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WEIGHTED NORM INEQUALITIES FOR MARTINGALES

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0. Introduction. Let $p > 1$ and $W \in L^1_{\mathrm{loc}}(R^n)$. B. Muckenhoupt [5] has recently proved that the inequality

$$(1) \qquad \int_{R^n} (f^*(x))^p W(x)dx \leqq C \int_{R^n} |f(x)|^p W(x)dx ,$$
$$f^*(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy$$

is valid for all $f \in L^p(W(x)dx)$ if and only if W satisfies the condition:

$$\sup_Q \left(\frac{1}{|Q|} \int_Q W(x)dx\right)\left(\frac{1}{|Q|} \int_Q W(x)^{-1/(p-1)}dx\right)^{p-1} < \infty .$$

Here, Q denotes a cube with sides parallel to the axes. This condition has already appeared many times in the literature in connection with several different questions. The purpose of this paper is to show that the analogue of his result holds even in the setting of martingale theory. This problem is indicated by C. Watari.

Let (Ω, F, P) be a probability space with a non-decreasing right continuous family (F_t) of sub σ-fields of F such that F_0 contains all P-null sets. Fix a random variable Z such that $Z > 0$ a.s. and $E[Z] = 1$. In our setting the above condition takes the form:

$$(A_p) \qquad \sup_t \operatorname{ess\cdot sup}_\omega Z_t E\left[\left(\frac{1}{Z}\right)^{1/(p-1)} \middle| F_t\right]^{p-1} < \infty$$

where $Z_t = E[Z|F_t]$. In Section 1 we show that, under some additional conditions, the condition (A_p) holds, if and only if, for every $L^p(d\hat{P})$-bounded martingale X over (F_t)

$$(2) \qquad \hat{E}[(X^*)^p] \leqq C_p \hat{E}[|X_\infty|^p]$$

where $X^* = \sup_t |X_t|$ and $\hat{E}[\cdot]$ denotes expectation over Ω with respect to the new weighted probability measure $d\hat{P} = ZdP$. In the later sections we deal only with continuous local martingales. If the martingale Z_t is continuous such that $Z_0 = 1$, then, as is well known, there is a unique continuous local martingale M such that $Z_t = \exp(M_t - (1/2)\langle M\rangle_t)$.

Conversely, for every continuous local martingale M, $\exp(M_t - (1/2)\langle M\rangle_t)$ is a positive continuous local martingale. Here $\langle M\rangle$ denotes the unique continuous increasing process such that $M^2 - \langle M\rangle$ is a local martingale. We show in Section 2 that if M belongs to the class BMO with respect to $d\hat{P}$, there holds

$$(3) \qquad \hat{E}[(X^*)^p] \leq C_p \hat{E}[\langle X\rangle_\infty^{p/2}]$$

for every P-continuous local martingale X over (F_t). In Section 3 we prove that Z satisfies the condition (A_p) for some $p > 1$ if and only if M is a BMO-martingale with respect to dP.

We thank heartily C. Watari and M. Kaneko with whom we had many helpful conversations.

1. The (A_p) condition. Throughout this paper we denote by C a positive constant and by C_p a positive constant depending on the indexed parameter p, both letters are not necessarily the same in each occurence. Let $p > 1$, and let q be the exponent conjugate to p. For simplicity, we assume that $\bigvee_{0 \leqq t < \infty} F_t = F$. As $d\hat{P} = Z_t dP$ on each F_t, an easy calculation shows that

$$(1) \qquad \hat{E}[X \mid F_t] = \frac{E[ZX \mid F_t]}{Z_t} \quad \text{a.s. under } dP \text{ and } d\hat{P}$$

for every $X \in L^1(d\hat{P})$.

THEOREM 1. *Let $1 < p < \infty$, and assume that $1/Z \in L^{q-1}(dP)$. If the inequality*

$$\hat{E}[(X^*)^p] \leqq C_p \hat{E}[|X_\infty|^p]$$

holds for all P-martingale X such that $\sup_t \hat{E}[|X_t|^p] < \infty$, then Z satisfies the condition (A_p).

PROOF. For any $A \in F$, we get from Hölder's inequality

$$E[|X_t|; A] \leqq E[|X_t|^p Z]^{1/p} E[Z^{-(q-1)}; A]^{1/q} .$$

The first term on the right hand side equals $\hat{E}[|X_t|^p]^{1/p}$ which is dominated by some constant C. Then, as $Z^{-(q-1)} \in L^1(dP)$, it is clear that the martingale X is uniformly integrable with respect to dP. That is, $\sup_t E[|X_t|^p] = E[|X_\infty|^p]$. Now, $\hat{E}[1/Z^q \mid F_t] = E[1/Z^{q-1} \mid F_t]/Z_t$ from (1) and so $E[1/Z^{q-1} \mid F_t] = Z_t \hat{E}[1/Z^q \mid F_t]$. Therefore $Z_t E[(1/Z)^{1/(p-1)} \mid F_t]^{p-1} = Z_t^p \hat{E}[1/Z^q \mid F_t]^{p-1}$. Now let T be any stopping time. For $A \in F_T$, put $N_t = E[I_A/Z^{q-1} \mid F_t]$. Then N is a P-martingale such that $N_\infty = I_A/Z^{q-1}$, and $N_t = Z_t \hat{E}[I_A/Z^q \mid F_t]$. It follows from the assumption that

$$\hat{E}[N_T^p] \leqq C_p \hat{E}\left[\frac{1}{Z^q}; A\right].$$

The left hand side equals $\hat{E}[Z_T^p \hat{E}[(1/Z^q)|F_T]^p; A]$. Thus we get

$$Z_T^p \hat{E}\left[\frac{1}{Z^q}\middle| F_T\right]^{p-1} \leqq C_p,$$

and an application of the section theorem concludes the proof.

THEOREM 2. *If Z satisfies (A_{p_0}) for some $p_0 > 1$, then the inequality*

$$\hat{E}[(X^*)^p] \leqq C_p \hat{E}[|X_\infty|^p], \quad p > p_0 \tag{2}$$

for every P-uniformly integrable martingale X.

PROOF. We may assume that $X_\infty \in L^p(d\hat{P})$. Denote by q_0 the exponent conjugate to p_0. As $X_t = Z_t \hat{E}[(1/Z)X_\infty | F_t]$, we get from Hölder's inequality

$$|X_t|^{p_0} \leqq \left\{Z_t^{p_0} \hat{E}\left[\left(\frac{1}{Z}\right)^{q_0}\middle| F_t\right]^{p_0-1}\right\} \hat{E}[|X_\infty|^{p_0}|F_t].$$

The first term on the right side equals $Z_t E[(1/Z)^{q_0-1}|F_t]^{p_0-1}$ which is dominated by some constant C. Then, applying the Doob inequality to the $\hat{P}$-martingale $\{\hat{E}[|X_\infty|^{p_0}|F_t]\}$, for every $p > p_0$,

$$\begin{aligned} \hat{E}[(X^*)^p] &\leqq C\hat{E}[\sup_t \hat{E}[|X_\infty|^{p_0}|F_t]^{p/p_0}] \\ &\leqq C_p \hat{E}[|X_\infty|^p] \end{aligned}$$

which completes the proof.

In particular, the inequality (2) is valid for every p if and only if Z satisfies (A_p) for every p.

THEOREM 3. *Let Z satisfy (A_p), and set $V_t = E[V|F_t]$ where $V = (1/Z)^{1/(p-1)}$. If there is some constant $k > 0$ such that*

$$\int_{A \cap \{V > \lambda > V_t\}} V dP \leqq C\lambda P(A, V > k\lambda), \quad A \in F_t, \quad \lambda > 0, \tag{3}$$

then there holds

$$\hat{E}[(X^*)^p] \leqq C_p \hat{E}[|X_\infty|^p]$$

for any P-uniformly integrable martingale X.

PROOF. From Theorem 3 it is sufficient to prove that (A_p) implies $(A_{p-\varepsilon})$ for some $\varepsilon > 0$. To see this, we first show the "reverse Hölder inequality":

$$E[V^{1+\delta}|F_t] \leqq CV_t^{1+\delta} \tag{4}$$

for some $\delta > 0$. For any $A \in F_t$, we have from (3)

$$\int_0^\infty d\lambda\left(\lambda^{\delta-1}\int_{A\cap\{V>\lambda>V_t\}} VdP\right) \leqq C\int_0^\infty \lambda^\delta P(A, V > k\lambda)d\lambda$$
$$\leqq CE\left[\int_0^{V/k} \lambda^\delta d\lambda;\ A\right]$$
$$\leqq \frac{C}{(1+\delta)k^{1+\delta}}\int_A V^{1+\delta}dP\,.$$

By the Fubini theorem, the left hand side equals

$$\int_{A\cap\{V>V_t\}} V\left(\int_{V_t}^V \lambda^{\delta-1}d\lambda\right)dP = \int_{A\cap\{V>V_t\}} V\frac{V^\delta - V_t^\delta}{\delta}dP$$
$$\geqq \frac{1}{\delta}\int_A V(V^\delta - V_t^\delta)dP$$

which is equal to $(1/\delta)\int_A (V^{1+\delta} - V_t^{1+\delta})dP$ since $A \in F_t$. Thus for sufficiently small $\delta > 0$

$$\int_A V^{1+\delta}dP \leqq \frac{1}{\delta\left(\frac{1}{\delta} - C\frac{1}{(1+\delta)k^{1+\delta}}\right)}\int_A V_t^{1+\delta}dP$$

which proves (4). Now, put $\varepsilon = (p-1)\delta/(1+\delta) > 0$. As $1+\delta = (p-1)/(p-\varepsilon-1)$, we get

$$E\left[\left(\frac{1}{Z}\right)^{1/(p-\varepsilon-1)}\middle| F_t\right]^{p-\varepsilon-1} = E[V^{1+\delta}\,|\,F_t]^{p-\varepsilon-1}$$
$$\leqq V_t^{(1+\delta)(p-\varepsilon-1)}\,.$$

The right hand side equals $E[(1/Z)^{1/(p-1)}\,|\,F_t]^{p-1}$. Consequently, the theorem is established.

These results are valid for discrete time martingales. It is proved in [2] that the condition (3) holds in the special case that the probability space is the d-dimensional unit cube Q_0 and the family of sub σ-fields is the sequence (F_n) of finite fields obtained by successive dyadic partitions of Q_0, but we don't know whether the inequality (3) is true in general.

2. Weighted norm inequalities for continuous martingales. In what follows, assume that the martingale Z_t is continuous, and we deal only with continuous local martingales. Now we state several lemmas used later.

LEMMA 1 (Ito's formula). *Let X^i be a continuous local martingale, A^i be a continuous process with bounded variation, ξ_0^i be a F_0-measurable* *random variable and* $\xi_t^i = \xi_0^i + X_t^i + A_t^i$ $(i = 1, 2, \cdots, d)$. *If* $F: R^d \to R$ *is a twice continuously differentiable function, then*

$$F(\xi_t) = F(\xi_0) + \sum_{i=1}^{d} \int_0^t F_{x_i}(\xi_s) dX_s^i + \sum_{i=1}^{d} \int_0^t F_{x_i}(\xi_s) dA_s^i + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t F_{x_i x_j}(\xi_s) d\langle X^i, X^j\rangle_s \tag{5}$$

where $\langle X^i, X^j\rangle = (1/2)(\langle X^i + X^j\rangle - \langle X^i\rangle - \langle X^j\rangle)$. (for example, see [6]).

Let M be the continuous local martingale defined by the stochastic integral $\int_0^t (1/Z_s) dZ_s$.

LEMMA 2. *If* X *is a* P*-continuous local martingale, then* $\hat{X} = X - \langle X, M\rangle$ *is a* $\hat{P}$*-continuous local martingale such that* $\langle \hat{X}\rangle = \langle X\rangle$ *under either probability measure. Conversely, if* X' *is a* $\hat{P}$*-continuous local matingale, then* $X' + \langle X', \hat{M}\rangle$ *is a* P*-continuous local martingale.*

PROOF. Because of (1)

$$\hat{E}[\hat{X}_{t+s} | F_t] = \frac{1}{Z_t} E[Z\hat{X}_{t+s} | F_t] = \frac{1}{Z_t} E[Z_{t+s}\hat{X}_{t+s} | F_t] \,.$$

Thus to prove that $\hat{X}$ is a $\hat{P}$-local martingale, it suffices to show that $Z\hat{X}$ is a P-local martingale. As $d\langle X, M\rangle_t = (1/Z_t) d\langle X, Z\rangle_t$, by applying Lemma 1 to the case such that $\xi_t^1 = Z_t$, $\xi_t^2 = \hat{X}_t = X_t - \langle X, M\rangle_t$ and $F(x_1, x_2) = x_1 x_2$ we get

$$Z_t \hat{X}_t = Z_0 \hat{X}_0 + \int_0^t \hat{X}_s dZ_s + \int_0^t Z_s dX_s$$

which is a P-local martingale. It is immediate to see that $\langle \hat{X}\rangle = \langle X\rangle$ under dP and $d\hat{P}$.

We are going to prove the later part of the lemma. As before, to see that $X' + \langle X', \hat{M}\rangle$ is a P-continuous local martingale, it is sufficient to prove that $(1/Z_t)(X_t' + \langle X', \hat{M}\rangle_t)$ is a $\hat{P}$-local martingale. As $1/Z_t$ is a $\hat{P}$-local martingale, by applying again Lemma 1 to the case such that $\xi_t^1 = 1/Z_t$, $\xi_t^2 = X_t' + \langle X', \hat{M}\rangle_t$ and $F(x_1, x_2) = x_1 x_2$, we get

$$\frac{1}{Z_t}(X_t' + \langle X', \hat{M}\rangle_t) = X_0' + \int_0^t (X_s' + \langle X', \hat{M}\rangle_s) d\frac{1}{Z_s} + \int_0^t \frac{1}{Z_s} dX_s'$$

which is a $\hat{P}$-local martingale. This completes the proof.

From Lemma 2, for every $\hat{P}$-continuous local martingale X', $X = X' + \langle X', \hat{M}\rangle$ is a P-local martingale and then $\hat{X} = X - \langle X, M\rangle$ is a $\hat{P}$-continuous local martingale. Thus $\hat{X} - X' = \langle X', \hat{M}\rangle - \langle X, M\rangle$ is also a $\hat{P}$-continuous local martingale zero at $t = 0$, so that $\hat{X} = X'$.

A P-continuous local martingale X belongs to the class $BMO(P)$ if

$$||X||_{B(P)}^2 = \sup_t \operatorname{ess\cdot sup}_\omega E[\langle X\rangle_\infty - \langle X\rangle_t | F_t] < \infty .$$

Similarly we define the class $BMO(\hat{P})$ relative to the measure $d\hat{P}$.

LEMMA 3. *Let $\hat{X}$ and $\hat{Y}$ be $\hat{P}$-continuous local martingales. Then:*
(i) (*Davis's inequality*)

$$\frac{1}{4\sqrt{2}}\hat{E}[\hat{X}^*] \leqq \hat{E}[\langle\hat{X}\rangle_\infty^{1/2}] \leqq 2\hat{E}[\hat{X}^*] .$$

(ii) (*Fefferman's inequality*)

$$\hat{E}\left[\int_0^\infty |d\langle\hat{X}, \hat{Y}\rangle_t|\right] \leqq \sqrt{2}||\hat{X}||_{B(\hat{P})}\hat{E}[\langle\hat{Y}\rangle_\infty^{1/2}]$$

(see [3]).

We are now in a position to state another weighted norm inequalities.

THEOREM 4. *Let X be any P-continuous local martingale. Then we have:*

$$\hat{E}[X^*] \leqq \sqrt{2}(4 + ||\hat{M}||_{B(\hat{P})})\hat{E}[\langle X\rangle_\infty^{1/2}] \tag{6}$$

$$\frac{1}{2}(1 - 2\sqrt{2}||\hat{M}||_{B(\hat{P})})\hat{E}[\langle X\rangle_\infty^{1/2}] \leqq \hat{E}[X^*] . \tag{7}$$

PROOF. Since $\hat{X} = X - \langle X, M\rangle$ is a $\hat{P}$-continuous local martingale with $\langle\hat{X}\rangle = \langle X\rangle$, we get from Lemma 3

$$\begin{aligned} 4\sqrt{2}\hat{E}[\langle X\rangle_\infty^{1/2}] &\geqq \hat{E}[\hat{X}^*] \\ &\geqq \hat{E}\left[X^* - \int_0^\infty |d\langle\hat{X}, \hat{M}\rangle_t|\right] \\ &\geqq \hat{E}[X^*] - \sqrt{2}||\hat{M}||_{B(\hat{P})}\hat{E}[\langle\hat{X}\rangle_\infty^{1/2}] \end{aligned}$$

which implies (6). Similarly, as $X = \hat{X} + \langle\hat{X}, \hat{M}\rangle$, we get

$$\hat{E}[X^*] \geqq \hat{E}[\hat{X}^*] - \sqrt{2}||\hat{M}||_{B(\hat{P})}\hat{E}[\langle\hat{X}\rangle_\infty^{1/2}] .$$

By Davis's inequality, the first term on the right hand side is larger than $(1/2)\hat{E}[\langle\hat{X}\rangle_\infty^{1/2}]$, so that the right hand side equals $(1/2 - \sqrt{2}||\hat{M}||_{B(\hat{P})}) \times \hat{E}[\langle X\rangle_\infty^{1/2}]$. This completes the proof.

Consider now a non-decreasing continuous function ϕ on $[0, \infty[$ with $\phi(0) = 0$. Suppose that $\Phi(t) = \int_0^t \phi(s)ds$ satisfies the growth condition: $\Phi(2t) \leqq c\Phi(t)$.

LEMMA 4 (A. Garsia). *Let A_t be a continuous increasing process. If there is a positive $\hat{P}$-integrable random variable Y such that*

$$\hat{E}[A_\infty - A_t | F_t] \leqq \hat{E}[Y | F_t]$$

for every t, then $\hat{E}[\Phi(A_\infty)] \leqq CE[\Phi(Y)]$. *Here, the choice of C depends only on the growth parameter of* Φ. (for example, see [1])

THEOREM 5. *Assume that* $\|\hat{M}\|_{B(\hat{P})} < 1/2\sqrt{2}$. *Then for any P-continuous local martingale X*

$$cE[\Phi(X^*)] \leqq \hat{E}[\Phi(\langle X\rangle_\infty^{1/2})] \leqq C\hat{E}[\Phi(X^*)] . \tag{8}$$

Here, the choice of c and C depends only on the growth parameter of Φ.

PROOF. At first we prepare some notations. Let T be any stopping time, and set

$$\Omega' = \{T < \infty\} , \quad F'_t = F_{T+t} , \quad dP' = \frac{dP}{P(\Omega')} .$$

Then $X'_t = X_{T+t} - X_T$ is a P'-continuous local martingale over (F'_t) such that $\langle X'\rangle_t = \langle X\rangle_{T+t} - \langle X\rangle_T$, so that $Z'_t = Z_{T+t}/Z_T = \exp(M'_t - (1/2)\langle M'\rangle_t)$. Let $d\hat{P}' = (Z/Z_T)dP'$ and $\hat{X}' = X' - \langle X', M'\rangle$. As before, $\hat{X}'$ is a $\hat{P}'$-continuous local martingale over (F'_t) such that $\langle \hat{X}'\rangle = \langle X'\rangle$. If S is an F'_t-stopping time, $S+T$ is an F_t-stopping time. Thus for any $A \in F'_S = F_{T+S}$, we get from (1)

$$\begin{aligned}\int_A \hat{E}'[\langle \hat{M}'\rangle_\infty - \langle \hat{M}'\rangle_S | F'_S] d\hat{P}' &= \int_A (\langle \hat{M}'\rangle_\infty - \langle \hat{M}'\rangle_S) d\hat{P}' \\ &= \int_A (\langle M\rangle_\infty - \langle M\rangle_{T+S}) \frac{Z}{Z_T} \frac{dP}{P(\Omega')} \\ &= \int_A \hat{E}[\langle \hat{M}\rangle_\infty - \langle \hat{M}\rangle_{T+S} | F_{T+S}] \frac{Z_{T+S}}{Z_T} \frac{dP}{P(\Omega')}\end{aligned}$$

the right hand side is dominated by

$$\int_A \|\hat{M}\|^2_{B(\hat{P})} \frac{Z_{T+S}}{Z_T} \frac{dP}{P(\Omega')} = \int_A \|\hat{M}\|^2_{B(\hat{P})} d\hat{P}' ,$$

that is, $\hat{E}'[\langle \hat{M}'\rangle_\infty - \langle \hat{M}'\rangle_S | F'_S] \leqq \|\hat{M}\|^2_{B(\hat{P})}$, so that from the section theorem we have $\|\hat{M}'\|_{B(\hat{P}')} \leqq \|\hat{M}\|_{B(\hat{P})} < 1/2\sqrt{2}$. Then from Theorem 4

$$c\hat{E}'[(X')^*] \leqq \hat{E}'[\langle X'\rangle_\infty^{1/2}] \leqq C\hat{E}'[(X')^*] .$$

Let $\Lambda \in F'_0 = F_T$. Substituting X' by $X'I_\Lambda$, we obtain

$$c\hat{E}'[(X')^* | F'_0] \leqq \hat{E}'[\langle X'\rangle_\infty^{1/2} | F'_0] \leqq C\hat{E}'[(X')^* | F'_0] .$$

Clearly, $X^* - X^*_T \leqq (X')^* \leqq 2X^*$ and $\langle X\rangle_\infty^{1/2} - \langle X\rangle_T^{1/2} \leqq \langle X'\rangle_\infty^{1/2} \leqq \langle X\rangle_\infty^{1/2}$, so that

$$\hat{E}[X^* - X_T^* | F_T] \leqq C\hat{E}[\langle X\rangle_\infty^{1/2} | F_T]$$
$$\hat{E}[\langle X\rangle_\infty^{1/2} - \langle X\rangle_T^{1/2} | F_T] \leqq 2C\hat{E}[X^* | F_T] .$$

Then by Garsia's lemma we get (8).

3. A characterization of *BMO*-martingales. In this section we study the relation between the condition (A_p) and the class BMO. We start with the next "John-Nirenberg type inequality".

LEMMA 5. *If X is a P-continuous martingale such that $||X||_{B(P)} < 1$, then for any F_t-stopping time T*

$$E[e^{\langle X\rangle_\infty - \langle X\rangle_T} | F_T] \leqq \frac{1}{1 - ||X||_{B(P)}^2} . \tag{9}$$

PROOF. If $E[\langle X\rangle_\infty - \langle X\rangle_t | F_t] \leqq c = ||X||_{B(P)}^2 < 1$ a.s. for all $t \geqq 0$, then the energy inequalities (see [4]) give $E[\langle X\rangle_\infty^n] \leqq c^n n!$, $n = 0, 1, \cdots$. Thus $E[e^{\langle X\rangle_\infty}] = \sum_{n=0}^\infty (1/n!) E[\langle X\rangle_\infty^n] \leqq \sum_{n=0}^\infty c^n = 1/(1 - ||X||_{B(P)}^2) < \infty$. Let T be any F_t-stopping time, and let $\Lambda \in F_T$. We may assume that $P(\Lambda) > 0$. Put $\Omega' = \Lambda$, and we adopt the same notations as in the proof of Theorem 5. Then for any $A \in F'_S = F_{T+S}$

$$\begin{aligned}\int_A E'[\langle X'\rangle_\infty - \langle X'\rangle_S | F'_S] dP' &= \int_A (\langle X\rangle_\infty - \langle X\rangle_{T+S}) \frac{dP}{P(\Omega')} \\ &= \int_A E[\langle X\rangle_\infty - \langle X\rangle_{T+S} | F_{T+S}] \frac{dP}{P(\Omega')} \\ &\leqq \int_A ||X||_{B(P)}^2 dP'\end{aligned}$$

from which $\hat{E}'[\langle X'\rangle_\infty - \langle X'\rangle_S | F'_S] \leqq ||X||_{B(P)}^2$. Therefore, according to the section theorem, $||X'||_{B(P')} \leqq ||X||_{B(P)} < 1$, so that we get

$$\begin{aligned}E[e^{\langle X\rangle_\infty - \langle X\rangle_T}; \Omega'] &= P(\Omega') E'[e^{\langle X'\rangle_\infty}] \\ &\leqq \frac{1}{1 - ||X'||_{B(P')}^2} P(\Omega') \\ &\leqq \frac{1}{1 - ||X||_{B(P)}^2} P(\Omega') ,\end{aligned}$$

which implies (9).

THEOREM 6. *Z satisfies the condition (A_p) for some $p > 1$ if and only if the P-continuous local martingale M belongs to the class $BMO(P)$.*

PROOF. Suppose firstly that $||M||_{B(P)} < \infty$, and choose $p > 1$ such that $||(\sqrt{p+1}/(p-1))M||_{B(P)} < 1$. Then we get for any stopping time T

$$
\begin{aligned}
Z_T E\Big[\Big(\frac{1}{Z}\Big)^{1/(p-1)}\Big|F_T\Big]^{p-1}
&= E\Big[\exp\Big(-\frac{1}{p-1}(M_\infty - M_T) + \frac{1}{2(p-1)}(\langle M\rangle_\infty - \langle M\rangle_T)\Big)\Big|F_T\Big]^{p-1} \\
&\leqq E\Big[\exp\Big(-\frac{1}{p-1}(M_\infty - M_T) - \frac{1}{(p-1)^2}(\langle M\rangle_\infty - \langle M\rangle_T)\Big) \\
&\quad \times \exp\Big(\frac{p+1}{2(p-1)^2}(\langle M\rangle_\infty - \langle M\rangle_T)\Big)\Big|F_T\Big]^{p-1} \\
&\leqq E\Big[\exp\Big(-\frac{2}{p-1}(M_\infty - M_T) - \frac{2}{(p-1)^2}(\langle M\rangle_\infty - \langle M\rangle_T)\Big)\Big|F_T\Big]^{(p-1)/2} \\
&\quad \times E\Big[\exp\Big(\frac{p+1}{(p-1)^2}(\langle M\rangle_\infty - \langle M\rangle_T)\Big)\Big|F_T\Big]^{(p-1)/2} .
\end{aligned}
$$

The first term on the right hand side is smaller that 1, and the second term is dominated by

$$
\frac{1}{\Big\{1 - \dfrac{p+1}{(p-1)^2}\,||M||^2_{B(P)}\Big\}^{(p-1)/2}} ,
$$

from (9). Therefore, according to the section theorem, Z satisfies (A_p).

On the other hand, for every $p > 1$, by using the Jensen inequality

$$
\begin{aligned}
Z_T E\Big[\Big(\frac{1}{Z}\Big)^{1/(p-1)}\Big|F_T\Big]^{p-1}
&= \exp\Big(M_T - \frac{1}{2}\langle M\rangle_T\Big) E\Big[\exp\Big(-\frac{1}{p-1}M_\infty + \frac{1}{2(p-1)}\langle M\rangle_\infty\Big|F_T\Big]^{p-1} \\
&\geqq \Big\{\exp\Big(\frac{1}{p-1}M_T - \frac{1}{2(p-1)}\langle M\rangle_T - \frac{1}{p-1}M_T \\
&\quad + \frac{1}{2(p-1)}E[\langle M\rangle_\infty | F_T]\Big)\Big\}^{p-1} \\
&= \exp\Big(\frac{1}{2}E[\langle M\rangle_\infty - \langle M\rangle_T | F_T]\Big)
\end{aligned}
$$

from which we get $||M||_{B(P)} < \infty$ if Z satisfies the condition (A_p) for some p. This completes the proof.

THEOREM 7. *If* $||M||_{B(P)} < 1/\sqrt{2}$, *then* $||\hat{M}||_{B(\hat{P})} \leqq (4/(1 - 2||M||^2_{B(P)}))^{1/4}$.

PROOF. Let T be any stopping time. As $x < 2e^{x/2}$ for every $x > 0$, we have

$$\hat{E}[\langle \hat{M}\rangle_\infty - \langle \hat{M}\rangle_T | F_T] = E\left[\frac{Z}{Z_T}(\langle M\rangle_\infty - \langle M\rangle_T)\middle| F_T\right]$$

$$= E\Big[\{\exp((M_\infty - M_T) - (\langle M\rangle_\infty - \langle M\rangle_T))\}$$

$$\times \left\{\exp\left(\frac{1}{2}(\langle M\rangle_\infty - \langle M\rangle_T)\right)\right\} \times (\langle M\rangle_\infty - \langle M\rangle_T) | F_T\Big]$$

$$\leqq 2E[\{\exp((M_\infty - M_T) - (\langle M\rangle_\infty - \langle M\rangle_T))\} \times \{\exp(\langle M\rangle_\infty - \langle M\rangle_T)\} | F_T]$$

$$\leqq 2E[\exp(2(M_\infty - M_T) - 2(\langle M\rangle_\infty - \langle M\rangle_T)) | F_T]^{1/2}$$

$$\times E[\exp(2(\langle M\rangle_\infty - \langle M\rangle_T)) | F_T]^{1/2}$$

by the Schwarz inequality. The right hand side is dominated by

$$\frac{2}{\sqrt{1 - 2\|M\|^2_{B(P)}}}$$

from the John-Nirenberg type inequality. Thus the theorem is established.

Finally, we give such an example that M does not belong to the class $BMO(P)$ even if Z is bounded. Let $B = (B_t, F_t)$ be a 1-dimensional Brownian motion such that $B_0 = 0$. Put $\tau = \inf\{t; B_t \geqq 1\}$ and $M_t = B_{t\wedge\tau}$. Then $Z_t = \exp(M_t - (1/2)\langle M\rangle_t) \leqq e$, but for each $t > 0$

$$\operatorname*{ess\cdot sup}_{\omega} E[(\tau - t) | F_t] I_{\{t<\tau\}} = +\infty .$$

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