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WEIGHTED PARTITIONS OF SPHERE MEASURES BY HYPERPLANES

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Abstract. In this paper we use the CS / TM scheme stated in [1] for the V.V. Makeev equipartition problem [7] to prove the existence of new weighted partitions. For the first time computations of the associated equivariant problem are done by codes written in the package Mathematica 5.0 [2].

1. Statement of the main result

Let H_1, H_2 and H_3 be planes in $\mathbb{R}^3$ through the origin. They are in the *fan position* if $H_1 \cap H_2 = H_1 \cap H_3 = H_2 \cap H_3$. Planes in the fan position cut the sphere S^2 in six parts $\sigma_1, .., \sigma_6$ which can be naturally oriented up to a cyclic permutation.

A measure μ is the *proper measure* μ if $\mu([a,b]) = 0$ for any circular arc $[a,b] \subset S^2$, and $\mu(U) > 0$ for each nonempty open set $U \subset S^2$.

We prove the existence of the following measure partitions.

Theorem 1. *Let μ be a proper Borel probability measure on the sphere S^2. Then there are three planes in the fan position such that the ratio of measure μ in angular sectors cut by planes is*

(A) $(1,1,2,1,1,2)$ *(C)* $(1,1,3,1,1,3)$

This result is a generalization of the Makeev result [7].

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2. Configuration space / Test map scheme

We use methods introduced in [1], and briefly review them. The CS / TM scheme is the standard method for solving measure partition problems.
A **k-fan** $(l; H_1, H_2, \ldots, H_k)$ in $\mathbb{R}^3$ is formed of an oriented line l through the origin and k closed half planes $H_1, H_2, \ldots, H_k$ which intersect along l. The point x on the sphere S^2 and k great semi circles emanating from x is equally called . Sometimes instead of great semicircles we use open angular sectors σ_i between l_i and l_{i+1} or tangent vectors t_i on l_i, $i = 1, \ldots, k$. The space of all k-fans in $\mathbb{R}^3$ or on the sphere S^2 is denoted by F_k.
The configuration space for a proper Borel probability measure μ on S^2 is given by

$$X_{\mu,n} = \{(x; t_1, \ldots, t_n) \in F_n \mid (\forall i = 1, \ldots, n)\, \mu(\sigma_i) = \tfrac{1}{n}\} \cong V_2(\mathbb{R}^3).$$

The test map. Fix a six-tuple $(\alpha_1, \alpha_2, \alpha_3, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^6$ such that $\alpha_1 + \alpha_2 + \alpha_3 = \frac{n}{2}$. Let W_n denote the hyperplane $\{x \in \mathbb{R}^n \mid x_1 + x_2 + \ldots + x_n = 0\}$. The test map $\Phi : X_{\mu,n} \to W_n$ is defined by

$$\Phi((x; t_1, \ldots, t_n)) = (\theta_1 - \tfrac{2\pi}{n}, \ldots, \theta_n - \tfrac{2\pi}{n}),$$

where $\theta_i = \measuredangle(t_i, t_{i+1})$ (assuming $t_{n+1} = t_1$).
The action. The dihedral group $\mathbb{D}_{2n} = \langle j, \varepsilon \mid \varepsilon^n = j^2 = 1,\ \varepsilon j = j\varepsilon^{n-1} \rangle$ acts both on the configuration space $X_{\mu,n}$ and on the hyperplane W_n in such a way that Φ becomes a $\mathbb{D}_{2n}$-map Precisely,

$$\begin{array}{ll} \varepsilon(x; t_1, \ldots, t_n) = (x; t_n, t_1, \ldots, t_{n-1}) & \varepsilon(x_1, \ldots, x_n) = (x_2, \ldots, x_n, x_1) \\ j(x; t_1, \ldots, t_n) = (-x; t_1, t_n, \ldots, t_2) & j(x_1, \ldots, x_n) = (x_n, \ldots, x_2, x_1) \end{array},$$

for $(x; t_1, \ldots, t_n) \in X_\mu$ and $(x_1, \ldots, x_n) \in W_n$.
The test space. The test space is the union $\bigcup \mathcal{A}_n \subset W_n$ of the smallest $\mathbb{D}_{2n}$-invariant arrangement $\mathcal{A}_n$, which contains the linear subspace $L \subset W_n$ given by equalities

$$x_1 + \ldots + x_{\frac{n}{2}} = x_{\alpha_1+1} + \ldots + x_{\alpha_1+\frac{n}{2}} = x_{\alpha_1+\alpha_2+1} + \ldots + x_{\alpha_1+\alpha_2+\frac{n}{2}} = 0.$$

The following basic proposition of the CS / TM scheme holds.

Proposition 2. *Let $(\alpha_1, \alpha_2, \alpha_3, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^6$ be such that $\alpha_1 + \alpha_2 + \alpha_3 = \frac{n}{2}$. If there is no $\mathbb{D}_{2n}$-equivariant map $V_2(\mathbb{R}^3) \to W_n \setminus \bigcup \mathcal{A}_n$ then for every proper Borel probability measure on the sphere S^2 there exist three planes in the fan position such that*

$$(\forall i \in \{1, .., 6\})\ \mu(\sigma_i) = \tfrac{\alpha_i}{n}.$$

As in the [1], the problem of the existence of the $\mathbb{D}_{2n}$-equivariant map $V_2(\mathbb{R}^3) \to W_n \setminus \bigcup \mathcal{A}_n$ can be substituted with the following equivalent problem.

Proposition 3. *The following maps exist or do not exist together:*

$$\mathbb{D}_{2n}\text{-}map \quad V_2(\mathbb{R}^3) \to W_n \setminus \bigcup \mathcal{A}_n \qquad and \quad \mathbb{Q}_{4n}\text{-}map \quad S^3 \to W_n \setminus \bigcup \mathcal{A}_n.$$

Therefore, to prove Theorem 1 it is enough to prove the following statement.

Theorem 4. *There is no $\mathbb{Q}_{4n}$-map $S^3 \to W_n \setminus \bigcup \mathcal{A}_n$, where $\mathcal{A}_n$ is the minimal $\mathbb{Q}_{4n}$(=$\mathbb{D}_{2n}$) arrangement containing subspace L defined*

(A) for $n = 8$ and $(\alpha_1, \alpha_2, \alpha_3, \alpha_1, \alpha_2, \alpha_3) = (1, 1, 2, 1, 1, 2)$, by equalities:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= x_2 + x_3 + x_4 + x_5 \\ &= x_3 + x_4 + x_5 + x_6; \end{aligned}$$

(B) for $n = 10$ and $(\alpha_1, \alpha_2, \alpha_3, \alpha_1, \alpha_2, \alpha_3) = (1, 1, 3, 1, 1, 3)$, by equalities:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= x_2 + x_3 + x_4 + x_5 + x_6 \\ &= x_3 + x_4 + x_5 + x_6 + x_7 = 0. \end{aligned}$$

3. Proof of Theorem 4

3.1. **The primary obstruction problem.** Both problems of the existence of $\mathbb{Q}_{32}$ and $\mathbb{Q}_{40}$ maps, respectively,

$$S^3 \to W_8 \setminus \bigcup \mathcal{A}_8 \text{ and } S^3 \to W_{10} \setminus \bigcup \mathcal{A}_{10}$$

are problems depending only on the primary obstruction. Indeed, the sphere S^3 is free 3-dimensional complex while both complements are 1-connected spaces. Therefore the relevant obstruction elements live, respectively, in the following equivariant cohomology groups

$$\mathfrak{H}^3_{\mathbb{Q}_{32}}(X, H_2(W_8 \setminus \bigcup \mathcal{A}_8, \mathbb{Z})) \text{ and } \mathfrak{H}^3_{\mathbb{Q}_{40}}(X, H_2(W_{10} \setminus \bigcup \mathcal{A}_{10}, \mathbb{Z})).$$

To identify the obstruction element, as in the papers [4],[1],[3] we use general position map scheme. Briefly, the procedure can be divided in four steps.

(1) *The sphere S^3 become a free $\mathbb{Q}_{4n}$ cell complex.* The description of concrete $\mathbb{Q}_{4n}$ cell structures of sphere S^3 can be found in [5] pp. 250-254,[3] and [4].

(2) *Definition of a general position $\mathbb{Q}_{4n}$-map $f : S^3 \to W_n$* (in the respect of the arrangement $\mathcal{A}_n$). The requirement of the general position means that:

(a) the 2-skeleton image does not intersect the arrangement $\bigcup \mathcal{A}_n$,

(b) intersection $f(S^3) \cap \bigcup \mathcal{A}_n$ is finite,

(c) all intersections of $f(S^3)$ and $\bigcup \mathcal{A}_n$ are transversal,

(d) $f^{-1}\left(f(S^3) \cap \bigcup \mathcal{A}_n\right) \subset \bigcup_{e \in S^3_{(3)}} \operatorname{relint}(e)$.

(3) *Computing the obstruction cocycle*

$$\mathfrak{O}_{\mathbb{Q}_{4n}}(f) \in \mathfrak{C}^3_{\mathbb{Q}_{4n}}(X, H_2(W_n \setminus \bigcup A_n, Z)).$$

The obstruction cocycle is computed via "intersection counting" formula

$$\mathfrak{O}_{\mathbb{Q}_{4n}}(f)(e) = \sum_{x \in f^{-1}(f(e) \cap (\bigcup \mathcal{A}_n))} \mathrm{I}(e, L_{f(x)}) \, \|f(x)\| \tag{1}$$

where e is a 3-cell of S^3. Here $\mathrm{I}(e, L_{f(x)})$ denotes the appropriate intersection number, while class $\|f(x)\|$ is a point or a broken point class. For notion of point and broken point classes consult [4], [3].

(4) *Describing of the obstruction cocycle in* $\mathfrak{H}^3_{\mathbb{Q}_{4n}}(X, H_2(W_n \setminus \bigcup A_n, Z))$. As the contrast to the paper [1] the identification of the obstruction element becomes much more difficult. The following decomposition and its geometric interpretation will be of the outmost importance for the testing whether the obstruction element vanishes or not.
Assume that the obstruction cocycle $\mathfrak{O}_{\mathbb{Q}_{4n}}(f)$ is computed. Since there is an isomorphism

$$\mathfrak{H}^3_{\mathbb{Q}_{4n}}(X, H_2(W_n \setminus \bigcup \mathcal{A}_n, Z)) \cong H_2(W_n \setminus \bigcup \mathcal{A}_n; \mathbb{Z})_{\mathbb{Q}_{4n}}$$

the objective is to identify the obstruction element inside the group of coinvariants $H_2(W_n \setminus \bigcup \mathcal{A}_n; \mathbb{Z})_{\mathbb{Q}_{4n}}$. Particularly, when we prove that the obstruction element does not vanish, we are not compelled to completely describe the obstruction element. The isomorphism (assuming $\mathbb{Z}$ coefficients)

$$\begin{aligned} H_2(W_n \setminus \bigcup \mathcal{A}_n) &\cong H^{(n-1)-2-1}(\bigcup \widehat{\mathcal{A}_n}) \\ &\cong \mathrm{Hom}(H_{n-4}\left(\bigcup \widehat{\mathcal{A}_n}\right); \mathbb{Z}) \oplus \mathrm{Ext}(H_{n-5}\left(\bigcup \widehat{\mathcal{A}_n}\right), \mathbb{Z}) \end{aligned} \tag{2}$$

(where $\widehat{\mathcal{A}_n}$ denotes the one-point compactification of the arrangement $\mathcal{A}_n$), and the decomposition

$$H_{n-4}(\bigcup \widehat{\mathcal{A}_n}) \cong \bigoplus_{V \in P} H_{n-4}\left(\Delta(P_{<V}) * S^{\dim V}\right) \tag{3}$$

$$\cong \bigoplus_{d=0}^{n-4} \bigoplus_{V \in P : \dim V = d} \tilde{H}_{n-5-d}(\Delta(P_{<V})) \tag{4}$$

allow the use of the homology of the arrangement instead of the appropriate homology of the complement. When the Ext part of the isomorphism (2) vanishes the isomorphism

$$\chi : H_2(W_n \setminus \bigcup \mathcal{A}_n; \mathbb{Z}) \to \mathrm{Hom}(H_{n-4}\left(\bigcup \widehat{\mathcal{A}_n}\right); \mathbb{Z})$$

has the geometric interpretation in the notion of the linking number. One should be very careful in using decompositions (2) and (4). The action of the group does not respect any of above isomorphisms. Particularly, the Poincaré duality map is an equivariant map up to the orientation character.

3.2. **Three steps.** The proof of both cases will follow the stages of the general position map algorithm.

(1) Definition of the general position map $f : S^3 \to W_n$. The sphere S^3 is a $\mathbb{Q}_{4n}$ simplicial complex $P^{(1)}_{2n} * P^{(2)}_{2n}$ where $P^{(i)}_{2n}$ is a simplicial representation of the sphere S^1 as an $2n$-gon. It is enough to define the image of the single vertex t and everything extends equivariantly.

(2) Computation of the singular set, i.e. the intersection of the image of the maximal cell (for details [5], [4], [3],)

$$e = [t, \epsilon t] * [jt, \epsilon jt] \cup [\epsilon t, \epsilon^2 t] * [jt, \epsilon jt] \cup ... \cup [\epsilon^{n-1} t, \epsilon^n t] * [jt, \epsilon jt]$$

and the union of the arrangement $\bigcup \mathcal{A}_n$. Then

$$\mathfrak{O}_{\mathbb{Q}_{4n}}(f)(e) = \sum_{x \in f^{-1}(f(e) \cap (\bigcup \mathcal{A}_n))} \mathrm{I}(e, L_{f(x)}) \, \|f(x)\|$$

(3) Identification of the cohomology class of the obstruction cocycle $\mathfrak{O}_{\mathbb{Q}_{4n}}(f)(e)$ in the group of coinvariants

$$H_2(W_n \setminus \bigcup \mathcal{A}_n; \mathbb{Z})_{\mathbb{Q}_{4n}} \cong H_{n-4}(\bigcup \widehat{\mathcal{A}_n}; \mathbb{Z})_{\mathbb{Q}_{4n}}.$$

The action of the group $\mathbb{Q}_{4n}$ on $H_{n-4}(\bigcup \widehat{\mathcal{A}_n}; \mathbb{Z})$ is defined by

$$(\forall x \in H_{n-4}(\bigcup \widehat{\mathcal{A}_n}; \mathbb{Z})) \ (\forall g \in \mathbb{Q}_{4n}) \ g * x = \deg(g) \ g^{-1} \cdot x$$

where $\cdot$ is the $\mathbb{Q}_{4n}$-action induced by the $\mathbb{Q}_{4n}$-action on the arrangement $\mathcal{A}_n$.

3.3. **Case** $(1,1,2,1,1,2)$**. (1)** Let us define a map $f : S^3 \to W_8$ on the vertex t by $f(t) = (-3, 3, -1, 1, 1, -2, 2, -1)$ and extend it equivariantly. Then $f(jt) = (-1, 2, -2, 1, 1, -1, 3, -3)$.

(2) $\mathcal{A}_8$ is the minimal $\mathbb{Q}_{32}$ arrangement containing the subspace L defined by

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= x_2 + x_3 + x_4 + x_5 = x_3 + x_4 + x_5 + x_6 \\ &= \sum_{i=1}^{8} x_i = 0. \end{aligned}$$

The arrangement $\mathcal{A}_8$ has four maximal elements L, ϵL, $\epsilon^2 L$ and $\epsilon^3 L$. That follows from set equalities $\epsilon^4 L = L$ and $\epsilon^2 L = jL$. The intersection $I = L \cap \epsilon L \cap \epsilon^2 L \cap \epsilon^3 L$ is a subspace of the codimension one in each of subspaces L, ϵL, $\epsilon^2 L$, $\epsilon^3 L$. Thus, the Hasse diagram of the arrangement $\mathcal{A}_8$ is as in the figure 1.

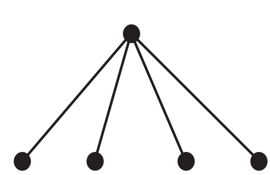

FIGURE 1. The Hasse diagram of the arrangement.

We intersect the f image of the maximal cell

$$\begin{aligned} e \;=\; & \big([t, \epsilon t] \cup [\epsilon t, \epsilon^2 t] \cup [\epsilon^2 t, \epsilon^3 t] \cup [\epsilon^3 t, \epsilon^4 t] \cup [\epsilon^4 t, \epsilon^5 t] \cup [\epsilon^5 t, \epsilon^6 t] \\ & \cup [\epsilon^6 t, \epsilon^7 t] \cup [\epsilon^7 t, \epsilon^8 t]\big) \\ & *[jt, \epsilon jt] \end{aligned}$$

with the test space $\bigcup \mathcal{A}_8 = L \cup \epsilon L \cup \epsilon^2 L \cup \epsilon^3 L$. Results of $8 \times 4 = 32$ intersections

$$f\left([\epsilon^i t, \epsilon^{i+1} t] * [jt, \epsilon jt]\right) \cap \epsilon^r L$$

can be summed in the following table:

		$f\left([\epsilon^i t, \epsilon^{i+1} t] * [jt, \epsilon jt]\right) \cap \epsilon^r L$		
i	**r**	preimage of the intersection point	intersection point in W_8	
1	2	$\frac{3}{7}jt + \frac{1}{14}\epsilon jt + \frac{1}{14}\epsilon t + \frac{3}{7}\epsilon^2 t$	$\left(-\frac{2}{7}, -\frac{2}{7}, -\frac{1}{2}, \frac{15}{14}, -\frac{2}{7}, -\frac{2}{7}, \frac{15}{14}, -\frac{1}{2}\right)$	$= p_1$
2	2	$\frac{20}{51}jt + \frac{23}{153}\epsilon jt + \frac{62}{153}\epsilon^2 t + \frac{8}{153}\epsilon^3 t$	$\left(-\frac{25}{153}, -\frac{1}{3}, -\frac{77}{153}, 1, -\frac{25}{153}, -\frac{1}{3}, \frac{16}{17}, -\frac{4}{9}\right)$	$= p_2$
2	1	$\frac{8}{153}jt + \frac{63}{153}\epsilon jt + \frac{23}{153}\epsilon^2 t + \frac{20}{51}\epsilon^3 t$	$\left(-\frac{77}{153}, -\frac{1}{3}, -\frac{25}{153}, -\frac{4}{9}, \frac{16}{17}, -\frac{1}{3}, -\frac{25}{153} 1, \right)$	$= p_3$
4	1	$\frac{1}{20}jt + \frac{11}{40}\epsilon jt + \frac{1}{10}\epsilon^4 t + \frac{23}{40}\epsilon^5 t$	$\left(\frac{1}{4}, -\frac{1}{5}, \frac{1}{2}, \frac{6}{5}, -\frac{3}{2}, -\frac{1}{5}, \frac{1}{2}, -\frac{11}{20}\right)$	$= p_4$
4	0	$\frac{23}{40}jt + \frac{1}{10}\epsilon jt + \frac{11}{40}\epsilon^4 t + \frac{1}{20}\epsilon^5 t$	$\left(-\frac{3}{2}, \frac{6}{5}, \frac{1}{2}, -\frac{1}{5}, \frac{1}{4}, -\frac{11}{20}, \frac{1}{2}, -\frac{1}{5}\right)$	$= p_5$
5	0	$\frac{1}{3}jt + \frac{1}{6}\epsilon jt + \frac{1}{6}\epsilon^5 t + \frac{1}{3}\epsilon^6 t$	$\left(-\frac{5}{3}, 1, \frac{1}{3}, \frac{1}{3}, 1, -\frac{5}{3}, \frac{1}{3}, \frac{1}{3}\right)$	$= p_6$
7	3	$\frac{1}{14}jt + \frac{3}{7}\epsilon jt + \frac{3}{7}\epsilon^7 t + \frac{1}{14}t$	$\left(\frac{1}{7}, -\frac{25}{14}, \frac{3}{2}, \frac{1}{7}, \frac{1}{7}, \frac{3}{2}, -\frac{25}{14}, \frac{1}{7}\right)$	$= p_7$

These are results of the Mathematica 5.0 code and can be obtained from [2]. Then

$$\mathfrak{O}_{\mathbb{Q}_{32}}(f)(e) = \sum_{i=1}^{7} \alpha_i \, \|p_i\| \tag{5}$$

for some $\alpha_i \in \{1, -1\}$. The obstruction cocycle can be described as in the Figure 2, (A). Orientations which correspond to intersection numbers α_i are not computed and so not indicated in the Figure.

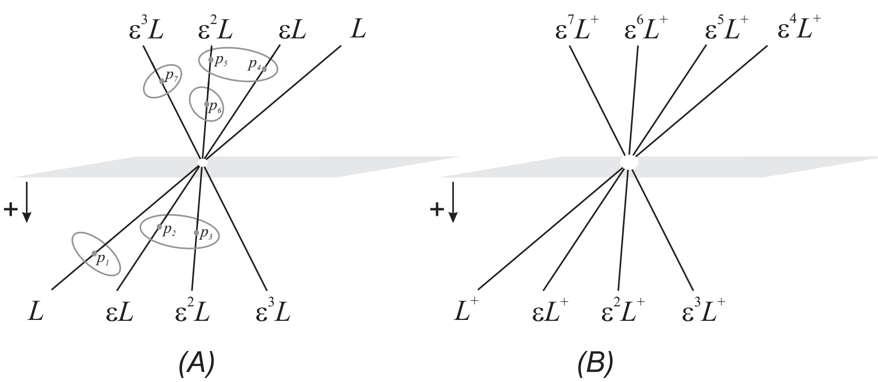

FIGURE 2. The obstruction cocycle.

Let us denote by L^+ and L^- the halfspaces of L defined by

$L^+ = \{x \in L \mid x_4 + x_5 + x_6 + x_7 > 0\}$ and $L^- = \{x \in L \mid x_4 + x_5 + x_6 + x_7 < 0\}$.

The action of the cyclic subgroup generated by ϵ is described in the Figure 2,(B). Following relations hold

$$\epsilon^4 L = L,\ \epsilon^4 L^+ = L^-,\ \epsilon^2 j L = L,\ \epsilon^2 j L^+ = L^+. \tag{6}$$

Moreover, the element ϵ acts on the subspace I by changing its orientation. From these relations one can read following equalities

$$\begin{aligned}\|p_1\| &= \beta_1\|p_2\| = \beta_2\epsilon^{-1}\|p_3\| = \beta_3\epsilon^{-5}\|p_4\| \\ &= \beta_4\epsilon^{-6}\|p_5\| = \beta_5\epsilon^{-6}\|p_6\| = \beta_6\epsilon^{-7}\|p_7\|\end{aligned}$$

where $\beta_i \in \{1,-1\}$.The obstruction element lives in the group of coinvariants $H_2(W_n \setminus \bigcup \mathcal{A}_n;\mathbb{Z})_{\mathbb{Q}_{4n}}$. Therefore, instead of the obstruction cocycle (5) we can use the cohomologues one

$$\mathfrak{O}'_{\mathbb{Q}_{32}}(f)(e) = \rho\|p_1\| \tag{7}$$

where $\rho \in \mathbb{Z}$ is odd.

(3) Let us first determine the ambient where the obstruction element lives.

Lemma 5. *(A)* $H_4(\bigcup\widehat{\mathcal{A}_8};\mathbb{Z}) \cong \mathbb{Z}^4$ *(B)* $H_4(\bigcup\widehat{\mathcal{A}_8};\mathbb{Z})_{\mathbb{Q}_{32}} \cong \mathbb{Z}_8$.

Proof. (A) The statement from Goresky-MacPherson formula 4 can be applied on this arrangement. We are going to do a little bit more. Let k denote an element of $H_4(\bigcup\widehat{\mathcal{A}_8};\mathbb{Z})$ geometrically represented by the union of halfspaces L^+ and ϵL^+. Then the group $H_4(\bigcup\widehat{\mathcal{A}_8};\mathbb{Z})$ is generated by elements

$$k,\ \epsilon\cdot k,\ \epsilon^2\cdot k,\ \epsilon^3\cdot k,\ \epsilon^4\cdot k,\ \epsilon^5\cdot k,\ \epsilon^6\cdot k,\ \epsilon^7\cdot k$$

which satisfy the following relation

$$k-\epsilon\cdot k+\epsilon^2\cdot k-\epsilon^3\cdot k+\epsilon^4\cdot k-\epsilon^5\cdot k+\epsilon^6\cdot k-\epsilon^7\cdot k=0. \tag{8}$$

The alternation of signs comes from the fact that ϵ changes the orientation of the intersection $I = L^+ \cap \epsilon\cdot L^+$.

(B) The relation (8) transforms in the following one for "$*$"-action

$$k+\epsilon * k+\epsilon^2 * k+\epsilon^3 * k+\epsilon^4 * k+\epsilon^5 * k+\epsilon^6 * k+\epsilon^7 * k=0.$$

In "$*$"-coinvariant $H_4(\bigcup\widehat{\mathcal{A}_8};\mathbb{Z})_{\mathbb{Q}_{32}}$ the relation becomes

$$8[k]=0.$$

□

The image of the cocycle (7) in the group $\mathrm{Hom}(H_4(\bigcup\widehat{\mathcal{A}_8};\mathbb{Z}) \cong H_4(\bigcup\widehat{\mathcal{A}_8},\mathbb{Z})$ is identified by the means of the geometric interpretation of the map χ. Let us fix the basis $\{k,\ -\epsilon\cdot k,\ \epsilon^2\cdot k,\ -\epsilon^3\cdot k,\ \epsilon^4\cdot k,\ -\epsilon^5\cdot k,\ \epsilon^6\cdot k\}$ of the group $H_4(\bigcup\widehat{\mathcal{A}_8};\mathbb{Z})$. Then

$$\chi(\|p_1\|)((-1)^i\epsilon^i\cdot k) = \begin{cases} \pm 1 & ,\ i=0 \\ 0 & ,\ otherwise \end{cases}.$$

This implies that $\chi(\|p_1\|)$ can be identified by $\pm k$ and consequently $\mathfrak{O}'_{\mathbb{Q}_{32}}(f)(e)$ can be identified by $\pm\rho k$. After passing to coinvariants we obtain

$$[\mathfrak{O}'_{\mathbb{Q}_{32}}(f)(e)] = \pm\rho[k] \neq 0$$

since ρ is an odd integer and $[k]$ is a generator of the group $H_4(\bigcup\widehat{\mathcal{A}_8};\mathbb{Z})_{\mathbb{Q}_{32}}\cong\ \mathbb{Z}_8$. Thus the obstruction element

$$[\mathfrak{O}_{\mathbb{Q}_{32}}(f)(e)] = [\mathfrak{O}'_{\mathbb{Q}_{32}}(f)(e)]\pm\rho[k] \neq 0$$

is not zero and we have proved the case (A) of the Theorem 4.

3.4. **Case** $(1,1,3,1,1,3)$**.** Since the proof goes in footsteps of the previous case, we just outline the computational parts which differs.
(1) Let $f : S^3 \to W_{10}$ be given by $f(t) = (1 - \frac{1}{10000}, 2, 3, 4, 5, 6, 7, 8, 9, -45 + \frac{1}{10000})$.
(2) The arrangement $\mathcal{A}_{10}$ is now minimal $\mathbb{Q}_{40}$ arrangement containing the subspace L defined by

$$\begin{aligned} x_1+x_2+x_3+x_4+x_5 &= x_2+x_3+x_4+x_5+x_6 = x_3+x_4+x_5+x_6+x_7 \\ &= \sum_{i=1}^{8} x_i = 0. \end{aligned}$$

The arrangement $\mathcal{A}_{10}$ has four maximal elements L, ϵL, $\epsilon^2 L$, $\epsilon^3 L$ and $\epsilon^4 L$. This follows from the set equality $L = \epsilon^5 L$. Let us determine the intersection of the f image of the maximal cell

$$e = \left([t,\epsilon t] \cup [\epsilon t, \epsilon^2 t] \cup [\epsilon^2 t, \epsilon^3 t] \cup\ \dots\ \cup [\epsilon^8 t, \epsilon^9 t]\right) * [jt, \epsilon jt]$$

with the test space $\bigcup\mathcal{A}_{10} = L\cup\epsilon L\cup\epsilon^2 L\cup\epsilon^3 L\cup\epsilon^4 L$. The results of $10\times 5 = 50$ intersections $f\left([\epsilon^i t, \epsilon^{i+1} t] * [jt, \epsilon jt]\right) \cap \epsilon^r L$ can be summed in the following way:

$$f\left([\epsilon^3 t, \epsilon^4 t] * [jt, \epsilon jt]\right) \cap L = \{q_1\},\ f\left([\epsilon^5 t, \epsilon^6 t] * [jt, \epsilon jt]\right) \cap \epsilon L = \{q_2\},$$
$$f\left([\epsilon^5 t, \epsilon^6 t] * [jt, \epsilon jt]\right) \cap \epsilon^2 L = \{q_3\}$$

and consequently

$$\operatorname{card}\left(f(e) \cap \bigcup \mathcal{A}_{10}\right) = 3.$$

The exact coordinates of intersection points as well as barycentric coordinates of its preimages can be found in [2]. Thus

$$\mathfrak{O}_{\mathbb{Q}_{40}}(f)(e) = \alpha_1 \|q_1\| + \alpha_2 \|q_2\| + \alpha_3 \|q_3\| \tag{9}$$

where $\alpha_i \in \{1, -1\}$.
The more precise description of the point classes requires detection of halfspaces which contain intersection points. Let

$$L^+ = \{x \in L \mid x_4 + x_5 + x_6 + x_7 + x_8 > 0\}$$

and

$$L^- = \{x \in L \mid x_4 + x_5 + x_6 + x_7 + x_8 < 0\}.$$

Then $q_1 \in L^+$, $q_2 \in \epsilon L^-$ and $q_3 \in \epsilon^2 L^+$. The element ϵ^5 stabilizing L interchanges halfspaces L^+ and L^-. Thus for some $\beta_1, \beta_2 \in \{-1, 1\}$,

$$\|q_1\| = \beta_1 \epsilon^{-6} \|q_2\| = \beta_3 \epsilon^{-2} \|q_3\|.$$

As in the case (A), instead of the cocycle (9) we analyze the cocycle

$$\mathfrak{O}'_{\mathbb{Q}_{40}}(f)(e) = \delta\,\|q_1\| \tag{10}$$

where δ is an odd integer.

(3) The structure of the intersection poset of the arrangement $\mathcal{A}_{10}$ is described in Figure 3. Let k be the element of $H_6(\bigcup\widehat{\mathcal{A}_{10}};\mathbb{Z})$ geometrically represented by the union of halfspaces L^+ and ϵL^+. The set $\{\epsilon^i\cdot k|i\in\mathbb{Z}\}$ is a bases of a $\mathbb{Q}_{40}$ submodule H of $H_6(\bigcup\widehat{\mathcal{A}_{10}};\mathbb{Z})$. Let $l\in H$ denote the element of $H_6(\bigcup\widehat{\mathcal{A}_{10}};\mathbb{Z})$ geometrically represented by the subspace L such that

$$l = k-\epsilon\cdot k+\epsilon^2\cdot k-\epsilon^3\cdot k+\epsilon^4\cdot k. \tag{11}$$

The decomposition of $\mathbb{Q}_{40}$ modules, or the exact sequence

$$0\to H'\to H_6(\bigcup\widehat{\mathcal{A}_8};\mathbb{Z})\xrightarrow{\xi} H\to 0$$

provides the following exact sequence of coinvarian groups (the coinvarian functor is right exact)

$$H'_{\mathbb{Q}_{40}}\to H_6(\bigcup\widehat{\mathcal{A}_8};\mathbb{Z})_{\mathbb{Q}_{40}}\xrightarrow{\xi_*} H_{\mathbb{Q}_{40}}\to 0$$

Lemma 6. *(A) $H_{\mathbb{Q}_{40}}=\mathbb{Z}_{10}$ (B) $\xi_*([\delta\,\|q_1\|])\neq 0$ (C) $[\mathfrak{O}_{\mathbb{Q}_{40}}(f)(e)]\neq 0$.*

Proof. (A) The relation (11) written in term of "$*$"-action is

$$l = k+\epsilon * k+\epsilon^2 * k+\epsilon^3 * k+\epsilon^4 * k. \tag{12}$$

The element ϵ^5 acts on W_{10} by changing its orientation. On the orthogonal complement $L^\perp$ of L the operator ϵ^5, for the basis $\{e_1+..+e_5,\ e_2+..+e_6,\ e_3+..+e_7,\ e_1+...+e_{10}\}$ of $L^\perp$, has the matrix

$$\Xi = \begin{pmatrix} -1 & 0 & 0 & 1\\ 0 & -1 & 0 & 1\\ 0 & 0 & -1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Since $\det\Xi=-1$, the element ϵ^5 changes the orientation of $L^\perp$ and consequently does not change the orientation on L. Thus

$$\epsilon^{-5} * l = \det(\epsilon^{-5})\ \epsilon^5 l = -\epsilon^5 l = -l. \tag{13}$$

The relations (12) and (13) imply that in coinvariants

$$[l]=5[k] \text{ and } 2[l]=0.$$

(B) The map χ helps to identify the ξ_* image of $[\delta\,\|q_1\|]$. Keeping in mind the geometric interpretation of the map χ and the definition of the submodule H, we get

$$\xi_*([\delta\,\|q_1\|]) = \pm\delta[k]\neq 0.$$

(C) This is the direct consequence of (10), (B) and the equality $[\mathfrak{O}_{\mathbb{Q}_{40}}(f)(e)] = [\mathfrak{O}'_{\mathbb{Q}_{40}}(f)(e)]$. □

The obstruction element $[\mathfrak{O}_{\mathbb{Q}_{40}}(f)(e)]$ is not zero, and the $\mathbb{Q}_{40}$ map $S^3 \to W_{10} \setminus \bigcup \mathcal{A}_{10}$ can not exist. The case (B) of Theorem 4 is proved

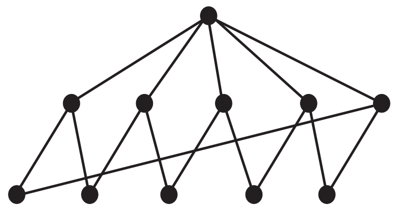

FIGURE 3. The Hasse diagram of the arrangement.

Concluding Remarks. The computational approach reviels the complexity of the problem and gives the opportunity of testing various hypothesis. We are free to conjecture the following statement.

Conjecture 7. *Let μ be a proper Borel probability measure on the sphere S^2. Then there are three planes in the fan position such that the ratio of measure μ in angular sectors cut by planes is (a, b, c, a, b, c), for arbitrary $a, b, c \in \mathbb{N}$.*

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