WEIGHTED MAXIMAL INEQUALITIES FOR MARTINGALES

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Abstract. The paper contains the study of sharp weighted versions of the classical Doob's weak-type estimates for real-valued martingales. As a by-product, some results concerning the structure of Muckenhoupt's classes are obtained. The proof rests on Bellman function method, i.e., it is based on the construction of a special function having appropriate concavity and majorization properties.

1. Introduction

Doob's maximal inequalities for martingales are of fundamental importance to the whole probability theory, and their extensions and applications to various areas of mathematics can be found in numerous papers in the literature. The purpose of this paper is to investigate the weighted versions of the weak-type estimates and, in particular, to determine the optimal values of the constants involved in these bounds. For related results, see e.g. [3], [5], [6], [8], [12], [13], [14] and references therein.

Let us start with introducing the background and notation. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, filtered by $(\mathcal{F}_t)_{t\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all the events of probability 0. Let X be an adapted, real-valued, uniformly integrable martingale with right-continuous trajectories that have limits from the left. Then $X^* = \sup_{s\geq 0}|X_s|$ denotes the maximal function of X; we will also use the notation $X_t^* = \sup_{0\leq s\leq t}|X_s|$ for the truncated maximal function. Assume that Y is a nonnegative, uniformly integrable martingale with continuous trajectories, satisfying $Y_0 \equiv 1$. For example, one can take an exponential martingale $\mathcal{E}(B)^\tau = \big(\exp\big[B_{\tau\wedge t} - \frac{1}{2}(\tau\wedge t)\big]\big)_{t\geq 0}$ corresponding to an adapted, standard Brownian motion B and some stopping time τ. Following Izumisawa and Kazamaki [7], we say that Y satisfies Muckenhoupt's condition (A_p) (where $1 < p < \infty$ is a fixed parameter), if

$$||Y||_{A_p} := \sup_\tau \left|\left|\mathbb{E}\Big[\big\{Y_\tau/Y_\infty\big\}^{1/(p-1)}\big|\mathcal{F}_\tau\Big]^{p-1}\right|\right|_\infty < \infty.$$

Furthermore, Y is said to satisfy Muckenhoupt's (A_1) condition, if

$$||Y||_{A_1} := \sup_\tau ||Y_\tau/Y_\infty||_\infty < \infty.$$

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Both suprema above are taken over all adapted stopping times τ. This notion is the probabilistic version of the classical condition of Muckenhoupt [9], who used it in the study of the L^p-boundedness of Hardy-Littlewood maximal operator on $\mathbb{R}^n$.

Any process Y as above gives rise to the probability measure $\mathbb{Q}$ defined by the equation $\mathrm{d}\mathbb{Q} = Y_\infty \mathrm{d}\mathbb{P}$, and thus it can be regarded as a weight. The principal goal of this paper is to study the weak-type estimates for X^* with respect to the measure $\mathbb{Q}$. In [13], Tsuchikura proved that for $1 \leq r < \infty$ and any X, Y as above, we have the inequality

$$||X^*||_{L^{r,\infty}(\mathbb{Q})} \leq ||Y||_{A_r}^{1/r} ||X_\infty||_{L^r(\mathbb{Q})}. \tag{1}$$

Here $||X^*||_{L^{r,\infty}(\mathbb{Q})} = \sup_{\lambda>0} \lambda \big[\mathbb{Q}(X^* > \lambda)\big]^{1/r}$ denotes the weak r-th norm of X^*. Then Uchiyama [14] proved that this inequality is sharp, by showing that

$$\sup_{X,\lambda} \left[\lambda^r \mathbb{Q}(X^* > \lambda)/||X_\infty||^r_{L^r(\mathbb{Q})}\right] = ||Y||_{A_r}. \tag{2}$$

This result has another remarkable implication, namely it shows that the weak-type (r, ∞) estimate holds true only for the weights satisfying the (A_r) condition. We will study the more general setting where the index of the Muckenhoupt's condition and the order of the norms may be different. To state our results, we need some more notation. For any p, $c > 1$, let $d_\pm(p, c)$ be the constants defined in Lemma 2.1 below and set

$$d(p, r, c) = \begin{cases} d_+(p, c) & \text{if } r \leq p, \\ d_-(p, c) & \text{if } r > p. \end{cases} \tag{3}$$

Now, for all $1 \leq p$, r, $c < \infty$, let

$$C(p, r, c) = \begin{cases} 1 & \text{if } c = 1, \\ \left[\left(1 - \dfrac{d(p, r, c)}{r-1}\right)^{r-1} (1 + d(p, r, c))\right]^{-1/r} & \text{if } p > 1,\ c > 1 \\ & \quad \text{and } r \in (d_+(p, c) + 1, \infty), \\ c\left(\dfrac{r-1}{cr-1}\right)^{(r-1)/r} & \text{if } p = 1,\ c > 1 \text{ and } r > 1, \\ c & \text{if } p = 1,\ c > 1 \text{ and } r = 1, \\ \infty & \text{in all the other cases.} \end{cases}$$

We will prove the following theorem.

Theorem 1.1. *Let $1 \leq p$, r, $c < \infty$. Then for any uniformly integrable martingale X and any weight Y satisfying $||Y||_{A_p} \leq c$, we have*

$$||X^*||_{L^{r,\infty}(\mathbb{Q})} \leq C(p, r, c)\, ||X_\infty||_{L^r(\mathbb{Q})}. \tag{4}$$

The inequality is sharp.

Here by sharpness of (4) we mean that for each p, r, c as above and any $C < C(p,r,c)$, there is a martingale X and a weight Y satisfying $||Y||_{A_p} \leq c$ such that

$$||X^*||_{L^{r,\infty}(\mathbb{Q})} > C||X_\infty||_{L^r(\mathbb{Q})}.$$

We have $C(r,r,c) = c^{1/r}$ and thus the result above contains Tsuchikura's inequality (1). Another case in which the constants $C(p,r,c)$ have a nice explicit form, corresponds to the choice $p = 2$. Indeed, one easily checks that $d_\pm(2,c) = \pm\sqrt{1-c^{-1}}$ and thus

$$C(2,r,c) = \begin{cases} 1 & \text{if } c = 1, \\ \left[\left(1 - \dfrac{\sqrt{1-c^{-1}}}{r-1}\right)^{r-1}\left(1+\sqrt{1-c^{-1}}\right)\right]^{-1/r} & \text{if } 1+\sqrt{1-c^{-1}} < r \leq 2 \\ & \qquad\text{and } c > 1, \\ \left[\left(1 + \dfrac{\sqrt{1-c^{-1}}}{r-1}\right)^{r-1}\left(1-\sqrt{1-c^{-1}}\right)\right]^{-1/r} & \text{if } r > 2 \text{ and } c > 1. \end{cases}$$

The above theorem, combined with Uchiyama's identity (2), yields the following interesting by-product concerning the structure of (A_p) weights. Namely, it follows directly from Jensen's inequality that if a weight satisfies the (A_p) condition for some p, then it automatically satisfies (A_r) for all $r > p$ (more precisely, we have $||Y||_{A_r} \leq ||Y||_{A_p}$). What may be a little unexpected, if $p > 1$, then such a weight satisfies (A_r) for some $r < p$ (see Kazamaki [8, Corollary 3.3]). The results obtained in this paper allow us to determine the precise range of the admissible parameters r. Here is the statement.

Theorem 1.2. *Suppose that Y is a weight satisfying $||Y||_{A_p} < \infty$ for a given $p > 1$. Then $||Y||_{A_r} < \infty$ for all $r \in (1 + d_+(p, ||Y||_{A_p}), \infty)$. The interval cannot be enlarged.*

Let us say a few words about the organization of the paper and the methodology that will be used. Tsuchikura's proof of (1) rests on a clever use of Hölder's inequality. To establish (4), much more effort is required. As we will see, the proof of Theorem 1.1 will rest on the existence of the so-called Bellman function, i.e., a certain special function having appropriate convexity and majorization properties (we refer the interested reader to [2] and [10]). Such an object is constructed and studied in the next section. In Section 3, we show how this special function can be used to establish the inequality (4). The final part of the paper is devoted to the optimality of the constant $C(\cdot,\cdot,\cdot)$.

2. A special function

Throughout this section, p, r and c are fixed number larger than 1. Consider the set

$$\mathcal{D} = \mathcal{D}_{p,c} = \{(x,y,z); x \geq 0,\ 1 \leq yz^{p-1} \leq c\}.$$

The purpose of this part of the paper is to find an appropriate function $B = B_{p,r,c} : \mathcal{D} \to \mathbb{R}$, which is locally convex and satisfies

$$B(x,y,z) \leq x^r y \qquad \text{if } yz^{p-1} = 1. \tag{5}$$

Here by local convexity of B we mean that this function is convex along any line segment which is contained in $\mathcal{D}$. Of course, there are many functions which enjoy the above properties. For example, $B \equiv 0$ is one of them. However, as we will see in the next section, the existence of such a function implies the validity of the estimate (4) with the constant $B(1,1,c^{1/(p-1)})^{-1/r}$ for all weights satisfying $||Y||_{A_p} \leq c$. Thus it is desirable to construct the largest function having the above properties, and we will succeed in finding one below.

We start with introducing an auxiliary function $F = F_{p,c} : [-1, p-1] \to \mathbb{R}$ by

$$F_{p,c}(s) = c^{1/(p-1)}(1+s)^{1/(p-1)}\left(1 - \frac{s}{p-1}\right).$$

Let us establish the following straightforward technical fact.

Lemma 2.1. *For any p, $c > 1$, the function $F_{p,c}$ is strictly increasing on $[-1,0]$ and strictly decreasing on $[0,p-1]$. Moreover, there are unique numbers $d_- = d_-(p,c) \in (-1,0)$ and $d_+ = d_+(p,c) \in (0,p-1)$ satisfying $F_{p,c}(d_-) = F_{p,c}(d_+) = 1$.*

Proof. We easily check that for $s \in (-1, p-1)$,

$$F'_{p,c}(s) = -\frac{pc^{1/(p-1)}}{(p-1)^2}(1+s)^{(p-2)/(p-1)}s, \tag{6}$$

which immediately yields the required monotonicity. Now the second part follows at once from the equalities $F_{p,c}(-1) = F_{p,c}(p-1) = 0$ and the inequality $F_{p,c}(0) = c^{1/(p-1)} > 1$. $\square$

Assume that $r > d_+ + 1$ and let $\varphi_\pm$ be given by the equality

$$\varphi_\pm\big(F_{p,c}(d_\pm s)\big) = \frac{\left(1 - d_\pm/(r-1)\right)^{r-1}(1+d_\pm)}{\left(1 - d_\pm s/(r-1)\right)^{r-1}(1+d_\pm s)}, \qquad s \in [0,1].$$

From the previous lemma we infer that both φ_+ and φ_- are defined on the interval $[1, c^{1/(p-1)}]$. Let

$$\varphi = \begin{cases} \varphi_+ & \text{for } r \leq p, \\ \varphi_- & \text{for } r > p. \end{cases}$$

Lemma 2.2. *The function $\varphi^{-1/(r-1)}$ is concave and nondecreasing.*

Proof. Let $g_\pm = \varphi_\pm^{-1/(r-1)}$. Directly from the definition of $\varphi_\pm$, we see that

$$g_\pm(F_{p,c}(d_\pm s)) = \alpha_\pm F_{r,c}(d_\pm s), \qquad s \in [0,1], \tag{7}$$

where

$$\alpha_\pm = \left(1 - \frac{d_\pm}{r-1}\right)^{-1}(1+d_\pm)^{-1/(r-1)}c^{-1/(r-1)}.$$

Differentiating both sides of (7) with respect to s and using the formula (6), we obtain

$$g'_\pm(F_{p,c}(d_\pm s)) = \alpha_\pm \frac{r(p-1)^2}{p(r-1)^2}\big[c(1+d_\pm s)\big]^{1/(r-1)-1/(p-1)}. \tag{8}$$

This gives the desired monotonicity of $\varphi^{-1/(r-1)}$. To prove the concavity, suppose first that $r \leq p$. Then $(\varphi^{-1/(r-1)})' = g'_+$, and since $d_+ > 0$, the right-hand side of (8) is a nondecreasing function of s. On the other hand, $s \mapsto F_{p,c}(d_+ s)$ is decreasing (see Lemma 2.1) and thus g'_+ is nonincreasing; thus $(\varphi^{-1/(r-1)})'' \leq 0$. If $r > p$, then the concavity is established by the same reasoning. $\square$

We are ready to define the special function B, the properties of which have been described at the beginning of this section. Let $B : \mathcal{D} \to \mathbb{R}$ be given by

$$B(x,y,z) = x^r y \varphi(y^{1/(p-1)} z). \tag{9}$$

It is not difficult to check the following interplay between B and the constant $C(p,r,c)$:

$$C(p,r,c)^{-r} = B(1,1,c^{1/(p-1)}) = \varphi(c^{1/(p-1)}). \tag{10}$$

Moreover, we have $F_{p,c}(d_\pm) = 1$, which implies $\varphi(1) = 1$ and thus $B(x,y,z) = x^r y$ for all $(x,y,z) \in \mathcal{D}$ satisfying $yz^{p-1} = 1$. Thus the majorization (5) holds true. It remains to prove the local convexity of B.

Lemma 2.3. *The function B is convex along any line segment contained in $\mathcal{D}$.*

Proof. The function B is continuous on $\mathcal{D}$ and of class C^∞ in the interior of this set, so it suffices to check that the Hessian matrix of B is positive-semidefinite. Obviously, we have $B_{xx} \geq 0$. Next, we derive that

$$\det \begin{bmatrix} B_{xx} & B_{xy} \\ B_{xy} & B_{yy} \end{bmatrix}(x,y,z) = r x^{2r-2} y^{1+2/(p-1)} \big[(r-1)\varphi\varphi'' - r(\varphi')^2\big],$$

where the functions φ, φ' and φ'' are evaluated at the point $y^{1/(p-1)}z$. The expression in the square brackets is nonnegative, because by Lemma 2.2,

$$(r-1)\varphi\varphi'' - r(\varphi')^2 = -\frac{(r-1)^2(\varphi^{-1/(r-1)})''}{\varphi^{(2r-1)/(1-r)}} \geq 0.$$

Finally, we have that the determinant of the Hessian is equal to zero, which follows from the fact that for each (x,y,z), the function B is linear along a certain line segment passing through this point. Specifically, recall the number $d = d(p,r,c)$ given by (3) and let $s_0 \in [0,1]$ be determined by the equation $F_{p,c}(ds_0) = y^{1/(p-1)}z$. It can be verified that for s belonging to a certain open interval containing 0, we have

$$B(x+Xs, y+Ys, z+Zs) = B(x,y,z) + As, \tag{11}$$

where

$$X = -\frac{x}{r-1-ds_0}, \quad Y = \frac{y}{1+ds_0}, \quad Z = -\frac{z}{p-1-ds_0}$$

and

$$A=\left(\frac{x}{r-1-ds_0}\right)^r\frac{y(1+d)\,(r-1-d)^{r-1}}{1+ds_0}.$$

Thus, we have checked that all the leading principal minors of D^2B are nonnegative, so the Hessian of B is positive-semidefinite, by a well-known fact from linear algebra. □

Of course, if $(B_i)_{i\in\mathcal{I}}$ is a family of all locally convex functions on $\mathcal{D}$ satisfying (5), then $(x,y,z)\mapsto\sup_{i\in\mathcal{I}}B_i(x,y,z)$ also has these properties. Thus we may think of the largest element from this class. It turns out that this extremal object coincides with the function B constructed above. This fact will be needed later, when we deal with the sharpness of the estimate (4).

THEOREM 2.4. *Let $r>d_++1$ and assume that $\mathcal{B}$ is the largest locally convex function on $\mathcal{D}$ satisfying* (5)*. Then $\mathcal{B}=B$.*

PROOF. It suffices to show that $\mathcal{B}\le B$ on $\mathcal{D}$. Observe that for any $\lambda>0$, the function $(x,y,z)\mapsto\lambda^{-r}\mathcal{B}(\lambda x,y,z)$ is locally convex on $\mathcal{D}$ and satisfies (5). Therefore, the function $(x,y,z)\mapsto\sup_{\lambda>0}\lambda^{-r}\mathcal{B}(\lambda x,y,z)$ also has these properties and since it majorizes $\mathcal{B}$ (take $\lambda=1$), the extremality of $\mathcal{B}$ implies

$$\mathcal{B}(x,y,z)=\sup_{\lambda>0}\lambda^{-r}\mathcal{B}(\lambda x,y,z),\qquad(x,y,z)\in\mathcal{D}.$$

A similar argument yields the identity

$$\mathcal{B}(x,y,z)=\sup_{\lambda>0}\lambda^{-1}\mathcal{B}(x,\lambda y,\lambda^{-1/(p-1)}z),\qquad(x,y,z)\in\mathcal{D}.$$

In consequence, $\mathcal{B}$ satisfies the homogeneity property

$$\mathcal{B}(\lambda x,\mu y,\mu^{-1/(p-1)}z)=\lambda^r\mu\mathcal{B}(x,y,z)\tag{12}$$

for all λ, $\mu>0$ and all $(x,y,z)\in\mathcal{D}$.

It is convenient to consider the cases $d_++1<r\le p$ and $r>p$ separately.

The case $d_++1<r\le p$. Let δ be a fixed positive number and let $\delta_-=\delta_-(\delta)\in(-1,0)$ be uniquely determined by the equation $F_{p,c}(\delta_-)=F_{p,c}(\delta)$, that is,

$$(1+\delta_-)^{1/(p-1)}\left(1-\frac{\delta_-}{p-1}\right)=(1+\delta)^{1/(p-1)}\left(1-\frac{\delta}{p-1}\right).\tag{13}$$

By straightforward differentiation, we check that

$$\lim_{\delta\to0}\delta_-(\delta)/\delta=-1.\tag{14}$$

Next, it is not difficult to verify that the line segment formed by the points

$$\left(1-\frac{s}{r-1},1+\delta+s,c^{1/(p-1)}\left(1-\frac{\delta+s}{p-1}\right)\right),$$

where $s \in (-\delta+\delta_-, d_+-\delta)$, is contained in $\mathcal{D}$. Therefore, using the convexity of $\mathcal{B}$ along this segment, we obtain

$$\begin{aligned}&\mathcal{B}\left(1,1+\delta,c^{1/(p-1)}\left(1-\frac{\delta}{p-1}\right)\right)\\ &\quad\leq \frac{\delta-\delta_-}{d_+-\delta_-}\mathcal{B}\left(1-\frac{d_+-\delta}{r-1},1+d_+,c^{1/(p-1)}\left(1-\frac{d_+}{p-1}\right)\right)\\ &\qquad+\frac{d_+-\delta}{d_+-\delta_-}\mathcal{B}\left(1+\frac{\delta-\delta_-}{r-1},1+\delta_-,c^{1/(p-1)}\left(1-\frac{\delta_-}{p-1}\right)\right).\end{aligned}\tag{15}$$

However, we have

$$c^{1/(p-1)}(1+d_+)^{1/(p-1)}\left(1-\frac{d_+}{p-1}\right)=1,$$

so the majorization (5) implies that the first summand on the right of (15) is bounded from above by

$$\frac{\delta-\delta_-}{d_+-\delta_-}\left(1-\frac{d_+-\delta}{r-1}\right)^r(1+d_+).$$

On the other hand, using (12), we see that the second summand is equal to

$$\begin{aligned}&\frac{d_+-\delta}{d_+-\delta_-}\left(1+\frac{\delta-\delta_-}{r-1}\right)^r\frac{1+\delta_-}{1+\delta}\mathcal{B}\left(1,1+\delta,\left(c\frac{1+\delta_-}{1+\delta}\right)^{1/(p-1)}\left(1-\frac{\delta_-}{p-1}\right)\right)\\ &\qquad=\frac{d_+-\delta}{d_+-\delta_-}\left(1+\frac{\delta-\delta_-}{r-1}\right)^r\frac{1+\delta_-}{1+\delta}\mathcal{B}\left(1,1+\delta,c^{1/(p-1)}\left(1-\frac{\delta}{p-1}\right)\right),\end{aligned}$$

where in the latter passage we have used (13). Plug the above two facts into (15) to obtain

$$\begin{aligned}&\left[1-\frac{d_+-\delta}{d_+-\delta_-}\left(1+\frac{\delta-\delta_-}{r-1}\right)^r\frac{1+\delta_-}{1+\delta}\right]\mathcal{B}\left(1,1+\delta,c^{1/(p-1)}\left(1-\frac{\delta}{p-1}\right)\right)\\ &\qquad\leq\frac{\delta-\delta_-}{d_+-\delta_-}\left(1-\frac{d_+-\delta}{r-1}\right)^r(1+d_+).\end{aligned}$$

The function $s\mapsto\mathcal{B}\left(1,1+s,c^{1/(p-1)}\left(1-s/(p-1)\right)\right)$ is convex in a neighborhood of zero and hence it is continuous there. Thus, dividing both sides above by 2δ and letting $\delta\downarrow 0$ yield, by (14),

$$\left(\frac{1}{d_+}-\frac{1}{r-1}\right)\mathcal{B}(1,1,c^{1/(p-1)})\leq\frac{1}{d_+}\left(1-\frac{d_+}{r-1}\right)^r(1+d_+),$$

or $\mathcal{B}(1,1,c^{1/(p-1)})\leq B(1,1,c^{1/(p-1)})$ (here we use the assumption $r>d_++1$). Next, the function

$$s\mapsto B\left(1-s/(r-1),1+s,c^{1/(p-1)}\left(1-\frac{s}{p-1}\right)\right),\qquad s\in[0,d_+],$$

is linear (compare this to (11)) and, as we have just proved, it majorizes the function

$$s\mapsto\mathcal{B}\left(1-s/(r-1),1+s,c^{1/(p-1)}\left(1-\frac{s}{p-1}\right)\right)$$

at the endpoints of the interval $[0,d_+]$. Thus, by the convexity of $\mathcal{B}$, we have

$$\begin{aligned}\mathcal{B}&\left(1-s/(r-1),1+s,c^{1/(p-1)}\left(1-\frac{s}{p-1}\right)\right)\\&\leq B\left(1-s/(r-1),1+s,c^{1/(p-1)}\left(1-\frac{s}{p-1}\right)\right)\end{aligned}$$

for $s\in[0,d_+]$, and hence, by (12) (which is also valid for B), we get $\mathcal{B}\leq B$ on the whole domain $\mathcal{D}$.

The case $r>p$. Here the reasoning is analogous. Namely, this time one has to pick a *negative* δ, define δ_- as the *positive* root of (13), and replace d_+ by d_-. The remaining arguments are word by word the same (the only change is that in the above limiting procedure, one divides both sides by -2δ and lets $\delta\uparrow 0$). $\square$

All the above considerations concerned the case $r>d_++1$. For the remaining values of the parameter r, we have the following fact.

THEOREM 2.5. *If $1<r\leq d_++1$, then for any function $B_0:\mathcal{D}\to\mathbb{R}$ which is locally convex and satisfies* (5)*, we have $B_0(1,1,c^{1/(p-1)})\leq 0$.*

PROOF. The argumentation is similar to that presented in the previous theorem. Let $\mathcal{B}$ be the largest locally convex function satisfying (5); then the homogeneity condition (12) is valid. Next, the line segment formed by the points

$$\left(1-\frac{s}{d_+-\delta},1+\delta+s,c^{1/(p-1)}\left(1-\frac{\delta+s}{p-1}\right)\right)$$

with $s\in(-\delta+\delta_-,d_+-\delta)$, is contained in $\mathcal{D}$. Therefore, by the convexity of $\mathcal{B}$,

$$\begin{aligned}\mathcal{B}&\left(1,1+\delta,c^{1/(p-1)}\left(1-\frac{\delta}{p-1}\right)\right)\\&\leq\frac{\delta-\delta_-}{d_+-\delta_-}\mathcal{B}\left(0,1+d_+,c^{1/(p-1)}\left(1-\frac{d_+}{p-1}\right)\right)\\&\quad+\frac{d_+-\delta}{d_+-\delta_-}\mathcal{B}\left(1+\frac{\delta-\delta_-}{d_+-\delta},1+\delta_-,c^{1/(p-1)}\left(1-\frac{\delta_-}{p-1}\right)\right).\end{aligned}$$

By (5), the first summand above is nonpositive, while the second equals

$$\left(\frac{d_+-\delta_-}{d_+-\delta}\right)^{r-1}\frac{1+\delta_-}{1+\delta}\mathcal{B}\left(1,1+\delta,c^{1/(p-1)}\left(1-\frac{\delta}{p-1}\right)\right),$$

in view of (12). Consequently, we obtain

$$\left[1-\left(\frac{d_+-\delta_-}{d_+-\delta}\right)^{r-1}\frac{1+\delta_-}{1+\delta}\right]\mathcal{B}\left(1,1+\delta,c^{1/(p-1)}\left(1-\frac{\delta}{p-1}\right)\right)\leq 0.$$

Now, if $r<d_++1$, we divide both sides by δ and let $\delta\to 0$, and if $r=d_++1$, we divide by δ^2 and let $\delta\to 0$. In both cases, we get the estimate which is equivalent to $\mathcal{B}(1,1,c^{1/(p-1)})\leq 0$. The proof is complete. $\square$

3. Proof of (4)

We split the proof into three cases.

3.1. The case $c = 1$.

Here the estimate (4) is evident. Indeed, the inequality $||Y||_{A_p} \leq 1$ means that the martingale Y is constant almost surely and thus $\mathbb{P} = \mathbb{Q}$. Consequently, the bound (4) reduces to

$$||X^*||_{L^{r,\infty}(\mathbb{P})} \leq ||X_\infty||_{L^r(\mathbb{P})},$$

which is the classical estimate of Doob [5].

3.2. The case $p > 1$, $c > 1$.

If $r \leq d_+(p,c) + 1$, then the bound is obvious, since the constant C is infinite. Therefore, we may and do assume that r is strictly larger than $d_+(p,c) + 1$. We start with the following related result for nonnegative submartingales.

Lemma 3.1. *Assume that X is a nonnegative local submartingale and Y is a weight satisfying $||Y||_{A_p} \leq c$. Then there is a nondecreasing sequence $(\eta_n)_{n\geq 0}$ of stopping times, satisfying $\lim_{n\to\infty} \eta_n = \infty$ almost surely, such that for $n = 0, 1, 2, \ldots$ and $t \geq 0$ we have*

$$\mathbb{Q}(X^*_{\eta_n\wedge t} > 1) \leq C(p,r,c)^r \, \mathbb{E} X^r_{\eta_n\wedge t} Y_\infty 1_{\{X^*_{\eta_n\wedge t}\geq 1\}}. \tag{16}$$

Proof. Introduce the stopping time $\tau = \inf\{t \geq 0; |X_t| \geq 1\}$. Let N, A be the local martingale part and the predictable finite variation part of X, uniquely determined by the Doob-Meyer decomposition $X = X_0 + N + A$ and the equations $N_0 = A_0 \equiv 0$ (see e.g. Dellacherie and Meyer [4] for details). Put $Z_t = \mathbb{E}\big\{Y_\infty^{-1/(p-1)} \big| \mathcal{F}_t\big\}$ and $S_t = (X_t, Y_t, Z_t)$, $t \geq 0$. Since $||Y||_{A_p} \leq c$, we have $Y_t Z_t^{p-1} \leq c$ for all t, that is, the process S takes values in the set $\mathcal{D}_{p,c}$. The plan is to compose the process S with the function from the previous section and apply Itô's formula. To guarantee the necessary smoothness of the function B, we slightly increase c, that is, we fix $\varepsilon > 0$ and let $B = B_{p,r,c+\varepsilon}$. Then the function B is of class C^∞ in the interior of $\mathcal{D}_{p,c+\varepsilon}$ and continuous on this domain. Furthermore, if S reaches the set $\{(x,y,z); x \geq 0,\, yz^{p-1} = 1\}$ (the "lower" boundary of $\mathcal{D}_{p,c+\varepsilon}$), then its second and third coordinate terminate, in view of Jensen's inequality, and the only possible movement of S is through its first coordinate X. But for any fixed y, z with $1 \leq yz^{p-1} \leq c$, the function $x \mapsto B(x,y,z)$ is of class C^2 on $(0,\infty)$. This analysis shows that we are allowed to apply Itô's formula. As the result, we obtain

$$B(S_t) = B(S_{\tau\wedge t}) + I_1 + I_2/2 + I_3 + I_4, \tag{17}$$

where, because of the path-continuity of Y,

$$\begin{aligned} I_1 &= \int_{\tau\wedge t+}^t B_x(S_{s-})\mathrm{d}A_s,\\ I_2 &= \int_{\tau\wedge t+}^t \left[D^2B(S_{s-})\mathrm{d}S_s, \mathrm{d}S_s\right]^c,\\ I_3 &= \sum_{\tau\wedge t<s\leq t} \Big[B(S_s) - B(S_{s-}) - B_x(S_{s-})\Delta X_s - B_z(S_{s-})\Delta Z_s\Big],\\ I_4 &= \int_{\tau\wedge t+}^t B_x(S_{s-})\mathrm{d}N_s + \int_{\tau\wedge t+}^t B_y(S_{s-})\mathrm{d}Y_s + \int_{\tau\wedge t+}^t B_z(S_{s-})\mathrm{d}Z_s. \end{aligned}$$

Here in the definition of I_2 we have used the shortened notation for second-order term from Itô's formula. More precisely, it should read

$$I_2 = \int_{\tau\wedge t+}^t B_{xx}(S_{s-})\mathrm{d}[N_s^c, N_s^c] + 2\int_{\tau\wedge t+}^t B_{xy}(S_{s-})\mathrm{d}[N_s, Y_s]^c + \dots$$

and so on. Let us analyze the terms I_1 through I_4 separately. We have $I_1 \geq 0$, because $B_x \geq 0$ and the process A is nondecreasing. The term I_2 is also nonnegative, which follows immediately from Lemma 2.3. Next, observe that $I_3 \geq 0$, which is again a direct consequence of that lemma, i.e., the function $u \mapsto B(X_{s-} + u\Delta X_s, Y_s, Z_{s-} + u\Delta Z_s)$ is convex on $[0, 1]$ and hence each summand in I_3 is nonnegative. To deal with I_4, note that the process

$$\left(\int_{0+}^t B_x(S_{s-})\mathrm{d}N_s + \int_{0+}^t B_y(S_{s-})\mathrm{d}Y_s + \int_{0+}^t B_z(S_{s-})\mathrm{d}Z_s\right)_{t\geq 0} \tag{18}$$

is a local martingale, and is localized by the sequence

$$\eta_n = \inf\{t \geq 0; X_t + Y_t + Z_t + |N_t| \geq n\}. \tag{19}$$

Thus, replacing in (17) the time t with $\eta_n \wedge t$, using the above facts and taking expectation of both sides give

$$\mathbb{E}B(S_{\eta_n\wedge t}) \geq \mathbb{E}B(S_{\eta_n\wedge\tau\wedge t}). \tag{20}$$

However,

$$\begin{aligned} \mathbb{E}B(S_{\eta_n\wedge t}) &= \mathbb{E}X^r_{\eta_n\wedge t}Y_{\eta_n\wedge t}\varphi\left(Y^{1/(p-1)}_{\eta_n\wedge t}Z_{\eta_n\wedge t}\right)\\ &= \mathbb{E}X^r_{\eta_n\wedge t}\mathbb{E}\left[Y_\infty|\mathcal{F}_{\eta_n\wedge t}\right]\varphi\left(Y^{1/(p-1)}_{\eta_n\wedge t}Z_{\eta_n\wedge t}\right)\\ &= \mathbb{E}X^r_{\eta_n\wedge t}Y_\infty\varphi\left(Y^{1/(p-1)}_{\eta_n\wedge t}Z_{\eta_n\wedge t}\right) \end{aligned}$$

and, similarly,

$$\mathbb{E}B(S_{\eta_n\wedge\tau\wedge t}) = \mathbb{E}X^r_{\eta_n\wedge\tau\wedge t}Y_\infty\varphi\left(Y^{1/(p-1)}_{\eta_n\wedge\tau\wedge t}Z_{\eta_n\wedge\tau\wedge t}\right).$$

Of course, on the set $\{\tau > \eta_n \wedge t\}$ we have

$$X^r_{\eta_n\wedge t}Y_\infty\varphi\left(Y^{1/(p-1)}_{\eta_n\wedge t}Z_{\eta_n\wedge t}\right) = X^r_{\eta_n\wedge\tau\wedge t}Y_\infty\varphi\left(Y^{1/(p-1)}_{\eta_n\wedge\tau\wedge t}Z_{\eta_n\wedge\tau\wedge t}\right). \tag{21}$$

However, by Lemma 2.2 and (10) we have

$$C(p,r,c+\varepsilon)^{-r}=\varphi((c+\varepsilon)^{1/(p-1)})\leq\varphi(s)\leq\varphi(1)=1$$

for $s\in[1,(c+\varepsilon)^{1/(p-1)}]$. Therefore, combining (20) and (21), we get

$$\mathbb{E}X^r_{\eta_n\wedge\tau\wedge t}Y_\infty 1_{\{\tau\leq\eta_n\wedge t\}}\leq C(p,r,c+\varepsilon)^r\,\mathbb{E}X^r_{\eta_n\wedge t}Y_\infty 1_{\{\tau\leq\eta_n\wedge t\}}. \tag{22}$$

On the other hand, by the definition of τ, we have $X^r_{\eta_n\wedge\tau\wedge t}1_{\{\tau\leq\eta_n\wedge t\}}\geq 1_{\{\tau\leq\eta_n\wedge t\}}$ and $\{X^*_{\eta_n\wedge t}>1\}\subseteq\{\tau\leq\eta_n\wedge t\}\subseteq\{X^*_{\eta_n\wedge t}\geq 1\}$, so (22) yields

$$\mathbb{Q}(X^*_{\eta_n\wedge t}>1)\leq C(p,r,c+\varepsilon)^r\mathbb{E}X^r_{\eta_n\wedge t}Y_\infty 1_{\{X^*_{\eta_n\wedge t}\geq 1\}}.$$

It remains to let $\varepsilon\to 0$ to get the claim. □

Next, we will need the following statement, already mentioned in Introduction (see Uchiyama [14], consult also Kazamaki [8, Theorem 3.15]).

THEOREM 3.2. *Fix a weight Y and let $1\leq r<\infty$. Suppose that for any bounded martingale X and any $\lambda>0$ we have*

$$\lambda^r\mathbb{Q}(X^*>\lambda)\leq C\mathbb{E}|X_\infty|^rY_\infty 1_{\{X^*\geq\lambda\}}, \tag{23}$$

where C depends only on Y and r. Then $||Y||_{A_r}<\infty$.

We are ready to establish the first part of Theorem 1.1.

Proof of (4). Assume first that the martingale X is bounded and let $\lambda>0$. Applying (16) to the nonnegative submartingale $(|X_t|/\lambda)_{t\geq 0}$ and letting $n\to\infty$, $t\to\infty$ yield (23) by Lebesgue's dominated convergence theorem. Thus $||Y||_{A_r}<\infty$, in view of Theorem 3.2.

Next, we drop the assumption on the boundedness of X. Of course, we may restrict ourselves to martingales satisfying $||X_\infty||_{L^r(\mathbb{Q})}<\infty$, since otherwise there is nothing to prove. For a given $\lambda>0$, apply (16) to the exponent $s\in(d_+(p,c)+1,r)$ and the nonnegative submartingale $(|X|_t/\lambda)_{t\geq 0}$. A careful inspection of the proof of the Lemma 3.1 shows that there is a sequence $(\eta_n)_{n\geq 0}$ which works simultaneously for all these submartingales; for instance, one may take

$$\eta_n=\inf\{t\geq 0;|X_t|+Y_t+Z_t+|N_t|\geq n\}.$$

Therefore, for all $n=0,\,1,\,2,\,\ldots$, $t\geq 0$ and $\lambda>0$ we have

$$\lambda^s\mathbb{Q}(X^*_{\eta_n\wedge t}>\lambda)\leq C(p,s,c)^s\mathbb{E}|X_{\eta_n\wedge t}|^sY_\infty 1_{\{X^*_{\eta_n\wedge t}\geq\lambda\}}.$$

Multiply both sides by λ^{r-s-1} and integrate over λ from 0 to $\alpha>0$. We obtain

$$\begin{aligned}||X^*_{\eta_n\wedge t}\wedge\alpha||^r_{L^r(\mathbb{Q})}&\leq\frac{r}{r-s}C(p,s,c)^s\mathbb{E}|X_{\eta_n\wedge t}|^s(X^*_{\eta_n\wedge t}\wedge\alpha)^{r-s}Y_\infty\\&\leq\frac{r}{r-s}C(p,s,c)^s||X_{\eta_n\wedge t}||^s_{L^r(\mathbb{Q})}||X^*_{\eta_n\wedge t}\wedge\alpha||^{r-s}_{L^r(\mathbb{Q})}.\end{aligned}$$

Letting $\alpha \to \infty$ yields the estimate

$$||X^*_{\eta_n\wedge t}||_{L^r(\mathbb{Q})} \leq \left(\frac{r}{r-s}\right)^{1/s} C(p,s,c)||X_{\eta_n\wedge t}||_{L^r(\mathbb{Q})}. \tag{24}$$

On the other hand, we have

$$\begin{aligned}|X_{\eta_n\wedge t}|^r &\leq \mathbb{E}\left[\left\{\frac{Y_{\eta_n\wedge t}}{Y_\infty}\right\}^{1/r}\cdot\left\{\frac{Y_\infty}{Y_{\eta_n\wedge t}}\right\}^{1/r}|X_\infty|\Big|\mathcal{F}_{\eta_n\wedge t}\right]\\ &\leq \mathbb{E}\left[\left\{\frac{Y_{\eta_n\wedge t}}{Y_\infty}\right\}^{1/(r-1)}\Big|\mathcal{F}_{\eta_n\wedge t}\right]^{r-1}\mathbb{E}\left[\frac{Y_\infty}{Y_{\eta_n\wedge t}}|X_\infty|^r\Big|\mathcal{F}_{\eta_n\wedge t}\right],\end{aligned}$$

where in the second passage we have exploited Hölder's inequality. Multiplying throughout by $Y_{\eta_n\wedge t}$ and taking expectation, we get

$$\mathbb{E}|X_{\eta_n\wedge t}|^r Y_\infty = \mathbb{E}|X_{\eta_n\wedge t}|^r Y_{\eta_n\wedge t} \leq ||Y||_{A_r}\mathbb{E}|X_\infty|^r Y_\infty.$$

Thus, by (24), the quantity $||X^*_{\eta_n\wedge t}||_{L^r(\mathbb{Q})}$ is bounded from above by a constant multiple of $||X_\infty||_{L^r(\mathbb{Q})}$. This implies $||X^*||_{L^r(\mathbb{Q})} < \infty$ in view of Lebesgue's monotone convergence theorem. Therefore, applying (16) to the nonnegative submartingale $(|X_t|/\lambda)_{t\geq 0}$ and letting $n\to\infty$, $t\to\infty$ give

$$\lambda^r\mathbb{Q}(X^*>\lambda) \leq C(p,r,c)^r\mathbb{E}|X_\infty|^r Y_\infty 1_{\{X^*\geq\lambda\}}$$

by Lebesgue's dominated convergence theorem. This yields the desired inequality (4). $\square$

3.3. **The case $p=1$, $c>1$.** If $r=1$, then the estimate (4) follows from Tsuchikura's bound (1), so we may assume that $r>1$. This case can be easily deduced from the previous one by a limiting procedure. Namely, as we have already mentioned in Introduction, we have $||Y||_{A_s} \leq ||Y||_{A_1} \leq c$ for any $s>1$, and thus, for any $1<s<r$,

$$||X^*||_{L^{r,\infty}(\mathbb{Q})} \leq C(s,r,c)||X_\infty||_{L^r(\mathbb{Q})} \tag{25}$$

by virtue of the previous case. The equation $F_{s,c}(d_-)=1$ is equivalent to

$$c(1+d_-)\left(1-\frac{d_-}{s-1}\right)^{s-1} = 1.$$

Hence, if $s\downarrow 1$, then $d_-(s,c)\to c^{-1}-1$ and

$$C(s,r,c) \to \left(\left(1-\frac{||Y||_{A_1}^{-1}-1}{r-1}\right)^{r-1}||Y||_{A_1}^{-1}\right)^{-1/r} = C(1,r,c).$$

Therefore (25) yields (4), and we are done.

4. Sharpness

We consider four cases separately.

4.1. **The case $c = 1$.** Here the sharpness of (4) is evident. For example, it suffices to consider the constant martingales $X = Y \equiv 1$.

4.2. **The case $p > 1$, $r \geq 1$, $c > 1$.** We will prove the following converse to the results of Section 3: the validity of (4) for a class of weights with (A_p)-norm not exceeding c implies the existence of a locally convex function $\mathbb{B}$ on $\mathcal{D}_{p,c}$ satisfying the majorization (5). Then, as we will see below, the lower bound for C will follow from the second part of Section 2.

We start with the following crucial observation. Suppose that Y is a weight satisfying $||Y||_{A_p} \leq c$ and define $Z = \big(\mathbb{E}\big[Y_\infty^{-1/(p-1)}|\mathcal{F}_t\big]\big)_{t\geq 0}$. Then (Y, Z) is a pair of uniformly integrable martingales, which takes values in $\{(y, z); 1 \leq yz^{p-1} \leq c\}$ and terminates at the lower boundary of this set, $Y_\infty Z_\infty^{p-1} = 1$ almost surely. The key fact is that the implication can be reversed. That is, if (Y, Z) is a pair of uniformly integrable martingales (with Y continuous and positive) taking values in $\{(y, z); 1 \leq yz^{p-1} \leq c\}$ and terminating at the lower boundary of this set, then $||Y||_{A_p} \leq c$. Indeed, for any stopping time τ,

$$\mathbb{E}\left[\left\{Y_\tau/Y_\infty\right\}^{1/(p-1)}|\mathcal{F}_\tau\right]^{p-1} = Y_\tau Z_\tau^{p-1} \leq c.$$

The next step is to introduce an abstract class of special processes. Namely, for any $c > 1$ and any (u, v) satisfying $1 \leq uv^{p-1} \leq c$, let $\mathcal{T}(u, v, c)$ consist of all positive, bounded continuous-path martingales ξ satisfying $||\xi||_{A_p} \leq c$, $\xi_0 = u$ and $\mathbb{E}\xi_\infty^{-1/(p-1)} = v$. Here we allow both the filtration and the probability space to vary.

Lemma 4.1. *The class $\mathcal{T}(u, v, c)$ is nonempty for each u, v and c.*

Proof. This is quite straightforward. Observe that there exists an interval $I \subset \{(y, z); 1 \leq yz^{p-1} \leq c\}$, passing through (u, v), with endpoints (y_-, z_-), (y_+, z_+) satisfying $y_\pm z_\pm^{p-1} = 1$. Let (ξ, ζ) be a continuous martingale starting from (u, v), taking values in this interval and terminating at its endpoints. Then ξ satisfies $\xi_0 = u$ and $||\xi||_{A_p} \leq c$ by the reasoning presented above. Furthermore, ζ is uniformly integrable, so $\mathbb{E}\xi_\infty^{-1/(p-1)} = \mathbb{E}\zeta_\infty = \zeta_0 = v$. This completes the proof. □

Now, for each $x \geq 0$, let $\mathcal{M}(x)$ denote the class of all bounded, nonnegative and continuous-path martingales starting from x. Define $\mathbb{B} : \mathcal{D}_{p,c} \to \mathbb{R}$ by the formula

$$\mathbb{B}(x, y, z) = \inf\Big\{\mathbb{E}X_\infty^r \xi_\infty; \xi \in \mathcal{T}(y, z, c),\ X \in \mathcal{M}(x)\Big\}.$$

This function has the following two properties.

Lemma 4.2. (i) *We have $\mathbb{B}(x, y, z) \leq x^r y$ if $yz^{p-1} = 1$.*
(ii) *The function $\mathbb{B}$ is locally convex.*

Proof. (i) This follows immediately from the fact that the constant martingale $X \equiv x$ belongs to $\mathcal{M}(x)$ and that the constant martingale $\xi \equiv y$ belongs to $\mathcal{T}(y, z, c)$ provided $yz^{p-1} = 1$.

(ii) This part is slightly more elaborate and can be regarded as a modification of the so-called "splicing argument" (see e.g. Burkholder [1] or the author [11]). Pick an interval $I \subset \mathcal{D}_{p,c}$, with endpoints (x_-, y_-, z_-) and (x_+, y_+, z_+), and choose a point (x, y, z) in its interior. Next, take $X^\pm \in \mathcal{M}\big(x_\pm\big)$, $\xi^\pm \in \mathcal{T}\big(y_\pm, z_\pm, c\big)$ and define $\zeta_t^\pm = \mathbb{E}((\xi_\infty^\pm)^{-1/(p-1)}|\mathcal{F}_t)$ for $t \geq 0$. We will "glue" X^- with X^+ and ξ^- with ξ^+. To do this, consider an independent, three-dimensional continuous-path martingale $(S_t, \phi_t, \psi_t)_{t\geq 0}$ starting from (x, y, z), taking values in I and terminating at the endpoints of this interval. Let

$$\tau = \inf \big\{t; (S_t, \phi_t, \psi_t) \in \{(x_-, y_-, z_-), (x_+, y_+, z_+)\}\big\}$$

and define

$$(X_t, \xi_t, \zeta_t) = \begin{cases} (S_t, \phi_t, \psi_t) & \text{if } t \leq \tau, \\ (X_{t-\tau}^-, \xi_{t-\tau}^-, \zeta_{t-\tau}^-) & \text{if } t > \tau,\ (S_\tau, \phi_\tau, \psi_\tau) = (x_-, y_-, z_-), \\ X_{t-\tau}^+, \xi_{t-\tau}^+, \zeta_{t-\tau}^+) & \text{if } t > \tau,\ (S_\tau, \phi_\tau, \psi_\tau) = (x_+, y_+, z_+). \end{cases}$$

In other words, up to time τ, the triple (X, ξ, ζ) behaves in the same manner as (S, ϕ, ψ), while for $t > \tau$ the evolution of the triple coincides with that of (X^-, ξ^-, ζ^-) or that of (X^+, ξ^+, ζ^+), depending on the value of (S, ϕ, ψ) at time τ. Let $(\overline{\mathcal{F}}_t)_{t\geq 0}$ denote the completion of the natural filtration of this new process. Then X is a bounded $(\overline{\mathcal{F}}_t)_{t\geq 0}$-martingale starting from x and the pair (ξ, ζ) forms a continuous-path bounded $(\overline{\mathcal{F}}_t)_{t\geq 0}$-martingale starting from (y, z) and satisfying $\xi_\infty \zeta_\infty^{p-1} = 1$ with probability 1. Consequently, by the reasoning from the beginning of this subsection, we get $||\xi||_{A_p} \leq c$. By the independence of (S, ϕ, ψ) from $X^\pm$ and $\xi^\pm$, we have

$$\mathbb{E}X_\infty^r \xi_\infty = s\mathbb{E}(X_\infty^-)^r \xi_\infty^- + (1-s)\mathbb{E}(X_\infty^+)^r \xi_\infty^+,$$

where $s = \mathbb{P}((S_\tau, \phi_\tau, \psi_\tau) = (x_-, y_-, z_-))$, that is, s satisfies the equality

$$(x, y, z) = s(x_-, y_-, z_-) + (1-s)(x_+, y_+, z_+).$$

However, $\mathbb{B}(x, y, z) \leq \mathbb{E}X_\infty^r \xi_\infty$, so taking infimum over all $\xi^\pm$ and $X^\pm$ gives

$$\mathbb{B}(x, y, z) \leq s\mathbb{B}(x_-, y_-, z_-) + (1-s)\mathbb{B}(x_+, y_+, z_+),$$

which is the desired convexity. □

Now the sharpness of (4) follows immediately from the considerations presented in the second half of Section 2. Namely, fix $c > 1$ and suppose that (4) holds with some constant $C > 0$ for all martingales X and all weights satisfying $||Y||_{A_p} \leq c$. Pick a martingale $X \in \mathcal{M}(1)$ and a weight Y as above, satisfying $\mathbb{E}Y_\infty^{-1/(p-1)} = c^{1/(p-1)}$ (i.e., an element of $\mathcal{T}(1, c^{1/(p-1)}, c)$). Then $\mathbb{Q}(X^* > \lambda) = 1$ for any $\lambda < 1$, so $||X^*||_{L^{r,\infty}} \geq 1$ and thus $C^r \mathbb{E}X_\infty^r Y_\infty \geq 1$. Taking infimum over all such X and Y, we get $\mathbb{B}(1, 1, c^{1/(p-1)}) \geq C^{-r}$. Now, if $r \leq d_+(p, c) + 1$, we get a contradiction. Namely, by Theorem 2.5 we have

$\mathbb{B}(1,1,c^{1/(p-1)}) \leq 0$ and hence the inequality (4) cannot hold with any finite constant C. On the other hand, if $r > d_+(p,c)+1$, then by Theorem 2.4 and (10),

$$C \geq \mathbb{B}(1,1,c^{1/(p-1)})^{-1/r} \geq B(1,1,c^{1/(p-1)})^{-1/r} = C(p,r,c),$$

so $C(p,r,c)$ is indeed the best possible.

4.3. **The case $p=1$, $r>1$, $c>1$.** This time we will construct an explicit example. Let B be a standard one-dimensional Brownian motion starting from the origin and let $\mathcal{E}(B) = \big(\exp(B_t - t/2)\big)_{t\geq 0}$ denote its exponential process. For a given $\delta \in (0, c-1)$, introduce the family $(\tau_n)_{n\geq 0}$ of stopping times by $\tau_0 \equiv 0$ and, inductively,

$$\tau_n = \inf\{t \geq \tau_{n-1}; \mathcal{E}(B)_t \leq (1+\delta)^n/c \text{ or } \mathcal{E}(B)_t = (1+\delta)^n\}$$

for $n = 1, 2, \ldots$. Clearly, we have

$$\mathbb{P}\big(\mathcal{E}(B)_{\tau_n} \leq (1+\delta)^n/c \text{ for some } n\big) = 1$$

(see (26) below). Furthermore, if $\mathcal{E}(B)_{\tau_n} \leq (1+\delta)^n/c$, then automatically $\mathcal{E}(B)_{\tau_n} \leq (1+\delta)^m/c$ for each $m > n$ and hence $\tau_m = \tau_n$. Let $\tau = \lim_{n\to\infty} \tau_n$ and consider the stopped process $\mathcal{E}(B)^\tau$. This process is a uniformly integrable martingale, since

$$\begin{aligned}\mathbb{P}((\mathcal{E}(B)^\tau)^* \geq (1+\delta)^n) = \mathbb{P}(\tau > \tau_n) &= \left(\frac{1-(1+\delta)/c}{(1+\delta)-(1+\delta)/c}\right)^n \\ &= \left(\frac{c-1-\delta}{(1+\delta)(c-1)}\right)^n,\end{aligned} \tag{26}$$

which implies $\mathbb{E}(\mathcal{E}(B)^\tau)^* < \infty$. Next, observe that $||\mathcal{E}(B)^\tau||_{A_1} \leq c$. Indeed, on the set $\{\tau_{n-1} < \tau = \tau_n\}$, we have $\mathcal{E}(B)^\tau_\infty = (1+\delta)^n/c$ and $(\mathcal{E}(B)^\tau)^* \leq (1+\delta)^n$, which implies $||(\mathcal{E}(B)^\tau)^*/\mathcal{E}(B)^\tau_\infty||_\infty \leq c$. Summarizing, $\mathcal{E}(B)^\tau$ is a weight satisfying the Muckenhoupt's (A_1) condition with the constant c.

Now, introduce the martingale $X = X^{(r)}$ by the stochastic integral

$$X_t = 1 + \int_{0+}^t K_s \mathrm{d}\mathcal{E}(B)^\tau_s,$$

where the predictable process K is given by

$$K_s = -\frac{1}{(r-1)\mathcal{E}(B)_{\tau_{n-1}}} \prod_{k=1}^{n-1} \left(\frac{r}{r-1} - \frac{\mathcal{E}(B)_{\tau_k}}{(r-1)\mathcal{E}(B)_{\tau_{k-1}}}\right)$$

for $s \in [\tau_{n-1}, \tau_n)$ and $K_s = 0$ for $s > \tau$ (for $n = 1$, we set the above product to be equal to 1). Then

$$\begin{aligned}X_{\tau_n} - X_{\tau_{n-1}} &= -\frac{\mathcal{E}(B)_{\tau_n} - \mathcal{E}(B)_{\tau_{n-1}}}{(r-1)\mathcal{E}(B)_{\tau_{n-1}}} \prod_{k=1}^{n-1} \left(\frac{r}{r-1} - \frac{\mathcal{E}(B)_{\tau_k}}{(r-1)\mathcal{E}(B)_{\tau_{k-1}}}\right) \\ &= \prod_{k=1}^{n} \left(\frac{r}{r-1} - \frac{\mathcal{E}(B)_{\tau_k}}{(r-1)\mathcal{E}(B)_{\tau_{k-1}}}\right) - \prod_{k=1}^{n-1} \left(\frac{r}{r-1} - \frac{\mathcal{E}(B)_{\tau_k}}{(r-1)\mathcal{E}(B)_{\tau_{k-1}}}\right),\end{aligned}$$

which yields the identity

$$X_{\tau_n} = \prod_{k=1}^{n} \left(\frac{r}{r-1} - \frac{\mathcal{E}(B)_{\tau_k}}{(r-1)\mathcal{E}(B)_{\tau_{k-1}}} \right)$$

for $n = 1, 2, \ldots$. In consequence, on the set $\{\tau_{n-1} < \tau = \tau_n\}$ we have

$$X_\infty = X_{\tau_n} = \left(\frac{r}{r-1} - \frac{1+\delta}{r-1} \right)^{n-1} \left(\frac{r}{r-1} - \frac{1+\delta}{(r-1)c} \right).$$

Since $\mathcal{E}(B)^\tau_\infty = (1+\delta)^n/c$ on this set, and

$$\mathbb{P}(\tau_{n-1} < \tau = \tau_n) = \mathbb{P}(\tau > \tau_{n-1}) - \mathbb{P}(\tau > \tau_n) = \left(\frac{c-1-\delta}{(1+\delta)(c-1)} \right)^{n-1} \frac{c\delta}{(1+\delta)(c-1)}$$

(see (26)), we conclude that

$$\begin{aligned} \mathbb{E}X^r_\infty \mathcal{E}(B)^\tau &= \sum_{n=1}^{\infty} \mathbb{E}\Big[X^r_\infty \mathcal{E}(B)^\tau 1_{\{\tau_{n-1}<\tau=\tau_n\}} \Big] \\ &= \left(\frac{r}{r-1} - \frac{1+\delta}{(r-1)c} \right)^r \frac{\delta}{c-1} \sum_{n=1}^{\infty} \left(1 - \frac{\delta}{r-1} \right)^{(n-1)r} \left(1 - \frac{\delta}{c-1} \right)^{n-1} \\ &= \left(\frac{r}{r-1} - \frac{1+\delta}{(r-1)c} \right)^r \frac{\delta}{c-1} \left[1 - \left(1 - \frac{\delta}{r-1} \right)^r \left(1 - \frac{\delta}{c-1} \right) \right]^{-1}. \end{aligned}$$

A straightforward analysis shows that if δ is sufficiently small, then the above expression can be made arbitrarily close to $((cr-1)/(r-1))^{r-1}\, c^{-r} = C(1,r,c)^{-r}$. Since X starts from 1, we have $||X^*||_{L^{r,\infty}(\mathbb{Q})} \geq 1$ and thus the constant $C(1,r,c)$ cannot be replaced in (4) by a smaller one.

4.4. **The case $p = r = 1$, $c > 1$.** Pick $s > 1$, $\delta \in (0, c-1)$ and consider the process $X = X^{(s)}$ and the weight $\mathcal{E}(B)^\tau$ as in the previous subsection. Repeating word by word the above calculations, we get that

$$\mathbb{E}X_\infty \mathcal{E}(B)^\tau = \left(\frac{s}{s-1} - \frac{1+\delta}{(s-1)c} \right) \frac{\delta}{c-1} \left[1 - \left(1 - \frac{\delta}{s-1} \right) \left(1 - \frac{\delta}{c-1} \right) \right]^{-1}.$$

If $\delta \to 0$, then the expression on the right converges to $(sc-1)/(c(s+c-2))$; this, in turn, tends to c^{-1} as $s \to 1$. Therefore, if s is sufficiently close to 1 and δ is sufficiently close to 0, then $\mathbb{E}X_\infty \mathcal{E}(B)^\tau$ can be as close to c^{-1} as we wish. On the other hand, $X_0 \equiv 1$, so $||X||_{L^{1,\infty}(\mathbb{Q})} \geq 1$ and hence the constant C is the best possible. This completes the proof.

4.5. **On Uchiyama's identity for $p \neq r$.** The final part of the paper is devoted to a very interesting question, raised by the referee, concerning the formula (2). Namely, given two different numbers p, $r > 1$ and an (A_p) weight Y, is it true that

$$C(p, r, ||Y||_{A_p}) = \sup_{X \in L^r(\mathbb{Q})} \Big[||X^*||_{L^{r,\infty}(\mathbb{Q})} / ||X||_{L^r(\mathbb{Q})} \Big]?$$

Of course, by the results of this paper, the left-hand side is not smaller than the expression on the right. Is the reverse bound also valid? The answer turns out to be negative. We will construct an appropriate example for $p=2$, $r=3$ and $||Y||_{A_p}=4/3$. Let Y be a Brownian motion starting from 1, stopped at the exit time from the interval $[1/2,3/2]$. Then the positive martingales Y, $Z=(8/3-4Y_t/3)_{t\geq 0}$ are bounded, the pair (Y,Z) takes values in the set $\{(y,z);1\leq yz\leq 4/3\}$ and $Y_0Z_0=4/3$. This implies $||Y||_{A_2}=4/3$, see the reasoning at the beginning of Subsection 4.2. Next, pick a uniformly integrable martingale X satisfying $||X_\infty||_{L^3(\mathbb{Q})}<\infty$. Then $||X^*||_{L^3(\mathbb{Q})}$ is also finite: see the proof of (4) in Section 3. Let

$$a=\sqrt{\frac{2}{3}}-\sqrt{2}<0,\qquad b=\frac{3}{2}\sqrt{2}-\frac{1}{2}\sqrt{\frac{2}{3}}>0$$

and consider the function $u:\mathbb{R}\times(-\infty,-b/a)\to\mathbb{R}$ given by $u(x,y)=|x|^3(ay+b)^{-2}$. This function satisfies $u(x,1/2)=|x|^3/2$ and $u(x,3/2)=3|x|^3/2$, so $u(X_\infty,Y_\infty)=|X_\infty|^3Y_\infty$ almost surely. Furthermore, u is of class C^2 and its Hessian matrix

$$D^2u(x,y)=\begin{bmatrix} 6|x|(ay+b)^{-2} & -6a|x|x(ay+b)^{-3}\\ -6a|x|x(ay+b)^{-3} & 6a^2|x|^3(ay+b)^{-4}\end{bmatrix}$$

is nonnegative definite. Hence, arguing as in the proof of Lemma 3.1, we show that the process $(u(X_t,Y_t))_{t\geq 0}$ is a local submartingale. In fact it is uniformly integrable, since Y is bounded and $X^*\in L^3(\mathbb{Q})$. Therefore, if we put $\tau=\inf\{t;|X_t|\geq 1\}$, then

$$\mathbb{E}|X_\infty|^3Y_\infty=\mathbb{E}u(X_\infty,Y_\infty)\geq\mathbb{E}u(X_\tau,Y_\tau)=\mathbb{E}\left\{\frac{|X_\tau|^3\mathbb{E}(Y_\infty|\mathcal{F}_\tau)}{Y_\tau(aY_\tau+b)^2}\right\}=\mathbb{E}\left\{\frac{|X_\tau|^3Y_\infty}{Y_\tau(aY_\tau+b)^2}\right\}.$$

However, we have $0<y(ay+b)^2\leq -4b^3/(27a)$ for $y\in[1/2,3/2]$ and $\mathbb{E}|X_\tau|^3Y_\infty\geq\mathbb{Q}(X^*>1)$, and hence

$$||X_\infty||^3_{L^3(\mathbb{Q})}\geq-\frac{27a}{4b^3}\mathbb{Q}(X^*>1)=\frac{81\sqrt{3}-81}{(3\sqrt{3}-1)^3}\mathbb{Q}(X^*>1).$$

Using homogenization and the fact that X was arbitrary, we get

$$\sup_{X\in L^3(\mathbb{Q})}\frac{||X^*||_{L^{3,\infty}(\mathbb{Q})}}{||X_\infty||_{L^3(\mathbb{Q})}}\leq\frac{3\sqrt{3}-1}{(81\sqrt{3}-81)^{1/3}}=1.076\ldots,$$

which is strictly smaller than $C(2,3,4/3)=(32/25)^{1/3}=1.085\ldots$ (see Introduction).

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