

# Weighted norm inequalities for functions of exponential type

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## 0. Introduction

A result by Plancherel—Polya [12] (cf. Boas [2], p. 101) says that if  $f \in \mathbf{R}_r$ , i.e. if its Fourier Transform is supported by  $\{\xi: |\xi| = (\xi_1^2 + \dots + \xi_n^2)^{1/2} \leq r\}$ , and if  $f \in L^p$  for some  $p > 0$  then

$$\sum_{k \in \mathbf{Z}^n} |f(k)|^p \leq C \int |f(x)|^p dx. \quad (0.1)$$

It has been recently remarked (cf. e.g. [1]) that a proof of this and other classical inequalities can be based on the ideas of Fefferman—Stein [6]. In fact, let us put

$$n_{r,\lambda} f(x) = \sup_{y \in \mathbf{R}^n} |f(x-y)| / (1+r|y|)^\lambda.$$

In [8] we proved (under the said conditions on  $f$ ) that

$$n_{r,\lambda} f \in L^p \quad (0.2)$$

provided  $\lambda > n/p$ . It is clear that (0.2) implies (0.1).

The purpose of the present paper is to extend the result (0.2) in the respect that we replace the Lebesgue measure by a general positive measure  $\mu$  on  $\mathbf{R}^n$ . That is, assuming  $f \in \mathbf{E}_r \cap L^p(d\mu)$  we want to prove

$$n_{r,\lambda} f \in L^p(d\mu). \quad (0.2')$$

It turns out that the relevant condition on  $\mu$  is that

$$\int \frac{d\mu(x)}{(1+r|x-a|)^{ns(1+\varepsilon)}} \leq C\mu(Q_R(a)) \quad (0.3)$$

for all  $\varepsilon > 0$  and  $a \in \mathbf{R}^n$ . Here  $Q_R(a)$  denotes the cube with center  $a$  and side  $R$ , and  $\lambda > ns/p$ ,  $R \approx 1/r$ . (For the precise statement see Main Theorem in Sec. 1.)

In a forthcoming publication we intend to use the present results in connection with Besov spaces (cf. [10]).

Finally I would like to express my gratitude to Prof. Peetre for his advice and interest.

### 1. The main result

The problem considered in this paper is to determine positive measures  $\mu$  such that, for some fixed  $r$  and  $p$  with  $0 < r, p < \infty$ :

- (i) There is a function  $f \in \mathbf{E}_r \cap L^p(d\mu)$ ,  $f \neq 0$ .
- (ii) If  $f \in \mathbf{E}_r$ , then  $\|n_{r,\lambda} f\|_{L^p(d\mu)} \leq C \|f\|_{L^p(d\mu)}$  for  $\lambda > ns/p$ , where  $s$  is a given positive number.

**Definition 1.1.**  $M_s(r, p)$  denotes all positive measures  $\mu$  on  $\mathbf{R}^n$  satisfying (i) and (ii) above.

Let  $|Q|$  and  $\mu(Q)$  denote the Lebesgue and the  $\mu$ -measure respectively of a cube  $Q$  on  $\mathbf{R}^n$ . We shall prove that  $M_s(r, p)$  can be described by the following family of measures:

**Definition 1.2.**  $A_s(R)$ ,  $s > 0$ , denotes all positive measures  $\mu$  on  $\mathbf{R}^n$  satisfying

$$\frac{\mu(Q)}{\mu(E)} \leq C |Q|^s$$

for some constant  $C = C(\mu)$  and all pairs of cubes  $E, Q$  such that  $E \subset Q$  and the side of  $E$  equals  $R$ .

Now the main result reads

**Main Theorem.** Let  $0 < r, p < \infty$  be given. Then there exist positive constants  $R_0$  and  $R_1$  such that

$$A_{s+\varepsilon}(R_0) \subset M_s(r, p) \subset A_{s+\varepsilon}(R_1)$$

for every  $\varepsilon > 0$ .

In applications we need a similar result which is independent of  $r$ . We therefore introduce another family of measures:

**Definition 1.3.**  $B_s$ ,  $s > 0$ , denotes all positive measures  $\mu$  on  $\mathbf{R}^n$  satisfying

$$\frac{\mu(Q)}{\mu(E)} \leq C \left( \frac{|Q|}{|E|} \right)^s$$

for some constant  $C = C(\mu)$  and all pairs of cubes  $E, Q$  such that  $E \subset Q$ .

The subsequent proof of the main theorem also gives with minor modifications the following

**Corollary.** Let  $\mu$  be a positive measure on  $\mathbf{R}^n$ . The following two conditions are equivalent:

- (1)  $\mu \in B_{s+\varepsilon}$  for each  $\varepsilon > 0$ .
- (2) For each  $r$  and  $p$  with  $0 < r, p < \infty$ 
  - (i) There is a function  $f \in \mathbf{E}_r \cap L^p(d\mu)$ ,  $f \neq 0$ .
  - (ii) If  $f \in \mathbf{E}_r$ , there is a constant  $C$ , independent of  $r$ , such that  $\|n_{n,\lambda} f\|_{L^p(d\mu)} \leq C \|f\|_{L^p(d\mu)}$  for all  $\lambda > ns/p$ .

### 2. Preliminaries

Later it will be convenient to have another characterization of  $A_{s+\varepsilon}(R)$  ( $B_{s+\varepsilon}$  can be described in a similar way):  $\mu$  satisfies  $A_{s+\varepsilon}(R)$  for each  $\varepsilon > 0$  if and only if there is a constant  $C_\varepsilon$  such that

$$\int \frac{d\mu(x)}{(1 + |x-a|/R)^{ns(1+\varepsilon)}} \leq C_\varepsilon \mu(E) \tag{2.1}$$

for any cube  $E$  with center  $a$  and side  $R$ .

Let us also compare  $A_s$  and  $B_s$  with two other conditions:

If  $d\mu(x) = w(x)dx$  and  $w(x)$  is a temperate weight function of order  $N$  in the sense of Hörmander (cf. [7], p. 34), then  $\mu \in A_{1+N/n}(R)$  for each  $R > 0$ .

If  $d\mu(x) = w(x)dx$  and  $w$  satisfies the so called  $A_\infty$ -condition, appearing in connection with for example weighted inequalities for the Hardy—Littlewood maximal operator and singular integrals (cf. [4], [9]), then  $\mu \in B_s$  for some  $s > 0$ . The converse is not true: It is easy to see that  $\mu \in B_s$  for some  $s > 0$  if and only if there is a constant  $C = C(\mu)$  such that

$$\mu(2Q) \leq C\mu(Q) \tag{2.2}$$

for any cube  $Q$ . (Here  $2Q$  is the cube with the same center as  $Q$  but with sides twice as long.) That (2.2) is not equivalent to  $A_\infty$  was shown in [5].

We also need the following fact. Let

$$m_\mu f(x) = \sup_{Q(x)} 1/\mu(Q(x)) \int_{Q(x)} |f| d\mu$$

where the supremum is taken over all cubes  $Q(x)$  with center  $x$ . From [3] we have

**Hardy—Littlewood maximal theorem.** Let  $\mu$  be a positive measure on  $\mathbf{R}^n$ . Then there is a constant  $C = C(n, p)$  such that

$$\|m_\mu f\|_{L^p(d\mu)} \leq C \|f\|_{L^p(d\mu)}$$

for every  $p > 1$ .

### 3. Proof of the main theorem

Let us first prove that there exists some  $R_0 > 0$  such that

$$A_{s+\varepsilon}(R_0) \subset M_s(r, p)$$

for every  $\varepsilon > 0$ . The inclusion will be an easy consequence of two lemmata based on ideas in [11].

**Lemma 3.1.** Suppose that  $f$  is a  $C^1$  function on  $\mathbf{R}^n$  and that  $\mu \in A_{s+\varepsilon}(R_0)$  for each  $\varepsilon > 0$ . Then there is a constant  $C = C(\varepsilon, \lambda, n)$  such that

$$n_{r,\lambda} f \cong C \{r^{-\lambda} (m_\mu |f|^a)^{1/a} + R_0 n_{r,\lambda}(\nabla f)\}$$

if  $R_0 \leq 1/r$  and  $a = (1 + \varepsilon)ns/\lambda$ .

*Proof:* By the mean value theorem

$$\begin{aligned} |f(x-y)| &\cong C \left\{ \left( \frac{1}{\mu(Q)} \int_Q |f|^a d\mu \right)^{1/a} + R_0 \sup_Q |\nabla f| \right\} \\ &\cong C \left\{ \left( \frac{\mu(Q^*)}{\mu(Q)} m_\mu |f|^a(x) \right)^{1/a} + R_0 (1+r(R_0+|y|))^\lambda n_{r,\lambda}(\nabla f)(x) \right\} \end{aligned}$$

where  $Q = Q_{R_0}(x-y)$  and  $Q^* = Q_{R_0+2|y|}(x)$ . Since  $\mu \in A_{s+\varepsilon}(R_0)$  and if  $R_0 \leq 1/r$  we get

$$|f(x-y)| \cong C \{r^{-\lambda} (m_\mu |f|^a(x))^{1/a} + R_0 n_{r,\lambda}(\nabla f)(x)\} \cdot (1+r|y|)^\lambda.$$

This clearly gives the desired inequality.

**Lemma 3.2.** Let  $f \in \mathbf{E}_r$ . Then

$$n_{r,\lambda}(\partial_i f) \cong C r n_{r,\lambda} f$$

where  $C = C(\lambda, n)$ .

*Proof:* Choose  $v \in \mathcal{S}$  with  $\hat{v}$  equal to 1 on  $|\xi| \leq 1$ . Then  $f = v_r * f$  and  $\partial_i f = r(\partial_i v)_r * f$  with  $v_r(x) = r^n v(rx)$ . Hence

$$\begin{aligned} |\partial_i f(x-y)| &\leq r^{n+1} \int |\partial_i v(rz)| (1+r|y+z|)^\lambda dz n_{r,\lambda} f(x) \leq \\ &\leq r \int |\partial_i v(z)| (1+|z|)^\lambda dz (1+r|y|)^\lambda n_{r,\lambda} f(x) \end{aligned}$$

where we have used the inequality

$$(1+u+v) \leq (1+u)(1+v), \quad u, v \geq 0.$$

With  $C = \max_i \int |\partial_i v(z)| (1+|z|)^\lambda dz$  our assertion follows.

Using (2.1) we see that any function  $f \in \mathbf{E}_r \cap \mathcal{S}$ ,  $f \neq 0$ , fulfils the first requirement of  $M_s(r, p)$ .

When proving the second we may assume  $\lambda \leq 2ns/p$ . Suppose  $n_{r,\lambda} f < \infty$ . Then by Lemma 3.1 and 3.2

$$n_{r,\lambda} f(x) \leq C(m_\mu |f|^a(x))^{1/a}.$$

Therefore, with  $\varepsilon > 0$  so small that  $p/a > 1$ , the Hardy—Littlewood maximal theorem (Sec. 2) gives

$$\|n_{r,\lambda} f\|_{L^p(d\mu)} \leq C \|f\|_{L^p(d\mu)}.$$

This is at least true if  $n_{r,\lambda} f < \infty$ . Since  $f \in \mathbf{E}_r$  does not grow faster than  $|x|^q$  as  $|x| \rightarrow \infty$  for some  $q$ , the general case follows if we study  $f(x) \prod_{i=1}^n (\sin \gamma x_i / \gamma x_i)^j$  for large  $j$  and let  $\gamma \rightarrow 0$ .

We shall also prove that

$$M_s(r, p) \subset A_{s+\varepsilon}(R_1)$$

for each  $\varepsilon > 0$ . A simple lemma will be needed:

**Lemma 3.3.** If  $\mu$  belongs to  $M_s(r, p)$  then  $(1 + |x - h|)^{-j} \in L_1(d\mu)$  for  $j > ns$  and for any  $h \in \mathbf{R}^n$ .

*Proof:* Let  $f$  denote the function given by  $M_s(r, p)$ . As  $|f(x - h)| \leq (1 + r|h|)^\lambda n_{r,\lambda} f(x)$  we can assume that  $f(h) = 1$ . The lemma is a consequence of the estimates

$$\int \frac{d\mu(x)}{(1 + r|x - h|)^{2p}} \leq \|n_{r,\lambda} f\|_{L^p(d\mu)}^2 \leq C \|f\|_{L^p(d\mu)}^2 < \infty.$$

To prove the inclusion, fix two cubes  $E, Q$  with  $E \subset Q$  and  $|E| = R_1^n$  and let  $\varepsilon > 0$  be given. Setting

$$f(x) = \prod_{i=1}^n \left\{ \sin \frac{r}{j}(x_i - a_i) \Big/ \frac{r}{j}(x_i - a_i) \right\}^j$$

where  $a$  is the center of  $E$  and  $j$  is an integer, we know that  $f \in \mathbf{E}_r$ . Now, using  $M_s(r, p)$ ,

$$\int \frac{d\mu(x)}{(1 + r|x - a|)^{2p}} \leq \|n_{r,\lambda} f\|_{L^p(d\mu)}^2 \leq C \|f\|_{L^p(d\mu)}^2.$$

Hence, with  $\lambda p = (1 + \varepsilon)ns$  and  $j \approx 2\lambda$ ,

$$\begin{aligned} \int \frac{d\mu(x)}{(1 + r|x - a|)^{(1 + \varepsilon)ns}} &\leq C \left\{ \int_{x \in E} d\mu(x) + \int_{x \notin E} |f(x)|^p d\mu(x) \right\} \leq \\ &\leq C\mu(E) + \frac{1}{2} \int \frac{d\mu(x)}{(1 + r|x - a|)^{(1 + \varepsilon)ns}} \end{aligned}$$

if we choose  $E$  (i.e.  $R_1$ ) large enough. By Lemma 3.3 both sides are finite. Thus

$$\int \frac{d\mu(x)}{(1 + r|x - a|)^{(1 + \varepsilon)ns}} \leq C\mu(E).$$

This completes the proof of the theorem.

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