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WEIGHTED NORM INEQUALITY FOR OPERATOR ON MARTINGALES

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1. Introduction. Let $\mathscr{M}$ be a family of martingales on a probability space (Ω, F, P). The norm inequalities for operators of matrix type on $\mathscr{M}$ were obtained by Burkholder, Davis and Gundy [2] [3]. Our purpose in this paper is to prove a weighted norm inequality similar to that of Burkholder-Davis-Gundy. Throughout the paper, we fix a BMO-martingale $M_n = \sum_{k=1}^{n} m_k$ such that $1 + m_k \geqq \varepsilon$ $(k \geqq 1)$ for some constant ε with $0 < \varepsilon \leqq 1$. Then the process Z given by the formula $Z_n = \prod_{k=1}^{n}(1 + m_k)$ is a positive uniformly integrable martingale and the weighted probability measure $d\hat{P} = (Z_\infty/Z_1)dP$ is equivalent to dP (see [6]).

THEOREM. *Let Φ be a non-decreasing continuous convex function on $[0, \infty[$ satisfying $\Phi(0) = 0$ and the growth condition $\Phi(2t) \leqq C\Phi(t)$ for all $t \geqq 0$. If U and V are two operators of matrix type on $\mathscr{M}$, then there exists a positive constant $C = C(U, V, \varepsilon, \Phi, M)$ such that the inequality*

$$\hat{E}[\Phi(U^{**}(X))] \leqq C\hat{E}[\Phi(V^*(X))] \tag{1}$$

holds for all $X \in \mathscr{M}$, where $\hat{E}[\ \ ]$ denotes the expectation over Ω with respect to $d\hat{P}$.

The result for the case $Z \equiv 1$ was established by Burkholder, Davis and Gundy [2, Theorem 2.3].

The following inequality was obtained in the continuous parameter case by Bonami and Lepingle [1] and Sekiguchi [9] independently.

COROLLARY. *Let us denote the square function operator by $S(X)$ and the maximal operator by X^*. Then the inequality*

$$c\hat{E}[\Phi(X^*))] \leqq \hat{E}[\Phi(S(X))] \leqq C\hat{E}[\Phi(X^*)] \tag{2}$$

is valid for all $X \in \mathscr{M}$.

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2. Preliminaries. The reader is assumed to be familiar with the martingale theory as is given in Meyer [7] and Neveu [8]. Throughout the paper, let us denote by c or C a positive constant and by $C(p)$ a positive constant depending only on the parameter p. Both letters are not necessarily the same at each occurrence.

1) Notations. Let (Ω, F, P) be a probability space with a non-decreasing sequence $(F_n)_{n\geqq 1}$ of sub σ-fields of F such that $\bigvee_{n=1}^{\infty} F_n = F$. Let $X = (X_n;\, n \geqq 1)$ be an (F_n)-adapted process and $(x_1, x_2, \cdots)$ be the difference sequence of X so that $X_n = \sum_{k=1}^{n} x_k$.

A matrix $(u_{jk};\, j \geqq 1, k \geqq 1)$ is said to be of type B-G (B-G stands for Burkholder and Gundy) if it has the following properties:

(a) Each entry u_{jk} is an F_{k-1}-measurable random variable.

(b) There is a constant $\alpha > 1$ such that for all $k \geqq 1$,

$$1/\alpha \leqq \sum_{j=1}^{\infty} u_{jk}^2 \leqq \alpha\ . \tag{3}$$

We define $U(X)$, $U_n(X)$, $U_n^*(X)$ and $U_n^{**}(X)$ for a matrix (u_{jk}) of type B-G as follows:

$$U(X) = \left(\sum_{j=1}^{\infty} \limsup_{n\to\infty} \left|\sum_{k=1}^{n} u_{jk}x_k\right|^2\right)^{1/2},$$
$$U_n(X) = \left(\sum_{j=1}^{\infty} \left|\sum_{k=1}^{n} u_{jk}x_k\right|^2\right)^{1/2},$$
$$U_n^*(X) = \sup_{i\leqq n} U_i(X)$$

and

$$U_n^{**}(X) = \left(\sum_{j=1}^{\infty} \sup_{i\leqq n} \left|\sum_{k=1}^{i} u_{jk}x_k\right|^2\right)^{1/2}.$$

We write simply $U^*(X)$ and $U^{**}(X)$ instead of $U_\infty^*(X)$ and $U_\infty^{**}(X)$. $U(X)$ is called an operator of matrix type which was introduced by Burkholder and Gundy [3]. In the same way, for another matrix (v_{jk}) of type B-G, we can define $V(X)$, $V^*(X)$ and $V^{**}(X)$ by using v_{jk} instead of u_{jk}. Typical examples, corresponding to the identity matrix or a single-row matrix, are $S_n(X) = (\sum_{k=1}^{n} x_k^2)^{1/2}$, $S(X) = (\sum_{k=1}^{\infty} x_k^2)^{1/2}$, $X_n^* = \sup_{k\leqq n} |X_k|$ and $X^* = \sup_k |X_k|$. Let us set $X_0 = U_0(X) = U_0^*(X) = 0$, $Z_0 = 1$ and $F_0 = F_1$ for convenience. Now we define $\hat{X}_n$ and $\hat{X}$ as follows:

$$\hat{x}_n = -x_n(Z_{n-1}/Z_n) = -x_n/(1 + m_n)\ ,$$
$$\hat{X}_n = \sum_{k=1}^{n} \hat{x}_k\ , \quad \hat{X} = (\hat{X}_n)_{n\geqq 1}\ .$$

In particular $\hat{m}_n = -m_n/(1 + m_n)$, $m_n = -\hat{m}_n/(1 + \hat{m}_n)$ and $(1 + m_n)(1 + \hat{m}_n) = 1$. So we obtain

$$(4)\qquad \hat{x}_n = -x_n(1+\hat{m}_n) = -x_n - x_n\hat{m}_n\ .$$

Let us denote by $\|X\|_{\mathrm{BMO}}$ the smallest positive constant c such that c^2 dominates $E[S(X)^2 - S_{n-1}(X)^2|F_n]$ P-a.s. for all $n \geqq 1$. BMO is the class of those martingales X which satisfy $\|X\|_{\mathrm{BMO}} < \infty$. We choose and fix a constant ε with $1/\varepsilon \geqq 1 + \|M\|_{\mathrm{BMO}} \geqq 1 + m_n \geqq \varepsilon > 0$. Then we get $\hat{M} \in \mathrm{BMO}(\hat{P})$ with $1/\varepsilon \geqq 1 + \hat{m}_n \geqq \varepsilon$, where $\mathrm{BMO}(\hat{P})$ is the BMO-class with respect to $\hat{P}$ (see [4]). Since the equality

$$(5)\qquad \hat{E}[Y|F_n] = E[Y(Z_\infty/Z_n)|F_n] \quad \text{a.s. under } P \text{ and } \hat{P}$$

holds for all $\hat{P}$-integrable random variable Y, it is easy to see that $\hat{X}$ is a $\hat{P}$-martingale for each martingale X. By (4) we have

$$(6)\qquad \varepsilon S(X) \leqq S(\hat{X}) \leqq (1/\varepsilon)S(X) \quad \text{a.s.}\ .$$

In this paper, unless otherwise stated, "a martingale" means "a martingale with respect to P".

2) Preliminary lemmas. To show our theorem, we need several lemmas.

LEMMA 1. *Let (a_{jk}) be a matrix of type* B-G. *Then there is a positive constant $C(\alpha)$ such that the inequality*

$$(7)\qquad \hat{E}\left[\left(\sum_{j=1}^{\infty} \sup_n \left|\sum_{k=1}^{n} a_{jk}x_k\hat{y}_k\right|^2\right)^{1/2}\right] \leqq C(\alpha)\|\hat{Y}\|_{\mathrm{BMO}(\hat{P})}\hat{E}[S(X)]$$

is valid for all $\hat{Y} \in \mathrm{BMO}(\hat{P})$, where $(\hat{y}_k)$ is the difference sequence of $\hat{Y}$.

PROOF. Let us fix a positive integer N. For any $\delta > 0$,

$$\begin{aligned}
&\hat{E}\left[\left(\sum_{j=1}^{\infty} \sup_{i\leqq N} \left|\sum_{k=1}^{i} a_{jk}x_k\hat{y}_k\right|^2\right)^{1/2}\right] \\
&\quad = \hat{E}\left[\left\{\sum_{j=1}^{\infty} \sup_{i\leqq N} \left|\sum_{k=1}^{i} x_k(S_k(X)+\delta)^{-1/2}(S_k(X)+\delta)^{1/2}a_{jk}\hat{y}_k\right|^2\right\}^{1/2}\right] \\
&\quad \leqq \hat{E}\left[\left\{\sum_{j=1}^{\infty} \sup_{i\leqq N} \left(\sum_{k=1}^{i} x_k^2/(S_k(X)+\delta)\right)\left(\sum_{k=1}^{i} a_{jk}^2(S_k(X)+\delta)\hat{y}_k^2\right)\right\}^{1/2}\right]
\end{aligned}$$

by Cauchy-Schwarz's inequality. Moreover,

$$x_k^2/(S_k(X)+\delta) = (S_k(X)^2 - S_{k-1}(X)^2)/(S_k(X)+\delta) \leqq 2(S_k(X) - S_{k-1}(X))\ .$$

Therefore the left hand side of (7) is dominated by

$$\begin{aligned}
&\sqrt{2}\,\hat{E}\left[\left\{\sum_{j=1}^{\infty} S_N(X)\left(\sum_{k=1}^{N} a_{jk}^2(S_k(X)+\delta)\hat{y}_k^2\right)\right\}^{1/2}\right] \\
&\quad = \sqrt{2}\,\hat{E}\left[S_N(X)^{1/2}\left\{\sum_{k=1}^{N}(S_k(X)+\delta)\hat{y}_k^2\left(\sum_{j=1}^{\infty} a_{jk}^2\right)\right\}^{1/2}\right]
\end{aligned}$$

$$\leqq C(\alpha)\hat{E}\left[S_N(X)^{1/2}\left\{\sum_{k=1}^{N}(S_k(X)+\delta)\hat{y}_k^2\right\}^{1/2}\right] \quad \text{(by (3))}$$

$$\leqq C(\alpha)\hat{E}[S_N(X)]^{1/2}\hat{E}\left[\sum_{k=1}^{N}(S_k(X)+\delta)\hat{y}_k^2\right]^{1/2}$$

by Schwarz's inequality. The last factor of the above expression is equal to

$$\hat{E}\left[\sum_{k=1}^{N}(S_k(X)+\delta)(S_k(\hat{Y})^2-S_{k-1}(\hat{Y})^2)\right]^{1/2}$$

$$=\hat{E}\left[\sum_{k=1}^{N}(S_k(X)-S_{k-1}(X))(S_N(\hat{Y})^2-S_{k-1}(\hat{Y})^2)+\delta S_N(\hat{Y})\right]^{1/2}$$

$$=\hat{E}\left[\sum_{k=1}^{N}(S_k(X)-S_{k-1}(X))\hat{E}[S_N(\hat{Y})^2-S_{k-1}(\hat{Y})^2\,|\,F_k]\right.$$

$$\left.+\,\delta\hat{E}[S_N(\hat{Y})\,|\,F_0]\right]^{1/2}\leqq \|\hat{Y}\|_{\mathrm{BMO}(\hat{P})}\hat{E}[S_N(X)+\delta]^{1/2}\,.$$

Letting $\delta\to 0$ and then $N\to\infty$, we obtain (7).

LEMMA 2. *The inequality*

$$\hat{E}\left[\sup_n\left|\sum_{k=1}^{n}x_k\hat{y}_k\right|\right]\leqq\sqrt{2}\,\|\hat{Y}\|_{\mathrm{BMO}(\hat{P})}\hat{E}[S(X)] \tag{8}$$

holds for all $\hat{Y}\in\mathrm{BMO}(\hat{P})$.

The inequality (8) is of Fefferman's type. The proof of Meyer [7, V-9, p. 337] is still valid in our case, where X is a semimartingale with respect to $\hat{P}$.

We are going to verify the inequality (1) in the case $\Phi(\lambda)=\lambda$.

LEMMA 3. *Let X be a martingale. Then*

$$\hat{E}[U^{**}(X)]\leqq C(\alpha,\varepsilon,M)\hat{E}[S(X)]\,. \tag{9}$$

PROOF. By (4) and Lemma 1, we have

$$\hat{E}[U^{**}(X)]\leqq\hat{E}[U^{**}(\hat{X})]+\hat{E}\left[\left(\sum_{j=1}^{\infty}\sup_n\left|\sum_{k=1}^{n}u_{jk}x_k\hat{m}_k\right|^2\right)^{1/2}\right]$$

$$\leqq\hat{E}[U^{**}(\hat{X})]+C(\alpha)\|\hat{M}\|_{\mathrm{BMO}(\hat{P})}\hat{E}[S(X)]\,.$$

Applying Burkholder-Davis-Gundy's theorem and (6), we have

$$\hat{E}[U^{**}(\hat{X})]\leqq C(\alpha)\hat{E}[S(\hat{X})]\leqq C(\alpha,\varepsilon)\hat{E}[S(X)]\,.$$

Thus (9) holds.

Here we define A_k, d_k and $D(X)=(D_n)_{n\geqq 1}$ as follows:

$$A_1=0,\quad A_k=\sum_{j=1}^{\infty}\left(\sum_{i=1}^{k-1}v_{ji}x_i\right)v_{jk},\quad d_k=2A_kx_k,\quad D_n=\sum_{k=1}^{n}d_k\,.$$

LEMMA 4. *There exists a positive constant* $C(\alpha, \varepsilon, M)$ *such that*

(10) $$\hat{E}[S(X)^2/(V^*(X)+\delta)] \leqq C(\alpha, \varepsilon, M)\{\hat{E}[V^*(X)+\delta] + \hat{E}[S(X)]\}$$

for every $\delta > 0$ *and for every martingale* X *with* $\hat{E}[D(X)^*] < \infty$.

PROOF. We define $\hat{H}$ and $\hat{G}$ as follows:

$$\hat{H}_n = \hat{E}[1/(V^*(X)+\delta)\,|\,F_n], \quad \hat{h}_n = \hat{H}_n - \hat{H}_{n-1}, \quad \hat{H} = (\hat{H}_n)_{n\geqq 1},$$

$$\hat{G}_n = \sum_{k=1}^{n} A_k \hat{h}_k \quad \text{and} \quad \hat{G} = (\hat{G}_n)_{n\geqq 1}.$$

Note that $\hat{H}$ is a $\hat{P}$-martingale dominated by $1/\delta$. First we show $\hat{G} \in \mathrm{BMO}(\hat{P})$. By Cauchy-Schwarz's inequality and (3),

$$|A_n| \leqq \left(\sum_{j=1}^{\infty}\left|\sum_{k=1}^{n-1} v_{jk}x_k\right|^2\right)^{1/2}\left(\sum_{j=1}^{\infty} v_{jn}^2\right)^{1/2} \leqq C(\alpha)\,V_{n-1}^*(X).$$

So we have

(11) $$|A_n\hat{H}_{n-1}| \leqq C(\alpha)\hat{E}[V_{n-1}^*(X)/(V^*(X)+\delta)\,|\,F_{n-1}] \leqq C(\alpha)$$

and $|A_n\hat{H}_n| \leqq C(\alpha)$. By using the above inequalities, we obtain

$$\begin{aligned}
\hat{E}[S(\hat{G})^2 - S_{n-1}(\hat{G})^2\,|\,F_n] &= \hat{E}\left[\sum_{k=n+1}^{\infty} A_k^2\hat{h}_k^2 \,\middle|\, F_n\right] + A_n^2(\hat{H}_n - \hat{H}_{n-1})^2 \\
&\leqq C(\alpha)\hat{E}\left[\sum_{k=n+1}^{\infty} V_{k-1}^*(X)^2\hat{h}_k^2 \,\middle|\, F_n\right] + 2(|A_n\hat{H}_n|^2 + |A_n\hat{H}_{n-1}|^2) \\
&\leqq C(\alpha)\hat{E}\left[\sum_{k=n+1}^{\infty} V_{k-1}^*(X)^2(\hat{H}_k^2 - \hat{H}_{k-1}^2) \,\middle|\, F_n\right] + C(\alpha) \\
&\leqq C(\alpha)\hat{E}[V^*(X)^2\hat{H}_\infty^2\,|\,F_n] + C(\alpha) \leqq C(\alpha)
\end{aligned}$$

from which

(12) $$\|\hat{G}\|_{\mathrm{BMO}(\hat{P})} \leqq C(\alpha)$$

follows. Secondly we modify $D_n\hat{H}_n$ as follows:

$$\begin{aligned}
D_n\hat{H}_n &= \left(\sum_{k=1}^{n} d_k\right)\left(\sum_{k=1}^{n}\hat{h}_k\right) = \sum_{k=1}^{n} D_{k-1}\hat{h}_k + \sum_{k=1}^{n}\hat{H}_{k-1}d_k + \sum_{k=1}^{n}\hat{h}_k d_k \\
&= \sum_{k=1}^{n} D_{k-1}\hat{h}_k + 2\sum_{k=1}^{n}\hat{H}_{k-1}A_k x_k + 2\sum_{k=1}^{n}\hat{h}_k A_k x_k \\
&= \sum_{k=1}^{n} D_{k-1}\hat{h}_k - 2\sum_{k=1}^{n}\hat{H}_{k-1}A_k\hat{x}_k - 2\sum_{k=1}^{n}\hat{H}_{k-1}A_k x_k\hat{m}_k + 2\sum_{k=1}^{n} x_k A_k\hat{h}_k.
\end{aligned}$$

Here $\sum D_{k-1}\hat{h}_k$ and $\sum \hat{H}_{k-1}A_k\hat{x}_k$ are $\hat{P}$-local martingales with value 0 at time 1, so there is a non-decreasing sequence $\{R_n\}$ of stopping times such that $\lim_{n\to\infty} R_n = \infty$ a.s. and $\hat{E}[\sum_{k=1}^{R_n} D_{k-1}\hat{h}_k] = \hat{E}[\sum_{k=1}^{R_n}\hat{H}_{k-1}A_k\hat{x}_k] = 0$. Therefore we obtain

$$
\begin{aligned}
|\hat{E}[D_{R_n}\hat{H}_{R_n}]| &\leqq 2\hat{E}\Big[\sum_{k=1}^{R_n}|\hat{H}_{k-1}A_k|\,|x_k\hat{m}_k|\Big] + 2\hat{E}\Big[\sum_{k=1}^{R_n}|x_kA_k\hat{h}_k|\Big] \\
&\leqq C(\alpha)\hat{E}\Big[\sum_{k=1}^{\infty}|x_k\hat{m}_k|\Big] + 2\hat{E}\Big[\sum_{k=1}^{\infty}|x_kA_k\hat{h}_k|\Big] \quad \text{(by (11))} \\
&\leqq C(\alpha)\|\hat{M}\|_{\mathrm{BMO}(\hat{P})}\hat{E}[S(X)] + 2\|\hat{G}\|_{\mathrm{BMO}(\hat{P})}\hat{E}[S(X)] \quad \text{(by Lemma 2)} \\
&\leqq C(\alpha, M)\hat{E}[S(X)] \quad \text{(by (12))} .
\end{aligned}
$$

Since $|D_n\hat{H}_n|$ is dominated by $(1/\delta)D(X)^*$, we get $|\hat{E}[D_\infty H_\infty]| \leqq C\hat{E}[S(X)]$ by the dominated convergence theorem. On the other hand,

$$
V_n(X)^2 - V_{n-1}(X)^2 - \sum_{j=1}^{\infty} v_{jn}^2 x_n^2 = 2\sum_{j=1}^{\infty}\Big(\sum_{k=1}^{n-1} v_{jk}x_k\Big)v_{jn}x_n = d_n ,
$$

so we find $\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} v_{jk}^2 x_k^2 = V_\infty(X)^2 - D_\infty$. Then

$$
\begin{aligned}
\hat{E}[S(X)^2/(V^*(X)+\delta)] &= \hat{E}\Big[\Big(\sum_{k=1}^{\infty}x_k^2\Big)\Big/(V^*(X)+\delta)\Big] \\
&\leqq C(\alpha)\hat{E}\Big[\Big(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}v_{jk}^2x_k^2\Big)\Big/(V^*(X)+\delta)\Big] \quad \text{(by (3))} \\
&\leqq C(\alpha)\hat{E}[(V_\infty(X)^2 - D_\infty)/(V^*(X)+\delta)] \\
&\leqq C(\alpha)\{\hat{E}[V_\infty(X)^2/(V^*(X)+\delta)] + |\hat{E}[D_\infty\hat{H}_\infty]|\} \\
&\leqq C(\alpha, \varepsilon, M)\{\hat{E}[V^*(X)+\delta] + \hat{E}[S(X)]\} .
\end{aligned}
$$

Therefore we obtain (10).

LEMMA 5. *The inequality*

$$
\hat{E}[U^{**}(X)] \leqq C(\alpha, \varepsilon, M)\hat{E}[V^*(X)] \tag{13}
$$

holds for all martingale X.

PROOF. Since $S(\hat{X})$ is locally $\hat{P}$-integrable for a $\hat{P}$-local martingale $\hat{X}$, $S(X)$ is also locally $\hat{P}$-integrable by (6). By using the stopping argument, we may assume $\hat{E}[S(X)] < \infty$ and $\hat{E}[S(D(X))] < \infty$. Applying Lemma 3 to the case where (u_{jk}) is a single-row matrix, we get $\hat{E}[D(X)^*] \leqq C\hat{E}[S(D(X))] < \infty$. By Schwarz's inequality and by Lemma 4, we have

$$
\begin{aligned}
\hat{E}[S(X)] &= \hat{E}[S(X)(V^*(X)+\delta)^{-1/2}(V^*(X)+\delta)^{1/2}] \\
&\leqq \hat{E}[S(X)^2/(V^*(X)+\delta)]^{1/2}\hat{E}[V^*(X)+\delta]^{1/2} \\
&\leqq c(\alpha, \varepsilon, M)\{\hat{E}[V^*(X)+\delta] + \hat{E}[S(X)]\}^{1/2}\hat{E}[V^*(X)+\delta]^{1/2} .
\end{aligned}
$$

Put $A = \hat{E}[S(X)]$ and $B = \hat{E}[V^*(X)+\delta]$. Then the above inequality is equal to $A \leqq c\{(B+A)B\}^{1/2}$, where $A < \infty$. Therefore there exists some constant c', depending only on c, such that $A \leqq c'B$, that is, $\hat{E}[S(X)] \leqq c'\hat{E}[V^*(X)+\delta]$. Letting $\delta \to 0$ and combining this inequality with (9), we obtain (13).

3. Proof of Theorem. By virtue of Neveu-Garsia's lemma (see [8, IX-3-5]), it is sufficient to prove that there is a positive constant c such that

$$\hat{E}[U^{**}(X) - U^{**}_{n-1}(X) \mid F_n] \leqq c\hat{E}[V^*(X) \mid F_n] \tag{14}$$

for every martingale X and $n \geqq 1$. By (5), the inequality (14) coincides with the following inequality:

$$E[(U^{**}(X) - U^{**}_{n-1}(X))(Z_\infty/Z_n) \mid F_n] \leqq cE[V^*(X)(Z_\infty/Z_n) \mid F_n] \tag{15}$$

for every martingale X over (F_n). Let Λ be an element of F_n. Set $d\bar{P} = (Z_\infty/Z_n)dP$ and $X'_k = \{X_{k+n-1} - X_{n-1}\}I_\Lambda$ for each martingale X over (F_n). Then $X' = (X'_k)_{k\geqq 1}$ is a martingale over $(F_{k+n-1})_{k\geqq 1}$ such that

$$U^{**}(X) - U^{**}_{n-1}(X) \leqq U^{**}(X') \quad \text{and} \quad V^*(X') \leqq 2V^*(X) \tag{16}$$

on Λ. Furthermore, it is easy to see that $M' \in \mathrm{BMO}$ with $1/\varepsilon \geqq 1 + m'_k \geqq \varepsilon$ for the BMO-martingale M with $1/\varepsilon \geqq 1 + m_k \geqq \varepsilon$. Therefore the inequality (13) is still valid for a martingale X' over $(F_{k+n-1})_{k\geqq 1}$ and for the weighted probability measure $d\bar{P}$ instead of $d\hat{P}$. Thus we get $\bar{E}[U^{**}(X')] \leqq C\bar{E}[V^*(X')]$, that is, $E[U^{**}(X')(Z_\infty/Z_n); \Lambda] \leqq CE[V^*(X')(Z_\infty/Z_n); \Lambda]$ for every X', where $\bar{E}[\ \ ]$ denotes the expectation over Ω with respect to $d\bar{P}$. By (16), we obtain

$$\begin{aligned} E[(U^{**}(X) - U^{**}_{n-1}(X))(Z_\infty/Z_n); \Lambda] &\leqq E[U^{**}(X')(Z_\infty/Z_n); \Lambda] \\ &\leqq CE[V^*(X')(Z_\infty/Z_n); \Lambda] \leqq CE[V^*(X)(Z_\infty/Z_n); \Lambda] \end{aligned}$$

for every martingale X. This holds for any $\Lambda \in F_n$, so that we get (15). Hence the theorem is established.

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