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WEIGHTED PALEY-WIENER SPACES

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1. Introduction

This paper explores the connections between three objects: Strictly positive weights on the real line, weighted L^2 spaces of entire functions, and entire functions generating such spaces, as depicted by de Branges [4].

Let H be a Hilbert space of entire functions whose norm is given by a positive weight function $w(x) > 0$:

$$\|f\|_w^2 = \int_{-\infty}^{\infty} |f(x)w(x)|^2 dx.$$

Assume that point evaluation is a bounded functional on H for each point $z \in \mathbb{C}$, and also that H has the following symmetry property: H is closed under the operations $f(z) \mapsto f(z)(z-\overline{\zeta})/(z-\zeta)$ (provided $f(\zeta) = 0$) and $f(z) \mapsto \overline{f(\overline{z})}$. These assumptions ensure that H is a Hilbert space of entire functions in the sense of de Branges [4]. According to de Branges' theory, there exists an entire function E belonging to the so-called Hermite-Biehler class (see below for definition) such that $H = H(E)$ isometrically; here $H(E)$ consists of all entire functions f such that both $f(z)/E(z)$ and $\overline{f(\overline{z})}/E(z)$ belong to H^2 of the upper half-plane, and the norm of f is given by

$$\|f\|_E^2 = \int_{-\infty}^{\infty} \frac{|f(x)|^2}{|E(x)|^2} dx.$$

The constructions of this paper rely crucially on the representation $H = H(E)$.

We define the *majorant* of such a space H as

$$M(z) = \sup\{|f(z)| : \ \|f\|_w \le 1, f \in H\}.$$

For $x \in \mathbb{R}$ the function $M(x)$ can be considered as a regularization of $1/w(x)$ by functions from H. We will say that the weight w is a *majorant weight* if $M(x)w(x)$ is uniformly bounded from below and above by positive constants (denoted $M(x)w(x) \simeq 1$) for some Hilbert space H as just described with norm $\|\cdot\|_w$. Such a space H will be called a *weighted Paley Wiener space*.

The model case is $H = PW_\tau$ ($\tau > 0$), where PW_τ is the classical Paley-Wiener space consisting of all entire functions of exponential type $\le \tau$ whose restrictions to the real line are square integrable. These spaces correspond to the trivial weight $w \equiv 1$. In a sense, we will show what is the widest class of weights and associated spaces for which results on sets of uniqueness, sampling, and interpolation for PW_τ

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can be extended in a direct and natural way, keeping the basic features of the theory intact.

Our first main theorem gives an explicit representation of all majorant weights:

Theorem 1. *A function $w(x) > 0$ is a majorant weight if and only if there exist a function $m(x) \simeq 1$ and a real entire function g such that*

$$(1) \qquad \log w(x) + g(x) + \int_{-\infty}^{\infty} [\log|1 - x/t| + (1 - \chi_{[-1,1]}(t))x/t] m(t)dt \in L^\infty,$$

where $\chi_{[-1,1]}$ is the characteristic function of the interval $[-1, 1]$.

Remark. For example, in the case of the classical space PW_τ one has $w(x) \equiv 1$ and thus can take $g = 0$, $m \equiv 1$.

There exists a trivial isometry by which the function g can be eliminated: Multiply all functions in H by a factor e^g, and replace the weight w by we^{-g}. Thus, without loss of generality, we may assume that $g \equiv 0$. However, it is convenient to accept a linear function; in what follows, we will assume that the function g of Theorem 1 is linear.

To a given majorant weight represented as in Theorem 1, with $g(x) = ax$, we associate a lower uniform density defined as

$$D^-(m) = \lim_{r \to +\infty} \frac{1}{r} \inf_x \int_x^{x+r} m(t)dt;$$

here the limit exists because the function $d(r) = \inf_x \int_x^{x+r} m(t)dt$ is superadditive, i.e., $d(r + s) \geq d(r) + d(s)$. (This lower uniform density is essentially the same as the one used by Beurling [2]; see Section 6 below.)

The following theorem gives a description of all weighted Paley-Wiener spaces with norm $\|\cdot\|_w$.

Theorem 2. *The family of all weighted Paley-Wiener spaces whose norms are given by a fixed majorant weight w represented as in Theorem 1 and with $g(x) = ax$ can be parametrized by all real numbers $b > -D^-(m)/\pi$. The space associated with a given b consists of all entire functions f satisfying $\|f\|_w < \infty$ and $\log|f(z)| \leq C_\varepsilon + \omega(z) + \varepsilon|z|$ for all $\varepsilon > 0$, where*

$$\omega(z) = a\Re z + b|\Im z| + \int_{-\infty}^{\infty} [\log|1 - z/t| + (1 - \chi_{[-1,1]}(t))\Re z/t] m(t)dt.$$

In other words, a weighted Paley-Wiener space is completely characterized by the function ω. Note that if $b \geq 0$, then clearly ω is subharmonic. On the other hand, if $b < 0$, then one can prove that ω differs from a subharmonic function by a bounded function. We may therefore assume without loss of generality that ω is subharmonic.

The weighted Paley-Wiener space associated with ω (from now on assumed to be subharmonic) will be denoted by L^2_ω.

It follows from Theorem 2 that weighted Paley-Wiener spaces consist of functions of order at most one (but not necessarily of finite type), or, more precisely, an entire function f from this space satisfies $\log|f(z)| \leq C|z|\log^+|z| + C_f$. Our inclusion of the linear term az in ω is partly motivated by this fact, as we obtain in this way all possible spaces of order one.

We introduce a terminology which reflects that we are dealing with entire functions with properties similar to those functions which are of exponential type and belong to the Cartwright class [10, Lecture 16]. Recall that a function F analytic in the upper half-plane belongs to the Nevanlinna class if $\log|F|$ has a positive harmonic majorant. By Nevanlinna's factorization theorem, every F in the Nevanlinna class can be factorized uniquely as

$$F(z) = B(z)e^{ihz}e^{G(z)},$$

with B a Blaschke product, h a real constant, and G analytic in the upper half-plane such that its real part is the Poisson integral of a measure μ satisfying

$$\int_{-\infty}^{\infty} \frac{d|\mu|(t)}{1+t^2} < \infty.$$

The number h is called the *mean type* of F. (See [4, pp. 22–30].) Now suppose ω is a harmonic function in the upper half-plane and let S be analytic such that $\log|S(z)| = \omega(z)$. If f is an entire function such that both $f(z)/S(z)$ and $\overline{f(\overline{z})/S(z)}$ belong to the Nevanlinna class and are of mean type h_1 and h_2, respectively, we say that f *is of* ω*-type* $\max(h_1, h_2)$. We note that L^2_ω consists of those entire functions f which are of nonpositive ω-type and whose restrictions to the real line satisfy $fe^{-\omega} \in L^2(\mathbb{R})$.

The regularity condition (1) on w leads to a very different characterization of weighted Paley-Wiener spaces. Given a sequence Σ of distinct complex numbers, we denote by $PW_\tau(\Sigma)$ the set of all functions in PW_τ vanishing on Σ; it is clear that $PW_\tau(\Sigma)$ is then a closed subspace of PW_τ. Using a multiplier lemma (which also plays a role in the proof of Theorem 1), we are able to associate weighted Paley-Wiener spaces with such subspaces of classical Paley-Wiener spaces:

Theorem 3. *To every weighted Paley-Wiener space L^2_ω, there exist $\tau > 0$, a sequence $\Sigma = \{\xi_k - i\}_{k\in\mathbb{Z}}$ with $\xi_{k+1} - \xi_k \simeq 1$, and an associated entire function F vanishing on Σ such that $f(z) \mapsto f(z)F(z)$ is a bijective mapping from L^2_ω to $PW_\tau(\Sigma)$, with norm equivalence $\|f\|_{L^2_\omega} \simeq \|fF\|_2$.*

An immediate consequence of this theorem is a Beurling-Malliavin-type density theorem concerning the zeros of functions in L^2_ω. In a similar way, we are able to transfer all known results about sampling and interpolation for PW_τ to the L^2_ω setting. Details about these results are presented in Section 6.

We have mentioned that $L^2_\omega = H(E)$ isometrically for some entire function E belonging to the Hermite-Biehler class. We denote this class by $\overline{HB}$; it consists of all entire functions E with no zeros in the upper half-plane and satisfying $|E(z)| \geq |E(\overline{z})|$ whenever $\Im z > 0$. More generally, to a given space L^2_ω, there exists a collection of entire functions $E \in \overline{HB}$ such that merely $L^2_\omega = H(E)$, where equality is understood in the sense of sets. (Incidentally, this implies equivalence between the two norms $\|\cdot\|_w$ and $\|\cdot\|_E$, by an argument based on the closed graph theorem.) We will now give a description of all E such that $L^2_\omega = H(E)$. (The space $H(E)$ will henceforth be referred to as a de Branges space; see Section 2 for a more detailed account of such spaces.)

Some additional notation, to be used throughout the paper, is required to state our fourth theorem. Write $E(x)$ as

$$E(x) = |E(x)|e^{-i\varphi(x)}, \tag{2}$$

where $\varphi(x)$, called the *phase function* of E, is a continuous branch of $-\operatorname{Im}\log E(x)$; since $E \in \overline{HB}$, $\varphi(x)$ is a nondecreasing function. Let Λ be the zero set of E and

$\lambda(x) = \xi(x) - i\eta(x)$ the point from Λ closest to the point x on the real line; this function is well defined for all real x, except for at most countably many points. We define $\lambda(x)$ for all x by requiring it to be left-continuous at each point. Also, set

$$\sigma(x) = \min(\eta(x), 1) \quad \text{and} \quad B_E(z) = \frac{\overline{E(\bar{z})}}{E(z)},$$

the latter being an inner function in the upper half-plane. Now our fourth main theorem reads as follows.

Theorem 4. *Let L^2_ω be a weighted Paley-Wiener space. Then a de Branges space $H(E)$ satisfies $H(E) = L^2_\omega$ if and only if the following five conditions are met:*

(i) *E is of ω-type* 0.
(ii) $\varphi'(x)|E(x)|^2 \simeq e^{2\omega(x)}$.
(iii) *There exists an $\varepsilon_0 > 0$ such that if $\Im\lambda_k \geq -\varepsilon_0$, then* $\operatorname{dist}(\lambda_k, \Lambda \setminus \{\lambda_k\}) \simeq 1$.
(iv) $\varphi'(x) \simeq \sigma(x)/\min(|x - \lambda(x)|^2, 1)$.
(v) *The two functions $|1 \pm B_E(x+i)|^2/\sigma(x)$ both satisfy the Muckenhoupt* (A_2) *condition.*

We shall see in Section 7 that if the zeros of E all lie in a horizontal strip, then condition (v) of Theorem 4 can be reduced to the statement that σ satisfies the Muckenhoupt (A_2) condition.

We end this introduction by pointing out some interesting similarities and connections with classical results. Let us consider the important case that w satisfies the logarithmic integral condition:

$$\int_{\mathbb{R}} \frac{|\log w(x)|}{1+x^2} dx < \infty.$$

It means in particular that the usual conjugation operator $f \mapsto \tilde{f}$ on the real line can be applied to $\log w$. Theorem 1 can then be rewritten in the following form: *w is a majorant weight if and only if it can be written as $w = e^{u+v}$, with $u, (\tilde{v})' \in L^\infty$.* Thus our condition has a certain similarity with the Helson-Szegö condition, and also with the following version of the Beurling-Malliavin multiplier theorem. Let $w(x) \geq 1$ be a measurable function on the real line, and denote by L^p_w the space of functions f such that $fw \in L^p$ ($p \geq 1$). Beurling and Malliavin determine those weight functions w such that (1) L^p_w contains Fourier transforms of measures supported in $[-a, a]$ for each $a > 0$, and (2) the translation operators $f(x) \mapsto f(x+t)$ are bounded on L^p_w: *w has properties* (1) *and* (2) *if and only if it can be written as $w = e^{u+v}$, with $u, v' \in L^\infty$.* Obviously, there is a large class of weights which are both majorant and of Beurling-Malliavin-type.

Problems concerning the relation between majorants and weights and problems about norm equivalence in spaces of analytic functions have been considered by many authors and for a number of different spaces. We refer the reader to [1], [13], where such problems are studied in connection with weighted approximation on the real line, and to [11], [6] in which weighted Bergman and Fock spaces with radial weights are considered.[1] In Section 7, we shall compare our Theorem 4 with a result of Volberg [17] about norm equivalence in so-called model spaces.

[1]It is an interesting open problem for the latter type of spaces to obtain a description of majorant weights in terms of Laplacians, without any a priori assumption about the weights.

Outline of the paper. We begin our study in Section 2 by exploring the distribution of the zeros of those functions E in the Hermite-Biehler class which generate weighted Paley-Wiener spaces. What we find is that the condition $M(x)w(x) \simeq 1$ imposes strict restrictions on the distribution and the location of the zeros of E. In Section 3, the results of Section 2 are used to prove the necessity part of Theorem 1. The sufficiency part is proved in Section 4 by using a certain multiplier lemma for entire functions. (Incidentally, this lemma has proved to be an efficient tool for other questions related to spaces of entire functions; see, e.g., [14].) Section 5 contains the proof of Theorem 2, and thus gives a description of all weighted Paley-Wiener spaces given by the same majorant weight. The lemma on multipliers is used again in Section 6, this time to prove Theorem 3. Section 6 also contains some examples of how Theorem 3 can be used to transfer function theoretic results from classical to weighted Paley-Wiener spaces. To be more specific, we prove results on sets of sampling, interpolation, and uniqueness. The last section (Section 7) presents the proof of Theorem 4. This proof makes use of an interpolation theorem of de Branges for the space $H(E)$, as well as a theorem from Section 6 describing so-called complete interpolating sequences for weighted Paley-Wiener spaces in terms of Muckenhoupt's (A_2) condition (cf. [12]). It is the latter relation which explains the appearance of the (A_2) condition in Theorem 4. $\square$

Let us make one remark on the notation of this paper: We will write $f \lesssim g$ whenever there is a constant K such that $f \leq Kg$; thus $f \simeq g$ if both $f \lesssim g$ and $g \lesssim f$.

2. Distribution of the zeros of E

This section is the most technical part of the paper. It makes a bridge between weighted Paley-Wiener spaces and the theory of de Branges spaces of entire functions. We begin with a brief summary of some important facts from de Branges' theory, which will play a crucial role in our exposition. We then obtain a description of the distribution of the zeros of those functions E from the Hermite-Biehler class $\overline{HB}$ for which $H(E)$ is a weighted Paley-Wiener space.

A Hilbert space H of entire functions is a *de Branges space* if the following conditions are met:

$(H1)$ Whenever f is in the space and has a nonreal zero ζ, the function $g(z) = f(z)(z-\bar{\zeta})/(z-\zeta)$ is in the space and has the same norm as f.

$(H2)$ For every nonreal ζ the linear functional defined on the space by $f \mapsto f(\zeta)$ is continuous.

$(H3)$ The function $f^*(z) = \overline{f(\bar{z})}$ belongs to the space whenever f belongs to the space and has the same norm as f.

The simplest model example is the classical Paley-Wiener space PW_τ $(= L^2_{\tau|\Im z|})$. A wider class of examples can be constructed via functions from $\overline{HB}$. To every function $E \in \overline{HB}$ we associate the space $H(E)$, which may be defined as in the introduction, but which we prefer to introduce in a slightly different way, in line with the definition of de Branges. We define $H(E)$ to be the set of all entire functions f satisfying

$$\|f\|_E^2 = \int_{-\infty}^{\infty} \frac{|f(t)|^2}{|E(t)|^2} dt < \infty,$$

and such that f is of nonpositive $\log|E|$-type. It is clear that $H(E)$ is a de Branges space. The following fundamental theorem of de Branges says that all de Branges spaces arise in this way [4, p. 57].

Theorem A. *A Hilbert space H, whose elements are entire functions, which satisfies $(H1)$, $(H2)$, and $(H3)$ and which contains a nonzero element is equal isometrically to some space $H(E)$.*

It follows from condition $(H2)$ that for each nonreal ζ there exists a reproducing kernel $K(\zeta, z)$ for the space $H(E)$. We shall need its representation, which also defines the kernel for real ζ [4, p. 50].

Theorem B. *For each $\zeta \in \mathbb{C}$ the function*

$$K(\zeta, z) = \frac{i}{2}\frac{E(z)\overline{E(\zeta)} - E^*(z)\overline{E^*(\zeta)}}{\pi(z - \bar{\zeta})}, \tag{3}$$

considered as a function of z, belongs to $H(E)$ and is the reproducing kernel of $H(E)$, i.e.,

$$\langle f, K(\cdot, \zeta)\rangle_E = \int_{-\infty}^{\infty} \frac{f(t)\overline{K(\zeta, t)}}{|E(t)|^2} dt = f(\zeta)$$

for each $f \in H(E)$.

Together with the elementary fact that $[M(z)]^2 = K(z, z)$, this yields an expression for the majorant via the function E. We shall use this expression only for real z. If $E(x) \neq 0$, then (3) yields

$$[M(x)]^2 = K(x, x) = \frac{1}{\pi}\varphi'(x)|E(x)|^2. \tag{4}$$

If $E(x) = 0$, then clearly $f(x) = 0$ for all $f \in H(E)$. Thus (4) holds for all $x \in \mathbb{R}$, if we declare the right-hand side to be zero when $E(x) = 0$.

There exists a well-known canonical factorization of functions in $\overline{HB}$ (see, e.g., [4], p. 20, or [10], Lecture 27). Suppose $E \in \overline{HB}$ and let $\Lambda = \{\lambda_k\}$ be the zero set of E, with $\lambda_k = \xi_k - i\eta_k$ ($\eta_k \geq 0$) and counting multiplicities in the usual way. Then

$$\sum_{\lambda_k \neq 0} \frac{\eta_k}{\xi_k^2 + \eta_k^2} < \infty,$$

and E can be represented as

$$E(z) = Cz^m e^{u(z)} e^{-i\alpha z} \prod_k (1 - z/\lambda_k) e^{z\Re(1/\lambda_k)}, \tag{5}$$

where $\alpha > 0$ and u is an entire function which is real on the real axis.

In what follows, we shall consider functions E which do not vanish on the real axis. Then $\varphi'(x)$ is well defined for all $x \in \mathbb{R}$ and, by (5),

$$\varphi'(x) = \alpha + \sum_k \frac{\eta_k}{(x - \xi_k)^2 + \eta_k^2} \tag{6}$$

for all $x \in \mathbb{R}$. Here α corresponds to the presence of an exponential factor $\exp(-i\alpha z)$ in the canonical representation of E. If we replace this factor by $\sin(\alpha(z + i))$, we obtain a function $E_1 \in \overline{HB}$ with $|E_1(x)| \simeq |E(x)|$ and the spaces $H(E)$ and $H(E_1)$ coincide and have equivalent norms. Therefore, without loss of generality, we will assume that $\alpha = 0$.

Now let w be a majorant weight, H the corresponding Hilbert space of entire functions and $E \in \overline{HB}$ (with $\alpha = 0$ in (5)) such that $H = H(E)$. Then

$$\|f\|_w^2 = \int_{-\infty}^{\infty} \frac{|f(x)|^2}{|E(x)|^2} dx \simeq \int_{-\infty}^{\infty} \frac{|f(x)|^2}{|E(x)|^2 \varphi'(x)} dx \tag{7}$$

holds for functions $f \in H$, as follows from the relation $M(x)w(x) \simeq 1$ and the expression for M given by (4). In particular, let us see what (7) means for two prototypes of functions in H. First, consider reproducing kernels corresponding to points from the real line: For each $\xi \in \mathbb{R}$, set $g_\xi(z) = K(\xi, z)/E(\xi)$. It follows from (4) that

$$\|g_\xi\|_w^2 = \frac{K(\xi,\xi)}{|E(\xi)|^2} = \frac{1}{\pi}\varphi'(\xi).$$

Using the explicit expression for $K(\xi, x)$ given by (3) as well as (7), we obtain the basic relation

$$\int_{-\infty}^{\infty} \frac{\sin^2(\varphi(\xi) - \varphi(x))}{\varphi'(x)(x-\xi)^2} dx \simeq \varphi'(\xi). \tag{8}$$

Second, define $f_k(z) = E(z)/(z - \lambda_k)$ for each $\lambda_k \in \Lambda$. Applying (7) with $f = f_k$, we get

$$\int_{-\infty}^{\infty} \frac{dx}{\varphi'(x)|x - \lambda_k|^2} \simeq \frac{1}{\eta_k}. \tag{9}$$

The results obtained below about the distribution of the zeros of E are mainly consequences of (8) and (9).

The chief result of this section is the following lemma, which shows the necessity of conditions (iii) and (iv) of Theorem 4.

Lemma 1. *Suppose w is a majorant weight and H is a Hilbert space of entire functions whose norm is $\|\cdot\|_w$. If $H = H(E)$, then the zero set Λ of E satisfies*

(iii) *There exists an $\varepsilon_0 > 0$ such that if $\Im\lambda_k \ge -\varepsilon_0$, then $dist(\lambda_k, \Lambda \setminus \lambda_k) \simeq 1$.*
(iv) $\varphi'(x) \simeq \sigma(x)/\min(1, |x - \lambda(x)|^2)$.

Lemma 1 is a consequence of a series of statements describing the properties of Λ. The first is:

Lemma 2. *There exist positive numbers M_0, ε_0 such that if $\varphi'(\xi) \le \varepsilon_0$ for some $\xi \in \mathbb{R}$, then there exists $\lambda_k = \xi_k - i\eta_k \in \Lambda$ satisfying*

$$|\xi - \xi_k| \le M_0 \quad \textit{and} \quad \eta_k \le M_0^2\, \varphi'(\xi). \tag{10}$$

Proof. Assume the lemma is false, i.e., that for each $\varepsilon > 0$ and $M > 0$ we have $\varphi'(\xi) \le \varepsilon$ for some $\xi \in \mathbb{R}$, but for each $k \in \mathbb{Z}$

$$|\xi - \xi_k| > M \quad \text{or} \quad \eta_k > M^2\varphi'(\xi). \tag{11}$$

Fix M and ε, and a corresponding point ξ. For each $h \in \mathbb{R}$ satisfying $|h| < M/10$, we claim that

$$\frac{1}{2}\varphi'(\xi) \le \varphi'(\xi + h) \le 2\varphi'(\xi). \tag{12}$$

Indeed, (6) yields $(\xi - \xi_k)^2 + \eta_k^2 \ge \eta_k/\varphi'(\xi)$ for each k. Together with (11) this implies $(\xi - \xi_k)^2 + \eta_k^2 \ge M^2,\ k \in \mathbb{Z}$. Therefore, if $|h| \le M/10$, we have

$$\frac{1}{2} \le \big((\xi + h - \xi_k)^2 + \eta_k^2\big)/\big((\xi - \xi_k)^2 + \eta_k^2\big) \le 2,$$

and so

$$\frac{1}{2}\frac{\eta_k}{(\xi-\xi_k)^2+\eta_k^2} \le \frac{\eta_k}{(\xi+h-\xi_k)^2+\eta_k^2} \le 2\frac{\eta_k}{(\xi-\xi_k)^2+\eta_k^2}$$

for $k \in \mathbb{Z}$. Summing over k, we obtain (12). Now take $Q = \min(\frac{\pi}{4\varphi'(\xi)}, \frac{1}{10}M)$. We have from (11) and the mean-value theorem that

$$\int_{-\infty}^{\infty} \frac{\sin^2(\varphi(\xi)-\varphi(x))}{\varphi'(x)(x-\xi)^2}dx \ge \int_{|x-\xi|\le Q} \frac{\sin^2(\varphi(\xi)-\varphi(x))}{\varphi'(x)(x-\xi)^2}dx \simeq Q\varphi'(\xi);$$

if M is sufficiently large and ε sufficiently small, then this contradicts (8). $\square$

Lemma 3. *There exists a constant C such that for all $L > 1$ the following statement holds: If, for some $j,k \in \mathbb{Z}$, $|\xi_j - \xi_k| \le L$ and $\eta_j, \eta_k \le L$, then*

$$\frac{1}{CL^2} \le \frac{\eta_k}{\eta j} \le CL^2.$$

Proof. Suppose that for some $L > 1$, $\gamma > 1$, and k, j we have $|\xi_j - \xi_k| \le L$ and $\eta_k > \gamma L^2 \eta_j$. We have then to prove that γ cannot be arbitrarily large. Consider the integrals

$$\begin{aligned} I_l = \int_{-\infty}^{\infty} \frac{dx}{|x-\lambda_l|^2\varphi'(x)} &= \int_{|\xi_j - x|<2L} + \int_{|\xi_j-x|\ge 2L} \\ &= I_l^{(1)} + I_l^{(2)} \simeq \frac{1}{\eta_l}, \quad l = j,k. \end{aligned}$$

The latter relation is just (9). Thus we have $I_k^{(1)}, I_k^{(2)} \lesssim 1/\eta_k$. Since $|x-\lambda_k| \simeq |x-\lambda_j|$ for $|\xi_j - x| > 2L$, we also have $I_j^{(2)} \simeq I_k^{(2)} \lesssim 1/\eta_k$. Since

$$I_j^{(1)} + I_j^{(2)} \simeq \frac{1}{\eta_j} > \frac{\gamma L^2}{\eta_k},$$

it follows that $I_j^{(1)} \gtrsim 1/\eta_j$ if γ is sufficently large. On the other hand, we may estimate $I_j^{(1)}$ from above. Set $\tau = L(\eta_j/\eta_k)^{1/2}$, and write

$$I_j^{(1)} = \left(\int_{|x-\xi_j|<\tau} + \int_{\tau<|x-\xi_j|<2L}\right)\frac{dx}{|x-\lambda_j|^2\varphi'(x)} = J_1 + J_2.$$

We use the inequality $\varphi'(x) \ge \eta_j|x-\lambda_j|^{-2}$ for estimating J_1, and $\varphi'(x) \ge \eta_k(3L)^{-2}$ ($|x-\xi_j| \le 2L$) for dealing with J_2, and obtain

$$I_j^{(1)} = J_1 + J_2 \lesssim \frac{L}{\sqrt{\eta_j\eta_k}}.$$

Our two estimates for $I_j^{(1)}$ imply $1/\eta_j \lesssim L/\sqrt{\eta_j\eta_k}$, which is incompatible with the assumption that $\eta_k > \gamma L^2 \eta_j$ if γ is sufficiently large. So γ is bounded, and the proof is complete. $\square$

Lemma 4. *For each $M > 0$ set $q(M) = \sup_x \#\{\lambda_k \in \Lambda : \ x < \xi_k \le x + M, \eta_k < M\}$. Then $q(M) - 1 \lesssim M\log(M+1)$.*

Proof. For small M the inequality $q(M)-1 \lesssim M\log(M+1)$ means that $q(M)=1$, i.e., that $\operatorname{dist}(\lambda_j, \Lambda \setminus \{\lambda_j\}) \gtrsim 1$ for η_j is sufficiently small. We begin by proving the latter statement. Let us assume the contrary, i.e., that we can choose $\lambda' = \xi' - i\eta', \lambda'' = \xi'' - i\eta'' \in \Lambda$ so that $\eta', \eta'' \leq \varepsilon$ and $|\lambda' - \lambda''| \leq 2\delta$. We will show that this yields a contradiction if ε and δ are sufficiently small.

By Lemma 3, $\eta' \simeq \eta''$. Therefore, we may assume $\eta' = \eta''(=\eta)$: If necessary, change the imaginary part of λ' from $-\eta'$ to $-\eta''$; this will only influence the constants in the equivalence relation (7). Assume for simplicity that $\lambda' = -\delta - i\eta$ and $\lambda'' = \delta - i\eta$. Consider the two test functions

$$g(z) = \frac{E(z)}{z-\lambda'} \quad \text{and} \quad h(z) = \frac{E(z)}{(z-\lambda')(z-\lambda'')},$$

which both clearly belong to $H(E)$. By applying (7) with $f = h$, we obtain

$$\int_{-\infty}^{\infty} \frac{dx}{\varphi'(x)|x-\lambda'|^2|x-\lambda''|^2} \simeq \int_{-\infty}^{\infty} \frac{dx}{|x-\lambda'|^2|x-\lambda''|^2} = \frac{\pi}{2\eta}\frac{1}{\eta^2+\delta^2}. \tag{13}$$

On the other hand,

$$\int_{-\infty}^{\infty} \frac{dx}{\varphi'(x)|x-\lambda'|^2|x-\lambda''|^2} = \int_{-1}^{1} + \int_{|x|>1} = I_1 + I_2.$$

We have

$$I_2 \lesssim \int_{|x|>1} \frac{dx}{\varphi'(x)|x-\lambda'|^2} \lesssim \int_{-\infty}^{\infty} \frac{dx}{\varphi'(x)|x-\lambda'|^2} \simeq \|g\|_w^2 \simeq \frac{1}{\eta},$$

$$I_1 \lesssim \frac{1}{\eta}\int_0^1 \frac{dx}{(x+\delta)^2+\eta^2} \lesssim \frac{1}{\eta\max(\eta,\delta)}.$$

So we obtain

$$I_1 + I_2 \lesssim \frac{1}{\eta\max(\eta,\delta)},$$

which contradicts (13) for η, δ sufficiently small.

We now know that there exists an $\varepsilon_0 > 0$ such that $q(M) = 1$ for $M \leq \varepsilon_0$. We next assume $M > \varepsilon_0$. Fix a square

$$Q(\xi, M) = \{z = x - iy : \ \xi < x \leq \xi + M, y < M\}.$$

By what was proved above, we need only bound the number of points from Λ to be found in

$$Q_{\varepsilon_0}(\xi, M) = \{z = x - iy : \ \xi < x \leq \xi + M, \varepsilon_0 \leq y < M\}.$$

To this end, set

$$p = \#(\Lambda \cap Q_{M/2}(\xi, M),$$

which is the number of points from Λ to be found in the lower half of the square $Q(\xi, M)$. We assume $p \geq 2$, and let $\lambda', \lambda'' \in \Lambda$ be two points from $Q_{M/2}(\xi, M)$. Then let h be as in the first part of the proof. On the one hand, we have

$$\|h\|_w^2 = \int_{-\infty}^{\infty} \frac{dx}{|x-\lambda'|^2|x-\lambda''|^2} \simeq \frac{1}{M^3},$$

independently of ξ. On the other hand, noting that $\varphi'(x) \gtrsim \frac{pM}{|x-\lambda'|^2}$, we obtain that

$$\|h\|_w^2 \simeq \int_{-\infty}^{\infty} \frac{dx}{\varphi'(x)|x-\lambda'|^2|x-\lambda''|^2} \lesssim \frac{1}{pM^2}.$$

Combining the two norm estimates, we get $p \lesssim M$.

We now make a dyadic decomposition: Split the upper half of $Q(\xi, M)$ into two smaller squares, and repeat the above argument for these smaller squares of side length $M/2$. We iterate the argument so that in the j-th step we are dealing with 2^j squares of side length $M/2^j$; we stop when $M/2^j \le \varepsilon_0$. Summing up the estimates in each step, we arrive at the estimate claimed in the lemma. □

Lemma 5. *For each sufficiently large $M > 0$ there exists $C > 0$ such that the inequality $\eta_j \le 1$ yields that each interval $[\xi_j - M, \xi_j]$ and $[\xi_j, \xi_j + M]$ contains points x at which $\varphi'(x) < C\eta_j$.*

Proof. Denote by Π the square which has as one of its sides the interval $[\xi_j - 5M, \xi_j + 5M]$, and which is located in the lower half-plane. Partition Λ as $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$, where

$$\Lambda_1 = \Lambda \cap \Pi, \quad \Lambda_2 = \{\lambda_k \in \Lambda \setminus \Pi, \eta_k \le 5M\eta_j\}, \quad \Lambda_3 = \Lambda \setminus (\Lambda_1 \cup \Lambda_2). \tag{14}$$

Write $\varphi'(x) = \psi_1(x) + \psi_2(x) + \psi_3(x)$, where

$$\psi_i(x) = \sum_{\lambda_k \in \Lambda_i} \frac{\eta_k}{(x-\xi_k)^2 + \eta_k^2}, \quad i = 1, 2, 3.$$

We will estimate each term $\psi_i(x)$ separately.

It follows from Lemma 4 that there exists a constant $N = N(M) > 0$ such that, for each $\xi \in \mathbb{R}$, the number of points $\lambda_k \in \Lambda$ with $|\xi_k - \xi| < 5M$ and $\eta_k < 5M$ does not exceed N. Therefore, for all $x \in [\xi_j - M, \xi_j + M]$, we have

$$\psi_2(x) = \sum_{l \ne 0} \sum_{|\xi_k - \xi_j - 10lM| < 5M,\ \eta_k < 5M\eta_j} \frac{\eta_k}{(x-\xi_k)^2 + \eta_k^2} \lesssim MN\eta_j \sum_{l=1}^{\infty} \frac{1}{(Ml)^2} \lesssim \eta_j.$$

Now by Lemma 3 there exists a constant $\gamma = \gamma(M) > 0$ such that $\eta_k < \gamma\eta_j$ if $\lambda_k \in \Lambda_1$, with γ independent of j. We also have $\#\Lambda_1 \le N$. Pick a point $x_0 \in [\xi_j - M, \xi_j]$ such that

$$\operatorname{dist}(x_0, \{\xi_k :\ \lambda_k \in \Lambda_1\}) > M/(10N).$$

Then $\psi_1(x_0) \le \gamma N (10N/M)^2 \eta_j$. Similarly, we can find a point $x_1 \in [\xi_j, \xi_j + M]$ such that

$$\psi_1(x_1) \le \gamma N (10N/M)^2 \eta_j \lesssim \eta_j.$$

The estimate for $\psi_3(x)$ follows from

Proposition 1. *For each sufficiently large M there exists $C > 0$ such that $\psi_3(\xi_j) \le C\eta_j$ for each j satisfying $\eta_j \le 1$.*

In the proof of Proposition 1, we shall use the following estimate.

Proposition 2. *Assume $\eta_j = \varepsilon \le 1$ and $\varphi'(x) \ge C\varepsilon$ for $|x - \xi_j| \le A$ for some positive numbers A, C such that $A \ge 1/\sqrt{C}$. Then*

$$J := \int_{|x-\xi_j| \le A} \frac{dx}{\varphi'(x)|x-\lambda_j|^2} \lesssim \frac{1}{\sqrt{C}} \frac{1}{\varepsilon}, \tag{15}$$

independently of ε, A, C.

Proof. Take $a > 0$. We claim that

$$J \leq \int_{|x-\xi_j|\leq a} \frac{dx}{\eta_j} + \frac{1}{C\eta_j} \int_{a<|x-\xi_j|\leq A} \frac{dx}{(x-\xi_j)^2} \lesssim \frac{a}{\eta_j} + \frac{1}{Ca\eta_j},$$

where the last estimate is uniform with respect to A. To prove the claim, it suffices to apply the inequalities $\varphi'(x) \geq \eta_j |x-\lambda_j|^{-2}$ for $|x-\xi_j| \leq a$ and $\varphi'(x) \geq C\varepsilon$ for $a \leq |x-\xi_j| \leq A$. Then we obtain (15) by putting $a = 1/\sqrt{C}$. □

Proof of Proposition 1. Assume the proposition is false: For arbitrary $M > 0$ and $C > 0$ we can find λ_j satisfying $\eta_j \leq 1$ and $\psi_3(\xi_j) \geq C\eta_j$. Let λ_k be the point (or one of the points) from Λ_3 which is closest to λ_j, and set $m = |\lambda_j - \lambda_k|$. Then a direct estimate shows that $\psi_3(x) \geq C\eta_j/10$ for $|x-\xi_j| \leq 2m$. Applying Proposition 2 with $A = 2m$ and assuming $1/\sqrt{C} \leq 2m$, we have

$$\int_{|x-\xi_j|\leq 2m} \frac{dx}{\varphi'(x)|x-\lambda_j|^2} \lesssim \frac{1}{\sqrt{C}} \cdot \frac{1}{\eta_j}.$$

On the other hand, we have $|x-\lambda_j| \simeq |x-\lambda_k|$ for $|x-\xi_j| \geq 2m$, and hence

$$\int_{|x-\xi_j|>2m} \frac{dx}{\varphi'(x)|x-\lambda_j|^2} \simeq \int_{|x-\xi_j|>2m} \frac{dx}{\varphi'(x)|x-\lambda_k|^2} \lesssim \frac{1}{\eta_k} \leq \frac{1}{5M\eta_j}.$$

It follows that

$$\int_{-\infty}^{\infty} \frac{dx}{\varphi'(x)|x-\lambda_j|^2} \lesssim \left(\frac{1}{\sqrt{C}} + \frac{1}{M} \right) \frac{1}{\eta_j},$$

where C and M can be chosen arbitrarily large. This is a contradiction in view of (9). □

Now we complete the proof of Lemma 5. Take M and C from Proposition 1. Since $\psi_3(x) \simeq \psi_3(\xi_j)$ for $|x-\xi_j| \leq M$, it follows that $\psi_3(x) \lesssim \eta_j$ for $|x-\xi_j| \leq M$. Collecting our estimates for ψ_1, ψ_2, and ψ_3, we see that there exists M_0 such that for $M \geq M_0$ and $\eta_j \leq 1$

$$\psi(x_i) \lesssim \eta_j,\ i = 0, 1.$$

Both M_0 and the constant in this estimate depend only on the constants in the relation $M(x)/w(x) \simeq 1$. □

Remark 1. We have in fact proved that $\psi_2(x), \psi_3(x) \lesssim \eta_j$ for $x \in [\xi_j - M, \xi_j + M]$. So the main contribution to $\varphi'(x)$ for all $x \in [\xi_j - M, \xi_j + M]$ comes from $\psi_1(x)$.

Remark 2. For $\lambda_k \in \Lambda$ we have from (5) that $\varphi'(\xi_k) > \eta_k^{-1}$. Therefore Lemma 2 and Lemma 5 are in a sense complementary: Lemma 2 says that if $\varphi'(\xi)$ is small enough, there exists λ_k with $|\xi_k - \xi| \lesssim 1$ and $\varphi'(\xi_k) \gtrsim 1/\varphi'(\xi)$. Under the assumption of Lemma 5, we have $\varphi'(\xi_j) > 1/\eta_j$ for $\eta_j < 1$ and obtain $\varphi'(x_i) \lesssim \eta_j$ for some x_i's with $|x_i - \xi_j| \lesssim 1$.

Proof of Lemma 1*, condition* (iii). We have already proved that $\operatorname{dist}(\lambda_j, \Lambda \backslash \{\lambda_j\}) \gtrsim 1$ when η_j is sufficiently small (see Lemma 4). The fact that also $\operatorname{dist}(\lambda_j, \Lambda \backslash \{\lambda_j\}) \lesssim 1$ in this case is now immediate: If not, then for arbitrarily large M and ε^{-1} one can find $\lambda_j \in \Lambda$ with $\eta_j < \varepsilon$ and such that Λ_1 in (14) consists of only one point λ_j. Remark 1 now yields that $\varphi'(x) \lesssim \eta_j < \varepsilon$ on a large interval, but this contradicts Lemma 2. □

To finish the proof of Lemma 1, we need to estimate the contribution to φ from the points which are located far away from the real axis. To this end, set

$$\Lambda_1 = \{\lambda_k = \xi_k - i\eta_k \in \Lambda;\ \eta_k \le 1\},\ \Lambda_2 = \Lambda \setminus \Lambda_1,$$

$$\vartheta_1(x) = \sum_{\lambda_k \in \Lambda_1} \frac{\eta_k}{(x-\xi_k)^2 + \eta_k^2},\ \vartheta_2(x) = \varphi'(x) - \vartheta_1(x).$$

Lemma 6. *We have* $\sup_x \vartheta_2(x) < \infty$.

Proof. Assume that for some $\xi \in \mathbb{R}$ and $C > 0$ we have $\vartheta_2(\xi) = C$. We shall see that this leads to a contradiction for sufficiently large C. To begin with, note that we may assume $\operatorname{dist}(\xi, \Lambda) \gtrsim 1$. This follows from condition (iii) of Lemma 1 (established above) and the fact that $\vartheta_2(x) \simeq \vartheta_2(x+t)$ if $|t| \le 1$. We may also assume that C is so large that $\operatorname{dist}(x, \Lambda) \gtrsim 1$ when $|x - \xi| \le 1/C$.

Let $\lambda^{(2)}(\xi)$ be the point (or one of the points) from Λ_2 closest to ξ, and set $|\lambda^{(2)}(\xi) - \xi| = m$. Then (9) and the definition of Λ_2 imply that

$$J := \int_{|\xi - x| \ge 2m} \frac{dx}{\varphi'(x)|x - \lambda^{(2)}(\xi)|^2} \le \int_{-\infty}^{\infty} \frac{dx}{\varphi'(x)|x - \lambda^{(2)}(\xi)|^2} \lesssim 1.$$

On the other hand, using that $\varphi' \ge \vartheta_2$ and taking (7) into account, we obtain

$$C \le \varphi'(\xi) \simeq \int_{-\infty}^{\infty} \frac{\sin^2(\varphi(x) - \varphi(\xi))}{\varphi'(x)(x-\xi)^2} dx = \int_{|x-\xi| \le 2m} + \int_{|x-\xi| > 2m} = I_1 + I_2.$$

We have $I_2 \lesssim J \lesssim 1$. Put

$$I_1 = \int_{|x-\xi| \le 1/C} + \int_{1/C < |x-\xi| \le 2m} = I_1^{(1)} + I_1^{(2)}.$$

A direct estimation shows that $C/10 \le \vartheta_2(x) \le 10C$ when $|x - \xi| \le 2m$, and since $\varphi' \ge \vartheta_2$, we get

$$I_1^{(2)} \lesssim \frac{1}{C} \int_{|x-\xi| > 1/C} \frac{dx}{(x-\xi)^2} \lesssim 1.$$

Using Lemma 4 and the assumptions that $\vartheta_2(\xi) = C$ and $\operatorname{dist}(x, \Lambda) \gtrsim 1$ for $|x - \xi| \le 1/C$, we obtain $\varphi'(x) \lesssim C$ for $|x - \xi| \le 1/C$. Thus by the mean value theorem, we have

$$\sin^2(\varphi(x) - \varphi(\xi)) \lesssim C^2 |x - \xi|^2$$

when $|x - \xi| \le 1/C$, and hence $I_1^{(1)} \lesssim 1$.

Combining our estimates for $I_1^{(1)}$, $I_1^{(2)}$, and I_2, we get

$$\int_{-\infty}^{\infty} \frac{\sin^2(\varphi(x) - \varphi(\xi))}{\varphi'(x)(x-\xi)^2} dx \lesssim 1,$$

which contradicts the inequality $\varphi'(x) \ge C$ if C is sufficiently large. □

Proof of Lemma 1*, condition* (iv). It is enough to prove that

$$\varphi'(x) \simeq \sigma(x)/|x - \lambda(x)|^2$$

when $\sigma(x) < \eta_0$ and otherwise $\varphi'(x) \simeq 1$. We observe that the first estimate follows from Lemma 5 and part (iii) of Lemma 1.

Assume that $\sigma(x) \ge \eta_0$. Then Lemmas 2 and 5 show that $\varphi'(x) \gtrsim 1$, while Lemmas 4 and 6 show that $\varphi'(x) \lesssim 1$. □

3. Proof of Theorem 1: Necessity

In this section, we shall prove the necessity of the representation for a majorant weight, as claimed in Theorem 1. Thus we begin by assuming we are given a majorant weight w and an associated Hilbert space H whose norm is $\|\cdot\|_w$.

We note that Theorem A guarantees the existence of a de Branges space such that $H = H(E)$ isometrically. This means in particular that Lemma 1 applies.

A simple and important special case is when $\varphi'(x) \simeq 1$, where φ is as in the previous section. Then the relation $M(x)w(x) \simeq 1$ and the expression for M given by (4) imply $w(x)^{-1} \simeq |E(x)|$, and so

$$\log w(x) + \log |E(x)| \in L^\infty.$$

The function $\log |E(x+i|y|)|$ is subharmonic because $E \in \overline{HB}$. Its Riesz measure is supported by the real line. Applying a Cauchy-Riemann equation to $\log |E(x+iy)|$, we find that the density of this Riesz measure with respect to one-dimensional Lebesgue measure is φ'. Therefore, we have a Riesz decomposition

$$\log |E(x+i|y|)| = u(z) + \frac{1}{\pi}\int_{-\infty}^{\infty} [\log |1 - z/t| + (1 - \chi_{[-1,1]}(t))x/t]\varphi'(t)dt,$$

where $u(z)$ is harmonic in the whole complex plane. The identity $u(x+iy) = u(x-iy)$ implies $u_y(x) = 0$, $x \in \mathbb{R}$, and so by the Cauchy-Riemann equations we have $v_x(x) = 0$ for each v which is a harmonic conjugate of u. Therefore, there exists a harmonic conjugate v of u vanishing on the real line. The condition of Theorem 1 will be met if we set $g(x) = u(x) + iv(x)$ and $m(x) = \varphi'(x)/\pi$.

Our goal is to reduce the general case to this special situation. It follows from Lemma 1 that the condition $\varphi'(x) \simeq 1$ is violated if and only if $\inf_j \eta_j = 0$. In particular, Lemma 1 implies:

Lemma 7. *Let E and ε_0 be as in Lemma* 1. *Then the product*

$$F(z) = E(z) \prod_{\eta_k < \varepsilon_0} \frac{1 - z/(\xi_k - i\varepsilon_0)}{1 - z/\lambda_k}$$

converges uniformly on compact sets to a function $F \in \overline{HB}$ and $F(x) = |F(x)|e^{-i\psi(x)}$ satisfies $\psi'(x) \simeq 1$ for $x \in \mathbb{R}$.

Hence the necessity of the condition in Theorem 1 in the general case follows from the following lemma.

Lemma 8. *With F as in Lemma* 7 *and σ defined* (*as earlier*) *via the zeros of E, we have*

$$w(x) \simeq \frac{1}{|F(x)|^2\sigma(x)},$$

and, for each $\varepsilon > 0$, there exists a function $\mu(x)$ for which $\|\mu\|_{L^\infty} < \varepsilon$ and a real number a such that

$$\log \sigma(x) + ax - \int_{-\infty}^{\infty} [\log |1 - x/t| + (1 - \chi_{[-1,1]}(t))x/t]\mu(t)dt \in L^\infty.$$

In Section 7, we shall obtain a more subtle estimate for σ.

Proof of Lemma 8. We use the properties of E described in Lemmas 1 and 3. Lemma 1 shows that $|F(x)|^2\sigma(x) \simeq |E(x)|^2\varphi'(x)$ and hence $|F(x)|^2\sigma(x)w(x) \simeq 1$. By Lemma 3, there exist an $L_0 > 1$ and a constant C such that

$$|\log\sigma(x) - \log\sigma(\xi)| \le 2\log|x-\xi| + C \tag{16}$$

if $|x-\xi| \ge L_0$. Now fix $\varepsilon > 0$ and choose a number $T > L_0$, $L = T^2$, such that $(\log T)/T < \varepsilon$ and

$$\frac{\log L}{T} < \varepsilon, \ \frac{T\log L}{L} < \varepsilon. \tag{17}$$

Define

$$h_1(x) = \begin{cases} -\log\sigma(kL), \ x = kL, \ k \in \mathbb{Z}, \\ \text{linear for } \ x \in (kL,(k+1)L), \ k \in \mathbb{Z}. \end{cases}$$

The function h_1 is piece-wise linear. The slope of each linear piece is estimated by (16), so that we get $\|h_1'\|_\infty < (2\log L + C)/L$. By the construction of h_1, we have

$$\frac{1}{\sigma(x)} = e^{u_1(x)+h_1(x)}\,, \quad u_1 \in L^\infty(\mathbb{R}).$$

By an appropriate smoothing of h_1, we can replace it by a smoother function h satisfying

$$\|h'\|_\infty < \frac{\log L}{L}, \ \|h''\|_\infty < \frac{\log L}{L}, \tag{18}$$

$$|h(x) - h(\xi)| \le 2\log|x-\xi| + \text{Constant}, \quad |x-\xi| \ge L_0,$$

and also

$$\sigma^{-1}(x) = e^{u(x)+h(x)}$$

for some $u \in L^\infty$.

We finally transform h appropriately. To this end, note that h can be extended continuously to a harmonic function in the upper half-plane, given there by

$$h(x+iy) = \frac{y}{\pi}\int \frac{h(t)}{(x-t)^2+y^2}dt.$$

We differentiate h in the vertical direction, and find by an integration by parts that

$$h_y(z) = -\Re\frac{1}{\pi}\int\frac{h'(t)}{z-t}dt.$$

If we can prove that in fact

$$\tilde{h'}(x) = \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{\infty}\frac{h'(t)}{x-t}dt$$

is well defined, and that $\tilde{h'} \in L^\infty$, then Green's theorem will give us that

$$h(z) = a\Re z + 2\int_{-\infty}^{\infty}[\log|1-z/t| + (1-\chi_{[-1,1]}(t))\Re z/t]\tilde{h'}(t)dt \tag{19}$$

for some real a, and $\Im z \ge 0$. If, in addition, we can make the L^∞ norm of $\tilde{h'}$ arbitrarily small by choosing ε sufficiently small, we have proved Lemma 8.

By (18) and (17), we have

$$\left|\text{p.v.} \int_{|x-t|<T} \frac{1}{x-t}\left(h'(x)+\int_x^t h''(\tau)d\tau\right)dt\right| \lesssim T\frac{\log L}{L} < \varepsilon.$$

Each of the integrals

$$\int_{x-t>T} \frac{h'(t)}{x-t}dt \quad \text{and} \quad \int_{x-t<-T} \frac{h'(t)}{x-t}dt$$

is absolutely convergent. Indeed, integration by parts gives

$$\begin{aligned}\left|\int_{x+T}^{\infty} \frac{h'(t)}{v-t}dt\right| &= \left|-\frac{h(x)-h(t)}{x-t}\Big|_{t=x+T} - \int_{x+T}^{\infty} \frac{h(x)-h(t)}{(x-t)^2}dt\right| \\ &\lesssim \frac{\log T}{T} + \int_T^{\infty} \frac{\log\tau}{\tau^2}d\tau < \varepsilon.\end{aligned}$$

Summing our estimates, we obtain $|\tilde{h}'(x)| < 3\varepsilon$, and the proof of Lemma 8 is complete. ☐

Formula (19) will be needed later. We state it in a slightly different form for future reference:

Lemma 9. *Set*

$$u(x+iy) = \frac{|y|}{\pi}\int \frac{\log\sigma(t)}{(x-t)^2+y^2}dt.$$

Then for each $\varepsilon > 0$ *we can find a function* μ *for which* $\|\mu\|_\infty < \varepsilon$ *and a real number* a *such that*

$$\left|u(z) + a\Re z - 2\int_{-\infty}^{\infty} [\log|1-z/t| + (1-\chi_{[-1,1]}(t))\Re z/t]\mu(t)dt\right| \lesssim 1$$

for all $z \in \mathbb{C}$.

4. A multiplier lemma and proof of Theorem 1: Sufficiency

The proof of the sufficiency of the condition in Theorem 1 is considerably easier than that of the necessity. It is a consequence of the following multiplier lemma.

Lemma 10. *Suppose* ω *is a subharmonic function of the form*

$$\omega(z) = \int_{-\infty}^{\infty} [\log|1-z/t| + (1-\chi_{[-1,1]}(t))\Re z/t]m(t)dt,$$

where $m(t) \simeq 1$. *Then there exists a function* $F \in \overline{HB}$ *with zero sequence* $\Sigma = \{\xi_k - i\}_{k\in\mathbb{Z}}$ *satisfying* $\xi_{k+1} - \xi_k \simeq 1$ *and such that*

$$|F(z)|e^{-\omega(z)} \simeq \frac{\operatorname{dist}(z,\Sigma)}{1+\operatorname{dist}(z,\Sigma)}$$

for all $z \in \mathbb{C}$.

Proof. Because $m \simeq 1$, we have $\omega(z) - \omega(z+i) = O(1)$, and so we may replace $\omega(z)$ by the function $z \mapsto \omega(z+i)$, which has its Riesz measure supported by the horizontal line $\Im z = -1$. Therefore, if we construct a function G with zeros on the real line and the desired properties, then we obtain F by setting $F(z) = G(z+i)$.

Partition the real line into a sequence of disjoint intervals $I_k = [x_k, x_{k+1})$, $k \in \mathbb{Z}$, with $x_0 = 0$, such that

$$\int_{I_k} m(t)dt = 1$$

for all k, and choose $\xi_k \in I_k$ so that

$$\xi_k = \int_{I_k} tm(t)dt.$$

Let δ_{ξ_k} denote a point mass at the point ξ_k, and set

$$\nu = \sum_k \delta_{\xi_k}$$

and $d\mu(t) = m(t)dt - d\nu(t)$. What we need to show is that

$$\left| \int_{-\infty}^{\infty} \log|1 - z/t| d\mu(t) \right| \simeq 1$$

when $\operatorname{dist}(z, \{\xi_k\}) \geq \varepsilon$ for every $\varepsilon > 0$.

Set $h(x) = \int_0^x d\mu(t)$ and $H(x) = \int_0^x h(t)dt$. Observe that

$$h(x_k) = H(x_k) = 0$$

for all k, by the construction of the sequence ξ_k, and consequently both h and H are bounded functions. Integrating twice by parts we get

$$\int_{-\infty}^{\infty} \log|1 - z/t| d\mu(t) = \Re \int_{-\infty}^{\infty} \frac{H(t)}{(z-t)^2} dt,$$

and the result follows. □

To prove the sufficiency part of Theorem 1, it is enough to choose $E = Fe^{g/2}$, where F is the function constructed in Lemma 9, and $H = H(E)$. Then $w \simeq 1/|E|^2$ and $M \simeq w$, since $\varphi' \simeq 1$.

5. Proof of Theorem 2

We begin by proving that each space L^2_ω of the form given in Theorem 2 is a weighted Paley-Wiener space. To this end, for given m and b as in the statement of that theorem, introduce the smoothed function

$$m_r(x) := \frac{1}{2r} \int_{-r}^{r} m(x-t)dt,$$

where $r > 0$ is chosen so large that $m_r(x) + \pi b \simeq 1$, and set

$$\omega_r(z) = a\Re z + \int_{-\infty}^{\infty} [\log|1 - z/t| + (1 - \chi_{[-1,1]}(t))\Re z/t](m_r(t) + \pi b)dt.$$

We see that $|\omega_r(z) - \omega(z)| \lesssim 1$, which means that ω can be replaced by ω_r. Now apply Lemma 10 to ω_r, so that we obtain a function $E \in \overline{HB}$ with zero sequence $\Sigma = \{\xi_k - i\}$ satisfying $\xi_{k+1} - \xi_k \simeq 1$ and such that

$$|E(z)|e^{-\omega(z)} \simeq \frac{\operatorname{dist}(z, \Sigma)}{1 + \operatorname{dist}(z, \Sigma)}.$$

It is plain that $H(E) \subset L^2_\omega$. To see that we also have $L^2_\omega \subset H(E)$, we recall the following classical inequality of Plancherel and Pólya [10, p. 50]: If g is analytic in the upper half-plane, continuous up to the real line, and $\log|g(z)| \lesssim \varepsilon|z|$ for all $\varepsilon > 0$, then

$$\int_{-\infty}^{\infty} |g(x+iy)|^2 dx \le \int_{-\infty}^{\infty} |g(x)|^2 dx \tag{20}$$

for all $y > 0$. In other words, if $g \in L^2(\mathbb{R})$, then $g \in H^2(\mathbb{C}^+)$. This means that if $f \in L^2_\omega$, then f/E and f^*/E belong to $H^2(\mathbb{C}^+)$ or equivalently $f \in H(E)$. Since $H(E)$ is a weighted Paley-Wiener space, we have shown that each space given in Theorem 2 is one too.

Before proceeding with the proof of Theorem 2, we note another consequence of the Plancherel-Pólya inequality.

Lemma 11. *Let E be the function constructed above. If $\varphi'(x)|E(x)|^2 \simeq e^{2\omega(x)}$ and the zeros of E satisfy conditions* (iii) *and* (iv) *of Lemma 1, then*

$$\|f\|_E \lesssim \|f\|_w$$

for all $f \in L^2_\omega$, where $w(x) = e^{-\omega(x)}$.

Proof. First set $g(z) = f(z)e^{-\omega(z)}$ and integrate (20) with respect to y from 0 to $1/2$; then set $g(z) = f^*(z)e^{-\omega(\overline{z})}$ and make the same integration. We then obtain

$$\int_{|y|<1/2} |f(z)|^2 e^{-2\omega(z)} dA(z) \le \|f\|_w^2$$

for $f \in L^2_\omega$, where dA denotes Lebesgue area measure on $\mathbb{C}$. A direct computation shows that $|\omega(z+iy) - \omega(z)| \le \pi|y|\|m+b\|_\infty$, and so we obtain

$$\int_{|y|<1/2} |f(z)|^2 e^{-2\omega(z+i/2)} dA(z) \le e^{\pi\|m+b\|_\infty} \|f\|_w^2. \tag{21}$$

We have then achieved that the integrand on the left-hand side is subharmonic in the strip $|\Im z| < 1/2$.

Choose a real sequence x_k, $k \in \mathbb{Z}$, so that $x_k < x_{k+1}$ and

$$\int_{x_k}^{x_{k+1}} \varphi'(x) dx = 1.$$

By the assumption about E, $x_{k+1} - x_k \simeq 1$. Choose $\varepsilon \le 1/2$ so that $\varepsilon \le x_{k+1} - x_k$ for all k. We then have

$$|f(x)|^2 e^{-2\omega(x+i/2)} \le \frac{1}{\varepsilon^2\pi} \int_{|x-z|<\varepsilon} |f(z)|^2 e^{-2\omega(z+i/2)} dA(z)$$

by subharmonicity. Multiplying by $\varphi'(x)$ and integrating with respect to x, we get

$$\int_{x_k}^{x_{k+1}} |f(x)|^2 e^{-2\omega(x+i/2)} \varphi'(x) dx \le \frac{1}{\varepsilon^2\pi} \int_{x_{k-1}<\Re z<x_{k+2}, |\Im z|<1/2} |f(z)|^2 e^{-2\omega(z+i/2)} dA(z).$$

Summing these inequalities, we obtain

$$\int_{-\infty}^{\infty} |f(x)|^2 e^{-2\omega(x+i/2)} \varphi'(x) dx \le \frac{3}{\varepsilon^2\pi} \int_{|\Im z|<1/2} |f(z)|^2 e^{-2\omega(z+i/2)} dA(z),$$

which proves the result, when taking into account (21) and the inequality

$$|\omega(x+i/2)-\omega(x)| \le \frac{\pi}{2}\|m+b\|_\infty.$$

$\square$

We continue with the proof of Theorem 2. What remains is to prove that any weighted Paley-Wiener space H with norm $\|\cdot\|_w$ can be expressed as stated in the theorem. Suppose such a space H is given. Take E and σ as in Lemma 1, and choose μ of Lemma 9 so that $m_0(x) = \phi'(x) + \mu(x) \simeq 1$. We prove first that $H = H(E) = L^2_{\omega_0}$, where

$$\omega_0(z) = a\Re z + \int_{-\infty}^{\infty} [\log|1-z/t| + (1-\chi_{[-1,1]}(x))\Re z/t] m_0(t)dt.$$

We begin by noting that the ω_0-type and the $\log|E|$-type are the same for any entire function f. To see this, we use the estimate

$$|E(z)| \simeq \min(1, \operatorname{dist}(z,\Lambda)) e^{\omega_0(z)-u(z)},$$

which follows from Lemmas 7 and 8, and deduce from this that

$$e^{u(z)}|E(z)|^2 \lesssim e^{2\omega_0(z)} \lesssim e^{-u(z)}|E(z)|^2.$$

The assertion then follows because Lemma 9 shows that $-u(z)$ is the Poisson integral of a nonnegative function.

Now if $f \in H(E)$, then f is of nonpositive ω_0-type, and so $f \in L^2_{\omega_0}$ since $\|f\|_w \simeq \|f\|_E$. On the other hand, if $f \in L^2_{\omega_0}$, we use Lemma 11, which says that $\|f\|_E \lesssim \|f\|_w$. Thus $f \in H(E)$ because f is of nonpositive $\log|E|$-type. We conclude that $H(E) = L^2_{\omega_0}$.

Next we prove that m_0 can be replaced by $m+b$. We consider the function

$$g(z) = \int_{-\infty}^{\infty} [\log|1-z/t| + (1-\chi_{[-1,1]}(x))\Re z/t](m(t) - m_0(t))dt,$$

which is bounded on $\mathbb{R}$ and harmonic off the real line. We will prove that

$$g(z) = h(z) + c|y|,$$

with $h(z) \simeq 1$ and c a real constant. By symmetry, it is enough to consider the upper half-plane. Set $(z = x+iy)$

$$k(z) = g(z) - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} g(t)dt.$$

A direct estimate shows that $|k(z)| \lesssim |y|$. Thus $k + C|y|$ is a positive harmonic function for a sufficiently large C, and so the representation for g follows from the canonical representation for positive harmonic functions in a half-plane [4, p. 7].

To prove the bound $b > -D^-(m)$, assume to the contrary that $b \le -D^-(m)$. Choose an $\varepsilon > 0$. Then for all sufficiently large R there exists x such that

$$\int_{x-R}^{x+R} (m(t)+b)dt \le \varepsilon R.$$

We may assume for convenience that $x = 0$. We apply Green's theorem,

$$\int_\Omega (v\Delta u - u\Delta v)dA(z) = \int_{\partial\Omega} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds,$$

with $u(z) = g(z) - c|y|$, $v(z) = R^2 - |z|^2$, and Ω the half disk $|z| < R$, $\Im z > 0$. We obtain from this

$$\left| \int_{-R}^{R} (m_0(t) - m(t) - b)(R^2 - t^2)dt \right| \lesssim R^2.$$

However, this integral can be estimated directly. Set

$$d(x) = \int_0^x (m_0(t) - m(t) - b)dt.$$

Then rewriting the integral and integrating by parts, we get

$$\int_{-R}^{R} (m_0(t) - m(t) - b)(R^2 - t^2)dt = \int_0^R (d'(t) + d'(-t))(R^2 - t^2)dt$$

$$= 2\int_0^R (d(t) - d(-t))tdt \geq \inf_t m_0(t)\frac{4}{3}R^3 - \varepsilon R^3 \gtrsim R^3$$

for sufficiently small ε. We have reached a contradiction, and the proof is finished.

6. Connection with classical Paley-Wiener spaces; Sampling, interpolating, and uniqueness sequences

In this section, we consider weighted L^p spaces as well, as they may be included in the discussion at no extra cost.

Assume we are given a subharmonic function ω as described in Theorem 2, with Laplacian supported by the real axis and Riesz measure $m(x)dx$, $m \simeq 1$, and with $b = 0$. We define

$$\|f\|_{\omega,p}^p = \int_{-\infty}^{\infty} |f(x)|^p e^{-p\omega(x)}\, dx$$

for $p < \infty$, and $\|f\|_{\omega,\infty} = \sup_{x\in\mathbb{R}} |f(x)|e^{-\omega(x)}$. We denote by L^p_ω $(0 < p \leq \infty)$ the space of entire functions f for which $\|f\|_{\omega,p} < \infty$ and such that $\log|f(z)| \leq C_\varepsilon + \omega(z) + \varepsilon|z|$ for all $\varepsilon > 0$. We say that L^p_ω is a *weighted Paley-Wiener space*. This definition is in accordance with our previous definition of a weighted Paley-Wiener space when $p = 2$. The L^p version of the Plancherel-Pólya inequality [10, p. 50] shows that L^p_ω is a Banach space when $1 \leq p \leq \infty$, and a complete metric space when $p < 1$. With this notation, the classical Paley-Wiener spaces are $L^p_{\tau|\Im z|}$, where $\tau > 0$ is the maximal type of functions in the space.

We will now prove Theorem 3. We begin by choosing $\tau > \sup_{x\in\mathbb{R}} m(x)$ so that $u = \tau|\Im z| - \omega$ is a subharmonic function with Riesz measure $\mu(x)dx$, $\mu \simeq 1$. Applying Lemma 10 to u, we get an entire function F satisfying $F \simeq e^u$ outside the vicinity of the zeros of F. Then clearly $f \in L^p_\omega$ if and only if $fF \in L^p_{\tau|\Im z|}$, and $\|f\|_{\omega,p} \simeq \|fF\|_p$. In other words, if we denote the zero set of F by Σ and the closed subspace of $L^p_{\tau|\Im z|}$ consisting of all functions $g \in L^p_{\tau|\Im z|}$ that vanish on Σ by $L^p_{\tau|\Im z|}(\Sigma)$, then L^p_ω may be associated with $L^p_{\tau|\Im z|}(\Sigma)$ through the transformation $f \mapsto fF$. Thus we have proved the following L^p-version of Theorem 3:

Theorem 3′. *To every weighted Paley-Wiener space L^p_ω, there exist $\tau > 0$, a sequence $\Sigma = \{\xi_k - i\}_{k\in\mathbb{Z}}$ with $\xi_{k+1} - \xi_k \simeq 1$, $\xi_k \in \mathbb{R}$, and an associated entire function F with zero set Σ such that $f(z) \mapsto f(z)F(z)$ is a bijective mapping from L^p_ω to $L^p_{\tau|\Im z|}(\Sigma)$, with norm equivalence $\|f\|_{\omega,p} \simeq \|fF\|_p$.*

Theorem $3'$ enables us to transfer function-theoretic results from classical to weighted Paley-Wiener spaces. We shall give a few examples, one of which will be needed in the proof of Theorem 4.

We begin by describing sampling and interpolation in L^p_ω. Given a sequence $\Gamma = \{\gamma_k\}$ of distinct points $\gamma_k = \alpha_k + i\beta_k$ from $\mathbb{C}$, we set

$$\|f|\Gamma\|^p_{\omega,p} = \sum_k |f(\gamma_k)|^p e^{-p\omega(\gamma_k)}(1+|\beta_k|)$$

for $p < \infty$ and $\|f|\Gamma\|_{\omega,\infty} = \sup_k |f(\gamma_k)|e^{-\omega(\gamma_k)}$, where we permit f to be a function defined on some set containing Γ. (We consider Γ both as a set and as a sequence.) In particular, a sequence of complex numbers (to be thought of as interpolation values) will sometimes be considered as a function on Γ: $a = \{a_k\} = \{a(\gamma_k)\}$. The space of sequences a such that $\|a|\Gamma\|_{\omega,p} < \infty$ will be denoted by ℓ^p_ω.

We say that Γ is

• a *sampling sequence* for L^p_ω if we have

$$\|f|\Gamma\|_{\omega,p} \simeq \|f\|_{\omega,p}$$

for $f \in L^p_\omega$;

• an *interpolating sequence* for L^p_ω if, for every sequence $a \in l^p_\omega$, there exists a solution $f \in L^p_\omega$ to the interpolation problem

$$f(\gamma_k) = a_k \quad \text{for every } \gamma_k \in \Gamma;$$

• a *complete interpolating sequence* for L^p_ω if Γ is both sampling and interpolating for L^p_ω.

Equivalently, Γ is a complete interpolating sequence for L^p_ω if and only if the interpolation problem $f(\gamma_k) = a_k$ has a unique solution $f \in L^p_\omega$ whenever $\|a|\Gamma\|_{\omega,p} < +\infty$. Such sequences exist only when $1 < p < \infty$.

It is convenient to introduce the following distance function:

$$\rho(z,\zeta) = \frac{|z-\zeta|}{1+|z-\bar{\zeta}|}.$$

We say that Γ is *ρ-separated* if there exists a number $\delta > 0$ such that $\rho(\Gamma_j, \gamma_k) \geq \delta$ whenever $j \neq k$.

Results concerning sampling and interpolation in classical Paley-Wiener spaces can be transferred by means of the following lemma.

Lemma 12. *With F, Σ, and τ as in Theorem* 3 *and $\Gamma \cup \Sigma$ ρ-separated, Γ is sampling for L^p_ω if and only if $\Gamma \cup \Sigma$ is sampling for $L^p_{\tau|\Im z|}$, and Γ is interpolating for L^p_ω if and only if $\Gamma \cup \Sigma$ is interpolating for $L^p_{\tau|\Im z|}$.*

For the proof of Lemma 12, we shall use the following lemma.

Lemma 13. *The sequence Σ from Theorem* $3'$ *is interpolating for each of the spaces* $L^p_{\tau|\Im z|}$, $0 < p \leq \infty$.

Lemma 13 is a consequence of an interpolation theorem of Beurling. We shall comment on its proof below.

Proof of Lemma 12. The following two implications follow directly from Theorem $3'$: If $\Gamma \cup \Sigma$ is sampling (interpolating) for $L^p_{\tau|\Im z|}$, then Γ is sampling (interpolating) for L^p_ω.

Assume now that $\Gamma\cup\Sigma$ is not sampling for $L^p_{\tau|\Im z|}$. Then we can find an $f\in L^p_{\tau|\Im z|}$ of norm 1 which is small on this sequence, i.e., for any $\varepsilon>0$ we can find such an f satisfying $\|f|\Gamma\cup\Sigma\|_{\tau|\Im z|,p}\le\varepsilon$. Since, by Lemma 13, Σ is interpolating for $L^p_{\tau|\Im z|}$, we can add a small function $g\in L^p_{\tau|\Im z|}$ ($\|g\|_p\lesssim\varepsilon$) such that $f+g$ vanishes on Σ. It follows that Γ cannot be sampling for L^p_ω.

Assume finally that Γ is interpolating for L^p_ω. We wish to show that then $\Gamma\cup\Sigma$ is interpolating for $L^p_{\tau|\Im z|}$. Suppose we are given a (the interpolation data on Γ) and b (the interpolation data on Σ). By Lemma 13, we may begin by solving the problem $g(\sigma_j)=b_j$ in $L^p_{\tau|\Im z|}$. We next solve the problem $h(\gamma_k)=(a_k-g(\gamma_k))/F(\gamma_k)$ in L^p_ω. Then $f=g+hF$ is a solution to the interpolation problem on $\Gamma\cup\Sigma$. □

Let us see how the results of [2] and [14] can be interpreted in the weighted case. We need then an extension of Beurling's notion of lower and upper uniform densities. In what follows, I denotes an arbitrary half-open interval of the form $[\alpha,\beta)$. Let μ and ν be two positive Borel measures on $\mathbb{R}$ both being *uniformly locally finite*, which we take to mean that there exist positive constants C_μ and C_ν such that $\mu(I)\le C_\mu|I|$ and $\nu(I)\le C_\nu|I|$ when $|I|\ge 1$. For all $r>0$ set

$$D^-_\mu(\nu,r)=\inf_{|I|=r}\frac{\nu(I)}{\mu(I)}\quad\text{ and }\quad D^+_\mu(\nu,r)=\sup_{|I|=r}\frac{\nu(I)}{\mu(I)}.$$

We observe that

$$D^-_\mu(\nu,s+t)\ge\frac{s}{s+t}D^-_\mu(\nu,s)+\frac{t}{s+t}D^-_\mu(\nu,t)$$

and

$$D^+_\mu(\nu,s+t)\le\frac{s}{s+t}D^+_\mu(\nu,s)+\frac{t}{s+t}D^+_\mu(\nu,t),$$

and, using that the two measures are uniformly locally finite, we deduce from this that the two limits

$$D^-_\mu(\nu)=\lim_{r\to\infty}D^-_\mu(\nu,r)\quad\text{ and }\quad D^+_\mu(\nu)=\lim_{r\to\infty}D^+_\mu(\nu,r)$$

exist.

If μ is the Riesz measure of a subharmonic function ω, we write D_ω instead of D_μ and we just drop the subindex μ when μ is Lebesgue measure. Also when $\nu=n_\Gamma$ is the counting measure of a sequence $\Gamma\subset\mathbb{R}$ we write $D_\mu(\Gamma)$ instead of $D_\mu(n_\Gamma)$. In particular, $D^+(\Gamma)$ and $D^-(\Gamma)$ are the classical upper and lower Beurling densities.

Suppose Γ is a ρ-separated sequence of complex numbers, and let A be a positive number. Define accordingly the following real sequence:

$$\Gamma(A)=\{\alpha_k:\ \alpha_k+i\beta_k\in\Gamma,|\beta_k|<A\}.$$

With the same abuse of notation, we define

$$D^-_\omega(\Gamma)=\lim_{A\to\infty}D^-_\omega(\Gamma(A))\quad\text{ and }\quad D^+_\omega(\Gamma)=\lim_{A\to\infty}D^+_\omega(\Gamma(A)).$$

We say that Γ satisfies the *two-sided Carleson condition* if for any disk D centered on the real line, we have

$$\sum_{\gamma_k\in D\cap\Gamma}|\Im\gamma_k|\le Cr(D),$$

where $r(D)$ is the radius of D and C is independent of D.

Beurling's classical theorems on real sampling and interpolating sequences [2] have recently been extended by Ortega-Cerdà and Seip [14]:

Theorem C. *A sequence Γ is sampling for $L^\infty_{\tau|\Im z|}$ if and only if it contains a ρ-separated subsequence Γ' satisfying $D^-(\Gamma') > \tau/\pi$.*

Theorem D. *A sequence Γ is interpolating for $L^\infty_{\tau|\Im z|}$ if and only if it is ρ-separated, satisfies the two-sided Carleson condition, and $D^+(\Gamma) < \tau/\pi$.*

Using Lemma 12, as well as the fact that

$$\left| n_{\Sigma_A}(I) - \int_I m(x)dx \right| \lesssim 1$$

for $A > 1$, we obtain from these theorems the following corollary:

Theorem 5. *A sequence Γ is sampling for L^∞_ω if and only if it contains a ρ-separated subsequence Γ' satisfying $D^-_\omega(\Gamma') > 1$. A sequence Γ is interpolating for L^∞_ω if and only if it is ρ-separated, satisfies the two-sided Carleson condition, and $D^+_\omega(\Gamma) < 1$.*

Note that since $D^+(\Sigma) < \tau/\pi$ and Σ is separated, Lemma 13 is a consequence of the sufficiency part of Beurling's original density theorem.

When $p < \infty$, it is clear that a sampling sequence must satisfy the two-sided Carleson condition. If this additional restriction is put on Γ, both results remain valid if $p = \infty$ is replaced by $p \leq 1$ (see [5]). However, for $1 < p < \infty$, only the sufficiency parts remain true. In this case, there exist sampling[2] and interpolating sequences Γ satisfying respectively $D^-_\omega(\Gamma) = 1$ and $D^+_\omega(\Gamma) = 1$. In particular, there exist complete interpolating sequences, the description of which is our next aim. From now on, we assume $1 < p < \infty$.

Our next theorem generalizes the main theorem of [12]. We should add that the problem of describing complete interpolating sequences for $L^2_{\tau|\Im z|}$ was investigated for the first time in [16]. We refer to [12] and [8] for the extensive history of the subject.

We need the classical Muckenhoupt (A_p) condition for a positive weight $v(x) > 0,\ x \in \mathbb{R}$:

$$\sup_I \left\{ \left(\frac{1}{|I|} \int_I v dx \right) \left(\frac{1}{|I|} \int_I v^{-\frac{1}{p-1}} dx \right)^{p-1} \right\} < \infty,$$

where I ranges over all intervals on the real line. The celebrated Hunt-Muckenhoupt-Wheeden theorem [3] asserts that the latter condition is necessary and sufficient for boundedness of the classical Hilbert operator

$$\mathcal{H} : f \mapsto (\mathcal{H}f)(t) = \frac{1}{i\pi} \int \frac{f(\tau)}{t - \tau} d\tau$$

[2]After this paper was written, a complete description of the real sampling sequences for the classical Paley-Wiener space $L^2_{\tau|\Im z|}$ was given by Ortega-Cerdà and Seip [15]. Their methods rely on de Branges' theory and lead to a corresponding description of the real sampling sequences for each weighted Paley-Wiener space.

on the weighted space consisting of all functions f satisfying

$$\|f\|_{w,p}^p := \int |f(t)w(t)|^p dt < \infty,$$

with $v = w^p$. It is this fact which underlies Theorem G below.

Suppose that Γ is a complete interpolating sequence for L^p_ω. It may be that $0 \in \Gamma$, in which case we assume that $\gamma_0 = 0$. The *generating function* of the sequence Γ is defined as

$$S(z) = (z - \gamma_0) \lim_{R\to\infty} \prod_{\substack{|\gamma_k|<R \\ k\neq 0}} (1 - \frac{z}{\gamma_k}) \exp\left\{ z \int_R^R \frac{m(x)(1 - \chi_{[-1,1](x)})dx}{x} \right\}. \tag{22}$$

The main theorem of [12] may now be stated as follows.

Theorem E. *A sequence $\Gamma = \{\gamma_k\}$ of distinct complex numbers $\gamma_k = \alpha_k + i\beta_k$ is a complete interpolating sequence for $L^p_{\tau|\Im z|}$, $1 < p < \infty$, if and only if the following three conditions are met:*

(i) Γ *is ρ-separated and satisfies the two-sided Carleson condition.*
(ii) (22) *converges compactwise to an entire function S of exponential type τ.*
(iii) $\{|S(x)|/\operatorname{dist}(x, \Gamma)\}^p$ $(x \in \mathbb{R})$ *satisfies the (A_p) condition.*

We obtain from this:

Theorem 6. *A sequence $\Gamma = \{\gamma_k\}$ of distinct complex numbers $\gamma_k = \alpha_k + i\beta_k$ is a complete interpolating sequence for L^p_ω if and only if the following three conditions are met:*

(i) Γ *is ρ-separated and satisfies the two-sided Carleson condition.*
(ii) (22) *converges compactwise to an entire function S of ω-type* 0.
(iii) $\{|S(x)|e^{-\omega}/\operatorname{dist}(x, \Gamma)\}^p$ $(x \in \mathbb{R})$ *satisfies the (A_p) condition.*

Proof. For general ω, Lemma 12 implies that Γ is a complete interpolating sequence for L^p_ω if and only if Γ is a complete interpolating sequence for $L^p_{\tau|\Im z|}$. Rewriting the corresponding condition on SF, replacing F by $e^{-\omega}$, we obtain Theorem 6 as a direct consequence of Theorem E. $\square$

Theorem 6 plays a crucial role in the proof of Theorem 4.

We end this section by proving a uniqueness theorem. This is done by interpreting Beurling-Malliavin densities in the L^p_ω setting.

Given a class K of entire functions, we say that a sequence of complex numbers $\Gamma = \{\gamma_k\}$ (counting multiplicities in the usual way) is a *uniqueness sequence for* K if there is no nontrivial function $f \in K$ vanishing on Γ. We are not able to characterize the uniqueness sequences for the individual spaces L^p_ω, but we shall get as close as the Beurling-Malliavin theorem permits us to get. Denote by $\mathcal{C}^-_\omega$ the collection of all entire functions of negative ω-type (see the Introduction for the definition of ω-type); it is clear that $\mathcal{C}^-_\omega$ contains all the spaces $L^p_{\omega-\varepsilon|\Im z|}$, $\varepsilon > 0$. We will assume that the Blaschke condition is fulfilled, i.e.,

$$\sum_k \left| \Im \frac{1}{\gamma_k} \right| < \infty, \tag{23}$$

because otherwise it is plain that Γ is a uniqueness sequence for all classes $\mathcal{C}^-_\omega$.

The Beurling-Malliavin density can be defined in many ways. We shall use the following definition from [9]. A partition $\{I_k\}_{k\in\mathbb{Z}}$ of the real line into intervals $I_k = [x_k(1-a_k), x_k(1+a_k))$ is said to be *fine* if $|x_k| \to \infty$ when $|k| \to \infty$, and

$$\sum_k a_k^2 < \infty.$$

Let μ and ν denote two positive Borel measures on $\mathbb{R}$. The *Beurling-Malliavin density* $\varrho_\mu(\nu)$ of ν with respect to μ is defined as the infimum of the numbers a for which there exists a fine partition $\{I_k\}$ such that

$$\nu(I_k) \le a\mu(I_k)$$

for all $k \in \mathbb{Z}$. If no such a exists, we set $\varrho_\mu(\nu) = \infty$. Here we have extended the usual Beurling-Malliavin density somewhat: In the classical situation, μ is just ordinary Lebesgue measure. In what follows, we are mainly interested in the case that $\mu = \mu_\omega$, where μ_ω is the absolutely continuous (Riesz) measure associated with ω:

$$\mu_\omega(S) = \int_S m(x)dx.$$

We will permit ourselves the same abuse of notation as we did when considering lower upper uniform densities: When $\mu = \mu_\omega$, we write $\varrho_\mu = \varrho_\omega$. Also, if Γ is a sequence of (not necessarily distinct) points from $\mathbb{R}$ and ν is the associated counting measure, we write $\varrho_\mu(\nu) = \varrho_\mu(\Gamma)$. If μ is ordinary Lebesgue measure, we drop the subscript and let $\varrho(\nu)$ denote the Beurling-Malliavin density of ν with respect of μ.

To deal with complex sequences, we map complex points γ_k from $\Gamma = \{\gamma_k\}$ to the real line according to the rule $1/\gamma_k^* = \Re(1/\gamma_k)$; thus points on the real line remain fixed while points on the imaginary line are thrown to infinity. This mapping carries those points γ_k with nonzero real part into a real sequence $\Gamma^* = \{\gamma_k^*\}$. The classical Beurling-Malliavin theorem may now be stated as follows:

Theorem F ([3]). *A sequence Γ satisfying the Blaschke condition* (23) *is a uniqueness sequence for $\mathcal{C}^-_{\tau|\Im z|}$ if and only if $\varrho(\Gamma^*) \ge \tau/\pi$.*

We obtain from it the following corollary:

Corollary 1. *A sequence satisfying the Blaschke condition* (23) *is a uniqueness sequence for $\mathcal{C}^-_\omega$ if and only if $\varrho_\omega(\Gamma^*) \ge 1$.*

Proof. We check that the theorem is a direct consequence of Theorem 3′ and Theorem F. If Σ and τ are as in Theorem 3′, it is clear that Γ is a uniqueness sequence for $\mathcal{C}^-_\omega$ if and only if $\Gamma \cup \Sigma$ is a uniqueness sequence for $\mathcal{C}^-_{\tau|\Im z|}$. Clearly, if Γ satisfies the Blaschke condition, so does $\Gamma \cup \Sigma$, and then Γ is a uniqueness sequence for $\mathcal{C}^-_{\tau|\Im z|}$ if and only if $\varrho_{\tau|\Im z|}(\Gamma^* \cup \Sigma^*) \ge 1$. In other words, it is enough to prove the identity

$$\varrho_{\tau|\Im z|}(\Gamma^* \cup \Sigma^*) = \varrho_\omega(\Gamma^*).$$

We denote the counting measures of Γ^* and $\Gamma^* \cup \Sigma^*$ by n_{Γ^*} and $n_{\Gamma\cup\Sigma^*}$, respectively, and note that the identity follows from the estimate

$$\left| n^*_\Gamma(I) + \int_I (\tau - m(t))dt - n_{\Gamma^*\cup\Sigma^*}(I) \right| \lesssim 1;$$

the latter estimate is a direct consequence of the construction of Σ. □

It is of interest to see how the uniform densities are related to the Beurling-Malliavin density. To this end, set $\omega_t(x) = \omega(x-t)$ and $\Gamma_t = \{\gamma_k - t\}_k$; then by a theorem of Beurling [2, p. 345] and Theorem 5, we have the relation

$$D_\omega^-(\Gamma) = \lim_{A\to\infty} \inf_{t\in\mathbb{R}} \varrho_{\omega_t}(\Gamma_t(A)),$$

provided Γ is ρ-separated. This relation makes precise the intuition that a sampling sequence is a uniform uniqueness sequence. On the other hand, if Γ satisfies the two-sided Carleson condition, we have

$$D_\omega^+(\Gamma) = \lim_{A\to\infty} \sup_{t\in\mathbb{R}} \varrho_{\omega_t}(\Gamma_t(A)),$$

which expresses that an interpolating sequence is "essentially" a uniform non-uniqueness sequence.

We finally note that the Beurling-Malliavin density can be used to relate two different weighted Paley-Wiener spaces in the following way:

Corollary 2. *The set of uniqueness sequences for $\mathcal{C}_{\omega_1}^-$ is contained in the set of uniqueness sequences for $\mathcal{C}_{\omega_2}^-$ if and only if $\varrho_{\omega_1}(\mu_{\omega_2}) \le 1$.*

The proof can be done by appropriate manipulations with fine partitions. However, it is probably most easily done by an appeal to a different formulation of the Beurling-Malliavin theorem, e.g., Theorem 2.1 of [9]. We omit the details.

7. Proof and discussion of Theorem 4

We shall make use of the following remarkable theorem of de Branges [4, p. 55].

Theorem G. *Let $H(E)$ be a de Branges space and φ a phase function associated with E. Suppose α is a real number and let $\Gamma = \{\gamma_k\}$ be the sequence of real numbers such that $\phi(\gamma_k) = \alpha + k\pi$, $k \in \mathbb{Z}$. Then if $e^{i\alpha}E - e^{-i\alpha}E^* \notin H(E)$, the normalized reproducing kernels $K(\gamma_k, z)/M(\gamma_k)$ constitute an orthonormal basis for $H(E)$; $e^{i\alpha}E - e^{-i\alpha}E^* \in H(E)$ holds for at most one α, modulo π.*

We shall see that if $H(E)$ is a weighted Paley-Wiener space, then $e^{i\alpha}E - e^{-i\alpha}E^*$ is never in $H(E)$, independent of α.

We begin by noting that the norms of L^2_ω and $H(E)$ are equivalent if $L^2_\omega = H(E)$ in the sense of sets, by the closed graph theorem. This is so because the identity map $f \mapsto f$ from L^2_ω to $H(E)$ is closed.

We prove the necessity of the five conditions. The necessity of (i) is obvious, and so is that of (ii) because it expresses the equivalence between the two majorants along the real line. The necessity of (iii) and (iv) follows from Lemma 1.

To prove the necessity of (v), observe that along with (i), (iii), and (iv), it says that the zero sequences of $1 \pm B_E$ both constitute complete interpolating sequences for L^2_ω. So if we can prove that these zero sequences are complete interpolating sequences for $H(E)$, we have proved the necessity of (v).

By Theorem G, we know that the zero sequence of at least one of the functions $1 \pm B_E$ is a complete interpolating sequence for $H(E)$. To prove that both zero sequences have this property, we need to show that none of the functions are in $L^2(\mathbb{R})$. We begin by noting that

$$\|1 \pm B_E\|_2^2 \gtrsim \int_{\mathbb{R}} \sigma(x)dx.$$

This is so because by (iv) there exists in either case an increasing sequence $x_k^\pm, k \in \mathbb{Z}$, such that $x_{k+1}^\pm - x_k^\pm \simeq 1$, and $|1 \pm B_E(x_k^\pm)| = 2$. Also by (iv), $\varphi'(x) \gtrsim 1/\sigma(x)$ so that

$$\int_{|x_k^\pm - t| < \sigma(x_k^\pm)} |1 \pm B_E(t)|^2 dt \gtrsim \sigma(x_k^\pm).$$

Summing over k, we obtain the desired estimate. But using (ii) and the fact that $E(z)/(z - \Gamma_0) \in H(E)$, we obtain

$$\int \frac{1}{\sigma(x)(1+x^2)} dx < \infty,$$

which implies that σ is not integrable.

To prove the converse implication, i.e., the sufficiency of the conditions (i)–(v), we begin by observing that (i)–(iv) along with Lemma 11 imply that $L^2_\omega \subset H(E)$. By Theorem 6, at least one of the two zero sequences $1 \pm B_E$ is a complete interpolating sequence for $H(E)$. Also, as above, (i)–(v) ensure that the same sequence is a complete interpolating sequence for L^2_ω. Since the majorants of the two spaces are equivalent for real x by (ii), this proves that $L^2_\omega = H(E)$.

The following four remarks shed some light on the contents of Theorem 4.

Remark 1. If all the zeros of E lie in a horizontal strip, then condition (v) of Theorem 4 can be replaced by the simpler and more convenient condition that σ be an (A_2) weight.

Let us prove this statement. Observe first that we may assume that the zeros of E are hyperbolically separated: Due to (iii) and (iv) of Theorem 4, we may distort the zeros of E vertically without changig the space $H(E)$ and in such a way that the zeros become hyperbolically separated. Now since the zeros of E are hyperbolically separated and because of (iii) and (iv) of Theorem 4, the zero sequence of E is a complete interpolating sequence for $H(E)$. This follows from a standard duality argument, along with Carleson's interpolation theorem, and the fact that the zeros of E constitute a set of uniqueness for $H(E)$.

Now if $H(E) = L^2_\omega$, it follows that Γ is a complete interpolating sequence for L^2_ω. By (ii) of Theorem 4 and (iii) of Theorem 6, σ is an (A_2) weight. Conversely, if σ is an (A_2) weight, then by (i) and (ii) of Theorem E and Theorem 6, Γ is a complete interpolating sequence for L^2_ω. We use Lemma 11 as above to check that $L^2_\omega \subset H(G)$, and check that the majorants of the two spaces are equivalent at Γ. It follows that $H(E) = L^2_\omega$.

Remark 2. If $\sigma(x) \to 0$ when $|x| \to \infty$ and $H(E) = L^2_\omega$, then all the zeros of E lie in a horizontal strip. For suppose there exists a sequence of indices k_j such that $\eta_{k_j} \to \infty$. We have from Lemma 9 that

$$\omega(x+iy) = \log|E(x+i|y|)| + u(x+iy),$$

where

$$\left| u(x+iy) - \frac{|y|}{\pi} \int \frac{\log \sigma(t)}{(x-t)^2 + y^2} dt \right| \lesssim 1.$$

Thus the majorant M of L^2_ω satisfies

$$M^2(\Gamma_{k_j}) \simeq \frac{|E(\xi_{j_k} + i\eta_{k_j})|^2 \exp(u(\xi_{j_k} + i\eta_{k_j}))}{\eta_{k_j}}.$$

On the other hand, since $L^2_\omega = H(E)$, we also have

$$M^2(\Gamma_{k_j}) \simeq \frac{|E(\xi_{j_k} + i\eta_{k_j})|^2}{\eta_{k_j}},$$

so that

$$\left| \frac{\eta_{k_j}}{\pi} \int \frac{\log \sigma(t)}{(\xi_{k_j} - t)^2 + \eta_{k_j}^2} dt \right| \lesssim 1.$$

But this is impossible if both $\eta_{k_j} \to \infty$ and $\sigma(t) \to 0$ when $|t| \to \infty$.

Remark 3. Let us consider some concrete E such that $H(E) = L^2_{\pi|\Im z|}$ but $|E(x)| \not\simeq 1$. (There are many such E.) Fix $\delta > 0$ and set

$$E_\delta(z) = (z+i) \prod_{k=1}^{\infty} \left(1 - \frac{z}{k - \delta - ik^{-4\delta}}\right) \left(1 - \frac{z}{-k + \delta - ik^{-4\delta}}\right);$$

the zero set of E_δ is denoted by Λ_δ. A direct estimate of the infinite product yields

$$|E_\delta(x)| \simeq (1 + |x|)^{2\delta} \operatorname{dist}(x, \Lambda_\delta).$$

We also have

$$\sigma(x) \simeq (1 + |x|)^{-4\delta} \qquad \text{and} \qquad \varphi'(x) = \frac{\sigma(x)}{\operatorname{dist}(x, \Lambda_\delta)^2}.$$

In this case, it is immediate that (i)–(iv) of Theorem 4 hold, with $\omega(z) = \pi|\Im z|$. To meet (v) of Theorem 4 as well, we need (by Remark 1) σ to be an (A_2) weight. Thus $H(E_\delta) = L^2_{\pi|\Im z|}$ if and only if $0 \le \delta < 1/4$.

Remark 4. Let L^2_ω be a weighted Paley-Wiener space, and suppose $E \in \overline{HB}$ is chosen so that $L^2_\omega = H(E)$. Then as observed above $B(z) = E^*(z)/E(z)$ is an inner function in $\mathbb{C}^+$, and it is seen that the mapping $f \mapsto f/E$ transforms $H(E)$ isometrically onto $K_\Theta = H^2 \ominus \Theta H^2$, where H^2 denotes the Hardy space H^2 of the upper half-plane $\mathbb{C}^+$. Thus Theorem 4 can be interpreted as a theorem about norm equivalence in so-called model spaces K_Θ for certain special inner functions Θ. The most comprehensive study of norm equivalence in model spaces can found in [17], where a necessary and sufficient condition is obtained when the weight function (corresponding to E/e^ω in our case) is bounded. This is in contrast to Theorem 4, in which the most interesting case occurs when E/e^ω is unbounded on $\mathbb{R}$. Thus, in a sense, Theorem 4 is complementary to [17, Theorem 2].

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