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WEIGHTED NORM INEQUALITIES FOR AVERAGING OPERATORS OF MONOTONE FUNCTIONS

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Abstract

We prove weighted norm inequalities for the averaging operator $Af(x) = \frac{1}{x}\int_0^x f$ of monotone functions.

1. Introduction

This paper is concerned with weighted Hardy type inequalities of the form

$$(*) \qquad \int_0^\infty (\frac{1}{x}\int_0^x f)^p w(x)dx \le c\int_0^\infty f(x)^p v(x)dx.$$

Muckenhoupt [6] has given necessary and sufficient conditions for $(*)$ to hold for arbitrary f.

In their paper [1] Ariño and Muckenhoupt studied the problem when the Hardy-Littlewood maximal operator is bounded on Lorentz spaces and observed that this leads to the study of $(*)$ for non-increasing f. There are more weights in this case than for general f [1]. They solved the problem for $w = v$ by the condition B_p, i.e., $w \in B_p$ if and only if $\int_r^\infty \left(\frac{r}{x}\right)^p w(x)dx \le c\int_0^r w(x)dx$, $r > 0$. The proof is rather lengthy and first establishes that B_p implies $B_{p-\epsilon}$ (Lemma 2.1 of [1]).

The purpose of this paper is

(i) to give a much shorter proof of a somewhat more general version of $(*)$ without B_p implies $B_{p-\epsilon}$,

(ii) to prove then B_p implies $B_{p-\epsilon}$ using an iterated version of $(*)$,

(iii) to investigate the reverse inequalities

$$\int_0^\infty f(x)^p w(x)dx \le c\int_0^\infty (\frac{1}{x}\int_0^x f)^p v(x)dx,$$

(iv) to study the same questions for non-decreasing functions, and finally

(v) to present some properties of B_p-weights suggested by the analogous properties of A_p-weights as, e.g. the $A_1 \cdot A_1^{1-p}$ factorization of an A_p-weight [3].

We point out that the double weight inequality $(*)$ has been characterized in a recent paper by E. Sawyer [7] for non-increasing functions with the q-norm of the averaging operator on the left and the p-norm on the right. It is also possible to prove some of our results by the methods developed in the paper by D.W. Boyd [2].

Throughout the paper we shall use the following notation. The symbol $f\uparrow$ $(f\downarrow)$ means $f:\mathbb{R}_+\to\mathbb{R}_+$ non-decreasing (non-increasing). For $f\downarrow$ we define $f^{-1}(t)=\inf\{\tau : f(\tau)\le t\}$ with an analogous statement for $f\uparrow$. In proving $(*)$ for monotone functions we may restrict ourselves to homeomorphisms since a general monotone function can be approximated by homeomorphisms. For $0<r<\infty$, let $\chi_r(x)=\chi_{[0,r]}(x)$ and $\chi^r(x)=\chi_{[r,\infty)}(x)$. By a weight w we mean any measurable $w:\mathbb{R}_+\to\mathbb{R}_+$.

2. Non-increasing functions

For the norm inequalities for the averaging operator $Af(x)=\dfrac{1}{x}\displaystyle\int_0^x f$ we need the following lemma.

Lemma 2.1. *Let $\varphi\downarrow$ and let W be a weight. Then*

(i)
$$\int_0^\infty\int_0^\infty \chi_{\varphi(y)}(x)W(x)dx\,dy=\int_0^\infty \varphi^{-1}(x)W(x)dx$$

(ii)
$$\int_0^\infty\int_0^\infty \chi^{\varphi(y)}(x)\left(\frac{\varphi(y)}{x}\right)^p W(x)dx\,dy$$
$$=\int_0^\infty\left\{\frac{1}{x^p}\int_0^x \varphi^{-1}(u)d(u^p)-\varphi^{-1}(x)\right\}W(x)dx.$$

Proof: (i) We interchange the order of integration and get

$$\int_0^\infty\int_0^{\varphi^{-1}(x)} W(x)dy\,dx=\int_0^\infty \varphi^{-1}(x)W(x)dx.$$

(ii) The left side is, after interchanging the order of integration,

$$\int_0^\infty\int_{\varphi^{-1}(x)}^\infty \frac{W(x)}{x^p}(\varphi(y))^p dy\,dx$$

and the inner integral in y is

$$\int_{\varphi^{-1}(x)}^\infty(\varphi(y))^p dy=\int_0^{x^p}\varphi^{-1}(t^{1/p})dt-x^p\varphi^{-1}(x)$$
$$=\int_0^x\varphi^{-1}(u)d(u^p)-x^p\varphi^{-1}(x).$$

This can be seen by comparing areas of the regions under the curve $t = (\varphi(y))^p$ or $y = \varphi^{-1}(t^{1/p})$. ■

Definition. For $1 \le p < \infty$ and n a positive integer we write $(w,v) \in B(p,n)$ if and only if there is $0 < c < \infty$ such that for every choice $0 < r_1, r_2, \cdots, r_n < \infty$,

$$\int_0^\infty \left\{ \prod_1^n \left(\chi_{r_j}(x) + \chi^{r_j}(x) \left(\frac{r_j}{x}\right)^p \right) \right\} w(x)dx \le c \int_0^\infty \left\{ \prod_1^n \chi_{r_j}(x) \right\} v(x)dx.$$

Remark. (i) In case $w = v$, we simply write $w \in B(p,n)$.

(ii) If $n = 1$, then $(w,v) \in B(p,1)$ means $\int_0^r w + \int_r^\infty \left(\frac{r}{x}\right)^p w(x)dx \le c \int_0^r v$, $r > 0$. Hence, if $v = w$, we get the equivalent condition

$$\int_r^\infty \left(\frac{r}{x}\right)^p w(x)dx \le c \int_0^r w$$

introduced in [**1**] as B_p.

(iii) The smallest c in the above expressions will be referred to as the $B_p(w)$-constant of w or the $B(p,n)$-constant of (w,v).

(iv) If we let $r_n \to \infty$ we see that $B(p,n) \subset B(p,n-1)$.

Theorem 2.2. *Let $1 \le p < \infty$ and let $f_j \downarrow$, $j = 1, \cdots, n$. Then*

$$\int_0^\infty \left\{ \prod_1^n \left(\frac{1}{x} \int_0^x f_j \right)^p \right\} w(x)dx \le c \int_0^\infty \left\{ \prod_{j=1}^n f_j \left(\frac{1}{x} \int_0^x f_j \right)^{p-1} \right\} v(x)dx$$

if and only if $(w,v) \in B(p,n)$ with c equal to the $B(p,n)$-constant of (w,v).

Proof: If $f_j = \chi_{r_j}$, $j = 1, \cdots, n$, then the norm inequality easily gives $(w,v) \in B(p,n)$. We do the converse for $n = 2$; the general case is obtained by repeating the argument.

Let $\varphi_j \downarrow$, $j = 1,2$, and let $r_j = \varphi_j(y_j)$, where $0 < y_1, y_2 < \infty$. We next integrate the condition $B(p,n)$ over $\{(y_1,y_2) : y_1, y_2 > 0\}$ and obtain

$$L \equiv \int_0^\infty \int_0^\infty \int_0^\infty \psi_1(x,y_1)\psi_2(x,y_2)w(x)dx\,dy_1\,dy_2$$
$$\le c \int_0^\infty \int_0^\infty \int_0^\infty \chi_{\varphi_1(y_1)}(x)\chi_{\varphi_2(y_2)}(x)v(x)dx\,dy_1\,dy_2 \equiv R,$$

where $\psi_j(x,y_j) = \chi_{\varphi_j(y_j)}(x) + \chi^{\varphi_j(y_j)}(x)\left(\frac{\varphi_j(y_j)}{x}\right)^p$. By Lemma 2.1,

$$\begin{aligned} R &= \int_0^\infty \int_0^\infty \varphi_1^{-1}(x)\chi_{\varphi_2(y_2)}(x)v(x)dx\,dy_2 \\ &= \int_0^\infty \varphi_1^{-1}(x)\varphi_2^{-1}(x)v(x)dx. \end{aligned}$$

The inner 2 integrals of L can be written as

$$\begin{aligned} &\int_0^\infty \int_0^{\varphi_1(y_1)} \psi_2(x,y_2)w(x)dx\,dy_1 \\ &+ \int_0^\infty \int_{\varphi_1(y_1)}^\infty \psi_2(x,y_2)\left(\frac{\varphi_1(y_1)}{x}\right)^p w(x)dx\,dy_1 = I_1 + I_2. \end{aligned}$$

By (i) of Lemma 2.1 with $W = \psi_2 w$, $I_1 = \int_0^\infty \varphi_1^{-1}(x)\psi_2(x,y_2)w(x)dx$. Similarly, by (ii) of Lemma 2.1,

$$I_2 = \int_0^\infty \left\{\frac{1}{x^p}\int_0^x \varphi_1^{-1}(u)d(u^p) - \varphi_1^{-1}(x)\right\}\psi_2(x,y_2)w(x)dx.$$

Hence $I_1 + I_2 = \int_0^\infty \left\{\frac{1}{x^p}\int_0^x \varphi_1^{-1}(u)d(u^p)\right\}\psi_2(x,y_2)w(x)dx$. We integrate this expression in y_2 and repeat the argument to get

$$L = \int_0^\infty \left\{\frac{1}{x^p}\int_0^x \varphi_1^{-1}(u)d(u^p)\right\}\left\{\frac{1}{x^p}\int_0^x \varphi_2^{-1}(u)d(u^p)\right\}w(x)dx.$$

We thus obtain

$$\begin{aligned} &\int_0^\infty \left\{\frac{1}{x^p}\int_0^x \varphi_1^{-1}(u)d(u^p)\right\}\left\{\frac{1}{x^p}\int_0^x \varphi_2^{-1}(u)d(u^p)\right\}w(x)dx \\ &\qquad \le c\int_0^\infty \varphi_1^{-1}(x)\varphi_2^{-1}(x)v(x)dx. \end{aligned}$$

We remark here that the constant c is the same as the c in $B(p,2)$.

We now let $\varphi_j^{-1}(u) = f_j(u)\left(\frac{1}{u}\int_0^u f_j\right)^{p-1}$, $j = 1,2$, and observe that

$$\begin{aligned} \frac{1}{x^p}\int_0^x \varphi_j^{-1}(u)d(u^p) &= p\frac{1}{x^p}\int_0^x f_j(u)\left(\int_0^u f_j\right)^{p-1}du \\ &= \frac{1}{x^p}\left(\int_0^x f_j\right)^p. \end{aligned}$$

This completes the proof of Theorem 2.2. ∎

Remark. It may be of interest to point out that there is an easy condition for equality in Theorem 2.2. Let

(i) $\int_0^\infty Af^p w = \int_0^\infty f Af^{p-1} v$,

(ii) $v(t) = pt^{p-1} \int_t^\infty \frac{w(x)}{x^p}\, dx$.

If (i) holds for $f\downarrow$, then (ii) follows. Simply let $f = \chi_t$ and differentiate the resulting equation $\int_0^t v = \int_0^t w + \int_t^\infty \left(\frac{t}{x}\right)^p w(x)dx$. Conversely, if (ii) holds, then (i) is valid for any $f : \mathbb{R}_+ \to \mathbb{R}_+$. This can be seen by replacing v in (i) by (ii) and then integrating by parts.

We state the special case $p = 1$ of Theorem 2.2 as

Corollary 2.3. *If $f_j \downarrow$, $j = 1, \cdots, n$, then*

$$\int_0^\infty \left\{ \prod_1^n \left(\frac{1}{x} \int_0^x f_j \right) \right\} w(x)dx \le c \int_0^\infty \left\{ \prod_{j=1}^n f_j(x) \right\} v(x)dx$$

if and only if $(w, v) \in B(1, n)$.

The case $w = v$ of Theorem 2.2 yields as a special case the Ariño-Muckenhoupt weighted norm inequality for non-increasing functions [1].

Corollary 2.4. *Let $1 \le p < \infty$ and $f_j \downarrow$, $j = 1, \cdots, n$. Then*

$$\int_0^\infty \left\{ \prod_{j=1}^n \left(\frac{1}{x} \int_0^x f_j \right)^p \right\} w(x)dx \le c \int_0^\infty \left\{ \prod_{j=1}^n f_j(x)^p \right\} w(x)dx$$

if and only if $w \in B(p, n)$.

Proof: The necessity follows from $f_j = \chi_{r_j}$, and for the sufficiency we apply Theorem 2.2 and use Hölder's inequality to obtain

$$\int_0^\infty \left\{ \prod_{j=1}^n f_j \right\} \cdot \prod_{j=1}^n \left(\frac{1}{x} \int_0^x f_j \right)^{p-1} w(x)dx$$

$$\le \left\{ \int_0^\infty \left\{ \prod f_j \right\}^p w \right\}^{1/p} \left\{ \int_0^\infty \left\{ \prod_{j=1}^n \left(\frac{1}{x} \int_0^x f_j \right)^p \right\} w \right\}^{1/p'}.$$

Divide by the last factor to obtain the norm inequality. ∎

Remark. (i) For a single weight the conditions $B(p, n)$ and B_p are equivalent, i.e., $w \in B(p, n)$ iff $w \in B_p$. Since the implication $B(p, n) \subset B_p$ was already observed in (iv) of the previous remark, we only need to show that $B_p \subset B(p,n)$. It is clear that if $u \downarrow$ and $w \in B_p$, then $uw \in B_p$. Let now $f_j \downarrow$, $j = 1, 2$, and let $w \in B_p$. Then $Af_2(x)^p w(x) \in B_p$, and hence

$$\int_0^\infty Af_1^p Af_2^p w \le c \int_0^\infty f_1^p Af_2^p w.$$

Since $f_1^p w \in B_p$, we can continue this inequality $\le c \int_0^\infty f_1^p f_2^p w$, i.e., $w \in B(p,2)$.

(ii) Results related to the above Corollaries can also be found in [**2**].

We will now show that an iterated version of Corollary 2.4 provides a short proof of B_p implies $B_{p-\epsilon}$, the basic Lemma in [1]. Similar ideas for the Hardy-Littlewood maximal operator and the "A_p implies $A_{p-\epsilon}$" case can be found in [4],[**5**].

Theorem 2.5. *Let $1 \le p < \infty$ and let $w \in B(p,1)$. Then there is $\epsilon > 0$ such that $w \in B(p-\epsilon, 1)$.*

Proof: Fix $r > 0$ and let $f = \chi_r$. If $A_n f(x)$ is the n-times iterated averaging operator, i.e., $A_0 f(x) = f(x)$, $A_1 f(x) = \frac{1}{x}\int_0^x f, \cdots$, then for $n \ge 1$,

$$A_n f(x) = \begin{cases} 1, & 0 < x \le r \\ \frac{r}{x}\sum_{j=0}^{n-1} \frac{1}{j!}\log^j\left(\frac{x}{r}\right), & x > r. \end{cases}$$

Since $w \in B(p,1)$ we have from Corollary 2.4,

$$\begin{aligned}\int_0^\infty A_n f(x)^p w(x)dx &\le c^n \int_0^\infty f(x)^p w(x)dx \\ &= c^n \int_0^r w(x)dx.\end{aligned}$$

For $x > r$,

$$A_n f(x)^p = \left(\frac{r}{x}\right)^p \left(\sum_{j=0}^{n-1}\frac{1}{j!}\log^j\left(\frac{x}{r}\right)\right)^p$$

$$\ge \left(\frac{r}{x}\right)^p \left(\sum_{j=0}^{n-1}\frac{1}{j!}\log^j\left(\frac{x}{r}\right)\right) \ge \left(\frac{r}{x}\right)^p \frac{1}{(n-1)!}\log^{n-1}\left(\frac{x}{r}\right),$$

where the next to the last inequality follows since $\sum_{j=0}^{n-1} \ge 1$. We substitute this in our norm inequality and get

$$\int_r^\infty \left(\frac{r}{x}\right)^p \frac{1}{(n-1)!}\log^{n-1}\left(\frac{x}{r}\right) w(x)dx \le c^n \int_0^r w(x)dx.$$

Let $s > c$. Then

$$\int_r^\infty \left(\frac{r}{x}\right)^p \sum_{n=1}^\infty \frac{1}{(n-1)!}\left(\frac{\log\frac{x}{r}}{s}\right)^{n-1} w(x)dx \leq C\int_0^r w(x)dx$$

or

$$\int_r^\infty \left(\frac{r}{x}\right)^{p-1/s} w(x)dx \leq C\int_0^r w(x)dx,$$

i.e. $w \in B\left(p - \dfrac{1}{s}, 1\right)$. ■

3. The case n = 1 and reverse inequalities

We begin by asking for which averaging operator is $(w, v) \in B(p, 1)$ a necessary and sufficient condition for a weighted norm inequality. The case $p = 1$ is handled by Corollary 2.3 with $Af(x) = \dfrac{1}{x}\displaystyle\int_0^x f$. For $1 \leq p < \infty$ we define

$$A_p f(x) = \left\{\frac{1}{x^p}\int_0^x f(u)^p d(u^p)\right\}^{1/p}.$$

Theorem 3.1. *If $f \downarrow$ and $1 \leq p < \infty$, then*

$$\int_0^\infty A_p f(x)^p w(x)dx \leq c\int_0^\infty f(x)^p v(x)dx$$

if and only if $(w, v) \in B(p, 1)$.

Proof: The necessity follows by taking $f = \chi_r$.

For the sufficiency simply let $\varphi^{-1}(u) = f(u)^p$ in the proof of Theorem 2.2.

We will now characterize the weights (w, v) for which the reverse inequality

$$\int_0^\infty f(x)^p w(x)dx \leq c\int_0^\infty \left(\frac{1}{x}\int_0^x f\right)^p v(x)dx$$

holds for $f \downarrow$. ■

Theorem 3.2. *Let $f \downarrow$ and $1 \leq p < \infty$. Then*

$$\int_0^\infty f(x)^p w(x)dx \leq c\int_0^\infty \left(\frac{1}{x}\int_0^x f\right)^p v(x)dx$$

if and only if $\int_0^r w \le c\Big(\int_0^r v + \int_r^\infty \left(\frac{r}{x}\right)^p v(x)dx\Big)$, $r > 0$, *with the same* c.

Proof: The necessity follows with $f = \chi_r$. For the sufficiency, let $\varphi \downarrow$ and let $r = \varphi(y)$. Then as in the proof of Theorem 2.2,

$$\int_0^\infty \int_0^{\varphi(y)} w(x)dx\,dy = \int_0^\infty \varphi^{-1}(x)w(x)dx$$

and

$$\begin{aligned}&\int_0^\infty \int_0^{\varphi(y)} v(x)dx\,dy + \int_0^\infty \int_{\varphi(y)}^\infty \frac{w(x)}{x^p}(\varphi(y))^p dx\,dy \\ &= \int_0^\infty \varphi^{-1}(x)v(x)dx + \int_0^\infty \frac{1}{x^p}\int_0^x \varphi^{-1}(u)d(u^p)v(x)dx \\ &- \int_0^\infty \varphi^{-1}(x)v(x)dx = \int_0^\infty \frac{1}{x^p}\int_0^x \varphi^{-1}(u)d(u^p)v(x)dx.\end{aligned}$$

We let now $\varphi^{-1}(u) = f(u)\Big(\frac{1}{u}\int_0^u f\Big)^{p-1}$ and obtain

$$\int_0^\infty f(x)\Big(\frac{1}{x}\int_0^x f\Big)^{p-1} w(x)dx \le c\int_0^\infty \Big(\frac{1}{x}\int_0^x f\Big)^p v(x)dx.$$

We complete the proof by noting that $\frac{1}{x}\int_0^x f \ge f(x)$ since $f \downarrow$. ■

We will now characterize the single weights, i.e., $w = v$, for which the above reverse inequality holds for a given $0 < c < 1$.

Theorem 3.3. *The following statements are equivalent for* $f \downarrow$, $0 < c < 1$, $1 < p < \infty$, *and* $w \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$.

(1) $\int_0^\infty f^p w \le c\int_0^\infty Af^p w$

(2) $B_{p'}(w(y^{1-p'})) \le \frac{c}{1-c}$.

Proof: (1) → (2). If $f = \chi_r$ we get

$$\int_0^r w \le c\left(\int_0^r w + \int_r^\infty \left(\frac{r}{x}\right)^p w(x)dx\right).$$

We let $x = y^{1-p'}$ and get

$$\begin{aligned}\int_0^r w(x)dx &= (p'-1)\int_{r^{1-p}}^\infty w(y^{1-p'})\frac{dy}{y^{p'}}, \\ r^p\int_r^\infty \frac{w(x)}{x^p}dx &= (p'-1)r^p\int_0^{r^{1-p}} w(y^{1-p'})dy.\end{aligned}$$

Hence

$$(1-c)(p'-1)\int_{r^{1-p}}^{\infty} w(y^{1-p'})\frac{dy}{y^{p'}} \le c(p'-1)r^p \int_0^{r^{1-p}} w(y^{1-p'})dy.$$

If we set $\rho = r^{1-p}$, then $r^p = \dfrac{1}{\rho^{p'}}$ and (2) follows.

(2) → (1). We have

$$\int_r^{\infty} \left(\frac{r}{y}\right)^{p'} w(y^{1-p'})dy \le \frac{c}{1-c}\int_0^r w(y^{1-p'})dy.$$

Let $y = x^{1-p}$. Then, again

$$\int_r^{\infty} \left(\frac{r}{y}\right)^{p'} w(y^{1-p'})dy = r^{p'}(p-1)\int_0^{r^{1-p'}} w(x)dx$$

$$\int_0^r w(y^{1-p'})dy = (p-1)\int_{r^{1-p'}}^{\infty} \frac{w(x)}{x^p}dx.$$

Thus, with $\rho = r^{1-p'}$ we get

$$\int_0^{\rho} w(x)dx \le \frac{c}{1-c}\int_{\rho}^{\infty} \left(\frac{\rho}{x}\right)^p w(x)dx.$$

We add $\dfrac{c}{1-c}\displaystyle\int_0^{\rho} w$ to both sides and get

$$\int_0^{\rho} w \le c\left(\int_0^{\rho} w + \int_{\rho}^{\infty} \left(\frac{\rho}{x}\right)^p w(x)dx\right).$$

Apply now Theorem 3.2. ■

Remark.. (2) of Theorem 3.3 reminds one of the duality $w \in A_p$ iff $w^{1-p'} \in A_{p'}$.

4. Non-decreasing functions

We will not dwell on the straightforward results of $f \uparrow$ that one gets from our previous results via the change of variables $x \to \dfrac{1}{x}$. In particular we have

Theorem 4.1. *If $f \uparrow$ and $1 \le p < \infty$, then*

$$\int_0^{\infty} \left(x\int_x^{\infty} f(u)\frac{du}{u^2}\right)^p w(x)dx \le c\int_0^{\infty} f(x)^p w(x)dx$$

if and only if $\displaystyle\int_0^r \left(\frac{x}{r}\right)^p w(x)dx \le c\int_r^{\infty} w(x)dx,\ r > 0.$

In order to see what type of results one has for the averaging operator $\dfrac{1}{x}\displaystyle\int_0^x f$ for $f \uparrow$ we need a lemma similar to Lemma 2.1.

Lemma 4.2. *Let $\varphi \uparrow$ with $\varphi(0) = 0$, and let W be a weight. Then*

$$\text{(i)} \qquad \int_0^\infty \int_0^\infty \chi^{\varphi(y)}(x) W(x) dx\, dy = \int_0^\infty \varphi^{-1}(x) W(x) dx$$

$$\text{(ii)} \qquad \begin{aligned} &\int_0^\infty \int_0^\infty \chi^{\varphi(y)}(x) \left(\frac{x - \varphi(y)}{x}\right)^p W(x) dx\, dy \\ &= \int_0^\infty \left\{ \frac{1}{x^p} \int_0^x \varphi^{-1}(x-u) d(u^p) \right\} W(x) dx. \end{aligned}$$

Proof: For (i) we simply interchange the order of integration. The left side of (ii) is $\int_0^\infty \int_0^{\varphi^{-1}(x)} \frac{W(x)}{x^p}(x - \varphi(y))^p dy\, dx$ and the inner integral is the same as

$$\int_0^{x^p} \varphi^{-1}(x - t^{1/p}) dt = \int_0^x \varphi^{-1}(x-u) d(u^p),$$

as can be seen by interpreting the integral as area under $t = (x - \varphi(y))^p$. ∎

Definition. Let n be a positive integer and $1 \le p < \infty$. We say that $(w, v) \in C(p, n)$ if and only if there is $0 < c < \infty$ such that for every choice $0 < r_1, r_2, \cdots, r_n < \infty$,

$$\int_0^\infty \left\{ \prod_{j=1}^n \chi^{r_j}(x) \right\} w(x) dx \le c \int_0^\infty \left\{ \prod_{j=1}^n \chi^{r_j}(x) \left(\frac{x - r_j}{x}\right)^p v(x) dx. \right.$$

Theorem 4.3. *Let $f_j \uparrow$, $j = 1, \cdots, n$. Then*

$$\int_0^\infty \left\{ \prod_1^n f_j(x) \right\} w(x) dx \le c \int_0^\infty \left\{ \prod_1^n \left(\frac{1}{x} \int_0^x f_j \right) \right\} v(x) dx$$

if and only if $(w, v) \in C(1, n)$.

Proof: The necessity follows by taking $f_j = \chi^{r_j}$. As in Theorem 2.2 we prove the converse for $n = 2$; the general case is obtained by repeating the argument. We let $\varphi_j \uparrow$, $\varphi_j(0) = 0$, and $r_j = \varphi_j(y_j)$, $j = 1, 2$, where $0 < y_1, y_2 < \infty$. We next integrate the $C(1, n)$ condition over all such (y_1, y_2) and obtain

$$\begin{aligned} L &\equiv \int_0^\infty \int_0^\infty \int_0^\infty \chi^{\varphi_1(y_1)}(x) \chi^{\varphi_2(y_2)}(x) w(x) dx\, dy_1\, dy_2 \\ &\le c \int_0^\infty \int_0^\infty \int_0^\infty \psi_1(x, y_1) \psi_2(x, y_2) v(x) dx\, dy_1\, dy_2 \equiv R, \end{aligned}$$

where $\psi_j(x, y_j) = \chi^{\varphi_j(y_j)}(x)\left(\frac{x-\varphi_j(y_j)}{x}\right)$. By (i) of Lemma 4.2,

$$L = \int_0^\infty \varphi_2^{-1}(x)\varphi_1^{-1}(x)w(x)dx,$$

and by (ii) with $p = 1$,

$$R = \int_0^\infty \left(\frac{1}{x}\int_0^x \varphi_1^{-1}\right)\left(\frac{1}{x}\int_0^x \varphi_2^{-1}\right)v(x)dx.$$

From this we get the theorem by letting $\varphi_j^{-1}(t) = f_j(t)$ if $f_j(0) = 0$. Otherwise, let $\epsilon_n(x) = nx$, if $0 \le x \le \frac{1}{n}$, and $\epsilon_n(x) = 1$, $x > \frac{1}{n}$. If $\varphi_{j,n}^{-1}(t) = \epsilon_n(t)f_j(t)$, then we get the weighted norm inequality for $\epsilon_n f_j$, and the final result by letting $n \to \infty$. ■

Corollary 4.4. *Let $f \uparrow$ and n a positive integer. Then*

$$\int_0^\infty f(x)^n w(x)dx \le c\int_0^\infty \left(\frac{1}{x}\int_0^x f\right)^n v(x)dx$$

if and only if $(w, v) \in C(1, n)$.

Proof: If $(w, v) \in C(1, n)$, then the inequality follows from Theorem 4.3 by letting $f_1 = f_2 = \cdots = f_n$. Conversely, let $f = \prod_1^n \chi^{r_j}$. Then $f = f^n$ and by Hölder's inequality

$$(\frac{1}{x}\int_0^x f)^n \le \prod_1^n \left(\frac{1}{x}\int_0^x \chi^{r_j}\right) = \prod_1^n \chi^{r_j}(x)\left(\frac{x-r_j}{x}\right). \quad ■$$

Remark. We were unable to find a characterization of

$$\int_0^\infty f(x)^p w(x)dx \le c\int_0^\infty \left(\frac{1}{x}\int_0^x f\right)^p v(x)dx$$

for $f \uparrow$ and p not a positive integer. However, as we shall see, $(w, v) \in C(p, 1)$ controls the averaging operator

$$A_p f(x) = \frac{1}{x^p}\int_0^x f(x-u)d(u^p).$$

We observe that, when p is a positive integer, then $\int_0^x f(x-u)d(u^p)$ is, apart from a multiplicative constant, the p-times iterated integral of f.

Theorem 4.5. *Let $f \uparrow$ and $1 \le p < \infty$. Then*

(i) $\int_0^\infty A_p f(x)w(x)dx \le c\int_0^\infty f(x)v(x)dx$ *if and only if* $\int_r^\infty \left(\frac{x-r}{x}\right)^p w(x)\,dx \le c\int_r^\infty v(x)dx$, $r > 0$.

(ii) $\int_0^\infty f(x)w(x)dx \le c\int_0^\infty A_p f(x)v(x)dx$ *if and only if* $\int_r^\infty w(x)dx \le c\int_r^\infty \left(\frac{x-r}{x}\right)^p v(x)dx$, $r > 0$, *i.e.*, $(w, v) \in C(p, 1)$.

Proof: (i) For the necessity let $f = \chi^r$. To prove the sufficiency, let $\varphi \uparrow$, $\varphi(0) = 0$, and $r = \varphi(y)$, $0 < y < \infty$. Then

$$L \equiv \int_0^\infty \int_{\varphi(y)}^\infty \frac{w(x)}{x^p}(x - \varphi(y))^p dx\,dy \le c\int_0^\infty \int_{\varphi(y)}^\infty v(x)dx\,dy \equiv R.$$

By Lemma 4.2, $R = \int_0^\infty \varphi^{-1}(x)v(x)dx$ and

$$L = \int_0^\infty \left\{\frac{1}{x^p}\int_0^x \varphi^{-1}(x-u)d(u^p)\right\}w(x)dx.$$

The proof can be completed by letting $\varphi^{-1}(x) = f(x)$ if $f(0) = 0$; otherwise let $\varphi^{-1}(x) = \epsilon_n(x)f(x)$ as in the proof of Theorem 4.3.

The proof of (ii) is the same as the one for (i). ■

5. More properties of weights

We begin with a "change of variables" result for B_p-weights.

Theorem 5.1. . *If $1 < q < p < \infty$ and $w \in B_q$, then $w\left(x^{\frac{p-1}{q-1}}\right) \in B_p$.*

Proof: We set $I_r = \int_r^\infty \left(\frac{r}{x}\right)^p w\left(x^{\frac{p-1}{q-1}}\right)dx$ and let $u = x^\alpha$, $\alpha = \dfrac{p-1}{q-1}$. Then

$$\begin{aligned} I_r &= c\int_{r^\alpha}^\infty \left(\frac{r}{u^{1/\alpha}}\right)^p w(u)u^{\frac{1-\alpha}{\alpha}}du \\ &= c\int_{r^\alpha}^\infty \frac{r^p}{u^{(p+\alpha-1)/\alpha}}w(u)du. \end{aligned}$$

We observe that $(p+\alpha-1)/\alpha = q$ and so

$$I_r = \int_{r^\alpha}^\infty \left(\frac{r^\alpha}{u}\right)^q w(u)du \cdot r^{p-\alpha q}.$$

Since $w \in B_q$ and $p - \alpha q = \dfrac{q-p}{q-1} < 0$, we see that

$$I_r \le c r^{\frac{q-p}{q-1}} \int_0^{r^\alpha} w(u)du = c r^{1-\alpha} \int_0^r w(x^\alpha) x^{\alpha-1} dx$$
$$\le c \int_0^r w(x^\alpha)dx. \blacksquare$$

The case $q = 1$ yields a slightly stronger result which we state as

Theorem 5.2. *If $w \in B_1$ and $\alpha \ge 1$, then $w(x^\alpha) \in B_1$ with $B_1(w) = B_1(w(x^\alpha))$.*

Proof: If $I_r = \int_r^\infty \left(\frac{r}{x}\right) w(x^\alpha)dx$ and $u = x^\alpha$, then

$$I_r = \frac{1}{\alpha} \int_{r^\alpha}^\infty \left(\frac{r}{u^{1/\alpha}}\right) w(u) u^{1/\alpha - 1} du = \frac{r^{1-\alpha}}{\alpha} \int_{r^\alpha}^\infty \left(\frac{r^\alpha}{u}\right) w(u)du$$
$$\le c r^{1-\alpha} \int_0^r w(x^\alpha) x^{\alpha-1} dx \le c \int_0^r w(x^\alpha)dx,$$

since $\alpha \ge 1$. $\blacksquare$

The next result reminds one of the important A_p-property, i.e., $w \in A_p \to w^\tau \in A_p$ for some $\tau > 1$.

Theorem 5.3. *If $w \in B_p$, then there is $\epsilon > 0$ such that $x^\epsilon w(x^{1+\epsilon}) \in B_p$.*

Proof: Choose $\epsilon > 0$ so that $w \in B_{p/1+\epsilon}$ (Theorem 2.5), and note that

$$\int_r^\infty \left(\frac{r}{x}\right)^p x^\epsilon w(x^{1+\epsilon})dx = \frac{1}{1+\epsilon} \int_{r^{1+\epsilon}}^\infty \frac{r^p}{u^{p/1+\epsilon}} w(u)du$$
$$= \frac{1}{1+\epsilon} \int_{r^{1+\epsilon}}^\infty \left(\frac{r^{1+\epsilon}}{u}\right)^{p/1+\epsilon} w(u)du \le \frac{c}{1+\epsilon} \int_0^{r^{1+\epsilon}} w(u)du$$
$$= c \int_0^r x^\epsilon w(x^{1+\epsilon})dx. \blacksquare$$

Corollary 5.4. *If $w \in B_p$, then there is $\epsilon > 0$ such that $w(x^{1+\epsilon}) \in B_p$.*

We are now ready to present a factorization theorem for B_p-weights similar to the factorization of $w \in A_p$ as $w = uv^{1-p}$, $u, v \in A_1$.

Theorem 5.5. *The following statements are equivalent for* $1 < p < \infty$.
(1) $w \in B_p$
(2) $w(x) = u(x) \cdot x^{p-1}$ *with* $u(x^{1/p}) \in B_1$.

Proof: (1) → (2). All we need to show is that $\frac{w(x^{1/p})}{x^{1/p'}} \equiv u(x^{1/p})$ is in B_1, and this follows from

$$\int_r^\infty (\frac{r}{x})\frac{w(x^{1/p})}{x^{1/p'}} = c\int_{r^{1/p}}^\infty \left(\frac{r}{t^p}\right)\frac{w(t)}{t^{p/p'}}t^{p-1}dt$$
$$= c\int_{r^{1/p}}^\infty \left(\frac{r^{1/p}}{t}\right)^p w(t)dt \le c\int_0^{r^{1/p}} w(t)dt = c\int_0^r w(t^{1/p})/t^{1/p'}dt.$$

(2) → (1). This is simply

$$\int_r^\infty (\frac{r}{x})^p u(x)x^{p-1}dx = \frac{1}{p}\int_{r^p}^\infty \left(\frac{r}{t^{1/p}}\right)^p u(t^{1/p})dt$$
$$= \frac{1}{p}\int_{r^p}^\infty \left(\frac{r^p}{t}\right) u(t^{1/p})dt \le \frac{c}{p}\int_0^{r^p} u(t^{1/p})dt =$$
$$c\int_0^r u(x)x^{p-1}dx. \blacksquare$$

Remark. By Theorem 5.2, if $u(x^{1/p}) \in B_1$, then $u(x) \in B_1$. Thus (2) can be written as $w = u \cdot \left(\frac{1}{x}\right)^{1-p}$, with $u \in B_1$. It is also clear that $\frac{1}{x} \in B_1$.

6. Weak type weights

We say that $w \in R_p$ iff $w\{A\chi_r > y\} \le \frac{c}{y^p}\int_0^r w$, $r > 0$, and we say that $w \in W_p$ iff for $f \downarrow$, $w\{Af > y\} \le \frac{c}{y^p}\int_0^\infty f^p w$. The "$R$" in R_p stands for "restricted".

We will study relationships among R_p, W_p, and B_p, and give a characterization of B_p.

Theorem 6.1. $w \in R_p$ *iff there is* $0 < c < \infty$ *so that for* $0 < r < s < \infty$,

$$\frac{1}{s^p}\int_0^s w \le c\frac{1}{r^p}\int_0^r w.$$

Proof: First assume that $w \in R_p$. The set $\{A\chi_r > y\} = (0, x_0)$, where $\frac{r}{x_0} = y$, $0 < y < 1$. Hence $\int_0^{r/y} w \le \frac{c}{y^p} \int_0^r w$ from which

$$\frac{1}{s^p} \int_0^s w \le \frac{c}{r^p} \int_0^r w, \quad s = \frac{r}{y} > r.$$

Conversely, for $0 < y < 1$, with the same notation as above,

$$\begin{aligned} w\{A\chi_r > y\} &= \int_0^{x_0} w = \frac{1}{y^p} \left(\frac{r}{x_0}\right)^p \int_0^{x_0} w \\ &\le \frac{c}{y^p} \int_0^r w. \ \blacksquare \end{aligned}$$

The next result shows how R_p and B_q are related.

Theorem 6.2. *If $w \in R_p$, then $w \in B_q$ for $q > p$.*

Proof: From Theorem 6.1, for $s > r$,

$$\left(\frac{r}{s}\right)^p \int_0^s w \le c \int_0^r w.$$

Let $t = \frac{r}{s}$. Then $t^p \int_0^{r/t} w \le c \int_0^r w$, or, if $0 < \epsilon < 1$,

$$t^{p-\epsilon} \int_0^{r/t} w \le ct^{-\epsilon} \int_0^r w, \quad 0 < t \le 1.$$

Hence

$$L \equiv \int_0^1 t^{p-\epsilon} \int_0^{r/t} w(x)dxdt \le c_\epsilon \int_0^r w.$$

We interchange the order of integration and see that

$$L \ge \int_r^\infty \int_0^{r/x} w(x)t^{p-\epsilon}dtdx = c \int_r^\infty w(x)(\frac{r}{x})^{p+1-\epsilon}dx.$$

Thus $w \in B_q$, $q = p + 1 - \epsilon$. $\blacksquare$

Example. Let $w(x) = x$. Then $w \in R_2$ but not in W_2 and thus not in B_2. For let $f(x) = \frac{1}{x \log \frac{1}{x}} \cdot \chi_{\frac{1}{e}}(x)$. Then $w\{Af > y\} = \infty$, but $\int f^2 w = \int_0^{1/e} \frac{dx}{x \log^2 \frac{1}{x}} < \infty$.

We will now show that the condition of Theorem 6.1 which characterizes R_p will, if slightly modified, characterize B_p. We begin with

Lemma 6.3. *Assume there exists* $1 < a < \infty$ *and* $0 < c = c_a < 1$ *such that* $\frac{1}{(ax)^p}\int_0^{ax} w \le c\frac{1}{x^p}\int_0^x w$, $x > 0$. *Then* $w \in B_p$.

Proof: For $0 < N < \infty$, let $w_N = w\chi_N$. Then w_N satisfies the same hypothesis as w with a constant $c = \max(c_a, 1/a^p) < 1$.

We have then $\frac{1}{a^p x^{p+1}}\int_0^{ax} w_N \le \frac{c}{x^{p+1}}\int_0^x w_N$. Hence for $0 < r < \infty$ fixed,

$$L \equiv \frac{1}{a^p}\int_r^\infty \frac{1}{x^{p+1}}\int_0^{ax} w_N(t)dtdx \le c\int_r^\infty \frac{1}{x^{p+1}}\int_0^x w_N(t)dtdx \equiv R.$$

We interchange the order of integration and see that

$$L \ge \frac{1}{a^p}\int_{ar}^\infty \int_{t/a}^\infty w_N(t)\frac{dx}{x^{p+1}}dt = \frac{1}{p}\int_{ar}^\infty \frac{w_N(t)}{t^p}\,dt,$$

$$R = c\{\int_0^r \int_r^\infty w_N(t)\frac{dx}{x^{p+1}}\,dt + \int_r^\infty \int_t^\infty w_N(t)\frac{dx}{x^{p+1}}\,dt\}$$

$$= c\{\frac{1}{p}\int_0^r \frac{w_N(t)}{r^p} + \frac{1}{p}\int_r^\infty \frac{w_N(t)}{t^p}\,dt\}.$$

The last integral $\int_r^\infty \frac{w_N(t)}{t^p}\,dt = \left(\int_r^{ar} + \int_{ar}^\infty\right)\frac{w_N(t)}{t^p}\,dt \le \frac{1}{r^p}\int_r^{ar} w_N(t)dt + \int_{ar}^\infty \frac{w_N(t)}{t^p}\,dt$. Hence $R \le c\{\frac{1}{p}\int_0^{ar}\frac{w_N(t)}{r^p}\,dt + \frac{1}{p}\int_{ar}^\infty \frac{w_N(t)}{t^p}\,dt\}$.

From this we obtain,

$$\frac{1}{p}(1-c)\int_{ar}^\infty \frac{w_N(t)}{t^p}\,dt \le \frac{c}{pr^p}\int_0^{ar} w_N(t)dt$$

or

$$\int_{ar}^\infty \left(\frac{ar}{t}\right)^p w_N(t)dt \le \frac{ca^p \cdot p}{1-c}\int_0^{ar} w_N(t)dt.$$

We complete the proof by letting $N \to \infty$. ■

Theorem 6.4. *Assume that* $w \in L^1_{\text{loc}}(\mathbb{R}_+)$. *Then* $w \in B_p$ *iff* $0 < \epsilon < 1$ *implies the existence of* $a_\epsilon > 1$ *such that for* $x > 0$,

$$\frac{1}{a^p x^p}\int_0^{ax} w \le \epsilon\frac{1}{x^p}\int_0^x w, \quad a \ge a_\epsilon.$$

Proof: By Lemma 6.3 we only need to prove the necessity. By Theorem 2.5, there is $\eta > 0$ such that $w \in B_{p-\eta}$. Thus for $a > 1$,

$$\frac{\frac{1}{a^p x^p}\int_0^{ax} w}{\frac{1}{x^p}\int_0^x w} = \frac{\frac{1}{(ax)^{p-\eta}}\int_0^{ax} w}{\frac{1}{x^{p-\eta}}\int_0^x w}\cdot\left(\frac{1}{a}\right)^\eta.$$

Since $w \in B_{p-\eta} \subset R_{p-\eta}$, by Theorem 6.1 the first factor $\le c$ and the proof is complete. ■

As an application of Theorem 6.4 we will prove

Theorem 6.5. *Let $w \in B_p$ and $W(x) = \int_0^x w$. Then for $0 < \alpha < \infty$, $W^\alpha \in B_{\alpha p+1}$.*

Proof: We do $\alpha = 1$ first. Let $0 < \epsilon < \dfrac{1}{p+1}$. Then for $a \geq a_\epsilon > 1$ we have $\dfrac{x^p}{(ar)^p} \int_0^{ar} w \leq \epsilon \int_0^x w = \epsilon W(x)$, $0 < x \leq r$. Thus

$$L \equiv \int_0^r \frac{x^p}{(ar)^p} \int_0^{ar} w \leq \epsilon \int_0^r W(x)dx,$$

and

$$\begin{aligned} L &= \frac{1}{(p+1)} \frac{r^{p+1}}{(ar)^p} W(ar) = \frac{1}{(p+1)} \frac{1}{a^{p+1}} (ar) W(ar) \\ &\geq \frac{1}{p+1} \frac{1}{a^{p+1}} \int_0^{ar} W, \end{aligned}$$

and so $W \in B_{p+1}$.

For the general case, since

$$W^\alpha(x) = \alpha \int_0^x W^{\alpha-1} w,$$

we only need to verify that $W^{\alpha-1} w \in B_{\alpha p}$. For some $0 < c < 1$ and $a > 1$ we have

$$\begin{aligned} \frac{1}{a^{p\alpha}} \int_0^{ax} W^{\alpha-1} w &= \frac{1}{\alpha a^{p\alpha}} W^\alpha(ax) \leq \frac{1}{\alpha} c W^\alpha(x) \\ &= c \int_0^x W^{\alpha-1} w. \ \blacksquare \end{aligned}$$

7. The equality $\mathbf{W_p = B_p}$

In this final section we will prove that $W_p = B_p$ for $1 < p < \infty$, a situation quite analogous to the A_p-case. I am indebted to Richard Bagby for the original proof of this property. We will present a somewhat simplified version based on some of our previous results. For the definitions of R_p, W_p see the beginning of section 6.

Lemma 7.1. *Let $w \in R_p$, $0 < a < \infty$, and $1 < s < \infty$. Then*

$$\int_a^{as} \left(\frac{a}{u}\right)^p w(u)du \leq c(1 + \log s) \int_0^a w.$$

Proof: We know that by Theorem 6.1,

$$\frac{1}{t^p}\int_0^{ta} w \le c\int_0^a w, \quad t \ge 1.$$

Hence $L \equiv \int_1^s \frac{1}{t^{p+1}}\int_0^{ta} w \le c\log s\int_0^a w$. We interchange the order of integration and get

$$L \ge \int_a^{as}\int_{u/a}^{s} w(u)\frac{dt}{t^{p+1}}du = \frac{1}{p}\int_a^{as} w(u)\left[\left(\frac{a}{u}\right)^p - \frac{1}{s^p}\right]du.$$

Hence

$$\begin{aligned}\frac{1}{p}\int_a^{as} w(u)\left(\frac{a}{u}\right)^p du &\le c\log s\int_0^a w + \frac{1}{p}\frac{1}{s^p}\int_a^{as} w\\ &\le c\log s\int_0^a w + c\int_0^a w,\end{aligned}$$

since $w \in R_p$. ■

Theorem 7.2. *$W_p = B_p$ for $1 < p < \infty$.*

Proof: The inclusion $B_p \subset W_p$ is obvious, and for the reverse inclusion we consider for $s > 1$ the function $f(x) = 1$, $0 \le x \le a; = a/x$, $a \le x \le sa$; and $= 0$, $x > sa$. Then $Af(as) = \frac{1+\log s}{s}$. Since $w \in W_p$ we have that

$$w\{Af(x) > y\} \le \frac{c}{y^p}\int_0^\infty f^p w.$$

If $y = \frac{1+\log s}{s}$, we get

$$\left(\frac{1+\log s}{s}\right)^p\int_0^{as} w \le c\left(\int_0^a w + \int_a^{as}\left(\frac{a}{u}\right)^p w(u)du\right) \le c(1+\log s)\int_0^a w$$

by Lemma 7.1. Thus

$$\frac{1}{s^p}\int_0^{sa} w \le c(1+\log s)^{1-p}\int_0^a w.$$

We choose s so large that $c(1+\log s)^{1-p} < 1$ and apply Theorem 6.4. ■

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