(iii) is an easy consequence of Hölder's inequality.

(iv) It is easily seen that $M\delta_0(x) \sim |x|^{-n}$ for Dirac's measures δ_0 with support $\{0\}$. Thus $|\cdot|^\alpha \in \mathcal{A}_1$ for $\alpha \in (-n, 0]$ by Lemma 2.2 (i). Further $(1+|\cdot|)^\alpha \sim \min(1, |\cdot|^\alpha) \in \mathcal{A}_1$ by (i). For $\beta \geqq 0$ and N sufficiently large $\phi(r) = \log^\beta(N+r)$ is nondecreasing and concave, since

$$\phi''(r) = \frac{\beta}{(N+r)^2} \log^{\beta-1}(N+r)\left(-1+\frac{\beta-1}{\log(N+r)}\right) \leqq 0$$

for all $r \geqq 0$. Thus $\log^\beta(N+|x|^{-1/2}) \sim \log^\beta(2+|x|^{-1})$ defines an $\mathcal{A}_1$-weight by (ii). Analogously $\phi(r) = \log^{-\beta}(N+r^{-1})$ is nondecreasing and concave; hence $\log^{-\beta}(N+r^{1/2}) \sim \log^{-\beta}(2+r) \in \mathcal{A}_1$. Finally (iii) and the fact that $\beta \geqq 0$ may be arbitrarily large imply that also products of the form

$$|x|^\alpha \log^{-\beta_1}(2+|x|),\ |x|^\alpha \log^{\beta_2}(2+|x|^{-1}),\ \log^{-\beta_1}(2+|x|) \cdot \log^{\beta_2}(2+|x|)$$

etc. lead to $\mathcal{A}_1$-weights for all $\alpha \in (-n, 0]$, $\beta_1, \beta_2 \geqq 0$.

(v) This is a consequence of (iv) and Lemma 2.2 (ii).

(vi) Let μ denote the k-dimensional Hausdorff measure

$$\mu(E) = \mathcal{H}^k(E \cap M) \quad \text{for } E \subset \boldsymbol{R}^n \text{ measurable.}$$

Then it is easily seen that $\mathcal{M}\mu(x) \sim \min(\operatorname{dist}(x, M)^{k-n}, \operatorname{dist}(x, M)^{-n})$; for $x \notin M$ choose the cube $Q_r(x)$, $r = 2\operatorname{dist}(x, M)$, yielding $\int_{Q_r(x)} d\mu \sim \min(r^k, 1)$. Taking the maximum of $(\mathcal{M}\mu)^\gamma$ and the $\mathcal{A}_1$-weight $(1+|\cdot|)^{(k-n)\gamma}$, $\gamma \in [0, 1)$, Lemma 2.2 (i) and Lemma 2.3 (i), (iv) show that $\operatorname{dist}(\cdot, M)^{(k-n)\gamma}$ is an $\mathcal{A}_1$-weight. Now Lemma 2.2 (ii) completes the proof. □

REMARK 2.4. (i) Lemma 2.3 implies that continuous functions w on $\boldsymbol{R}^n$ with a finite number of zeros and singularities $x^1, \cdots, x^m \in \boldsymbol{R}^n$ such that in a neighborhood of x^j

$$w(x) \sim |x-x^j|^{\alpha_j}, \quad -n < \alpha_j < n(q-1)$$

or

$$w(x) \sim |\log|x-x^j||^{\beta_j}, \quad \beta_j \in \boldsymbol{R},\ 1 \leqq j \leqq m,$$

are $\mathcal{A}_q$-weights, $1 < q < \infty$. At infinity

$$w(x) \sim |x|^{\alpha}, \ -n < \alpha < n(q-1), \text{ or } w(x) \sim \log^{\beta}|x|, \quad \beta \in \boldsymbol{R},$$

is allowed. Additionally, for a finite number of compact Lipschitzian manifolds M^j of dimension $k_j \in \{1, \cdots, n-1\}$, $j=1, \cdots, m$,

$$w(x) \sim \operatorname{dist}(x, M_j)^{\alpha_j}, \quad -(n-k_j) < \alpha_j < (n-k_j)(q-1),$$

or

$$w(x) \sim |\log \operatorname{dist}(x, M_j)|^{\beta_j}, \quad \beta_j \in \boldsymbol{R},$$

in a neighborhood of M_j is allowed.

(ii) Anisotropic weights in $\mathcal{A}_2$ of the form

$$w(x) = (1+|x|)^{\alpha}(1+|x|-x_1)^{\beta}$$

when $|\alpha+\beta|<3$, $|\beta|<1$, are considered in [4] concerning fluid flow past an obstacle in $\boldsymbol{R}^3$.

(iii) For later use (see Corollary 3.7) we mention a further property of the weight $|\cdot|^{\alpha}$ for all $\alpha>-n$:

$$\int_{Q_r(x)} |y|^{\alpha} dy \sim r^n \max{}^{\alpha}(r, |x|) \tag{2.4}$$

where $r>0$, $x \in \boldsymbol{R}^n$.

For the proof we may replace $Q_r(x)$ by the ball $B_r(x)$. First let $r<|x|/2$. Since $|y| \sim |x|$ for all $y \in B_r(x)$,

$$\int_{B_r(x)} |y|^{\alpha} dy \sim r^n |x|^{\alpha} = r^n \max{}^{\alpha}(r, |x|).$$

If $r>|x|/2$, we find a bounded sector $\Sigma=\Sigma_{r,x}$ around the axis of direction x and with opening angle $\delta>0$ independent of r, x such that

$$(B_{2(r+|x|)/3}(0) \backslash \overline{B_{(r+|x|)/2}(0)}) \cap \Sigma \subset B_r(x) \subset B_{r+|x|}(0).$$

Thus

$$\int_{B_r(x)} |y|^{\alpha} dy \leqq c \int_0^{r+|x|} s^{\alpha+n-1} ds \sim (r+|x|)^{\alpha+n}$$

and

$$\int_{B_r(x)} |y|^{\alpha} dy \geqq c \int_{(r+|x|)/2}^{2(r+|x|)/3} s^{\alpha+n-1} dx \sim (r+|x|)^{\alpha+n}.$$

Summarizing we obtain (2.4).

In analogy to (2.4) one has for all $\alpha>-n$

$$\int_{Q_r(x)} (1+|y|)^{\alpha} dy \sim r^n(1+\max(r, |x|))^{\alpha}. \tag{2.5}$$

If $r<|x|/2$, we use the same argument as before. For $r>|x|/2$ the integral is bounded from above by

$$c\int_0^{r+|x|}(1+s)^\alpha s^{n-1}ds\sim\begin{cases}(r+|x|)^n, & r+|x|<1\\ (r+|x|)^{n+\alpha}+1, & r+|x|\geqq 1\end{cases}$$

and from below by

$$c\int_{(r+|x|)/2}^{2(r+|x|)/3}(1+s)^\alpha s^{n-1}ds\sim\begin{cases}(r+|x|)^n, & r+|x|<1\\ (r+|x|)^{n+\alpha}, & r+|x|\geqq 1.\end{cases}$$

Summarizing we are led to (2.5).

For an exterior domain $\Omega\subset\boldsymbol{R}^n$ with boundary at least in $C^{1,1}$ we are interested in the behavior of the Stokes resolvent problem at infinity rather than near the boundary. Therefore we introduce a restricted class of $\mathcal{A}_q$-weights on Ω.

DEFINITION 2.5. Let $\Omega\subset\boldsymbol{R}^n$ be an exterior domain. Then for $1\leqq q<\infty$

$$\mathcal{A}_q(\Omega)=\left\{\begin{array}{l} w\in\mathcal{A}_q;\ \text{there is a bounded domain } G=G(w)\subset\Omega \text{ and an } \varepsilon>0\\ \text{such that } \{x\in\Omega;\ \operatorname{dist}(x,\partial\Omega)<\varepsilon\}\subset G \text{ and } w\in C^0(\bar{G}),\ w|_{\bar{G}}>0\end{array}\right\}.$$

Obviously the bounded domain G can be chosen such that ∂G has the same regularity as $\partial\Omega$.

3. Weighted function spaces.

DEFINITION 3.1. Let $\Omega=\boldsymbol{R}^n$ or let $\Omega\subset\boldsymbol{R}^n$ be an exterior domain with Lipschitz boundary. For given $1<q<\infty$ and $w\in\mathcal{A}_q$ (or $w\in\mathcal{A}_q(\Omega)$ if $\Omega\neq\boldsymbol{R}^n$) define

$$L_w^q(\Omega)=\left\{u\in L_{\mathrm{loc}}^1(\bar{\Omega});\ \|u\|_{q,w}=\left(\int_\Omega|u|^q w\,dx\right)^{1/q}<\infty\right\}$$

$$H_w^{k,q}(\Omega)=\{u\in H_{\mathrm{loc}}^{k,1}(\bar{\Omega});\ \|(u,\nabla u,\cdots,\nabla^k u)\|_{q,w}<\infty\},$$

$k=1, 2, \cdots$, and

$$H_{0,w}^{k,q}(\Omega)=\text{closure of } C_0^\infty(\Omega) \text{ in } H_w^{k,q}(\Omega).$$

Further let the *homogeneous Sobolev space*

$$\dot{H}_w^{1,q}(\Omega)=\{u\in L_{\mathrm{loc}}^1(\bar{\Omega});\ \nabla u\in L_w^q(\Omega)^n\}$$

be endowed with the (semi-) norm $\|\nabla\cdot\|_{q,w}$ and let

$$\hat{H}_w^{1,q}(\Omega)=\text{closure of } C_0^\infty(\bar{\Omega}) \text{ in } \dot{H}_w^{1,q}(\Omega).$$

Here $C_0^\infty(\bar{\Omega})=\{u|_\Omega;\ u\in C_0^\infty(\boldsymbol{R}^n)\}$. The dual space of $\hat{H}_{w'}^{1,q'}(\Omega)$, where $q'=q/(q-1)$

and $w'=w^{-1/(q-1)}$, is denoted by

$$\hat{H}_w^{-1,q}(\Omega)=\hat{H}_{w'}^{1,q'}(\Omega)^*$$

and endowed with the norm

$$\|F\|_{-1,q,w}=\sup\left\{\frac{|\langle F,\varphi\rangle|}{\|\nabla\varphi\|_{q',w'}};\ 0\neq\varphi\in\hat{H}_{w'}^{1,q'}(\Omega)\right\}$$

for $F\in\hat{H}_w^{-1,q}(\Omega)$.

Let us discuss some properties of the spaces introduced in Definition 3.1. To show that $C_0^\infty(\Omega)$ is dense in $L_w^q(\Omega)$ it suffices to consider a characteristic function $u=\chi_E$ where $E\subset\Omega$ is a measurable set with $|E|<\infty$. By Lemma 2.2 (v) there is an $\varepsilon>0$ such that $w^{1+\varepsilon}\in\mathcal{A}_q$; in particular $w^{1+\varepsilon}\in L_{\mathrm{loc}}^1(\boldsymbol{R}^n)$. Since $u\in L^r(\Omega)$ where $r=q(1+\varepsilon)/\varepsilon>q$, Hölder's inequality implies that

$$\int|u-\varphi|^q w\,dx\leqq\left(\int|u-\varphi|^r dx\right)^{q/r}\left(\int w^{r/(r-q)}dx\right)^{(r-q)/r}$$

for all $\varphi\in C_0^\infty(\Omega)$. Now the density of $C_0^\infty(\Omega)$ in $L^r(\Omega)$ yields the density of $C_0^\infty(\Omega)$ in $L_w^q(\Omega)$. In particular $L_w^q(\Omega)$ is *separable* for all $1\leqq q<\infty$. Since obviously

$$L_w^q(\Omega)^*=L_{w'}^{q'}(\Omega),\quad 1\leq q<\infty,$$

we conclude that $L_w^q(\Omega)$ is a *reflexive* Banach space for all $q\in(1,\infty)$ and all $w\in\mathcal{A}_q$, $w\in\mathcal{A}_q(\Omega)$.

Since $L_w^q(\boldsymbol{R}^n)$ is not translation invariant, it is not straightforward to prove the standard approximation properties of Friedrichs' mollifiers on the weighted space $L_w^q(\boldsymbol{R}^n)$; see Remark 3.4 below. Anticipating this result we get that $C_0^\infty(\boldsymbol{R}^n)$ is dense in $H_w^{k,q}(\boldsymbol{R}^n)$ and that

$$H_{0;w}^{k,q}(\Omega)=\{u\in H_w^{k,q}(\Omega);\ u|_{\partial\Omega}=0\text{ in the sense of traces}\}$$

when $\Omega\neq\boldsymbol{R}^n$ and $\partial\Omega$ is sufficiently smooth.

In the space $\dot{H}_w^{1,q}(\Omega)$ the term $\|\nabla\cdot\|_{q,w}$ defines a seminorm yielding the same value for two functions that differ only by an additive constant. Thus the quotient space $\dot{H}_w^{1,q}(\Omega)/\boldsymbol{C}$ defines a *separable* and *reflexive* Banach space since it can be considered as a closed subspace of $L_w^q(\Omega)^n$. For simplicity we write $\dot{H}_w^{1,q}(\Omega)$ instead of $\dot{H}_w^{1,q}(\Omega)/\boldsymbol{C}$ having in mind that the elements of $\dot{H}_w^{1,q}(\Omega)$ are equivalence classes; analogously an equivalence class $[u]\in\dot{H}_w^{1,q}(\Omega)$ is simply denoted by u. By Corollary 4.3 and Lemma 5.1 below

$$\dot{H}_w^{1,q}(\Omega)=\hat{H}_w^{1,q}(\Omega)$$

for all $1<q<\infty$, $w\in\mathcal{A}_q$ or $w\in\mathcal{A}_q(\Omega)$, respectively.

Moreover, if some $g\in L^1_{loc}(\bar{\Omega})$ is given, the linear functional

$$\langle g, \cdot\rangle : \varphi \mapsto \langle g, \varphi\rangle := \int_\Omega g\varphi\, dx, \quad \varphi\in C_0^\infty(\bar{\Omega}),$$

is well defined; if it is bounded with respect to $\|\nabla\varphi\|_{q',w'}$, i.e., $|\langle g, \varphi\rangle|\leqq C\|\nabla\varphi\|_{q',w'}$ with C independent of φ, then g defines a functional in $\hat{H}_w^{-1,q}(\Omega)$ and we simply write $g\in\hat{H}_w^{-1,q}(\Omega)$.

Next we present the main tools for the subsequent sections, namely estimates of singular integral operators and multiplier operators in $L_w^q(\boldsymbol{R}^n)$.

THEOREM 3.2. *Let T be a singular integral operator defined by*

$$Tf(x) = \lim_{\varepsilon\to 0}\int_{\boldsymbol{R}^n\setminus B_\varepsilon(x)} f(y)k(x-y)dy, \quad f\in L_w^q(\boldsymbol{R}^n),$$

where k is a singular kernel of the type $k(x)=\omega(x)/|x|^n$ with $\omega\in C^1(\boldsymbol{R}^n\setminus\{0\})$, $\omega(x)=\omega(x/|x|)$, $\int_{\partial B_1(0)}\omega\, do=0$. Then

$$T: L_w^q(\boldsymbol{R}^n)\to L_w^q(\boldsymbol{R}^n)$$

is a continuous linear operator for all $q\in(1, \infty)$ and all weights $w\in\mathcal{A}_q$.

PROOF. See [**8**, Theorem IV 3.1].

Theorem 3.2 is a special case of the following main theorem on weighted estimates of multiplier operators. Let $S(\boldsymbol{R}^n)$ denote the *Schwartz space* of rapidly decreasing functions and $S'(\boldsymbol{R}^n)$ its dual, the space of tempered distributions. Further let $\mathcal{F}=\wedge$ denote the Fourier transform on $\boldsymbol{R}^n$ and $\mathcal{F}^{-1}$ the inverse Fourier transform. Given a bounded (multiplier) function $m(\xi)$ in the phase space, the multiplier operator T is defined by

$$Tf(x) = \mathcal{F}^{-1}(m\hat{f})(x), \quad f\in S(\boldsymbol{R}^n).$$

Then there holds the following theorem, see [**13**, Theorem 2] or [8, Theorem IV 3.9].

THEOREM 3.3. *Let for some $s\in(1, \infty)$ the multiplier function $m\in L^\infty(\boldsymbol{R}^n)$ satisfy the condition*

$$\sup_{R>0} R^{sk-n}\int_{R<|\xi|<2R}|\nabla^k m(\xi)|^s d\xi \leqq C_m < \infty \tag{3.1}$$

for $k\in\{0, \cdots, n\}$. Then for every $q\in(1, \infty)$ and weight $w\in\mathcal{A}_q$ the multiplier operator T defined by $\widehat{Tf}=m\hat{f}$ is a bounded linear operator from $L_w^q(\boldsymbol{R}^n)$ to $L_w^q(\boldsymbol{R}^n)$; further the operator norm of T only depends on n, q, C_m and the $\mathcal{A}_q$-constant of w.

Note that the well-known multiplier condition

$$|\xi|^k |\nabla^k m(\xi)| \leqq c_m, \quad k = 0, 1, \cdots, n, \ 0 \neq \xi \in \boldsymbol{R}^n, \tag{3.2}$$

immediately implies that m also satisfies (3.1).

REMARK 3.4. As a first application of Theorem 3.3 we investigate Friedrichs' mollification procedure in $L^q_w(\boldsymbol{R}^n)$. Let $0 \leqq \varphi \in C^\infty_0(\boldsymbol{R}^n)$ with $\int \varphi\, dx = 1$ and let $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$, $\varepsilon > 0$. Obviously $\hat{\varphi}_\varepsilon \in S(\boldsymbol{R}^n)$, $\hat{\varphi}_\varepsilon(\xi) = \hat{\varphi}(\varepsilon\xi)$, and $m_\varepsilon = \hat{\varphi}_\varepsilon$ satisfies (3.2) and consequently also (3.1) with a constant $c(m_\varepsilon)$ independent of $\varepsilon > 0$. Thus by Theorem 3.3 the mollification operators

$$J_\varepsilon : L^q_w(\boldsymbol{R}^n) \to C^\infty(\boldsymbol{R}^n),$$
$$J_\varepsilon u(x) = \varphi_\varepsilon * u(x) = \int u(x-y)\varphi_\varepsilon(y)dy,$$

map $L^q_w(\boldsymbol{R}^n)$ into $L^q_w(\boldsymbol{R}^n)$ for all $1 < q < \infty$, $w \in \mathcal{A}_q$, and are uniformly bounded in $\varepsilon > 0$: there is a constant $C > 0$ such that

$$\|J_\varepsilon u\|_{q,w} \leqq C\|u\|_{q,w}$$

for all $u \in L^q_w(\boldsymbol{R}^n)$, $\varepsilon > 0$. Since $C^\infty_0(\boldsymbol{R}^n)$ is dense in $L^q_w(\boldsymbol{R}^n)$ we easily conclude that

$$J_\varepsilon u \to u \text{ in } L^q_w(\boldsymbol{R}^n) \quad \text{as } \varepsilon \to 0$$

for every $u \in L^q_w(\boldsymbol{R}^n)$.

Also the following interpolation result is based on Theorem 3.3.

THEOREM 3.5. *Let $\Omega = \boldsymbol{R}^n$ or $\Omega \subset \boldsymbol{R}^n$ be an exterior domain and let $1 < q < \infty$, $w \in \mathcal{A}_q$ or $w \in \mathcal{A}_q(\Omega)$, respectively. Then there is a constant $c = c(q, w, \Omega) > 0$ such that for all $u \in H^{2,q}_w(\Omega)$ and all $\varepsilon \in (0, 1)$*

$$\|\nabla u\|_{q,w} \leqq c\Big(\varepsilon\|\nabla^2 u\|_{q,w} + \frac{1}{\varepsilon}\|u\|_{q,w}\Big). \tag{3.3}$$

PROOF. First let $\Omega = \boldsymbol{R}^n$ and $u \in C^\infty_0(\boldsymbol{R}^n)$. Then for any $\varepsilon > 0$

$$\widehat{\nabla u}(\xi) = i\xi\hat{u} = \frac{i\xi}{1/\varepsilon + \varepsilon|\xi|^2}\Big(\frac{1}{\varepsilon}\hat{u} - \varepsilon\widehat{\Delta u}\Big).$$

Due to the estimates $|\xi| \leqq (1/\varepsilon + \varepsilon|\xi|^2)/2$ and $\varepsilon(\xi)^2 \leqq 1/\varepsilon + \varepsilon|\xi|^2$ the multiplier functions $m_j(\xi) = \xi_j(1/\varepsilon + \varepsilon|\xi|^2)^{-1}$, $j = 1, \cdots, n$, satisfy the conditions (3.2) and also (3.1) with a bound independent of $\varepsilon > 0$. Thus Theorem 3.3 yields the estimate (3.3) for every $u \in C^\infty_0(\boldsymbol{R}^n)$ and every $\varepsilon > 0$. Since $C^\infty_0(\boldsymbol{R}^n)$ is dense in $H^{2,q}_w(\boldsymbol{R}^n)$, Theorem 3.5 is proved for $\Omega = \boldsymbol{R}^n$.

For an exterior domain $\Omega\subset\boldsymbol{R}^n$ choose a cut-off function $\varphi\in C_0^\infty(\boldsymbol{R}^n;[0,1])$ satisfying $\varphi\equiv 1$ in a neighborhood of $\partial\Omega$ and $\operatorname{supp}\varphi\cap\Omega\subset G$, where the bounded domain $G=G(w)\subset\Omega$ is given by Definition 2.5. Further recall the interpolation inequality [6]

$$\|\nabla u\|_{L^q(G)}\leq c\Big(\varepsilon\|\nabla^2 u\|_{L^q(G)}+\frac{1}{\varepsilon}\|u\|_{L^q(G)}\Big) \tag{3.4}$$

for $u\in H^{2,q}(G)$ and $\varepsilon\in(0,1)$. Applying (3.4) to $u\varphi$ and (3.3) to $u(1-\varphi)\in H_w^{2,q}(\boldsymbol{R}^n)$ we get for $\varepsilon\in(0,1)$ that

$$\begin{aligned}\|\nabla u\|_{L_w^q(\Omega)}\leqq&\frac{c}{\varepsilon}(\|u\varphi\|_{L^q(G)}+\|u(1-\varphi)\|_{q,w})\\&+c\varepsilon(\|\nabla^2(u\varphi)\|_{L^q(G)}+\|\nabla^2(u(1-\varphi))\|_{q,w}).\end{aligned}$$

Obviously the term in the first brackets is bounded by $\|u\|_{L_w^q(\Omega)}$ while the second term is bounded from above by

$$c\|\nabla^2 u\|_{L_w^q(\Omega)}+c\|\nabla u\|_{L^q(G)}+c\|u\|_{L_w^q(\Omega)}.$$

Now a further application of (3.4) yields the assertion. □

Besides Theorem 3.2 on singular integrals we need estimates of weakly singular integral operators. By these estimates we will prove embedding properties of the spaces $\hat{H}_w^{1,q}(\Omega)$ and $H_w^{2,q}(\Omega)$. For $\Omega=\boldsymbol{R}^n$ and $u\in C_0^\infty(\boldsymbol{R}^n)$ the identity

$$u(x)=c_n\int_{\boldsymbol{R}^n}\frac{x-y}{|x-y|^n}\cdot\nabla u(y)dy,\quad x\in\boldsymbol{R}^n, \tag{3.5}$$

leads to the estimate $|u(x)|\leqq c_n I_1(|\nabla u|)(x)$ where I_1 denotes the fractional integral operator

$$I_1(g)(x)=\int_{\boldsymbol{R}^n}|x-y|^{1-n}g(y)dy. \tag{3.6}$$

To estimate I_1 on weighted spaces we refer to [**16**, Theorem 1 (A)] yielding the following result.

THEOREM 3.6. *Let $1<q\leqq r<\infty$, $w\in\mathcal{A}_q$ and $v\in\mathcal{A}_r$. Assume that for some $\delta>1$ there is a constant $C>0$ such that*

$$|Q|^{1/n+1/r-1/q}\Big(\fint_Q v^\delta\Big)^{1/\delta r}\Big(\fint_Q (w')^\delta\Big)^{1/\delta q'}\leqq C \tag{3.7}$$

for all cubes $Q\subset\boldsymbol{R}^n$. Then

$$I_1: L_w^q(\boldsymbol{R}^n)\to L_v^r(\boldsymbol{R}^n)$$

is a bounded linear operator.

COROLLARY 3.7. (i) *Let* $1<q<\infty$, $w\in\mathcal{A}_q$, *and for* $r\geqq q$ *let* $v\in\mathcal{A}_r$, *where*

$$v(x)=w(x)^{r/q}|x-x_0|^{-\gamma r},\quad x\in\boldsymbol{R}^n, \tag{3.8}$$

with $x_0\in\boldsymbol{R}^n$ *fixed and*

$$\gamma=n\Big(\frac{1}{n}+\frac{1}{r}-\frac{1}{q}\Big)\geqq 0. \tag{3.9}$$

Then for all $u\in\hat{H}_w^{1,q}(\boldsymbol{R}^n)$ *there exists some constant* $u_\infty\in\boldsymbol{C}$ *such that*

$$\int_{\boldsymbol{R}^n}|u-u_\infty|^r v\,dx\leqq c\int_{\boldsymbol{R}^n}|\nabla u|^q w\,dx \tag{3.10}$$

with a constant $c>0$ *independent of* u. *To be more precise, for every equivalence class* $[u]\in\hat{H}_w^{1,q}(\boldsymbol{R}^n)$ *and every representative* $u\in[u]$ *there is a unique* $u_\infty\in\boldsymbol{C}$ *satisfying* $u-u_\infty\in L_v^r(\boldsymbol{R}^n)$ *and* (3.10). *In this sense we get the embedding*

$$\hat{H}_w^{1,q}(\boldsymbol{R}^n)\subset L_v^r(\boldsymbol{R}^n).$$

The same result holds if $v=w^{r/q}|\cdot-x_0|^{-\gamma r}$ *is replaced by* $v=w^{r/q}(1+|\cdot|)^{-\gamma r}$.

(ii) *Let* $\Omega\subset\boldsymbol{R}^n$ *be an exterior domain and additionally assume that* $w\in\mathcal{A}_q(\Omega)$, $v\in\mathcal{A}_r(\Omega)$. *Then for every* $u\in\hat{H}_w^{1,q}(\Omega)$ *there exists some constant* $u_\infty\in\boldsymbol{C}$ *such that* (3.10) *holds where the domain of integration is* Ω *instead of* $\boldsymbol{R}^n$.

PROOF. In order to apply Theorem 3.6 let us check the condition (3.7). Since by assumption $w\in\mathcal{A}_q$ and $v\in\mathcal{A}_r$, Lemma 2.2 (iv) yields the existence of some $\delta>1$ such that $w'\in\mathcal{A}_{q'}$ and v satisfy the reverse Hölder condition (RH_δ). Thus the left-hand side J of (3.7) satisfies

$$J\leqq c|Q|^{1/n+1/r-1/q}\Big(\fint_Q v\Big)^{1/r}\Big(\fint_Q w'\Big)^{1/q'}.$$

Let $Q=Q_R(x)$, $R>0$. Since by assumption $\gamma\geqq 0$, inequality (2.4) and (3.8) yield with $\varepsilon=(1+1/q-1/r)^{-1}\in(0,1]$

$$\begin{aligned}J&\leqq c(\max(R,|x-x_0|)^{\varepsilon\gamma})^{1/\varepsilon}\Big(\fint_Q v\Big)^{1/r}\Big(\fint_Q w'\Big)^{1/q'}\\&\leqq c\Big(\fint_Q v^{-\varepsilon/r}w^{\varepsilon/q}\Big)^{1/\varepsilon}\Big(\fint_Q v\Big)^{1/r}\Big(\fint_Q w'\Big)^{1/q'}.\end{aligned}$$

Now by Hölder's inequality with exponents q/ε, $q/(q-\varepsilon)$, the $\mathcal{A}_q$-condition for w and the $\mathcal{A}_r$-condition for v yield

$$J\leqq c\Big(\fint_Q v^{-\varepsilon q/(r(q-\varepsilon))}\Big)^{(q-\varepsilon)/\varepsilon q}\Big(\fint_Q w\Big)^{1/q}\Big(\fint_Q v\Big)^{1/r}\Big(\fint_Q w'\Big)^{1/q'}\leqq C$$

with a constant $C>0$ independent of Q.

Thus Theorem 3.6 applied to (3.5), (3.6) yields for arbitrary $u\in C_0^\infty(\boldsymbol{R}^n)$ the estimate

$$\|u\|_{r,v}\leqq c\|\nabla u\|_{q,w} \tag{3.11}$$

with a constant $c>0$ independent of u. Let $u\in\hat{H}_w^{1,q}(\boldsymbol{R}^n)$ and let (u_k) in $C_0^\infty(\boldsymbol{R}^n)$ be a sequence such that $\|\nabla u_k-\nabla u\|_{q,w}\to 0$ as $k\to\infty$. Since by (3.11) (u_k) is a Cauchy sequence in $L_v^r(\boldsymbol{R}^n)$, there is a $\tilde{u}\in L_v^r(\boldsymbol{R}^n)$ such that $u_k\to\tilde{u}$ in $L_v^r(\boldsymbol{R}^n)$ as $k\to\infty$ and also in $L_{\mathrm{loc}}^1(\boldsymbol{R}^n)$. Thus we easily conclude that $\tilde{u}\in H_{\mathrm{loc}}^{1,1}(\boldsymbol{R}^n)$ and $\nabla\tilde{u}=\nabla u$. Hence there exists a $u_\infty\in\boldsymbol{C}$ such that $u-u_\infty=\tilde{u}\in L_v^r(\boldsymbol{R}^n)$ and satisfies (3.10).

If in (3.8) $|x-x_0|$ is replaced by $1+|x|$, we use (2.5) instead of (2.4) to prove (3.7) and proceed as before.

To prove (ii) for $u\in\hat{H}_w^{1,q}(\Omega)$ let $m\in\boldsymbol{C}$ be defined by the condition $\int_G(u-m)dx=0$ where $G=G(w)\subset\Omega$ is the bounded domain of Definition 2.5. By a well-known extension theorem and Poincaré's inequality there is an extension $\tilde{u}\in\hat{H}_w^{1,q}(\boldsymbol{R})$ of $u-m$ such that

$$\|\nabla\tilde{u}\|_{L^q(\boldsymbol{R}^n\setminus\Omega)}\leqq c\|(u-m,\ \nabla u)\|_{L^q(G)}\leqq c\|\nabla u\|_{L^q(G)}\,.$$

Then the first part of the proof yields a $\tilde{u}_\infty\in\boldsymbol{C}$ such that

$$\|\tilde{u}-\tilde{u}_\infty\|_{L_v^r(\boldsymbol{R}^n)}\leqq c\|\nabla\tilde{u}\|_{L_w^q(\boldsymbol{R}^n)}\leqq c\|\nabla u\|_{L_w^q(\Omega)}\,.$$

Thus (3.10) is proved with $u_\infty=\tilde{u}_\infty+m$. □

In the final corollary we consider explicit examples of radial weight functions $w=|\cdot|^\alpha\in\mathcal{A}_q$, $v=|\cdot|^\beta\in\mathcal{A}_r$ such that the embedding $\hat{H}_w^{1,q}(\Omega)\subset L_v^r(\Omega)$ holds.

COROLLARY 3.8. *Let $1<q<\infty$ and $q-n<\alpha<n(q-1)$. Further let $r\geqq q$ satisfy (3.9) and let $\beta\in\boldsymbol{R}$ be defined by*

$$1+\frac{\beta+n}{r}=\frac{\alpha+n}{q}\,. \tag{3.12}$$

Then, with $w(x)=|x-x_0|^\alpha$ or $w(x)=(1+|x|)^\alpha$ and $v(x)=|x-x_0|^\beta$ or $v(x)=(1+|x|)^\beta$, respectively, the embedding $\hat{H}_w^{1,q}(\boldsymbol{R}^n)\hookrightarrow L_v^r(\boldsymbol{R}^n)$ is continuous. A similar result holds for an exterior domain $\Omega\subset\boldsymbol{R}^n$ with $x_0\notin\partial\Omega$.

PROOF. By the assumptions on α and by Lemma 2.3 (v), $w\in\mathcal{A}_q$. Further $-n<\beta<(n-1)r-n<n(r-1)$ yielding $v\in\mathcal{A}_r$. Since (3.12) and (3.9) yield (3.8), Corollary 3.7 proves the assertion. For an exterior domain the condition $x_0\notin\partial\Omega$ is only needed to guarantee that $|\cdot-x_0|^\alpha\in\mathcal{A}_q(\Omega)$ and $|\cdot-x_0|^\beta\in\mathcal{A}_r(\Omega)$. □

4. The whole space problem.

We start with the investigation of the weak Laplacian

$$
\begin{aligned}
&-\Delta_{q,w}:\hat{H}_w^{1,q}(\boldsymbol{R}^n)\rightarrow\hat{H}_w^{-1,q}(\boldsymbol{R}^n),\\
&\langle-\Delta_{q,w}u,\varphi\rangle:=\int\nabla u\cdot\nabla\varphi\,dx,
\end{aligned}
\tag{4.1}
$$

for all $u\in\hat{H}_w^{1,q}(\boldsymbol{R}^n)$, $\varphi\in\hat{H}_{w'}^{1,q'}(\boldsymbol{R}^n)$ with $1<q<\infty$, $w\in\mathcal{A}_q$ and $w'=w^{-1/(q-1)}\in\mathcal{A}_{q'}$. For simplicity we write $\hat{H}_w^{1,q}(\boldsymbol{R}^n)=\hat{H}_w^{1,q}$, $L_w^q(\boldsymbol{R}^n)=L_w^q$ etc. and $\int u$ instead of $\int_{\boldsymbol{R}^n}u(x)dx$. We will need the following

LEMMA 4.1. (i) *For every $q\in(1,\infty)$ and every $w\in\mathcal{A}_q$ the space L_w^q is continuously embedded into the space S' of tempered distributions.*

(ii) *If $u\in L_w^q$ is harmonic, then $u=0$. The same result holds when $u\in L_{w_1}^{q_1}+L_{w_2}^{q_2}$, where $q_i\in(1,\infty)$ and $w_i\in\mathcal{A}_{q_i}$, $i=1, 2$.*

(iii) *The space ΔC_0^∞ is dense in L_w^q.*

PROOF. (i) Given $u\in L_w^q$ and $\varphi\in S$

$$
\begin{aligned}
|\langle u,\varphi\rangle| &\leqq \|u\|_{q,w}\|\varphi\|_{q',w'}\\
&\leqq \|u\|_{q,w}\Big(\int\frac{w'}{(1+|\cdot|)^{nq'}}\Big)^{1/q'}\|\varphi(1+|\cdot|)^n\|_\infty .
\end{aligned}
$$

Since $w'\in\mathcal{A}_{q'}$, the integral $\int w'(1+|\cdot|)^{-nq'}<\infty$ by Lemma 2.2 (iii). Furthermore $\varphi\mapsto\|\varphi(1+|\cdot|)^n\|_\infty$ is a seminorm on S. This proves that $u\in S'$ and that the embedding $L_w^q\subset S'$ is continuous.

(ii) For $w\in\mathcal{A}_q$ Lemma 2.2 (v) and (iii) applied to $w'\in\mathcal{A}_{q'}$ yield an $\varepsilon>0$ such that $\int(1+|\cdot|)^{-n(q'-\varepsilon)}w'<\infty$. Then for an harmonic $u\in L_w^q$ the mean value formula and Hölder's inequality imply that

$$
\begin{aligned}
|u(0)| &\leqq \frac{c}{R^n}\int_{B_R}|u|\,dx \leqq \frac{c}{R^n}\Big(\int_{B_R}|u|^q w\,dx\Big)^{1/q}\Big(\int_{B_R}w'dx\Big)^{1/q'}\\
&\leqq \frac{c}{R^{n\varepsilon/q'}}\|u\|_{q,w}\Big(\int_{\boldsymbol{R}^n}(1+|\cdot|)^{-n(q'-\varepsilon)}w'\Big)^{1/q'};
\end{aligned}
$$

here we used that $1+|x|\leqq 2R$ for all $x\in B_R=\{y\in\boldsymbol{R}^n;\,|y|<R\}$ and $R\geqq 1$. Letting $R\rightarrow\infty$ yields $u(0)=0$. An analogous argument is possible for arbitrary $x\in\boldsymbol{R}^n$. Thus $u\equiv 0$. If $u=u_1+u_2\in L_{w_1}^{q_1}+L_{w_2}^{q_2}$, the above estimates are applied to

$$
u(0)=\frac{c}{R^n}\int_{B_R}u_1dx+\frac{c}{R^n}\int_{B_R}u_2dx
$$

to show that $u\equiv 0$.

(iii) Let $u\in L^{q'}_{w'}=(L^q_w)^*$ satisfy $\int u\Delta\varphi=0$ for all $\varphi\in C_0^\infty$. By Weyl's lemma u is harmonic; hence $u\equiv 0$ due to part (ii) proved just before. Now Hahn-Banach's theorem implies that ΔC_0^∞ is dense in L^q_w. □

THEOREM 4.2. (i) *For all* $q\in(1, \infty)$ *and* $w\in\mathcal{A}_q$ *the operator* $-\Delta=-\Delta_{q,w}$: $\hat{H}^{1,q}_w\to\hat{H}^{-1,q}_w$ *in* (4.1) *is an isomorphism. Furthermore* $-\Delta_{q',w'}$ *coincides with the adjoint operator* $(-\Delta_{q,w})^*$ *of* $-\Delta_{q,w}$.

(ii) *If* $F\in\hat{H}^{-1,q_1}_{w_1}\cap\hat{H}^{-1,q_2}_{w_2}$ *for* $q_i\in(1, \infty)$, $w_i\in\mathcal{A}_{q_i}$, $i=1, 2$, *then the weak solution* u *of* $-\Delta u=F$ *satisfies* $u\in\hat{H}^{1,q_1}_{w_1}\cap\hat{H}^{1,q_2}_{w_2}$.

PROOF. Due to the inequality

$$|\langle-\Delta_{q,w}u, \varphi\rangle| = \left|\int\nabla u\cdot\nabla\varphi\right| \leqq \|\nabla u\|_{q,w}\|\nabla\varphi\|_{q',w'}$$

the operator $-\Delta_{q,w}$ is continuous and $\|-\Delta_{q,w}\|_{-1,q,w}\leqq\|\nabla u\|_{q,w}$ for all $u\in\hat{H}^{1,q}_w$. For the moment take the reversed inequality

$$\|\nabla u\|_{q,w} \leqq c\|-\Delta_{q,w}u\|_{-1,q,w}, \quad u\in\hat{H}^{1,q}_w, \tag{4.2}$$

for granted. This implies that $-\Delta_{q,w}$ is injective and has a closed range. By symmetry $(-\Delta_{q,w})^*=-\Delta_{q',w'}$ and $-\Delta_{q',w'}$ satisfies an inequality analogous to (4.2). Thus also $-\Delta_{q',w'}$ is injective with closed range. Consequently $-\Delta_{q,w}$ is an isomorphism by the closed range theorem.

To prove (4.2) recall that in terms of Fourier transforms

$$\partial_i\partial_j\varphi = \mathcal{F}^{-1}(m\widehat{\Delta\varphi}), \quad m(\xi) = \frac{\xi_i\xi_j}{|\xi|^2},$$

for $\varphi\in C_0^\infty$, $i, j=1, \cdots, n$. Then Theorem 3.3 yields a constant $c>0$ such that

$$\|\nabla^2\varphi\|_{q',w'} \leqq c\|\Delta\varphi\|_{q',w'} \quad \text{for all } \varphi\in C_0^\infty.$$

Hence for all $0\neq\varphi\in C_0^\infty$, $i=1, \cdots, n$,

$$\|-\Delta_{q,w}u\|_{-1,q,w} \geqq \frac{|\langle-\Delta_{q,w}u, \partial_i\varphi\rangle|}{\|\nabla\partial_i\varphi\|_{q',w'}} \geqq \frac{1}{c}\frac{\left|\int\partial_i u\Delta\varphi\right|}{\|\Delta\varphi\|_{q',w'}}.$$

Since by Lemma 4.1 (iii) ΔC_0^∞ is dense in $L^{q'}_{w'}$, we conclude that $\|\partial_i u\|_{q,w}\leqq c\|-\Delta_{q,w}u\|_{-1,q,w}$. This proves (4.2).

(ii) Given $F\in\hat{H}^{-1,q_1}_{w_1}\cap\hat{H}^{-1,q_2}_{w_2}$ there are solutions $u_i\in\hat{H}^{1,q_i}_{w_i}$, $i=1, 2$, of $-\Delta u_i=F$. Then $\Delta(u_1-u_2)=0$ in the weak sense, and u_1-u_2 as well as $\nabla u_1-\nabla u_2\in(L^{q_1}_{w_1})^n+(L^{q_2}_{w_2})^n$ are harmonic. Consequently $\nabla u_1-\nabla u_2=0$ by Lemma 4.1 (ii) yielding a unique solution $u=u_1=u_2\in\hat{H}^{1,q_1}_{w_1}\cap\hat{H}^{1,q_2}_{w_2}$ of $-\Delta u=F$. □

COROLLARY 4.3. *For all* $q\in(1, \infty)$ *and* $w\in\mathcal{A}_q$ *the space* C_0^∞ *is dense in* $\dot{H}^{1,q}_w$; *thus* $\hat{H}^{1,q}_w=\dot{H}^{1,q}_w$.

PROOF. Given $u\in\dot{H}^{1,q}_w$ define the functional $F\in\hat{H}^{-1,q}_w$ by $\langle F, \varphi\rangle=\int\nabla u\cdot\nabla\varphi$ for $\varphi\in C_0^\infty\subset\hat{H}^{1,q'}_{w'}$. By Theorem 4.2 there exists a unique $\tilde{u}\in\hat{H}^{1,q}_w$ such that $\int\nabla\tilde{u}\cdot\nabla\varphi=\int\nabla u\cdot\nabla\varphi$ for all $\varphi\in C_0^\infty$. Thus $u-\tilde{u}$ and also $\nabla u-\nabla\tilde{u}\in(L^q_w)^n$ are harmonic. Then Lemma 4.1 (ii) yields $\nabla u=\nabla\tilde{u}$ proving that $\hat{H}^{1,q}_w=\dot{H}^{1,q}_w$. □

COROLLARY 4.4. (i) *Let $q\in(1, \infty)$ and $w\in\mathcal{A}_q$. Then $(L^q_w)^n$ has an algebraic and topological decomposition*

$$(L^q_w)^n = L^q_{w,\sigma}\oplus\nabla\hat{H}^{1,q}_w,\quad L^q_{w,\sigma} = \overline{C^\infty_{0,\sigma}(\boldsymbol{R}^n)}^{\|\cdot\|_{q,w}},$$

where $C^\infty_{0,\sigma}(\boldsymbol{R}^n)=\{u\in C_0^\infty(\boldsymbol{R}^n)^n ; \operatorname{div} u=0\}$. In particular there exists a bounded projection operator $P_{q,w}:(L^q_w)^n\rightarrow L^q_{w,\sigma}$ with kernel $\nabla\hat{H}^{1,q}_w$ and range $L^q_{w,\sigma}$.

(ii) *$(P_{q,w})^*=P_{q',w'}$ and $(L^q_{w,\sigma})^*=L^{q'}_{w',\sigma}$.*

(iii) *If $u\in(L^{q_1}_{w_1})^n\cap(L^{q_2}_{w_2})^n$ for $q_i\in(1, \infty)$, $w_i\in\mathcal{A}_{q_i}$, $i=1, 2$, then $P_{q_1,w_1}u=P_{q_2,w_2}u$.*

We omit the proof of Corollary 4.4 since it is based on Theorem 4.2 and parallels the proof of Theorem 1.3 about the Helmholtz decomposition in an exterior domain, see Section 5.

Now we pass to the *generalized Stokes resolvent problem*

$$\lambda u-\Delta u+\nabla p = f,\quad \operatorname{div} u=g \quad \text{on } \boldsymbol{R}^n \tag{4.3}$$

in L^q_w, $1<q<\infty$, $w\in\mathcal{A}_q$, where $f\in(L^q_w)^n$. Assuming $u\in(H^{2,q}_w)^n$ and $p\in\hat{H}^{1,q}_w$, we get that $g\in H^{1,q}_w$. Furthermore the estimate

$$|\langle g, \varphi\rangle| = \left|-\int u\cdot\nabla\varphi\right| \leqq \|u\|_{q,w}\|\nabla\varphi\|_{q',w'},\quad \varphi\in C_0^\infty,$$

implies that necessarily $g\in\hat{H}^{-1,q}_w$ and $\|g\|_{-1,q,w}\leqq\|u\|_{q,w}$.

THEOREM 4.5. (i) *Let $q\in(1, \infty)$, $w\in\mathcal{A}_q$ and $\varepsilon\in(0, \pi/2)$. Then for every $f\in(L^q_w)^n$, $g\in H^{1,q}_w\cap\hat{H}^{-1,q}_w$ and $\lambda\in S_\varepsilon$ problem (4.3) has a unique solution $(u, p)\in(H^{2,q}_w)^n\times\hat{H}^{1,q}_w$. This solution satisfies the a priori estimates*

$$\|(\lambda u, \nabla^2 u, \nabla p)\|_{q,w} \leqq C_\varepsilon(\|f, \nabla g)\|_{q,w}+\|\lambda g\|_{-1,q,w}), \tag{4.4}$$

$$\|(\lambda u, -\Delta u+\nabla p)\|_{q,w} \leqq C_\varepsilon(\|f\|_{q,w}+\|\lambda g\|_{-1,q,w}) \tag{4.5}$$

with a constant $C_\varepsilon>0$ independent of f, g, λ, u and p.

(ii) *If additionally $f\in(L^{\bar{q}}_{\bar{w}})^n$ and $g\in H^{1,\bar{q}}_{\bar{w}}\cap\hat{H}^{-1,\bar{q}}_{\bar{w}}$ for some $\bar{q}\in(1, \infty)$ and $\bar{w}\in\mathcal{A}_{\bar{q}}$, then also $u\in(H^{2,\bar{q}}_{\bar{w}})^n$ and $p\in\hat{H}^{1,\bar{q}}_{\bar{w}}$.*

PROOF. (i) Since $g\in\hat{H}^{-1,q}_w$, Theorem 4.2 yields a unique $P\in\hat{H}^{1,q}_w$ such that $\Delta_{q,w}P=g$ and $\|\nabla P\|_{q,w}\leqq c\|g\|_{-1,q,w}$. To see that $\nabla P\in(H^{2,q}_w)^n$ when $g\in H^{1,q}_w\cap\hat{H}^{-1,q}_w$, let $P_i\in\hat{H}^{1,q}_w$ be the solution of $\Delta P_i=-\partial_i g$, $i=1, \cdots, n$; note that $\partial_i g\in$

$\hat{H}_w^{-1,q}$ and $\|\partial_i g\|_{-1,q,w} \leqq \|g\|_{q,w}$. Thus for all $\varphi \in C_0^\infty$

$$\int P_i \Delta\varphi = -\int \nabla P_i \cdot \nabla\varphi = -\int \partial_i g \ \varphi = \int g \partial_i \varphi$$
$$= \int \nabla P \cdot \nabla \partial_i \varphi = \int \partial_i P \cdot \Delta\varphi .$$

Since by Lemma 4.1 (iii) ΔC_0^∞ is dense in L_w^q, we conclude that $P_i = \partial_i P \in L_w^q$, $\nabla \partial_i P = \nabla P_i \in (L_w^q)^n$ and finally $\|\nabla^2 P\|_{q,w} \leqq c\|g\|_{q,w}$. Analogously $\nabla^3 P \in (L_w^q)^{n^3}$ and $\|\nabla^3 P\|_{q,w} \leqq c\|\nabla g\|_{q,w}$. Thus $u_g := \nabla P \in (H_w^{2,q})^n$, $\operatorname{div} u_g = g$ and

$$\|u_g\|_{q,w} \leqq c\|g\|_{-1,q,w}, \quad \|\nabla^2 u_g\|_{q,w} \leqq c\|\nabla g\|_{q,w} .$$

Next Theorem 4.2 yields a unique solution $p \in \hat{H}_w^{1,q}$ of the equation $-\Delta p = -\operatorname{div} f + (\lambda - \Delta) g \in \hat{H}_w^{-1,q}$ satisfying

$$\|\nabla p\|_{q,w} \leqq c(\|f\|_{q,w} + \|\lambda g\|_{-1,q,w} + \|\nabla g\|_{q,w})$$

since $\|\operatorname{div} f\|_{-1,q,w} \leqq \|f\|_{q,w}$ etc..

Finally we solve the problem

$$(\lambda - \Delta) v = F \quad \text{with } F = f - (\lambda - \Delta) u_g - \nabla p \in (L_w^q)^n. \tag{4.6}$$

Using Fourier transform and considering only F in the dense subspace $S(\boldsymbol{R}^n)^n$ of $(L_w^q)^n$, we get that

$$\lambda \hat{v} = m(\xi) \hat{F}, \quad \widehat{\sqrt{|\lambda|} \partial_j v} = m_j(\xi) \hat{F}, \quad \widehat{\partial_j \partial_k v} = m_{jk}(\xi) \hat{F}$$

with the multiplier functions

$$m(\xi) = \frac{\lambda}{\lambda + |\xi|^2}, \quad m_j(\xi) = \frac{i\xi_j \sqrt{|\lambda|}}{\lambda + |\xi|^2}, \quad m_{jk}(\xi) = \frac{-\xi_j \xi_k}{\lambda + |\xi|^2},$$

$1 \leqq j, k \leqq n$. To apply Theorem 3.3 note that these multiplier functions are in $L^\infty(\boldsymbol{R}^n)$ and in $C^\infty(\boldsymbol{R}^n \setminus \{0\})$. Concerning the condition (3.2) for m_{jk}, say, let $\omega = \lambda/|\lambda|$, $\xi' = \xi/|\lambda|$. Since $|\arg \omega| < \pi - \varepsilon$ and $|\omega| = 1$, there is a constant $c_\varepsilon > 0$ such that $1 + |\xi'|^2 \leqq c_\varepsilon |\omega + |\xi'|^2|$ for all $\xi' \in \boldsymbol{R}^n$. Thus

$$|m_{jk}(\xi)| = \frac{|\xi'_j| |\xi'_k|}{|\omega + |\xi'|^2|} \leqq C(\varepsilon), \quad \xi \in \boldsymbol{R}^n,$$

with a constant $C(\varepsilon) > 0$ independent of $\lambda \in S_\varepsilon$. Analogously (3.2) is proved for derivatives of m_{jk}. Hence the conditions (3.2) and also (3.1) are satisfied with constants on the right-hand side which are independent of $\lambda \in S_\varepsilon$. Now Theorem 3.3 yields a solution $v \in (H_w^{2,q})^n$ of (4.6) satisfying $\|(\lambda v, \sqrt{|\lambda|} \nabla v, \nabla^2 v)\|_{q,w} \leqq C_\varepsilon \|F\|_{q,w}$, if $F \in S(\boldsymbol{R}^n)^n$. Since S is dense in L_w^q, this result is easily extended to arbitrary $F \in (L_w^q)^n$, and we conclude that

$$\|(\lambda v, \sqrt{|\lambda|}\nabla v, \nabla^2 v)\|_{q,w} \leqq C_\varepsilon(\|(f, \nabla g)\|_{q,w}+\|\lambda g\|_{-1,q,w}).$$

Further an investigation of the equation $(\lambda-\Delta)h=\operatorname{div} F=0$ with $h=\operatorname{div} v\in L^q_w$ in S' implies that $\operatorname{div} v=0$.

Summarizing we see that $(u=v+u_g, p)\in(H^{2,q}_w)^n\times\hat{H}^{1,q}_w$ is a solution of (4.3) satisfying the estimate (4.4). To prove uniqueness let (u, p) be a solution of $(\lambda-\Delta)u+\nabla p=0$ and $\operatorname{div} u=0$. Since $u, \nabla p\in S'(\boldsymbol{R}^n)^n$ Fourier analysis and Lemma 4.1 immediately yield $u=\nabla p=0$.

To prove the estimate (4.5) let $f'\in(L^{q'}_{w'})^n=((L^q_w)^n)^*$ and let $(u', p')\in(H^{2,q'}_{w'})^n \times\hat{H}^{1,q'}_{w'}$ be the solution of $(\lambda-\Delta)u'+\nabla p'=f'$, $\operatorname{div} u'=0$ satisfying an estimate analogous to (4.4). Since C_0^∞ is dense in $H^{2,q}_w$, $H^{2,q'}_{w'}$ and in $\hat{H}^{1,q}_w$, $\hat{H}^{1,q'}_{w'}$, integration by parts yields

$$\langle u, f'\rangle=\langle u, \lambda u'-\Delta u'+\nabla p'\rangle=\langle f, u'\rangle-\langle g, p'\rangle.$$

Thus

$$\begin{aligned}|\langle u, f'\rangle| &\leqq \|f\|_{q,w}\|u'\|_{q',w'}+\|g\|_{-1,q,w}\|\nabla p'\|_{q',w'}\\ &\leqq \frac{C_\delta}{|\lambda|}(\|f\|_{q,w}+\|\lambda g\|_{-1,q,w})\|f'\|_{q',w'}.\end{aligned}$$

Since f' was arbitrary, we are led to (4.5).

(ii) Assuming $f\in(L^q_w)^n\cap(L^{\bar q}_{\bar w})^n$ and $g\in H^{1,q}_w\cap H^{1,\bar q}_{\bar w}\cap\hat{H}^{-1,q}_w\cap\hat{H}^{-1,\bar q}_{\bar w}$ there are solutions $(u, p)\in(H^{2,q}_w)^n\times H^{1,q}_w$ and $(\bar u, \bar p)\in(H^{2,\bar q}_{\bar w})^n\times\hat{H}^{1,\bar q}_{\bar w}$. Consequently $v=u-\bar u \in S'(\boldsymbol{R}^n)^n$ and $\nabla p-\nabla\bar p\in S'(\boldsymbol{R}^n)^n$ solve the homogeneous equation $(\lambda-\Delta)v+(\nabla p-\nabla\bar p)=0$, $\operatorname{div} v=0$. Again Fourier analysis and Lemma 4.1 yield $v=\nabla p-\nabla\bar p =0$. Now Theorem 4.5 is completely proved. □

Obviously Theorem 1.1 is a particular case of Theorem 4.5. Analogously Theorem 1.4 (i) and (ii) are easy consequences. For results on analytic semigroups see [**6**].

PROOF OF THEOREM 1.4 (iii). For $z\in\boldsymbol{C}$, $|Re\, z|<1$, the fractional power A^z of the Stokes operator $A=A_{q,w}$ is a closed operator defined by

$$A^z f=\frac{\sin\pi z}{\pi}\Big(\int_0^1\lambda^{z+1}(\lambda+A)^{-1}g\,d\lambda-\frac{1}{z+1}g+\frac{1}{z}f-\frac{1}{z-1}Af -\int_1^\infty\lambda^{z-2}A(\lambda+A)^{-1}Af\,d\lambda\Big) \tag{4.7}$$

for $f=Ag\in\mathcal{D}(A)\cap A\mathcal{D}(A)$ and $g\in\mathcal{D}(A)$. Furthermore

$$A^z f=\frac{\sin\pi z}{\pi}\int_0^\infty\lambda^{z-1}(\lambda+A)^{-1}Af\,d\lambda \tag{4.8}$$

for $f\in\mathcal{D}(A)$, $0<Re\, z<1$; see [**10, 12**]. By (4.5) all integrals converge absolutely in $L^q_{w,\sigma}$. Let $f\in\Delta C^\infty_{0,\sigma}\subset\mathcal{D}(A)\cap A\mathcal{D}(A)$ which is dense in $L^q_{w,\sigma}$. Using Fourier transform in S we get for $0<Re\,z<1$ that $A^z f=\mathcal{F}^{-1}(m\hat{f})$ where

$$m(\xi)=m(z\,;\,\xi)=\frac{\sin\pi z}{\pi}\int_0^\infty\lambda^{z-1}\frac{|\xi|^2}{\lambda+|\xi|^2}d\lambda=|\xi|^{2z}.$$

Since (4.7), (4.8) are analytic in z we conclude that

$$A^{it}f=\mathcal{F}^{-1}(|\xi|^{2it}\hat{f}(\xi))$$

for $f\in\mathcal{D}(A)\cap A\mathcal{D}(A)$, $t\in\boldsymbol{R}$. Now the estimate

$$|\xi|^k|\nabla^k|\xi|^{2it}|\leqq c_k(1+|t|)^k\quad\text{for } 0\neq\xi\in\boldsymbol{R}^n,\ t\in\boldsymbol{R},\ k\in\boldsymbol{N},$$

and Theorem 3.3 yield $\|A^{it}f\|_{q,w}\leqq c(1+|t|)^n\|f\|$. In particular A^{it} is a bounded operator on $L^q_{w,\sigma}$ satisfying $\|A^{it}\|\leqq ce^{\delta|t|}$ for every $\delta>0$ with a constant $c=c(\delta)>0$. □

5. The exterior domain problem.

Let $\Omega\subset\boldsymbol{R}^n$ be an exterior domain with at least $C^{1,1}$-boundary $\partial\Omega$. From Sections 2 and 3 recall the Definitions 2.1, 2.5 of weights $w\in\mathcal{A}_q$ or $w\in\mathcal{A}_q(\Omega)$, $1\leqq q<\infty$, and of the weighted spaces $L^q_w(\Omega)$ with norm $\|\cdot\|_{q,w}$, $H^{k,q}_w(\Omega)$, $H^{k,q}_{0,w}(\Omega)$, $\dot{H}^{1,q}_w(\Omega)$, $\hat{H}^{1,q}_w(\Omega)$ and $\hat{H}^{-1,q}_w(\Omega)$ with norm $\|\cdot\|_{-1,q,w}$. In this section the integral $\int_\Omega u(x)dx$ is also written in the form $\int u$.

LEMMA 5.1. *Let $1<q<\infty$ and $w\in\mathcal{A}_q(\Omega)$. Then $\dot{H}^{1,q}_w(\Omega)=\hat{H}^{1,q}_w(\Omega)$, i.e., $C^\infty_0(\bar{\Omega})$ is dense in $\dot{H}^{1,q}_w(\Omega)$.*

PROOF. Let $u\in\dot{H}^{1,q}_w(\Omega)$, i.e., $u\in L^q_{\mathrm{loc}}(\bar{\Omega})$ and $\nabla u\in L^q_w(\Omega)^n$. Since $w>0$ on some suitable bounded domain G, see Definition 2.5, and consequently $u\in Hu^{1,q}(G)$, there is an extension $\tilde{u}\in\dot{H}^{1,q}_w(\boldsymbol{R}^n)$ of u. Then Corollary 4.3 yields a sequence (u_k) in $C^\infty_0(\boldsymbol{R}^n)$ such that $u_k\to\tilde{u}$ in $\dot{H}^{1,q}_w(\boldsymbol{R}^n)$. Now the sequence $(u_k|_{\bar{\Omega}})$ in $C^\infty_0(\bar{\Omega})$ implies that $u\in\hat{H}^{1,q}_w(\Omega)$. □

To construct the Helmholtz decomposition in weighted spaces on the exterior domain $\Omega\subset\boldsymbol{R}^n$ we first investigate the weak Neumann problem.

THEOREM 5.2. (i) *Let $1<q<\infty$ and $w\in\mathcal{A}_q(\Omega)$. Then for every $F\in\hat{H}^{-1,q}_w(\Omega)$ the weak Neumann problem*

$$\int_\Omega\nabla p\cdot\nabla\varphi\,dx=\langle F,\varphi\rangle,\quad\varphi\in\hat{H}^{1,q'}_{w'}(\Omega),\tag{5.1}$$

has a unique solution $p\in\hat{H}^{1,q}_w(\Omega)$. Furthermore

$$\|\nabla p\|_{q,w}\leqq c\|F\|_{-1,q,w}\tag{5.2}$$

with a constant $c=c(\Omega, q, w)>0$.

(ii) *If* $F\in\hat{H}_{w_1}^{-1,q_1}(\Omega)\cap\hat{H}_{w_2}^{-1,q_2}(\Omega)$ *for weights* $w_i\in\mathcal{A}_{q_i}(\Omega)$, $q_i\in(1, \infty)$, $i=1, 2$, *then the weak solution* u *of* (5.1) *satisfies* $u\in\hat{H}_{w_1}^{1,q_1}(\Omega)\cap\hat{H}_{w_2}^{1,q_2}(\Omega)$.

PROOF. Given $F\in\hat{H}_w^{-1,q}(\Omega)$ and $p\in\hat{H}_w^{1,q}(\Omega)$ satisfying (5.1) we first prove the preliminary estimate

$$\|\nabla p\|_{q,w}\leqq c(\|F\|_{-1,q,w}+\|p\|_{q,G}) \tag{5.3}$$

where $G=G(w)$ is the bounded domain associated with the weight $w\in\mathcal{A}_q(\Omega)$.

Choose cut-off functions $\psi_1, \psi_2\in C^\infty(\boldsymbol{R}^n; [0, 1])$ such that $\psi_1+\psi_2=1$ and

$$\psi_1(x)=\begin{cases}1\\0\end{cases}\text{ in a neighborhood of }\begin{cases}\Omega\cap(\boldsymbol{R}^n\setminus G)\\ \boldsymbol{R}^n\setminus\Omega\end{cases}.$$

Thus $\operatorname{supp}\nabla\psi_i\subset G$, $i=1, 2$, is compact. Consider a test function $\varphi\in C_0^\infty(\boldsymbol{R}^n)$ and define $\tilde{\varphi}=\varphi-|G|^{-1}\int_G\varphi\,dx$. Note that Poincaré's inequality yields $\|\tilde{\varphi}\|_{q,G}\leqq c\|\nabla\varphi\|_{q,G}$. Since obviously $p\psi_1\in\hat{H}_w^{1,q}(\boldsymbol{R}^n)$ and

$$\begin{aligned}&\int_{\boldsymbol{R}^n}\nabla(p\psi_1)\cdot\nabla\varphi\,dx\\ &=\int_\Omega\nabla p\cdot\nabla(\psi_1\tilde{\varphi})dx-\int_\Omega\nabla p\cdot\tilde{\varphi}\nabla\psi_1dx+\int_\Omega p\nabla\psi_1\cdot\nabla\varphi\,dx\\ &=\int_\Omega\nabla p\cdot\nabla(\psi_1\tilde{\varphi})dx+\int_G p\operatorname{div}(\tilde{\varphi}\nabla\psi_1)dx+\int_G p\nabla\psi_1\cdot\nabla\varphi\,dx,\end{aligned} \tag{5.4}$$

the inequality (4.2) yields

$$\|\nabla(p\psi_1)\|_{q,w,\boldsymbol{R}^n}\leqq c(\|F\|_{-1,q,w}+\|p\|_{q,G}).$$

Analogously (5.4) holds with ψ_1 replaced by ψ_2 and $\int_{\boldsymbol{R}^n}\nabla(p\psi_1)\cdot\nabla\varphi\,dx$ replaced by $\int_G\nabla(p\psi_2)\cdot\nabla\varphi\,dx$ for all $\varphi\in C^\infty(\bar{G})$. Then a well-known variational inequality on $H^{1,q}(G)$, see [17], yields

$$\|\nabla(p\psi_2)\|_{q,G}\leqq c(\|F\|_{-1,q,w}+\|p\|_{q,G}).$$

This proves (5.3).

Assume that (5.2) is not true. Then there is a sequence (p_k) in $\hat{H}_w^{1,q}(\Omega)$ with $\|\nabla p_k\|_{q,w}=1$, but $\|F_k\|_{-1,q,w}\to 0$ as $k\to\infty$; here F_k is defined by (5.1). Since $\hat{H}_w^{1,q}(\Omega)$ is separable and reflexive, we may assume without loss of generality that (p_k) converges weakly in $\hat{H}_w^{1,q}(\Omega)$ to some $p\in\hat{H}_w^{1,q}(\Omega)$. Obviously

$$\int_\Omega\nabla p\cdot\nabla\varphi\,dx=0\quad\text{for all }\varphi\in\hat{H}_{w'}^{1,q'}(\Omega). \tag{5.5}$$

Below we will show that $p=0$ is the unique solution of (5.5) in $\hat{H}_w^{1,q}(\Omega)$.

Taking this result for granted we are led to a contradiction as follows. Since the functions p_k are uniquely determined only up to constants we may assume that $\int_G p_k dx=0$ for all $k\in\boldsymbol{N}$ and $\int_G p\,dx=0$. Thus due to Poincaré's inequality (p_k) is a bounded sequence in $H^{1,q}(G)$, and owing to the compact embedding $H^{1,q}(G)\subset L^q(G)$ we may assume that $p_k\to p=0$ in $L^q(G)$. Then (5.3) yields the contradiction $1\leq c\|p\|_{q,G}=0$. This proves (5.2).

It remains to prove that (5.5) yields $p=0$. Since $-\Delta p=0$ in the sence of distributions and $N\cdot\nabla p|_{\partial\Omega}=0$ where N denotes the normal vector on $\partial\Omega$, $p\in H^{2,r}(G)$ for all $1<r<\infty$ due to the local regularity theory. Further for all $\varphi\in C_0^\infty(\boldsymbol{R}^n)$ and $1<r<\infty$

$$\left|\int_{\boldsymbol{R}^n}\nabla(p\phi_1)\cdot\nabla\varphi\,dx\right| = \left|\int_G p\,\mathrm{div}(\tilde{\varphi}\nabla\phi_1)dx+\int_G p\nabla\phi_1\cdot\nabla\varphi\,dx\right|$$
$$\leq c(r,\ p)\|\nabla\varphi\|_{r',\boldsymbol{R}^n};$$

whence $-\Delta_{r,1}(p\phi_1)\in\hat{H}_1^{-1,r}(\boldsymbol{R}^n)$, and Theorem 4.2 (ii) yields $p\phi_1\in\hat{H}^{1,2}(\boldsymbol{R}^n)$. Thus $p\in\hat{H}^{1,2}(\Omega)$, and inserting $\varphi=p$ in (5.5) implies that $\nabla p=0$. Hence up to a constant $p=0$.

So far we proved the a priori estimate (5.2). This means that the weak Neumann operator

$$N_{q,w}:\hat{H}_w^{1,q}(\Omega)\to\hat{H}_w^{-1,q}(\Omega),\quad \langle N_{q,w}p,\ \varphi\rangle=\int\nabla p\cdot\nabla\varphi$$

is injective and has a closed range. Due to symmetry we easily see that $(N_{q,w})^*=N_{q',w'}$. Then the closed range theorem implies that $N_{q,w}$ is an isomorphism.

The assertion (ii) is an easy consequence of Theorem 4.2 (ii) and the local regularity theory. □

PROOF OF THEOREM 1.3. Let $1<q<\infty$, $w\in\mathcal{A}_q(\Omega)$ and let $u\in L_w^q(\Omega)^n$ be given. By Theorem 5.2 there is a unique $p\in\hat{H}_w^{1,q}(\Omega)$ such that

$$\int\nabla p\cdot\nabla\varphi=\int u\cdot\nabla\varphi\quad\text{for all }\varphi\in\hat{H}_{w'}^{1,q'}(\Omega).$$

Then the Helmholtz projection $P_{q,w}u$ of u is defined by

$$P_{q,w}u=u-\nabla p\in L_w^q(\Omega)^n.$$

Obviously $P_{q,w}:L_w^q(\Omega)^n\to L_w^q(\Omega)^n$ is a linear continuous projection with range

$$\mathcal{R}_{q,w}=\mathcal{R}(P_{q,w})=\left\{v\in L_w^q(\Omega)^n;\ \int v\cdot\nabla\varphi=0\text{ for all }\varphi\in\hat{H}_{w'}^{1,q'}(\Omega)\right\}$$

and kernel $\nabla\hat{H}^{1,q}_{w}(\Omega)$. Further (5.2) yields the estimate

$$\|\nabla p\|_{q,w} \leqq c\left\|\int u\cdot\nabla(\cdot)\right\|_{-1,q,w} \leqq c\|u\|_{q,w}.$$

Let us prove that $(\mathcal{R}_{q,w})^*=\mathcal{R}_{q',w'}$. Since $\mathcal{R}_{q',w'}\subset(\mathcal{R}_{q,w})^*$ is trivial, choose any $\phi\in(\mathcal{R}_{q,w})^*$. By Hahn-Banach's theorem ϕ has an extension to a continuous linear functional on $L^q_w(\Omega)^n$. Thus there is a $u\in L^{q'}_{w'}(\Omega)^n$ such that $\langle\phi, v\rangle=\int_\Omega u\cdot v\,dx$ for all $v\in\mathcal{R}_{q,w}$. Using the Helmholtz decomposition $u=P_{q',w'}u+\nabla p$ with $p\in\hat{H}^{1,q'}_{w'}(\Omega)$ we get that

$$\langle\phi, v\rangle=\int P_{q',w'}u\cdot v+\int\nabla p\cdot v=\int P_{q,w}u\cdot v$$

for all $v\in\mathcal{R}_{q,w}$. Thus ϕ may be identified with $P_{q',w'}u\in\mathcal{R}_{q',w'}$.

The assertion $(P_{q,w})^*=P_{q',w'}$ follows immediately from the identity

$$\int v\cdot(P_{q,w})^*u=\int(P_{q,w}v)\cdot u=\int P_{q,w}v\cdot P_{q',w'}u=\int v\cdot P_{q',w'}u$$

for all $v\in L^q_w(\Omega)^n$, $u\in L^{q'}_{w'}(\Omega)^n$.

To show that $C^\infty_{0,\sigma}(\Omega)$ is dense in $\mathcal{R}_{q,w}$ let $\phi\in\mathcal{R}_{q',w'}=(\mathcal{R}_{q,w})^*$ vanish on $C^\infty_{0,\sigma}(\Omega)$, i.e.,

$$\int\phi\cdot\varphi=0 \quad \text{for all } \varphi\in C^\infty_{0,\sigma}(\Omega).$$

By de Rham's well known argument [**15**] we conclude that there is some $p\in\hat{H}^{1,q'}_{w'}(\Omega)$ such that $\phi=\nabla p$. For an elementary proof see [**17**, Lemma 2.1]. Thus

$$\int\phi\cdot u=\int\nabla p\cdot u=0 \quad \text{for all } u\in\mathcal{R}_{q,w}.$$

Hence $C^\infty_{0,\sigma}(\Omega)$ is dense in $\mathcal{R}_{q,w}$, i.e., $\mathcal{R}_{q,w}=L^q_{w,\sigma}(\Omega)$.

The assertion (ii) is an easy consequence of Theorem 5.2 (ii). □

The next lemma yields a regularity property of the Helmholtz decomposition which is needed later on.

LEMMA 5.3. *Let $1<q<\infty$, $w\in\mathcal{A}_q(\Omega)$ and $f\in L^q_w(\Omega)^n$ satisfying $\nabla\operatorname{div}f\in L^q_w(\Omega)^n$ and $N\cdot f=0$ on $\partial\Omega$; here N denotes the exterior normal vector on $\partial\Omega$. Further let $f=f_0+\nabla p$ with $f_0\in L^q_{w,\sigma}(\Omega)$, $p\in\hat{H}^{1,q}_w(\Omega)$ be the Helmholtz decomposition of f. Then $\nabla^2p\in L^q_w(\Omega)^{n^2}$ and $f\in L^q_w(\Omega)$.*

PROOF. Due to the proof of Theorem 1.3 we know that p is the weak solution of the Neumann problem

$$\Delta p=\operatorname{div} f\in\hat{H}_w^{-1,q}(\Omega),\quad N\cdot\nabla p=0 \text{ on } \partial\Omega.$$

Since $\nabla \operatorname{div} f\in L_w^q(\Omega)^n$ and $w\in\mathcal{A}_q(\Omega)$, $\operatorname{div} f\in L^q(G)$ and the boundary condition $N\cdot\nabla p=N\cdot f=0$ is well defined in the weak sense. Thus elliptic regularity theory yields $p\in H_{\mathrm{loc}}^{2,q}(G\cup\partial\Omega)$; see [**5**] for details when the boundary is only of class $C^{1,1}$. Recall the cut-off function $\phi_1\in C^\infty(\boldsymbol{R}^n;[0,1])$ from the proof of Theorem 5.2 satisfying $\phi_1=0$ in $\boldsymbol{R}^n\backslash\Omega$ and $\operatorname{supp}\nabla\phi_1\subset G$. Then $u=\nabla(p\phi_1)\in L_w^q(\boldsymbol{R}^n)^n$ satisfies

$$\Delta u=\phi_1\nabla \operatorname{div} f+(\nabla\phi_1)\operatorname{div} f+\nabla(2\nabla\phi_1\cdot\nabla p+p\Delta\phi_1)$$

on $\boldsymbol{R}^n$. By assumption and the previous results $\Delta u\in L_w^q(\boldsymbol{R}^n)^n$. Next consider the resolvent problem

$$(I-\Delta)v=u-\Delta u\in L_w^q(\boldsymbol{R}^n)^n.$$

Using multiplier theory in the same way as in Section 4 for the corresponding Stokes resolvent we get that this equation has a unique solution $v\in H_w^{2,q}(\boldsymbol{R}^n)^n$. Since Fourier analysis shows that the equation $(I-\Delta)w=0$ has only the trivial solution $w=0$ in $L_w^q(\boldsymbol{R}^n)\subset S'(\boldsymbol{R}^n)$, we conclude that $u=v\in H_w^{2,q}(\boldsymbol{R}^n)^n$. In particular $\nabla^2 p\in L_w^q(\Omega)^{n^2}$ and consequently $\operatorname{div} f\in L_w^q(\Omega)$. □

As for the whole space problem we consider the generalized Stokes resolvent problem where $g=\operatorname{div} u$ may be non-zero. Again $u\in H_w^{2,q}(\Omega)^n\cap H_{0,w}^{1,q}(\Omega)^n$ yields $g\in H_w^{1,q}(\Omega)\cap\hat{H}_w^{-1,q}(\Omega)$ and $\|g\|_{-1,q,w}\leq\|u\|_{q,w}$.

For a bounded domain G we define the space $\hat{H}^{1,q}(G)=H^{1,q}(G)/\boldsymbol{C}$ as quotient space with norm $\|\nabla\cdot\|_q$, and we let

$$\hat{H}^{-1,q}(G)=(\hat{H}^{1,q'}(G))^*,\quad q'=\frac{q}{q-1},$$

be the dual space. If $g\in H^{1,q}(G)$ with $\int_G g\,dx=0$, then the mapping $\varphi\mapsto\int_G g\varphi\,dx$, $\varphi\in\hat{H}^{1,q'}(G)$, defines an element of $\hat{H}^{-1,q}(G)$; we simply write $g\in H^{-1,q}(G)$ for this functional. Thus $\{g\in H^{1,q}(G);\int_G g\,dx=0\}\subset\hat{H}^{-1,q}(G)$ and even

$$H^{1,q}(G)\cap\hat{H}^{-1,q}(G)=\left\{g\in H^{1,q}(G);\int_G g\,dx=0\right\}.$$

LEMMA 5.4. (i) *Let $G\subset\boldsymbol{R}^n$ be a bounded domain with boundary of class $C^{1,1}$ and let $1<q<\infty$, $0<\varepsilon<\pi/2$. Then for every $f\in L^q(G)^n$, $g\in H^{1,q}(G)\cap\hat{H}^{-1,q}(G)$ and $\lambda\in S_\varepsilon$ the Stokes resolvent problem*

$$\lambda u-\Delta u+\nabla p=f,\quad \operatorname{div} u=g \text{ in } G,\quad u=0 \text{ on } \partial G$$

has a unique solution $(u,p)\in H^{2,q}(G)^n\times\hat{H}^{1,q}(G)$. Further

$$\|(\lambda u,\nabla^2 u,\nabla p)\|_{q,G}\leq C_\varepsilon(\|(f,\nabla g)\|_q+\|\lambda g\|_{\hat{H}^{-1,q}(G)})$$

with a constant $C_\varepsilon > 0$ *independent of* f, g, λ *and* u, p.

(ii) *If additionally* $f \in L^{\bar q}(G)^n$ *and* $g \in H^{1,\bar q}(G) \cap \hat H^{-1,\bar q}(G)$ *for some* $\bar q \in (1, \infty)$ *then even* $u \in H^{2,q}(G)^n \cap H^{2,\bar q}(G)^n$ *and* $p \in \hat H^{1,q}(G) \cap \hat H^{1,\bar q}(G)$.

PROOF. For (i) see [5, Theorem 1.2]. If in (ii) $\bar q < q$, then the solution (u, p) coincides with the unique solution in $H^{2,\bar q}(G)^n \times \hat H^{1,\bar q}(G)$ since G is bounded.

THEOREM 5.5. *Let* $\Omega \subset \boldsymbol{R}^n$ *be an exterior domain with boundary of class* $C^{1,1}$, *let* $1 < q < \infty$ *and* $w \in \mathcal{A}_q(\Omega)$.

(i) *For every* $f \in L^q_w(\Omega)$, $g \in H^{1,q}_w(\Omega) \cap \hat H^{-1,q}_w(\Omega)$ *and* $\lambda \in S_\varepsilon$, $0 < \varepsilon < \pi/2$, *the generalized Stokes resolvent problem*

$$\lambda u - \Delta u + \nabla p = f, \quad \operatorname{div} u = g \ \textit{in} \ \Omega, \quad u = 0 \ \textit{on} \ \partial\Omega \tag{5.6}$$

has a unique solution $(u, p) \in H^{2,q}_w(\Omega)^n \times \hat H^{1,q}_w(\Omega)$. *Furthermore* (u, p) *satisfies the a priori estimates*

$$\|(\lambda u, \nabla^2 u, \nabla p)\|_{q,w} \leqq C(\|(f, \nabla g)\|_{q,w} + \|\lambda g\|_{-1,q,w}) \tag{5.7}$$

$$\|(\lambda u, -\Delta u + \nabla p)\|_{q,w} \leqq C(\|f\|_{q,w} + \|\lambda g\|_{-1,q,w}) \tag{5.8}$$

with a constant $C = C(\Omega, q, w, \varepsilon, \delta) > 0$ *when* $\lambda \in S_\varepsilon$ *is restricted by* $|\lambda| \geqq \delta$ *with given* $\delta > 0$.

(ii) *The constant* C *in* (5.7) *is independent of* $\delta > 0$ *under the following conditions*: $n \geqq 3$ *and there exists some* $s \geqq q$ *such that* $\gamma := n(2/n + 1/s - 1/q) \geqq 0$ *and*

$$w^{s/q} |\cdot - x_0|^{-\gamma s} \in \mathcal{A}_s(\Omega), \ x_0 \notin \partial\Omega, \quad \textit{or} \quad w^{s/q}(1 + |\cdot|)^{-\gamma s} \in \mathcal{A}_s(\Omega). \tag{5.9}$$

This condition is satisfied for $n \geqq 3$ *if*

$$w = |\cdot - x_0|^\alpha, \ x_0 \notin \partial\Omega, \quad \textit{or} \quad w = (1 + |\cdot|)^\alpha \tag{5.10}$$

with

$$2q - n < \alpha < n(q-1). \tag{5.11}$$

(iii) *The constant* C *in* (5.8) *is independent of* $\delta > 0$ *under the following conditions*: $n \geqq 3$ *and there exists some* $s \in (1, q]$ *such that* $\tilde\gamma := n(2/n + 1/q - 1/s) \geqq 0$ *and*

$$w^{s/q} |\cdot - x_0|^{\tilde\gamma s} \in \mathcal{A}_s(\Omega), \ x_0 \notin \partial\Omega, \quad \textit{or} \quad w^{s/q}(1 + |\cdot|)^{\tilde\gamma s} \in \mathcal{A}_s(\Omega). \tag{5.12}$$

This condition is satisfied for $n \geqq 3$ *if* $w = |\cdot - x_0|^\alpha$, $x_0 \notin \partial\Omega$, *or* $w = (1 + |\cdot|)^\alpha$ *with*

$$n \geqq 3 \quad \textit{and} \quad -n < \alpha < n(q-1) - 2q. \tag{5.13}$$

REMARK 5.6. The weight w in both cases of (5.10) and of (5.12) may be multiplied by the logarithmic terms

$$\log^{\beta}(2+|\cdot|) \quad \text{and} \quad \log^{\beta}(2+|\cdot-x_1|^{-1}) \quad \text{with } x_1\notin\partial\Omega,\ \beta\in\boldsymbol{R}.$$

We divide the proof of Theorem 5.5 in several parts and in a sequence of lemmata.

LEMMA 5.7. *For a given solution $(u, p)\in H_w^{2,q}(\Omega)^n\times\hat{H}_w^{1,q}(\Omega)$ of (5.6) it holds the preliminary a priori estimate*

$$\begin{aligned}\|(\lambda u, \nabla^2 u, \nabla p)\|_{q,w} \leqq C(&\|f, \nabla g)\|_{q,w}+\|\lambda g\|_{-1,q,w}\\ &+\|(u, \nabla u, p)\|_{q,G}+\|\lambda u\|_{H^{1,q'}(G)^*})\end{aligned} \tag{5.14}$$

with a constant $C=C(\Omega, G, w, q, \varepsilon)>0$ independent of $\lambda\in S_\varepsilon$. Here G is the bounded domain associated with the weight $w\in\mathcal{A}_q(\Omega)$ in Definition 2.5, and $H^{1,q'}(G)^*$ is the dual space of $H^{1,q'}(G)$.

PROOF. Recall the cut-off functions ϕ_1, $\phi_2=1-\phi_1$ from the proof of Theorem 5.2. Then for $\phi=\phi_1$ consider $(\phi u, \phi p)$ as an element of $H_w^{2,q}(\boldsymbol{R}^n)^n\times\hat{H}_w^{1,q}(\boldsymbol{R}^n)$ solving in $\boldsymbol{R}^n$ the system

$$(\lambda-\Delta)(\phi u)+\nabla(\phi p)=F(\phi), \quad \operatorname{div}(\phi u)=G(\phi) \tag{5.15}$$

where

$$\begin{aligned}F(\phi)&=\phi f-2\nabla\phi\cdot\nabla u-(\Delta\phi)u+p\nabla\phi\\ G(\phi)&=\phi g+u\cdot\nabla\phi.\end{aligned} \tag{5.16}$$

Since $\operatorname{supp}\nabla\phi\subset G$ and $g=\operatorname{div} u$,

$$\|(F(\phi), \nabla G(\phi))\|_{q,w,\boldsymbol{R}^n} \leqq c(\|(f, \nabla g)\|_{q,w,\Omega}+\|(u, \nabla u, p)\|_{q,G}). \tag{5.17}$$

To estimate $G(\phi)\in\hat{H}_w^{-1,q}(\boldsymbol{R}^n)$ let $\varphi\in C_0^\infty(\boldsymbol{R}^n)$ and $\tilde{\varphi}=\varphi-|G|^{-1}\int_G\varphi\,dx$. Then the identity

$$\langle G(\phi), \varphi\rangle=-\int_{\boldsymbol{R}^n}u\phi\cdot\nabla\tilde{\varphi}\,dx=-\int_\Omega u\cdot\nabla(\phi\tilde{\varphi})dx+\int_G(u\cdot\nabla\phi)\tilde{\varphi}\,dx$$

and Poincaré's inequality $\|\tilde{\varphi}\|_{q,G}\leqq c\|\nabla\varphi\|_{q,G}$ yield

$$|\langle G(\phi), \varphi\rangle| \leqq c(\|g\|_{-1,q,w}+\|u\|_{H^{1,q'}(G)^*})\|\nabla\varphi\|_{q',w',\boldsymbol{R}^n}. \tag{5.18}$$

Thus

$$\|\lambda G(\phi)\|_{\hat{H}_w^{-1,q}(\boldsymbol{R}^n)} \leqq c(\|\lambda g\|_{-1,q,w}+\|\lambda u\|_{H^{1,q'}(G)^*}). \tag{5.19}$$

Analogously for $\phi=\phi_2$ we consider $(\phi u, \phi p)\in H^{2,q}(G)^n\times H^{1,q}(G)$ as a solution of (5.15) in the bounded domain G satisfying the boundary condition $\phi u=0$ on ∂G. As before we see that $\|(F(\phi), \nabla G(\phi))\|_{q,G}$ and $\|\lambda G(\phi)\|_{\hat{H}^{-1,q}(G)}$ are bounded by the right-hand sides of (5.17), (5.19), respectively. Summarizing these estimates, (4.4) and Lemma 5.4, the a priori estimate (5.14) is proved. □

LEMMA 5.8. *Define the operator* $S_{q,w}(\lambda)$ *from* $H_w^{2,q}(\Omega)^n\times\hat{H}_w^{1,q}(\Omega)$ *to* $L_w^q(\Omega)^n\times\hat{H}_w^{-1,q}(\Omega)$ *by*

$$\mathcal{D}(S_{q,w}(\lambda))=(H_w^{2,q}(\Omega)^n\cap H_{0,w}^{1,q}(\Omega)^n)\times\hat{H}_w^{1,q}(\Omega)$$

and

$$S_{q,w}(\lambda)(u,\ p)=((\lambda-\Delta)u+\nabla p,\ \operatorname{div}u).$$

Then $S_{q,w}(\lambda)$ *is injective and its range* $\mathcal{R}(S_{q,w}(\lambda))$ *is dense in* $L_w^q(\Omega)^n\times\hat{H}_w^{-1,q}(\Omega)$ *for all* $\lambda\in S_\varepsilon$.

PROOF. To prove the injectivity let $(u,\ p)\in H_w^{2,q}(\Omega)^n\times\hat{H}_w^{1,q}(\Omega)$ be a solution of

$$(\lambda-\Delta)u+\nabla p=0,\quad \operatorname{div}u=0 \text{ in } \Omega,\quad u=0 \text{ on } \partial\Omega.$$

If $q=2$ and $w\equiv1$, take the scalar product in $L^2(\Omega)^n$ of $(\lambda-\Delta)u+\nabla p=0$ with u and use integration by parts to see that $\lambda\int|u|^2+\int|\nabla u|^2=0$. Thus $u\equiv0$ and $\nabla p\equiv0$. Now let $q\geqq2$ and $w\in\mathcal{A}_q(\Omega)$ arbitrary. Using the cut-off function $\phi=\phi_1,\ \phi_2$ from the previous proofs, $(\phi u,\ \phi p)$ is a solution of (5.15) with $f=0$, $g=0$ in $\boldsymbol{R}^n$ or in G together with $u|_{\partial G}=0$, respectively. Since $\operatorname{supp}\nabla\phi_1\subset G$ and $w\in\mathcal{A}_p(\Omega)$ we easily get that $F(\phi_1)\in L_w^q(\boldsymbol{R}^n)^n\cap L^2(\boldsymbol{R}^n)^n$, $G(\phi_1)\in H_w^{1,q}(\boldsymbol{R}^n)\cap H^{1,2}(\boldsymbol{R}^n)$ and, see (5.18), (5.19), that $G(\phi_1)\in\hat{H}_w^{-1,q}(\boldsymbol{R}^n)\cap\hat{H}^{-1,2}(\boldsymbol{R}^n)$. Then Theorem 4.5 (ii) yields $\phi_1u\in H_w^{2,q}(\boldsymbol{R}^n)^n\cap H^{2,2}(\boldsymbol{R}^n)^n$ and $\phi_1p\in\hat{H}_w^{1,q}(\boldsymbol{R}^n)\cap\hat{H}^{1,2}(\boldsymbol{R}^n)$. Analogously $\phi_2u\in H^{2,2}(G)^n$ and $\phi_2p\in H^{1,2}(G)$ by Lemma 5.4. Thus $(u,\ p)\in H^{2,2}(\Omega)^n\times\hat{H}^{1,2}(\Omega)$ and $u\equiv0$, $\nabla p\equiv0$ as before. Finally let $1<q<2$ and $w\in\mathcal{A}_q(\Omega)$. By Sobolev's embedding theorem $H^{1,q}(G)\subset L^{s_1}(G)$ where $s_1>q$ is defined by $1/n+1/s_1=1/q$. Hence it is easily seen that $F(\phi_1)\in L_w^q(\boldsymbol{R}^n)^n\cap L^{s_1}(\boldsymbol{R}^n)^n$, $G(\phi_1)\in H_w^{1,q}(\boldsymbol{R}^n)\cap H^{1,s_1}(\boldsymbol{R}^n)$ and $G(\phi_1)\in\hat{H}_w^{-1,q}(\boldsymbol{R}^n)\cap\hat{H}^{-1,s_1}(\boldsymbol{R}^n)$. Thus $\phi_1u\in H^{2,s_1}(\boldsymbol{R}^n)^n$ and $\phi_1p\in\hat{H}^{1,s_1}(\boldsymbol{R}^n)$ by Theorem 4.5 (ii). Analogously $\phi_2u\in H^{2,s_1}(G)^n$, $\phi_2p\in H^{1,s_1}(G)$ by Lemma 5.4. If $s_1<2$ this procedure is repeated a finite number of times to get exponents $q<s_1<\cdots<s_k$ with $s_k>2$ such that $(u,\ p)\in H^{2,s_k}(\Omega)^n\times\hat{H}^{1,s_k}(\Omega)$. Thus the problem is reduced to the case $q\geqq2$, and the injectivity of $S_{q,w}(\lambda)$ is proved.

To show the density of the range of $S_{q,w}(\lambda)$ we first restrict ourselves to solenoidal vector fields, i.e., we introduce the operator

$$\begin{aligned}&S_{q,w}^0(\lambda)(u,\ p)=(\lambda-\Delta)u+\nabla p,\\&\mathcal{D}(S_{q,w}^0(\lambda))=\{(u,\ p)\in\mathcal{D}(S_{q,w}(\lambda));\ \operatorname{div}u=0\}\end{aligned}\tag{5.20}$$

with range in $L_w^q(\Omega)^n$.

First let $q=2$ and $w\equiv1$. Applying the lemma of Lax-Milgram in the space $V=\{u\in H_0^{1,2}(\Omega)^n;\ \operatorname{div}u=0\}$ to the variational problem

$$\lambda\int u\cdot\bar{v}+\int\nabla u\cdot\nabla\bar{v}=\int f\cdot\bar{v}\quad\text{for all } v\in V$$

we find for every $f\in L^2(\Omega)^n$ a unique solution $u\in V$. Then de Rham's theorem yields a $p\in L^2_{loc}(\bar{\Omega})$ such that (u, p) is a weak solution of $S^0_{2,1}(\lambda)(u, p)=f$. Since $f\in L^2(\Omega)^n$, local regularity theory implies that $(u, p)\in H^{2,2}_{loc}(\bar{\Omega})^n\times H^{1,2}_{loc}(\bar{\Omega})$. Finally writing $(u\phi_1, p\phi_1)$ as a solution of the inhomogeneous Stokes system with right-hand side in $L^2(\boldsymbol{R}^n)^n\times H^{1,2}(\boldsymbol{R}^n)$, the Calderón-Zygmund inequality implies that $(u\phi_1, p\phi_1)\in H^{2,2}(\boldsymbol{R}^n)^n\times\hat{H}^{1,2}(\boldsymbol{R}^n)$. This proves $(u, p)\in\mathcal{D}(S^0_{2,1}(\lambda))$ and $\mathcal{R}(S^0_{2,1}(\lambda))=L^2(\Omega)^n$.

If $q\neq 2$ and $w\in\mathcal{A}_q(\Omega)$ is arbitrary, let $f\in C_0^\infty(\Omega)^n$. By the previous step there is a unique $(u, p)\in\mathcal{D}(S^0_{2,1}(\lambda))$ such that $S^0_{2,1}(\lambda)(u, p)=f$. Repeating the regularity arguments of the proof of the injectivity assertion, we conclude that even $(u, p)\in\mathcal{D}(S^0_{q,w}(\lambda))$. Thus

$$\mathcal{R}(S^0_{q,w}(\lambda)) \text{ is dense in } L^q_w(\Omega)^n. \tag{5.21}$$

To complete the proof let $(f', g')\in L^{q'}_{w'}(\Omega)^n\times\hat{H}^{1,q'}_{w'}(\Omega)$, the dual space of $L^q_w(\Omega)^n\times\hat{H}^{-1,q}_w(\Omega)$, and suppose that

$$\langle S_{q,w}(\lambda)(u, p), (f'g')\rangle = 0 \quad \text{for all } (u, p)\in\mathcal{D}(S_{q,w}(\lambda)),$$

i.e., $\int_\Omega(\lambda u-\Delta u+\nabla p)\cdot f'dx+\langle g', \operatorname{div} u\rangle=0$. Restricting to $(u, p)\in\mathcal{D}(S^0_{q,w}(\lambda))$, (5.21) yields $f'=0$. Thus

$$0=\langle g', \operatorname{div} u\rangle=-\int_\Omega u\cdot\nabla g'$$

for all $u\in C_0^\infty(\Omega)^n$. Since $C_0^\infty(\Omega)$ is dense in $L^q_w(\Omega)$ we conclude that $\nabla g'=0$, or $g'=0$ in $\hat{H}^{1,q'}_{w'}(\Omega)$. Now Hahn-Banach's theorem completes the proof of Lemma 5.8. □

PROOF OF INEQUALITY (5.7) AND OF THEOREM 5.5 (ii). We start with the case where $\lambda\in S_\varepsilon$ is restricted by $|\lambda|\geq\delta$ for some positive δ. Assuming that (5.7) is false there are sequences (u_k, p_k) in $H^{2,q}_w(\Omega)^n\times\hat{H}^{1,q}_w(\Omega)$ and (λ_k) in S_ε, $|\lambda_k|\geq\delta$, satisfying (5.6) with (f_k) in $L^q_w(\Omega)^n$, (g_k) in $H^{1,q}_w(\Omega)\cap\hat{H}^{-1,q}_w(\Omega)$ such that

$$\begin{aligned}&\|(\lambda_k u_k, \nabla^2 u_k, \nabla p_k)\|_{q,w}=1 \quad \text{for all } k\in\boldsymbol{N}\\&\|(f_k, \nabla g_k)\|_{q,w}+\|\lambda_k g_k\|_{-1,q,w}\to 0 \quad \text{as } k\to\infty.\end{aligned} \tag{5.22}$$

Without loss of generality we may suppose that $\int_G p_k dx=0$, $k\in\boldsymbol{N}$, and that (λ_k) converges to some $\lambda\in\bar{S}_\varepsilon$ or to ∞ as $k\to\infty$. In both cases the interpolation inequality (3.3) implies that (u_k) in $H^{2,q}_w(\Omega)^n$ is bounded. Since $L^q_w(\Omega)$ and $\hat{H}^{1,q}_w(\Omega)$ are separable and reflexive we may assume—by suppressing the notation of subsequences—that

$$u_k \rightharpoonup u \text{ in } H^{2,q}_w(\Omega)^n \cap H^{1,q}_{0,w}(\Omega)^n$$
$$\lambda_k u_k \rightharpoonup V \text{ in } L^q_w(\Omega)^n, \quad \nabla p_k \rightharpoonup \nabla p \text{ in } L^q_w(\Omega)^n \tag{5.23}$$

for $k\to\infty$; here $\rightharpoonup$ denotes weak convergence, $u\in H^{2,q}_w(\Omega)^n$, $u=0$ on $\partial\Omega$, $V\in L^q_w(\Omega)^n$, $p\in \hat{H}^{1,q}_w(\Omega)$ and $\int_G p\,dx=0$. Further by (5.22)

$$V-\Delta u+\nabla p=0, \quad \operatorname{div} u=\operatorname{div} V=0 \text{ in } \Omega$$
$$u=0 \text{ on } \partial\Omega, \quad V\cdot N=0 \text{ on } \partial\Omega. \tag{5.24}$$

If $\lambda_k\to\lambda\in\bar{S}_\varepsilon$, $|\lambda|\geqq\delta$, then (5.23) yields $V=\lambda u$. Thus (u, p) solves the homogeneous Stokes resolvent system and consequently $u=0$, $p=0$ by Lemma 5.8. Then the compact embeddings $H^{2,q}(G)\subset H^{1,q}(G)$, $H^{1,q}(G)\subset L^q(G)$ and $L^q(G)\subset H^{1,q'}(G)^*$, the assumption $w|_{\bar{G}}>0$ and (5.23) yield $\|(u_k, \nabla u_k, p_k)\|_{q,G}\to 0$ and $\|\lambda_k u_k\|_{H^{1,q'}(G)^*}\to 0$ as $k\to\infty$. By (5.14), (5.22) this leads to a contradiction. If $|\lambda_k|\to\infty$ as $k\to\infty$, then $u=0$ and (5.24) reduces to the Helmholtz decomposition $0=V+\nabla p$ in $L^q_w(\Omega)^n$. Hence by Theorem 1.3 $V=0$, $p=0$. Using the above compact embeddings we are again led to a contradiction. This proves (5.7) when $|\lambda|\geqq\delta$.

Next we consider (5.7) for $|\lambda|\to 0$ under the assumption (5.9). Set

$$v=w^{s/q}|\cdot-x_0|^{-\gamma s} \quad \text{or} \quad v=w^{s/q}(1+|\cdot|)^{-\gamma s},$$

respectively. Before constructing a contradiction as above let us prove the embeddings

$$H^{2,q}_w(\Omega)\subset H^{1,r}_{\tilde{w}}(\Omega)\subset L^s_v(\Omega) \tag{5.25}$$

where

$$\tilde{w}=(w^{1/2q}v^{1/2s})^r \quad \text{and} \quad \frac{1}{r}=\frac{1}{2}\Big(\frac{1}{q}+\frac{1}{s}\Big).$$

Obviously $q\leqq r\leqq s$ and by (5.9) $\gamma/(2n)=1/n+1/r-1/q=1/n+1/s-1/r\geqq 0$. Further $\tilde{w}\in\mathcal{A}_r(\Omega)$, since for every cube $Q\subset\mathbf{R}^n$

$$\Big(\fint_Q\tilde{w}\Big)^{1/r}\Big(\fint_Q\tilde{w}^{-r'/r}\Big)^{1/r'}$$
$$\leqq\Big(\fint_Q w\Big)^{1/2q}\Big(\fint_Q v\Big)^{1/2s}\Big(\fint_Q w^{-q'/q}\Big)^{1/2q'}\Big(\fint_Q v^{-s'/s}\Big)^{1/2s'}\leqq C$$

due to Hölder's inequality and the assumptions $w\in\mathcal{A}_q(\Omega)$ and $v\in\mathcal{A}_s(\Omega)$. Thus Corollary 3.7 yields the continuous embeddings $H^{1,q}_w(\Omega)\subset L^r_{\tilde{w}}(\Omega)$ and $H^{1,r}_{\tilde{w}}(\Omega)\subset L^s_v(\Omega)$. Note that the constant $u_\infty\in C$ in (3.10) vanishes when e.g., $u\in L^q_w(\Omega)$ and $u-u_\infty\in L^r_{\tilde{w}}(\Omega)$ is known; in this case $u_\infty\in L^q_w(\mathbf{R}^n)+L^r_{\tilde{w}}(\mathbf{R}^n)$ is harmonic, and Lemma 4.1 (ii) shows that $u_\infty=0$. Thus (5.25) is proved.

Now let $(u_k, p_k, f_k, g_k, \lambda_k)$ be a sequence satisfying (5.22) and $\lim_{k\to\infty}\lambda_k=0$. By (5.25) (∇u_k) is bounded in $L^r_{\tilde{w}}(\Omega)^{n^2}$ and (u_k) is bounded in $L^s_v(\Omega)^n$. Thus—by suppressing the notation of subsequences—we may assume the following weak convergences:

$$u_k \rightharpoonup u \quad \text{in } L^s_v(\Omega), \quad \nabla u_k \rightharpoonup \nabla u \text{ in } L^r_{\tilde{w}}(\Omega)$$
$$\nabla^2 u_k \rightharpoonup \nabla^2 u \text{ in } L^q_w(\Omega), \quad \nabla p_k \rightharpoonup \nabla p \text{ in } L^q_w(\Omega).$$

Further $\lambda_k u_k \rightharpoonup V=\lambda u=0$ in $L^q_w(\Omega)$ and $u=0$ on $\partial\Omega$. Hence (u, p) solves the Stokes system

$$-\Delta u+\nabla p = 0, \quad \operatorname{div} u = \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{5.26}$$

or using again the cut-off functions $\phi=\phi_1, \phi_2\in C^\infty(\boldsymbol{R}^n; [0, 1])$,

$$\begin{aligned} -\Delta(u\phi)+\nabla(p\phi) &= -2\nabla u\cdot\nabla\phi-u\Delta\phi+p\nabla\phi \\ \operatorname{div}(u\phi) &= u\cdot\nabla\phi. \end{aligned} \tag{5.27}$$

For $\phi=\phi_1$ consider (5.27) as a Stokes system on $\boldsymbol{R}^n$. Since $\tilde{w}\in\mathcal{A}_r(\Omega)$, $r\leqq s$ and $\operatorname{supp}\nabla\phi_1\subset G$, the right-hand side is contained in $L^r(\boldsymbol{R}^n)^n\times H^{1,r}(\boldsymbol{R}^n)$ and has a compact support in G. Using Fourier analysis we can write down an explicit solution $(\tilde{u}, \tilde{p})$ of (5.27) which satisfies $(\nabla^2\tilde{u})\in L^r(\boldsymbol{R}^n)^{n^3}$, $\nabla\tilde{p}\in L^r(\boldsymbol{R}^n)^n$ due to Theorem 3.3. Then Lemma 4.1 (ii) implies that $\tilde{p}$ coincides with $p\phi_1$ up to a constant and $\tilde{u}$ coincides with $u\phi_1$ up to a linear polynomial, see [5] for details. For $\phi=\phi_2$ we consider (5.27) as a Stokes system on G together with the boundary condition $u\phi_2=0$ on ∂G. Here classical L^r-theory yields $u\phi_2\in H^{2,r}(G)^n$ and $p\phi_2\in H^{1,r}(G)$. Combining both results and using standard embedding theorems we see that the right-hand side of (5.27) is even contained in $L^t(\boldsymbol{R}^n)\times H^{1,t}(\boldsymbol{R}^n)$ for all $t\in(1, \infty)$ if $r\geqq n$ and for all $t\in(1, rn/(n-r))$ if $r<n$. Repeating the previous arguments, if necessary a finite number of times, we arrive at the regularity assertion

$$\|(\nabla^2 u, \nabla p)\|_t < \infty \quad \text{for all } t \in (1, \infty).$$

Further the embedding $\hat{H}^{1,t}(\Omega)\subset L^\tau(\Omega)$ with $1/\tau=1/t-1/n$ for $t<n$ yields, since $\nabla u\in L^r_{\tilde{w}}(\Omega)^{n^2}$ and $u\in L^s_v(\Omega)^n$,

$$\|\nabla u\|_{t_1}+\|u\|_{t_2} < \infty \quad \text{for all } t_1 > \frac{n}{n-1},\ t_2 > \frac{n}{n-2}.$$

Since $n\geqq 3$, $\nabla u\in L^2(\Omega)^{n^2}$ and, by the same embedding theorem, $p-m\in L^2(\Omega)$ for a suitable constant $m\in\boldsymbol{C}$. Now an approximation argument and integration by parts shows that

$$0 = \int(-\Delta u+\nabla p)\bar{u} = \int|\nabla u|^2-\int(p-m)\operatorname{div} u\,.$$

Since $\operatorname{div} u=0$, we get $\nabla u=0$ and even $u=0$, for $u=0$ on $\partial\Omega$. Furthermore $\nabla p=0$. This will lead to the same contradiction to (5.14), (5.22) as before.

It remains to consider the case (5.10), (5.11). Choose any $s\geqq q$ such that $\gamma=n(2/n+1/s-1/q)\geqq 0$. Then, if e.g., $w(x)=|x|^{\alpha}$, we get $v(x)=|x|^{\alpha s/q-\gamma s}$. But $v\in\mathcal{A}_s(\Omega)$, since the inequality $2q-n<\alpha<n(q-1)$ yields the inequality $-n<\alpha s/q-\gamma s<n(s-1)$. By Lemma 2.3 (v) the same arguments hold when w is multiplied by a logarithmic factor.

Now (5.7) and Theorem 5.5 (ii) are completely proved. □

PROOF OF INEQUALITY (5.8) AND OF THEOREM 5.5 (iii). Set

$$v=w^{s/q}|\cdot-x_0|^{\tilde{\gamma}s}\quad\text{or}\quad v=w^{s/q}(1+|\cdot|)^{\tilde{\gamma}s},$$

respectively. The proof rests on an elementary duality argument; cf. the proof of (4.5). However it is important to recall that the range $\mathcal{R}(S^0_{q',w'}(\lambda))$ is dense in $L^{q'}_{w'}(\Omega)^n$, see (5.21). Thus (5.7) yields (5.8). To prove (iii) let $x_0=0$ and note that the assumptions on $w\in\mathcal{A}_q(\Omega)$ yield $w'\in\mathcal{A}_{q'}(\Omega)$, $s'\geqq q'$, $\gamma'=\tilde{\gamma}=n(2/n+1/s'-1/q')\geqq 0$ and

$$v'=v^{-s'/s}=w^{-s'/q}|\cdot|^{-\gamma's'}=(w')^{s'/q'}|\cdot|^{-\gamma's'}\in\mathcal{A}_{s'}(\Omega).$$

Hence w' satisfies (5.9), and (5.7) holds for q', w' with a constant C independent of δ. Then the duality argument yields (5.8) for q, w with C independent of δ. If e.g., $w(x)=|x|^{\alpha}$, $-n<\alpha<n(q-1)-2q$, then $w'(x)=|x|^{-\alpha q'/q}$ and $2q'-n<-\alpha(q'/q)<n(q'-1)$. Thus (5.11) is satisfied for q', w'. This completes the proof of Theorem 5.5 (iii).

PROOF OF SURJECTIVITY OF $S_{q,w}(\lambda)$. For fixed $\lambda\in S_\varepsilon$ the range of the operator $S^0_{q,w}(\lambda)$ is dense in $L^q_w(\Omega)^n$, see (5.20), (5.21). Furthermore (5.7) implies that $\mathcal{R}(S^0_{q,w}(\lambda))$ is closed in $L^q_w(\Omega)^n$. Thus $\mathcal{R}(S^0_{q,w}(\lambda))=L^q_w(\Omega)^n$.

Next consider the operator $S_{q,w}(\lambda)$ and define the space

$$D_{q,w}=\{\operatorname{div} u\,;\, u\in H^{2,q}_w(\Omega)^n,\ u=0 \text{ on } \partial\Omega\}.$$

To prove that $\mathcal{R}(S_{q,w}(\lambda))=L^q_w(\Omega)^n\times D_{q,w}$ let $(f,g)\in L^q_w(\Omega)^n\times D_{q,w}$. By definition there is some $u_0\in H^{2,q}_w(\Omega)^n$ with $u_0=0$ on $\partial\Omega$ and $g=\operatorname{div} u_0$. By the above results there is a unique $(u_1,p)\in H^{2,q}_w(\Omega)^n\times\hat{H}^{1,q}_w(\Omega)$ satisfying

$$(\lambda-\Delta)u_1+\nabla p=f-(\lambda-\Delta)u_0,\quad \operatorname{div} u_1=0 \text{ in } \Omega,\quad u_1=0 \text{ on } \partial\Omega.$$

Then $(u_0+u_1,p)\in\mathcal{D}(S_{q,w}(\lambda))$ and $S_{q,w}(\lambda)((u_0+u_1,p))=(f,g)$.

Thus it suffices to prove that

$$D_{q,w}=H^{1,q}_w(\Omega)\cap\hat{H}^{-1,q}_w(\Omega).\tag{5.28}$$

By definition and (5.7) $D_{q,w}$ is a closed subset of $H^{1,q}_w(\Omega)\cap\hat{H}^{-1,q}_w(\Omega)$ with respect to the norm $\|\nabla g\|_{q,w}+\|g\|_{-1,q,w}$. To show that the inclusion is also dense with respect to this norm we identify $D_{q,w}$ with

$$E_{q,w}=\{(\nabla\operatorname{div}u,\ u);\ u\in H_w^{2,q}(\Omega)^n,\ u=0 \text{ on } \partial\Omega\}$$

equipped with the norm $\|\nabla\operatorname{div}u\|_{q,w}+\|u\|_{q,w}$. To introduce an analogous identification for $H_w^{1,q}(\Omega)\cap\hat{H}_w^{-1,q}(\Omega)$ consider $g\in\hat{H}_w^{-1,q}(\Omega)$. Since $\hat{H}_{w'}^{1,q'}(\Omega)$ is isomorphic to a closed subspace of $L_{w'}^{q'}(\Omega)^n$, Hahn-Banach's theorem yields some $u\in L_w^q(\Omega)^n$ with $\|u\|_{q,w}=\|g\|_{-1,q,w}$ satisfying $g=\operatorname{div}u$, i.e., $\langle g,\varphi\rangle=-\int u\cdot\nabla\varphi$ for all $\varphi\in\hat{H}_{w'}^{1,q'}(\Omega)$. If additionally $g\in\hat{H}_w^{1,q}(\Omega)$ and consequently $g=\operatorname{div}u\in L^q(G)$, Gauss' integral theorem yields $u\cdot N=0$ on $\partial\Omega$ where N denotes the normal vector on $\partial\Omega$. Then Lemma 5.3 implies that even $\operatorname{div}u\in L_w^q(\Omega)$ and consequently $g\in H_w^{1,q}(\Omega)$. Thus the density of $D_{q,w}$ in $H_w^{1,q}(\Omega)\cap\hat{H}_w^{-1,q}(\Omega)$ is equivalent to the assertion that

$$E_{q,w} \text{ is dense in } Y_{q,w}=\{(\nabla\operatorname{div}u,\ u)\in L_w^q(\Omega)^n\times L_w^q(\Omega)^n;\ N\cdot u|_{\partial\Omega}=0\} \tag{5.29}$$

with respect to $\|\cdot\|_{q,w}+\|\cdot\|_{q,w}$.

To prove that assertion consider any functional $(F,H)\in L_{w'}^{q'}(\Omega)^{2n}=(L_w^q(\Omega)^{2n})^*$ vanishing on $E_{q,w}$, i.e.,

$$\int F\cdot\nabla\operatorname{div}u+\int H\cdot u=0 \quad \text{for all } u\in H_w^{2,q}(\Omega)^n,\ u|_{\partial\Omega}=0.$$

Choosing $u\in C_0^\infty(\Omega)^n$ implies that $\nabla\operatorname{div}F=-H\in L_{w'}^{q'}(\Omega)^n$ in the sense of distributions. In particular $\operatorname{div}F\in L^{q'}(G)$. Next note that by Lemma 5.4 $\operatorname{div}u|_{\partial\Omega}$ takes on all values $g|_{\partial\Omega}$, $g\in C_0^\infty(G\cup\partial\Omega)$, when u runs through all of $H_w^{2,q}(\Omega)^n$ with zero boundary values. Thus Gauss' integral theorem implies that $F\cdot N=0$ on $\partial\Omega$. Hence $\operatorname{div}F\in L_{w'}^{q'}(\Omega)$ by Lemma 5.3. To finish the proof it suffices to show that the functional (F,H) vanishes on $Y_{q,w}$. Let $u\in L_w^q(\Omega)^n$ satisfy $\nabla\operatorname{div}u\in L_w^q(\Omega)^n$, $u\cdot N=0$ on $\partial\Omega$, and note that $\operatorname{div}u\in L_w^q(\Omega)$ by Lemma 5.3. Then a standard approximation argument justifies the following integration by parts:

$$\int F\cdot\nabla\operatorname{div}u+\int H\cdot u=-\int\operatorname{div}F\operatorname{div}u+\int H\cdot u=\int(\nabla\operatorname{div}F+H)\cdot u=0.$$

Thus Hahn-Banch's theorem completes the proof of (5.29) and also of (5.28). □

We note that the proof of Theorem 5.5 and in particular of Theorem 1.2 is complete.

PROOF OF THEOREM 1.5. The a priori estimate (5.7) with $\lambda=1$ yields that the Stokes operator $A_{q,w}$ is a closed operator on $\mathcal{D}(A_{q,w})\subset L_{w,\sigma}^q(\Omega)$ for every $w\in\mathcal{A}_q(\Omega)$, $1<q<\infty$. Further for every $\lambda\in\boldsymbol{C}\backslash\bar{\boldsymbol{R}}_-$ the inverse $(\lambda+A_{q,w})^{-1}$ exists and is continuous. To show that $(A_{q,w})^*=A_{q',w'}$ it suffices to prove $[(1+A_{q,w})^{-1}]^* =(1+A_{q',w'})^{-1}$. Let $f\in L^q_{w,\sigma}(\Omega)$, $f'\in L^{q'}_{w',\sigma}(\Omega)$ and let $(u, p)\in\mathcal{D}(S^0_{q,w}(1))$, $(u', p')\in\mathcal{D}(S^0_{q',w'}(1))$—see (5.20)—be defined by $u-\Delta u+\nabla p=f$, $u'-\Delta u'+\nabla p'=f'$. Then integration by parts yields

$$((1+A_{q,w})^{-1}f,\ f')=(u,\ f')=(f,\ u')=(f,\ (1+A_{q',w'})^{-1}f').$$

Thus $(A_{q,w})^*=A_{q',w'}$.

To prove (ii) we define the linear operator

$$T_{q,w}(\lambda)\colon L^q_w(\Omega)^n\to L^q_w(\Omega)^n,\quad T_{q,w}(\lambda)f=(\lambda+A_{q,w})^{-1}P_{q,w}f,$$

i.e., $u=T_{q,w}(\lambda)f$ is part of the solution (u, p) of the resolvent problem $(\lambda-\Delta)u+\nabla p=f$, $\operatorname{div} u=0$ in Ω, $u|_{\partial\Omega}=0$. By Theorem 5.5

$$\|T_{q,w}(\lambda)\|\leqq\frac{c}{|\lambda|},\quad c=c(\delta),\quad \text{for } \lambda\in S_\varepsilon,\ |\lambda|\geqq\delta>0. \tag{5.30}$$

Moreover under the assumption (1.4) the constant c in (5.30) is independent of $\delta>0$ for $T_{q,w_0}(\lambda)$ and $T_{q,w_1}(\lambda)$. Then the complex interpolation in the spaces $L^q_{w_0}(\Omega)$ and $L^q_{w_1}(\Omega)$, see [**1**, Theorem 5.5.3, Corollary 5.5.4], yields (5.30) for $T_{q,w}(\lambda)$ with c independent of δ where $w=w_0^{1-\theta}w_1^\theta$.

Concerning the weights $|x|^\alpha$, $(1+|x|)^\alpha$ etc. let $-n<\alpha<n(q-1)$. Then there are $\theta\in[0, 1]$ and $\alpha_0\in(2q-n,\ n(q-1))$, $\alpha_1\in(-n,\ n(q-1)-2q)$ such that $\alpha=(1-\theta)\alpha_0+\theta\alpha_1$. Now the same interpolation argument as before yields (5.30) for $T_{q,w}(\lambda)$, $w(x)=|x|^\alpha$ etc., with c independent of δ.

For the results on the semigroup $\{e^{-tA_{q,w}};\ t\geqq0\}$ we refer to [**6**]. The proof of Theorem 1.4 is complete. □

REMARK 5.9. Obviously complex interpolation theory for the spaces $L^{q_0}_{w_0}(\Omega)$ and $L^{q_1}_{w_1}(\Omega)$ with $q_0\neq q_1$, see [1], yields a more general result on the boundedness of the semigroup $\{e^{-tA_{q,w}};\ t\geqq0\}$. Let $n\geqq3$, $1<q_0,\ q_1<\infty$, $w_0\in\mathcal{A}_{q_0}(\Omega)$, $w_1\in\mathcal{A}_{q_1}(\Omega)$ and let

$$\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1},\quad w=(w_0^{(1-\theta)/q_0}w_1^{\theta/q_1})^q,\quad \theta\in(0, 1).$$

Further assume that there exist $s_0\geqq q_0$ and $1<s_1\leqq q_1$ with

$$\gamma_0=n\Big(\frac{2}{n}+\frac{1}{s_0}-\frac{1}{q_0}\Big)\geqq0,\quad w_0^{s_0/q_0}|\cdot-x_0|^{-\gamma_0 s_0}\in\mathcal{A}_{s_0}(\Omega),$$

$$\gamma_1=n\Big(\frac{2}{n}+\frac{1}{q_1}-\frac{1}{s_1}\Big)\geqq0,\quad w_1^{s_1/q_1}|\cdot-x_0|^{\gamma_1 s_1}\in\mathcal{A}_{s_1}(\Omega),$$

where $x_0\notin\partial\Omega$. Then $A_{q,w}$ satisfies a resolvent estimate uniformly for $\lambda\in S_\varepsilon$ and $\{e^{-tA_{q,w}};\ t\geqq0\}$ is a bounded analytic semigroup. The same result holds when the term $|\cdot-x_0|$ is replaced by $1+|\cdot|$.

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