[13] R. C. James, *Bases and reflexivity of Banach spaces*, Ann. of Math. (2) 52 (1950), pp. 518–527.
[14] — *Uniformly non-square Banach spaces*, Ann. of Math. 80 (1964), pp. 542–550.
[15] — *Some self-dual properties of normed linear spaces*, Annals of Math. Studies 69 (1972), pp. 159–175.
[16] — *Super-reflexive spaces with bases*, Pacific J. Math. 41 (1972), pp. 409–419.
[17] W. B. Johnson, H. P. Rosenthal, and M. Zippin, *On bases, finite dimensional decompositions, and weaker structures in Banach spaces*, Israel J. Math 9 (1971), pp. 488–506.
[18] M. I. Kadec and M. G. Snobar, *Certain functionals on the Minkowski compactum*, Mat. Zametki 10 (1971), pp. 453–457 (in Russian).
[19] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in $\mathscr{L}_p$ spaces and their applications*, Studia Math. 29 (1968), pp. 275–326.
[20] — and H. P. Rosenthal, *The $\mathscr{L}_p$ spaces*, Israel J. Math. 7 (1969), pp. 325–349.
[21] B. Maurey, *Théorèmes de factorisation pour les opérateurs linéaires à valeurs dans les espaces L^p*, Société Mathématique de France (1974).
[22] A. Persson and A. Pietsch, *p-nucleare und p-integrale Abbildungen in Banachräumen*, Studia Math. 33 (1969), pp. 19–62.
[23] H. P. Rosenthal, *On subspaces of L_p*, Ann. of Math. 97 (1973), pp. 344–373.
[24] J. J. Schäffer and K. Sundaresan, *Reflexivity and the girth of spheres*, Math. Ann. 184 (1970), pp. 163–168.
[25] I. I. Tseitlin, *On a particular case of the existence of a compact linear operator which is not nuclear*, Funk. Anal. i Pril. 6 (1973), p 102.
[26] L. Tzafriri, *Reflexivity in Banach lattices and their subspaces*, Journal of Functional Anal. 10 (1972), pp. 1–18.

THE OHIO STATE UNIVERSITY

Received June 11, 1973 (711)

STUDIA MATHEMATICA, T. LI. (1974)

Weighted norm inequalities for maximal functions and singular integrals

by

R. R. COIFMAN (St. Louis, Mo.) and C. FEFFERMAN (Princeton, N. J.)

Abstract. We present simplified proofs of the weighted-norm inequalities of R. Hunt, B. Muckenhoupt and R. Wheeden, concerning singular integrals and maximal functions. The inequalities in question are

$$\int_{R^n} |Tf(x)|^p \omega(x)\,dx = C \int_{R^n} |f(x)|^p \omega(x)\,dx,$$

where T denotes either a singular integral operator, or the maximal function of Hardy and Littlewood, and ω satisifes appropriate (necessary and sufficient) conditions.

§ 1. This note is concerned with the problem of identifying those weight functions $\omega(x)$ on R^1 for which the Hilbert transform $Tf(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(y)\,dy}{x-y}$ is bounded on $L^p(\omega(x)\,dx)$, that is

$$(1)\qquad \int_{R^1} |Tf(x)|^p \omega(x)\,dx \leqslant C \int_{R^1} |f(x)|^p \omega(x)\,dx \quad \text{for all } f.$$

Until recently, the only significant partial result known was that of Helson and Szegö [6]: Inequality (1) holds for $p = 2$ if and only if $\omega = e^{b_1+Tb_2}$ for functions $b_1, b_2 \in L^\infty$ with $\|b_2\|_\infty < \pi/2$. Unfortunately, there is no obvious way to tell whether a given ω can be so represented, so that even for L^2, the problem of inequality (1) remained open. Attempts to generalize the Helson–Szegö theorem to L^p $(p \neq 2)$ were only partly successful.

Surprisingly, there is a simple necessary and sufficient condition for inequality (1) to hold. It was B. Muckenhoupt who made the key discovery, by studying the analogue of (1) for the maximal function

$$f^*(x) = \sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y)|\,dy \quad \text{in } R^n.$$

(Here, Q denotes a cube with sides parallel to the axes.)

THEOREM I (Muckenhoupt [8]). *Let* $p>1$ *and* $\omega\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$. *The inequality*

$$(2)\qquad \int_{\mathbf{R}^n} (f^*(x))^p\,\omega(x)\,dx \leqslant C\int_{\mathbf{R}^n} |f(x)|^p\,\omega(x)\,dx$$

is valid for all $f\in L^p(\omega(x)\,dx)$, *if and only if* ω *satisfies the condition*

$$(\mathrm{A}_p)\qquad \sup_Q\left(\frac{1}{|Q|}\int_Q \omega\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega^{-\frac{1}{p-1}}\,dx\right)^{p-1}<\infty,$$

where the supremum is taken over all cubes Q.

Shortly after the proof of Theorem I, R. Hunt, B. Muckenhoupt, and R. Wheeden [7] overcame considerable technical problems to prove

THEOREM II. *For any* p $(1<p<\infty)$ *and any positive* $\omega\in L^1_{\mathrm{loc}}(\mathbf{R}^1)$, *inequality* (1) *is equivalent to* (A_p).

In particular, (A_2) and the Helson–Szegö condition are equivalent, i.e.

COROLLARY. *A real-valued function* f *on* $\mathbf{R}^1$ *may be written in the form* $f=f_1+Tf_2$ *with* $f_1\in L^\infty$, $\|f_2\|_\infty<\pi/2$ *if and only if*

$$\sup_Q\left(\frac{1}{|Q|}\int_Q \exp(f(x))\,dx\right)\left(\frac{1}{|Q|}\int_Q \exp(-f(x))\,dx\right)<\infty.$$

The corollary sharpens the one-dimensional case of results in [4] on the duality of H^1 and BMO.

In this note, we present greatly simplified proofs of Theorems I and II. The ideas and methods discussed here are the fruit of discussion and collaboration among R. Gundy, R. Hunt, B. Muckenhoupt, R. Wheeden and the authors. This paper could be considered a summary of our joint efforts.

We note in retrospect that the (A_p) condition has already appeared many times in the literature in connection with several different questions. (See, e.g. Rosenblum [12], and the work of Serrin [13], Murthy and Stampacchia [11], and others on partial differential equations.) Much of this earlier work can probably be sharpened by means of Theorems I and II and the related results discussed below.

In sequel, we assume that the reader knows the first two chapters of Stein's book [14].

§ 2. We now proceed to prove Theorems I and II.

Proof of Theorem I. That (2) implies (A_p) is easy. We simply fix a cube Q and a function $f\geqslant 0$, and observe that

$$f^*(x)\geqslant\left(\frac{1}{|Q|}\int_Q f(y)\,dy\right)\chi_Q(x).$$

If condition (2) is valid, we obtain $\left(\int_Q \omega(x)\,dx\right)(m_Q(f))^p\leqslant C\int_Q f^p(x)\omega(x)\,dx$ where $m_Q(f)=\frac{1}{|Q|}\int_Q f(y)\,dy$. Thus,

$$(3)\qquad m_Q(f)\leqslant C\left(\frac{1}{\int_Q\omega\,dx}\int_Q f^p\omega\,dx\right)^{1/p}.$$

Substituting $f=\omega^{-\frac{1}{p-1}}$, we obtain (3) at once.

To prove that (A_p) implies (2), we first note that (A_p) implies (3). This follows from replacing f by $(f\omega^{\frac{1}{p}})\omega^{-\frac{1}{p}}$ in the definition of $m_Q(f)$ and applying Hölder's inequality. Now taking the supremum in (3) over all cubes Q containing a given point x, we find that

$$(4)\qquad f^*(x)\leqslant C[M_\omega(f^p)(x)]^{1/p},$$

where $M_\omega f(x)=\sup_{x\in Q}\frac{1}{\int_Q\omega(y)\,dy}\int_Q|f(y)|\,\omega(y)\,dy$.

We are in position to invoke a simple variant of the maximal theorem.

LEMMA 1. *Let* μ *be a positive measure on* $\mathbf{R}^n$, *so that* $\mu(I^*)\leqslant C\mu(I)$ *for any cube* I. (I^* *denotes the double of* I.) *Form the maximal function* $M\mu(f)(x)=\sup_{x\in Q}\frac{1}{\mu(Q)}\int_Q|f(y)|\,d\mu(y)$. *Then*

$$\int_{\mathbf{R}^n}(M\mu(f)(x))^r\,d\mu(x)\leqslant C_r\int_{\mathbf{R}^n}|f(x)|^r\,d\mu(x)\qquad \textit{for any } r>1.$$

The proof in Stein [14] for the case μ = Lebesgue measure works in general with trivial changes. (See also [2].)

Now take $d\mu(x)=\omega(x)\,dx$. That $\mu(I^*)\leqslant C\mu(I)$ is just the special case $Q=I^*$, $f=\chi_I$ of (3). Lemma 1 yields $\int_{\mathbf{R}^n}(M_\omega f(x))^r\omega\,dx\leqslant C\int_{\mathbf{R}^n}|f|^r\omega\,dx$ for $r>1$, which together with (4) implies

$$(5)\qquad \int_{\mathbf{R}^n}(f^*(x))^{p_1}\omega(x)\,dx\leqslant C_{p_1}\int_{\mathbf{R}^n}|f(x)|^{p_1}\omega(x)\,dx\qquad\text{for every } p_1>p,$$

whenever $\omega(x)$ satisfies (A_p).

In Section 3 below, we will prove the following result.

LEMMA 2. *Suppose that* ω *satisfies* (A_p). *Then* ω *also satisfies* $(\mathrm{A}_{p-\varepsilon})$ *for some* $\varepsilon>0$.

From this and from (5), we see at once that (A_p) implies (2). Thus, modulo Lemma 2, the proof of Theorem I is complete. ■

Proof of Theorem II. It is easy to show that (1) implies (A_p). Let Q_1 and Q_2 be the two halves of a single interval Q_0, and take a function

$f \geqslant 0$ supported in Q_1. Then $|Tf(x)| \geqslant C\left(\frac{1}{|Q_1|}\int_{Q_1} f(y)\,dy\right)\chi_{Q_2}(x)$, so that if (1) holds, it follows that

$$\Big(\int_{Q_2}\omega(x)\,dx\Big)\big(m_{Q_1}(f)\big)^p \leqslant C\int_{Q_1} f^p\,\omega(x)\,dx.$$

Taking $f = 1$ we obtain $\int_{Q_2}\omega(x)\,dx \leqslant C\int_{Q_1}\omega(x)\,dx$ and interchanging Q_1 and Q_2 we get $\int_{Q_1}\omega(x)\,dx \leqslant C\int_{Q_2}\omega(x)\,dx$. Taking now $f = \omega^{-\frac{1}{p-1}}$ we obtain condition (A_p).

It remains to show that (A_p) implies (1). Rather than restrict ourselves to the Hilbert transform, we shall work with a general singular integral operator T: $f \to K * f$ in $\mathbf{R}^n$, with a convolution kernel K satisfying the standard conditions:

(a) $$\|\hat{K}\|_\infty \leqslant C.$$

(b) $$|K(x)| \leqslant \frac{C}{|x|^n}.$$

(c) $$|K(x)-K(x-y)| \leqslant \frac{C|y|}{|x|^{n+1}} \quad \text{for } |y| < \frac{|x|}{2}.$$

Our result on singular integrals is the following.

THEOREM III. *Suppose that the weight function ω satisfies (A_∞). There are positive constants C, $\delta > 0$ so that given any cube Q and any measurable subset $E \subseteq Q$, $\frac{\omega(E)}{\omega(Q)} \leqslant C\left(\frac{|E|}{|Q|}\right)^\delta$. (Here $\omega(A) = \int_A \omega(x)\,dx$ for $A \subseteq \mathbf{R}^n$.) Then*

$$\int_{\mathbf{R}^n}|Tf(x)|^p\,\omega(x)\,dx \leqslant C_p\int_{\mathbf{R}^n}(f^*(x))^p\,\omega(x)\,dx \qquad (0 < p < \infty).$$

From Theorems I and III we see that

$$\int_{\mathbf{R}^n}|Tf(x)|^p\,\omega(x)\,dx \leqslant C_p\int_{\mathbf{R}^n}|f(x)|^p\omega(x)\,dx$$

whenever $\omega(x)$ satisfies (A_p) and (A_∞). In Section 3 we shall prove

LEMMA 3. (A_p) *implies* (A_∞).

Thus, (A_p) implies (1), not only for the Hilbert transform, but for arbitrary singular integrals in $\mathbf{R}_n$. Modulo Theorem III and Lemma 3, the proof of Theorem II is complete.

Proof of Theorem III. We shall work with the "maximal operator" $T^*f(x) = \sup_{Q_x}\left|\int_{\mathbf{R}^n - Q_x} K(x-y)f(y)\,dy\right|$, where Q_x ranges over all cubes centered at x. The basic real-variable fact concerning T^* is the weak-type inequality

(6) $$|\{x \in \mathbf{R}^n \mid T^*f(x) > \alpha\}| \leqslant \frac{C}{\alpha}\int_{\mathbf{R}^n}|f(x)|\,dx. \qquad \text{(See Stein [14], p. 42.)}$$

By combining (6) with the (A_∞) condition, we shall prove

(7) $$\omega(\{T^*f > 2\alpha \text{ and } f^* \leqslant \gamma\alpha\}) \leqslant C\gamma^\delta\,\omega(\{T^*f > \alpha\}).$$

Once we know this, Theorem III is easy, for

$$\int_{\mathbf{R}^n}(T^*f)^p\,\omega\,dx = C\int_0^\infty \alpha^{p-1}\,\omega(\{T^*f > 2\alpha\})\,d\alpha$$

$$\leqslant C\int_0^\infty \alpha^{p-1}\,\omega(\{f^* > \gamma\alpha\})\,d\alpha + C\gamma^\delta\int_0^\infty \alpha^{p-1}\,\omega(\{T^*f > \alpha\})\,d\alpha \qquad \text{(by (8))}$$

$$= C(\gamma)\int_{\mathbf{R}^n}(f^*)^p\,\omega\,dx + C\gamma^\delta\int_{\mathbf{R}^n}(T^*f)^p\,\omega\,dx.$$

Taking γ so small that $C\gamma^\delta \leqslant \frac{1}{2}$, we obtain the conclusion of Theorem III. Thus, Theorem III reduces to estimate (7). What follows is a proof of (7).

By Whitney's lemma (see [14], p. 16), the open set $U_\alpha = \{T^*f > \alpha\}$ breaks up as a disjoint union of cubes $\{Q_i\}$ in such a way that the distance from Q_i to $\mathbf{R}^n - U_\alpha$ is comparable to d_i = diameter (Q_i). Thus, there are points $x_i \in \mathbf{R}^n - U_\alpha$ such that distance $(x_i, Q_i) < 2d_i$. Let $\bar{Q}_i$ be the cube centered at x_i, with diameter $20d_i$. Note that $Q_i^* \leqslant \bar{Q}_i$.

The main step in our proof of (7) is to show that

(8) $$|\{x \in Q_i \mid T^*f(x) > 2\alpha \text{ and } f^*(x) \leqslant \gamma\alpha\}| \leqslant C\gamma|Q_i|.$$

In proving (8) we may assume that $f^*(\xi_i) \leqslant \gamma\alpha$ for at least one point $\xi_i \in Q_i$ (for otherwise there is nothing to prove), and also that γ is small (since (8) is trivial for $C \geqslant \gamma^{-1}$).

Now write $f = f_1 + f_2$ where $f_1 = f\chi_{\bar{Q}_i}$ and $f_2 = f\chi_{\mathbf{R}^n - \bar{Q}_i}$. Since $\xi_i \in Q_i \subseteq \bar{Q}_i$, it follows that

$$\frac{1}{|\bar{Q}_i|}\int_{\mathbf{R}^n}|f_1(y)|\,dy = \frac{1}{|\bar{Q}_i|}\int_{\bar{Q}_i}|f(y)|\,dy \leqslant f^*(\xi_i) \leqslant \gamma\alpha,$$

so that the weak-type inequality (6) yields

(9) $$\left|\left\{T^*f_1 > \frac{\alpha}{2}\right\}\right| \leqslant \frac{2C}{\alpha}\int_{\mathbf{R}^n}|f_1(y)|\,dy \leqslant C\gamma|Q_i|.$$

Next we shall prove that

(10) $$T^*f_2(x) \leqslant \alpha + C\gamma\alpha \qquad \text{for } x \in Q_i.$$

We fix a cube Q_x centered at x, and let Q_{x_i} be the same size cube centered at x_i. Then

$$\Big|\int_{R^n-Q_x} K(x-y)f_2(y)\,dy\Big|$$

$$\leqslant \Big|\int_{R^n-Q_{x_i}} K(x-y)f_2(y)\,dy\Big| + \int_{Q_{x_i}\Delta Q_x} |K(x-y)|\,|f_2(y)|\,dy$$

(where Δ denotes the symmetric difference)

$$\leqslant \Big|\int_{R^n-Q_{x_i}} K(x_i-y)f_2(y)\,dy\Big| + \int_{R^n-Q_{x_i}} |K(x_i-y)-K(x-y)|\,|f_2(y)|\,dy +$$

$$+\int_{Q_{x_i}\Delta Q_x} |K(x-y)|\,|f_2(y)|\,dy$$

$$= A_1+A_2+A_3.$$

Now $A_1 = |\int_{R^n-\tilde{Q}} K(x_i-y)f(y)\,dy|$ (where $\tilde{Q} = \tilde{Q}_i \cup Q_{x_i}$ is a cube centered at x_i) $\leqslant T^*f(x_i) \leqslant a$, since $x_i \notin U_a$. Standard arguments using inequalities (c) and (b) show that $A_2, A_3 \leqslant Cf^*(\xi)$ for any point $\xi \in Q_i$. In particular, since $f^*(\xi_i) \leqslant \gamma a$, we known that $A_2+A_3 \leqslant C\gamma a$, so that altogether, $|\int_{R^n-Q_x} K(x-y)f_2(y)\,dy| \leqslant a + C\gamma a$. Since Q_x was an arbitrary cube centered at x, we have proved estimate (10).

From (9) and (10) we have $|\{x \in Q_i |\, T^*f(x) > a/2 + a + C\gamma a\}| \leqslant C\gamma|Q_i|$, which proves (8) for all $\gamma \leqslant \frac{1}{2}C$. Thus, (8) holds.

Now estimate (7) is trivial. From (8) and (A_∞) we see that $\omega(\{x \in Q_i |\, T^*f(x) > 2a$ and $f^*(x) \leqslant \gamma a\}) \leqslant C\gamma^\delta \omega(Q_i)$. Adding in i yields $\omega(\{x \in U_a |\, T^*f(x) > 2a$ and $f^*(x) \leqslant \gamma a\}) \leqslant C\gamma^\delta \omega(U_a)$. Since $U_a = \{T^*f > a\}$, estimate (7) is proved, and with it, Theorem III. ∎

§ 3. To complete the proofs of Theorems I–III, it remains only to show that Lemmas 2 and 3 are valid. Both lemmas are in fact simple corollaries of the following result.

THEOREM IV. *Let ω satisfy (A_p), where $1 < p < \infty$. Then the "reverse Hölder inequality"*

$$\left(\frac{1}{|Q|}\int_Q (\omega(x))^{1+\delta}\,dx\right)^{\frac{1}{1+\delta}} \leqslant C\left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right) \tag{11}$$

holds for all cubes Q, with constants C, $\delta > 0$ independent of Q.

Proof of Lemma 2. Observe that $v(x) = (\omega(x))^{-\frac{1}{p-1}}$ satisfies (A_q), where $1/p + 1/q = 1$. Applying Theorem IV to v, we see that ω satisfies $(A_{-\varepsilon})$ with $\varepsilon = (p-1)\dfrac{\delta}{1+\delta}$.

Proof of Lemma 3. Just estimate $\int_{R^n} \chi_E(x)\omega(x)\,dx$ using Hölder's inequality and (11).

Proof of Theorem IV. We first claim that the condition

$$(A'_\infty) \qquad |\{x \in Q|\ \omega(x) > \beta m_Q(\omega)\}| > a|Q|, \qquad \text{where } m_Q(\omega) = \frac{1}{|Q|}\int_Q \omega\,dx,$$

holds for some positive constants a, β. To see this, set $E' = \{x \in Q|\ \omega(x) \leqslant \beta m_Q(\omega)\}$ and observe that

$$\frac{1}{\beta}\left(\frac{|E'|}{|Q|}\right)^{p-1} = m_Q(\omega)\left(\frac{1}{|Q|}\int_{E'} (\beta m_Q(\omega))^{-\frac{1}{p-1}}\,dx\right)^{p-1}$$

$$\leqslant m_Q(\omega)\left(\frac{1}{|Q|}\int_{E'} (\omega(x))^{-\frac{1}{p-1}}\,dx\right)^{p-1}$$

$$\leqslant m_Q(\omega)\left(\frac{1}{|Q|}\int_Q \omega^{-\frac{1}{p-1}}\,dx\right)^{p-1} \leqslant C$$

(the last inequality is the (A_p) condition). Taking β small enough, we obtain (A'_∞).

Next, we shall prove that for any cube Q and any number $\lambda > m_Q(\omega)$, we have

$$\int_{\{x \in Q|\,\omega(x) > \lambda\}} \omega(x)\,dx \leqslant C\lambda|\{\omega(x) > \beta\lambda\}|. \tag{12}$$

This is the main point in our proof of Theorem IV. To prove it, we use the Calderón–Zygmund lemma (see Stein [14], p. 17) to produce a family $\{Q_i\}$ of pairwise disjoint subcubes of Q, with the properties

$$\omega(x) \leqslant \lambda \quad \text{for almost every } x \in Q - \bigcup_i Q_i. \tag{13}$$

$$\lambda < \frac{1}{|Q_i|}\int_{Q_i} \omega(x)\,dx \leqslant 2^n\lambda. \tag{14}$$

From (13), (14), and (A'_∞) we obtain

$$\int_{\{x \in Q|\,\omega(x) > \lambda\}} \omega(x)\,dx \leqslant \sum_i \int_{Q_i} \omega(x)\,dx \leqslant 2^n\lambda \sum_i |Q_i|$$

$$\leqslant \frac{2^n\lambda}{a}\sum_i |\{x \in Q_i|\ \omega(x) > \beta m_{Q_i}(\omega)\}|$$

$$\leqslant \frac{2^n}{a}\lambda\sum_i |\{x \in Q_i|\ \omega(x) > \beta\lambda\}| \leqslant C\lambda|\{x \in Q|\ \omega(x) > \beta\lambda\}|,$$

which proves (12).

Now the proof of Theorem IV is easy. Multiplying both sides of (12) by $\lambda^{\delta-1}$ and integrating, we find that

$$\int_{m_Q(\omega)}^{\infty} \lambda^{\delta-1}\Big(\int_{\{x\in Q|\omega(x)>\lambda\}} \omega(x)\,dx\Big)d\lambda \leqslant C\int_0^{\infty} \lambda^{\delta}|\{x\in Q|\ \omega(x)>\beta\lambda\}|\,d\lambda$$

$$= \frac{C'}{1+\delta}\int_Q \omega^{1+\delta}\,dx.$$

By Fubini's theorem, the left hand side equals

$$\int_{\{x\in Q|\omega(x)>m_Q(\omega)\}} \omega(x)\Big(\int_{m_Q(\omega)} \omega(x)\lambda^{\delta-1}d\lambda\Big)dx$$

$$= \int_{\{x\in Q|\omega(x)>m_Q(\omega)\}} \omega(x)\Big[\frac{\omega^{\delta}(x)}{\delta} - \frac{m_Q^{\delta}(\omega)}{\delta}\Big]dx$$

$$\geqslant \frac{1}{\delta}\int_Q \omega^{1+\delta}\,dx - \frac{(m_Q(\omega))^{1+\delta}}{\delta}|Q|.$$

Therefore, $\Big(\frac{1}{\delta} - \frac{C'}{1+\delta}\Big)\frac{1}{|Q|}\int_Q \omega^{1+\delta}\,dx \leqslant \frac{(m_Q(\omega))^{1+\delta}}{\delta}$, and (11) follows if we take δ small enough. ■

We conclude this section with a few remarks concerning (A_∞). Let μ_1, μ_2 be positive measures on $\boldsymbol{R}^n$, satisfying $\mu_j(Q^*) \leqslant C\mu_j(Q)$ for every cube Q. We say that μ_1 is comparable to μ_2 if there exist constants $\alpha, \beta \in (0,1)$ such that whenever E is a measurable subset of a cube Q, $\frac{\mu_2(E)}{\mu_2(Q)} < \alpha$ implies $\frac{\mu_1(E)}{\mu_1(Q)} < \beta$.

LEMMA 5. *The following are equivalent:*

(15) $\frac{\mu_2(E)}{\mu_2(Q)} \leqslant C\Big(\frac{\mu_1(E)}{\mu_1(Q)}\Big)^{\delta}$ *for all* $E \subseteq Q \subseteq \boldsymbol{R}^n$, *with* $C, \delta > 0$ *independent of* E *and* Q.

(16) μ_2 *is comparable to* μ_1.

(17) μ_1 *is comparable to* μ_2.

(18) $d\mu_2(x) = \omega_1(x)\,d\mu_1(x)$, *where*

$$\Big(\frac{1}{\mu_1(Q)}\int_Q \omega_1^{1+\delta}\,d\mu_1\Big)^{\frac{1}{1+\delta}} \leqslant C\frac{1}{\mu_1(Q)}\int_Q \omega_1\,d\mu_1 \text{ for every cube } Q.$$

Moreover comparability is an equivalence relation.

To prove Lemma 5, one shows that (15) ⇒ (16) ⇒ (17) ⇒ (18) ⇒ (15).

The proof of Theorem IV shows that (17) ⇒ (18), and the other implications are easy.

Setting μ_1 = Lebesgue measure, we see from Lemma 5 that (A_∞), (A'_∞), and (16) are equivalent. Moreover, we can now deduce the following result of Muckenhoupt [9].

THEOREM V. *Any weight function ω satisfying (A_∞) already satisfies (A_p) for some $p < \infty$.*

Proof. Set $d\mu_1(x) = \omega(x)\,dx$ and μ_2 = Lebesgue measure. Condition (A_∞) implies (17) at once, so we know from Lemma 5 that (18) holds also. However, since in this case $\omega_1 = \frac{1}{\omega}$, (18) simply asserts that (A_p) holds for $p = \frac{1}{\delta}+1$. ■

To conclude we would like to point out that recently other weighted norm inequalities have been proved; for the Lusin area function [5], for fractional integral operators [10] and for the commutator integral of Calderón [1]. R. Hunt and Wo-Sang Young have also shown that the arguments described here yield the weighted norm inequalities for the maximal partial sum operator for Fourier series.

References

[1] R. R. Coifman, *Distribution function inequalities for singular integrals*, Proc. Nat. Acad. Sci. USA 69 (1972), pp. 2838–2839.

[2] — and G. Weiss, *Analyse Harmonique non commutative sur certains espaces Homogènes*, Berlin 1971.

[3] D. E. Edmunds and L. A. Peletier, *A Harnack inequality for weak solutions of degenerate quasilinear elliptic equations*, J. London Math. Soc. (2) 5, 1972.

[4] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. 129 (1972), pp. 137–193.

[5] R. F. Gundy and R. L. Wheeden, *Weighted integral inequalities for the nontangential maximal function, Lusin area integral, and Walsh–Paley series*, Studia Math. 49 (1974), pp. 107–124.

[6] H. Helson and G. Szegö, *A Problem in prediction theory*, Ann. Math. Pura Appl. 51 (1960).

[7] R. A. Hunt, B. Muckenhoupt, and R. L. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, to appear in Trans. Amer. Math. Soc.

[8] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), pp. 207–226.

[9] — *The equivalence of two conditions for weight functions*, Studia Math. 49 (1974), pp. 101–106.

[10] — and R. L. Wheeden, *Weighted norm inequalities for fractional integrals*, to appear in Trans. Amer. Math. Soc.

[11] M. R. V. Murthy and G. Stampacchia, *Boundary value problems for some degenerate elliptic operators*, Ann. Math. Pura Appl. 80 (1968), pp. 1–122.
[12] M. Rosenblum, *Summability of Fourier series in $L^p(d\mu)$*, Trans. Amer. Math. Soc. 105 (1962), pp. 32–42.
[13] J. Serrin, *Local behaviour of quasi linear equations*, Acta Math. III (1964), pp. 247–302.
[14] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton 1970.

Received August 1, 1973 (713)

STUDIA MATHEMATICA, T. LI. (1974)

Centered operators

by

BERNARD B. MORREL (Athens, Ga.)
and
PAUL S. MUHLY* (Iowa City, Iowa)

Abstract. An operator T on a Hilbert space is called a centered operator in case the sequence $\ldots T^2(T^*)^2, TT^*, T^*T, (T^*)^2T^2, \ldots$ consists of mutually commuting operators. In this paper, all centered operators are completely described up to unitary equivalence and criteria are given for deciding when one is irreducible. Roughly speaking, it is shown that the most general centered operator is a direct sum of unilateral weighted shifts (backward, forward, or truncated) with commuting operator weights and a weighted translation operator acting on a space of vector-valued functions.

§ 1. Introduction. A computation reveals that if T is a weighted shift (unilateral or bilateral, forward or backward), then the operators in the sequence $\ldots, T^2(T^*)^2, TT^*, T^*T, (T^*)^2T^2, \ldots$ are mutually commuting operators. Following [10], we shall take this property as the defining property of a class of operators called *centered operators* and, answering the question raised in [10], we shall establish the extent to which this property determines the class of weighted shifts.

In the next section we show that the partial isometry in the polar decomposition of a centered operator is a power partial isometry (i.e., all of its positive powers are partial isometries). This fact coupled with the work of Halmos and Wallen [5] enables us to show that a centered operator can be written as a direct sum whose summands are either weighted shifts (with operator weights) or quasi-invertible centered operators. (Recall that a quasi-invertible operator is one with zero kernel and dense range.) We then show, in Section 3, that every quasi-invertible centered operator may be written as the direct sum of operators which are essentially weighted translation operators on spaces of vector-valued functions. In Section 4, we exhibit a complete set of unitary invariants for centered operators, while in Section 5, we derive conditions for a centered operator to be irreducible. Our concluding Section 6 is devoted to questions for future investigation.

* Supported in part by a grant from the National Science Foundation.