Weighted Regularization of Maxwell Equations in Polyhedral Domains

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Abstract. *We present a new method of regularizing time harmonic Maxwell equations by a divergence part adapted to the geometry of the domain. This method applies to polygonal domains in two dimensions as well as to polyhedral domains in three dimensions. In the presence of reentrant corners or edges, the usual regularization is known to produce wrong solutions due the non-density of smooth fields in the variational space. We get rid of this undesirable effect by the introduction of special weights inside the divergence integral. Standard finite elements can then be used for the approximation of the solution. This method proves to be numerically efficient.*

Introduction

We consider the solution $(\mathbf{E},\mathbf{H})$ of the time-harmonic three-dimensional Maxwell equations in a domain Ω filled with an homogeneous medium, subject to perfect conductor boundary conditions

$$\left\{\begin{array}{lll} \mathbf{curl}\,\mathbf{E} - i\omega\,\mu\mathbf{H} = 0 \quad \text{and} \quad \mathbf{curl}\,\mathbf{H} + i\omega\,\varepsilon\mathbf{E} = \mathbf{J} & \text{in} & \Omega \\ \mathbf{E}\times\boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{H}\cdot\boldsymbol{n} = 0 & \text{on} & \partial\Omega\,. \end{array}\right. \tag{0.1}$$

For a divergence-free current density $\mathbf{J}$, both electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$ are also divergence-free, and the choice of a variational space to set this problem in variational form is in no way unique.

The finiteness of the electromagnetic energy requires that both the electric and the magnetic field belong to $\mathrm{H}(\mathbf{curl}\,;\Omega)$, where

$$\mathrm{H}(\mathbf{curl}\,;\Omega) = \big\{\boldsymbol{u}\in\mathscr{D}'(\Omega)^3 \mid \ \boldsymbol{u}\in\mathrm{L}^2(\Omega)^3,\ \mathbf{curl}\,\boldsymbol{u}\in\mathrm{L}^2(\Omega)^3\big\}. \tag{0.2}$$

For simplicity we assume $\varepsilon\mu = 1$, and we set $\boldsymbol{f} = i\omega\mu\mathbf{J}$. In order to obtain a variational formulation, we can eliminate the magnetic field from equations (0.1). We obtain formally the equation

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \omega^2\mathbf{E} = \boldsymbol{f}.$$

The "minimal" choice for the electric variational space would be

$$\big\{\boldsymbol{u}\in \mathrm{H}(\mathbf{curl}\,;\Omega)\mid \quad \boldsymbol{u}\times\boldsymbol{n}\big|_{\partial\Omega}=0 \quad\text{and}\quad \operatorname{div}\boldsymbol{u}=0\big\}.$$

A conforming discretization would then impose the use of divergence-free elements.

A "maximal" and more widely used choice for the electric variational space is

$$\mathring{\mathrm{H}}(\mathbf{curl}\,;\Omega)=\big\{\boldsymbol{u}\in \mathrm{H}(\mathbf{curl}\,;\Omega)\mid \quad \boldsymbol{u}\times\boldsymbol{n}\big|_{\partial\Omega}=0\big\}. \tag{0.3}$$

The corresponding variational formulation for problem (0.1) is then

$$\boldsymbol{u}\in\mathring{\mathrm{H}}(\mathbf{curl}\,;\Omega),\ \ \forall\boldsymbol{v}\in\mathring{\mathrm{H}}(\mathbf{curl}\,;\Omega),\ \ \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}-\omega^2\,\boldsymbol{u}\cdot\boldsymbol{v}=\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,. \tag{0.4}$$

But the associated operator $\boldsymbol{u}\mapsto \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}-\omega^2\boldsymbol{u}$ is not elliptic and the equation $\operatorname{div}\boldsymbol{u}=0$ is an independent constraint for $\omega=0$. This is in relation with the fact that the corresponding eigenvalue problem has an infinite dimensional eigenspace for $\omega=0$ formed by all gradient fields $\mathbf{E}=\mathbf{grad}\,\varphi$ with $\varphi\in\mathring{\mathrm{H}}^1(\Omega)$. A well-known strategy for finite element computations of the Maxwell eigen-frequencies is the use of "spurious-free" elements whose classical representatives are the two families of NEDELEC's edge elements [27, 28]. These elements are **curl** conforming but not div conforming, and, roughly speaking, they reproduce at the discrete level the splitting into a large kernel space and a space where the lowest Maxwell eigen-frequencies are approximated [6, 5, 21, 8, 25].

There are good reasons why one may prefer a discretization of the Maxwell problem by more standard and more widely used elements, e.g. nodal elements where the compatibility conditions between neighboring elements are pointwise and scalar. A well-known strategy consists then of regularizing the operator by adding a term containing the divergence, that is, to transform it into an elliptic system. The classical way of doing this, *cf* LEIS [23], and more precisely for this application HAZARD-LENOIR [20] (and also their references), is to introduce the variational space

$$\mathrm{X}_N:=\big\{\boldsymbol{u}\in\mathring{\mathrm{H}}(\mathbf{curl}\,;\Omega)\mid \quad \operatorname{div}\boldsymbol{u}\in \mathrm{L}^2(\Omega)\big\}. \tag{0.5}$$

and to note that, since the solutions of (0.1) are divergence-free, the electric field $\mathbf{E}$ is the solution $\boldsymbol{u}$ of

$$\boldsymbol{u}\in\mathrm{X}_N,\ \ \forall\boldsymbol{v}\in\mathrm{X}_N,\ \ \int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v}+\operatorname{div}\boldsymbol{u}\,\operatorname{div}\boldsymbol{v}-\omega^2\,\boldsymbol{u}\cdot\boldsymbol{v}=\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,. \tag{0.6}$$

In any convex domain, the formulation (0.6) can be discretized by a Galerkin method using nodal finite elements: The discrete spaces consist of functions which are piecewise polynomial and **curl** and div conforming, hence continuous across the interfaces of the mesh; therefore the discrete spaces are contained in $\mathrm{H}^1(\Omega)^3$. In a convex domain Ω, this fact is innocent since X_N coincides with H_N, the subspace of $\boldsymbol{u} \in \mathrm{H}^1(\Omega)^3$ satisfying the tangential boundary condition $\boldsymbol{u} \times \boldsymbol{n} = 0$ on $\partial\Omega$. But if the domain has reentrant corners or edges, H_N no longer coincides with X_N, and even the codimension of H_N in X_N is infinite. And, worse, H_N is *closed* in X_N for the topology of this latter space, *cf* [10, 17]. As any discrete space based on $\mathbf{curl}$ - div conforming elements is contained in H_N, this makes the approximation by such a method *impossible*, see [13].

The new idea that we will develop in this paper is the introduction of suitable *intermediate* spaces between the spaces (0.3) and (0.5), coupled with the corresponding modification of the bilinear form in (0.4), so that

1. The subspace H_N is dense,
2. The associated operator is elliptic,
3. The solution of the new problem coincides with that of (0.1).

More precisely, we are looking for spaces Y such that H_N is dense in

$$\big\{\boldsymbol{u} \in \mathring{\mathrm{H}}(\mathbf{curl}\,;\Omega) \mid \quad \mathrm{div}\,\boldsymbol{u} \in \mathrm{Y}\big\}, \tag{0.7}$$

and such that the new bilinear form

$$(\boldsymbol{u},\boldsymbol{v}) \longmapsto \int_\Omega (\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} - \omega^2\,\boldsymbol{u}\cdot\boldsymbol{v})\;\mathrm{d}\boldsymbol{x} + \langle\,\mathrm{div}\,\boldsymbol{u}, \mathrm{div}\,\boldsymbol{v}\rangle_{\mathrm{Y}}$$

defines an elliptic operator and a problem whose solution is the electric field $\mathbf{E}$ in (0.1).

The requirement of ellipticity imposes that Y is a L^2-type space. Since it is understood that H_N is contained in the space (0.7), $\mathrm{L}^2(\Omega)$ should be contained in Y. As, moreover, any element in $\mathring{\mathrm{H}}(\mathbf{curl}\,;\Omega)$ has its divergence in $\mathrm{H}^{-1}(\Omega)$, we should also have the embedding $\mathrm{Y} \subset \mathrm{H}^{-1}(\Omega)$. Therefore we will concentrate on *weighted* L^2 spaces

$$\big\{\varphi \in \mathrm{L}^2_{\mathrm{loc}}(\Omega) \mid \quad \underline{w}\,\varphi \in \mathrm{L}^2(\Omega)\big\} \tag{0.8}$$

with a weight $\underline{w} \in \mathscr{C}^\infty(\Omega)$, positive in Ω, bounded on $\overline{\Omega}$ – therefore the weighted space (0.8) contains $\mathrm{L}^2(\Omega)$. We will see in particular that the weight satisfying the previous requirements 1.-3. must have an inverse $\underline{w}^{-1}$ unbounded in the neighborhood of the reentrant edges of Ω. Thus the partial differential operator is not uniformly elliptic in Ω, but degenerates near non-convex edges. This unboundedness of the inverse contrasts with regularizations used in [20] and [3, 4, 19] where a bounded weight with bounded inverse is used. On the other hand, our choice of a weight which tends to 0 in a neighborhood of reentrant edges goes in the same direction as a numerical method consisting of setting the weight to 0 in a few layers of elements around reentrant edges and to 1 elsewhere [29].

It makes sense to consider the question also in two dimensions. The above formalism carries over with only minimal obvious changes. The space $\mathring{\mathrm{H}}(\mathrm{curl};\Omega)$ is then defined using the scalar curl operator:

$$\mathring{\mathrm{H}}(\mathrm{curl}\,;\Omega) = \big\{\boldsymbol{u} \in \mathscr{D}'(\Omega)^2 \mid \ \boldsymbol{u} \in \mathrm{L}^2(\Omega)^2,\ \mathrm{curl}\,\boldsymbol{u} \in \mathrm{L}^2(\Omega),\ \boldsymbol{u}\times\boldsymbol{n}\big|_{\partial\Omega} = 0\big\},$$

and the variational formulation of the 2D Maxwell problem corresponding to (0.4) is

$$\boldsymbol{u} \in \mathring{\mathrm{H}}(\mathrm{curl}\,;\Omega), \quad \forall \boldsymbol{v} \in \mathring{\mathrm{H}}(\mathrm{curl}\,;\Omega), \quad \int_\Omega \mathrm{curl}\,\boldsymbol{u}\cdot\mathrm{curl}\,\boldsymbol{v} - \omega^2\,\boldsymbol{u}\cdot\boldsymbol{v} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,. \tag{0.9}$$

This problem is interesting in its own right as a model for TE modes in wave guides. An equivalent problem arises in the study of irrotational fluids in 2D. More importantly, it is useful as a test case for the detailed analysis and the implementation of numerical algorithms.

In particular, it is well known that on non-convex polygons the codimension of H_N in X_N corresponds to the number of non-convex corner points, and the non-convergence of standard conforming finite element methods based on the regularization (0.6) has been studied in detail [13].

Our proposed new algorithm based on weighted regularizations can be very easily implemented in two dimensions, and its convergence can be studied theoretically and experimentally in every detail.

It can also be compared to other algorithms that exist for the 2D problem, for example the singular function method studied in [1, 7]. Compared to this method, our weighted regularization method seems simpler to implement, and it has the big advantage to allow a generalization to three dimensions which does not require precise calculations of the three-dimensional singular functions.

We organize our paper as follows. We first state the functional framework in §1, defining a family of regularized problems (each of which corresponds to a different choice for the space Y), and giving criteria for equivalence with problem (0.1), and for the density of smooth functions in the variational spaces. This requires only weak assumptions on the domain, viz Lipschitz regularity of the boundary. The criterion for the density of smooth functions relies on the density of smooth functions in the domain of a Laplace operator in special spaces. We study this property when Y is realized as a weighted space, first in two-dimensional polygonal domains (§2) and then in three-dimensional corner domains (§3).

We obtain a criterion for the density of smooth functions when the domain of the corresponding Laplacian can be characterized as a weighted Sobolev space (like those of KONDRAT'EV [22] or for more general geometries those in [26, Ch.9]). We summarize the results of the previous sections in §4, exhibiting a class of *admissible weights* $\underline{w}$ providing spaces Y such that conditions 1.-3. above are satisfied. Examples of such weights are provided by $\underline{d}_0^\gamma$ where $\underline{d}_0$ is the distance to the set of reentrant edges and corners of Ω and γ belongs to an interval $(\gamma_{\min}, 1]$, where $\gamma_{\min}$ depends on the domain Ω.

The rest of the paper is devoted to an investigation of the performance of nodal finite elements associated with a variational formulation regularized by an admissible weight. Our error estimate is simply based on the decomposition of the solution $\boldsymbol{u}$ of (0.6) into

$\boldsymbol{w} + \mathbf{grad}\,\varphi$ where $\boldsymbol{w}$ belongs to H_N and the potential φ is a singular function of the Laplace-Dirichlet problem. In §5, we give a precise characterization of the functional properties of $\boldsymbol{w}$ and φ. Relying on this, we exhibit in §6 general *sufficient conditions* for the convergence of finite element methods with a certain convergence rate. One of these conditions requires that the finite element space contains "sufficiently many gradients" to approach $\mathbf{grad}\,\varphi$ by a gradient. In §7, we show that these conditions are satisfied by standard families of nodal finite elements satisfying the usual classical assumptions. The error analysis requires, apart from the theoretical analysis of the corner and edge singularities of the solution, only very standard finite element estimations, combined to get estimates in weighted Sobolev spaces.

We end our paper by more practical results for two-dimensional domains: first we prove that any nodal elements based on $\mathbb{P}_4$ triangles or $\mathbb{Q}_3$ rectangles satisfy the general conditions in §6. Second, we provide in §8 results of numerical experiments in an L-shaped domain with $\mathbb{Q}_p$ rectangular elements, based upon the FEM library MÉLINA developed by D. MARTIN [24]. These results clearly show that our method works as expected, and even better since we already see correct convergence rates with $\mathbb{Q}_2$ rectangles. Even for $p = 1$ we seem to get a convergent algorithm.

Finally, we want to emphasize that the conditions on the families of finite element spaces that we introduce are sufficient, but (apparently) not necessary. Moreover, although one condition is related to the presence of gradients in the finite element spaces, the Galerkin method itself only uses the nodal $\mathscr{C}^0$ elements and not the $\mathscr{C}^1$ densities which serve in the proof of the convergence estimate. In this paper, we perform only a few first steps of a finite element analysis. We expect that this error analysis can be extended to show higher convergence rates for refined methods, using non-uniform meshes and/or p or h - p versions of the finite element method.

1 Reduction to a Laplacian

In this section, Y denotes a (separable) Hilbert space with scalar product $\langle\cdot,\cdot\rangle_{\mathrm{Y}}$ such that

$$\mathrm{L}^2(\Omega) \subset \mathrm{Y} \subset \mathrm{H}^{-1}(\Omega). \tag{1.1}$$

We define the corresponding "electric regularized space" $\mathrm{X}_N[\mathrm{Y}]$ by

$$\mathrm{X}_N[\mathrm{Y}] = \big\{\boldsymbol{u} \in \mathring{\mathrm{H}}(\mathbf{curl}\,;\Omega) \mid \quad \operatorname{div}\boldsymbol{u} \in \mathrm{Y}\big\}, \tag{1.2}$$

with the norm

$$\|\boldsymbol{u}\|^2_{\mathrm{X}_N[\mathrm{Y}]} = \|\,\mathbf{curl}\,\boldsymbol{u}\|^2_{\mathrm{L}^2(\Omega)^3} + \|\operatorname{div}\boldsymbol{u}\|^2_{\mathrm{Y}} + \|\boldsymbol{u}\|^2_{\mathrm{L}^2(\Omega)^3}\,.$$

In this section we only suppose that Ω has a Lipschitz boundary. We define the variational formulation corresponding to each space Y and prove the equivalence with problem (0.1) subject to the density of the range of a certain Helmholtz-type operator. Then, applying a classical decomposition result by BIRMAN-SOLOMYAK [3], we prove a similar decomposition for our new variational spaces, which is then used as a basis for the density criterion.

While we state and prove the results for the 3-dimensional problems, they are also valid, with the obvious minimal changes, for the 2-dimensional problems.

1.a Equivalent problems

The variational problem associated with the space $\mathrm{X}_N[\mathrm{Y}]$ is:

$$\left\{\begin{array}{l} \boldsymbol{u} \in \mathrm{X}_N[\mathrm{Y}], \quad \forall \boldsymbol{v} \in \mathrm{X}_N[\mathrm{Y}], \\ \displaystyle\int_\Omega (\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} - \omega^2\,\boldsymbol{u}\cdot\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} + \langle\,\mathrm{div}\,\boldsymbol{u}, \mathrm{div}\,\boldsymbol{v}\rangle_{\mathrm{Y}} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}. \end{array}\right. \tag{1.3}$$

The equivalence between problems (0.1) and (1.3) uses classical arguments and relies on an assumption about a Helmholtz-type operator with frequency ω. In the standard case when $\mathrm{Y} = \mathrm{L}^2(\Omega)$, this operator is simply $\Delta + \omega^2\mathrm{Id}$. In the present more general case, we have to define two operators:

1. The corresponding Laplace-Dirichlet operator is denoted $\Delta^{\mathrm{Dir}}[\mathrm{Y}]$ and defined as

$$\Delta^{\mathrm{Dir}}[\mathrm{Y}]: \begin{array}{rcl} \mathrm{D}(\Delta^{\mathrm{Dir}}[\mathrm{Y}]) := \ \big\{\varphi \in \mathring{\mathrm{H}}^1(\Omega) \mid \Delta\varphi \in \mathrm{Y}\big\} & \longrightarrow & \mathrm{Y} \\ \varphi & \longmapsto & \Delta\varphi. \end{array} \tag{1.4}$$

   The assumption $\mathrm{Y} \subset \mathrm{H}^{-1}(\Omega)$ makes this definition natural, that is:

$$q = \Delta\varphi \quad \Longleftrightarrow \quad \varphi \in \mathring{\mathrm{H}}^1(\Omega),\ \forall\psi \in \mathring{\mathrm{H}}^1(\Omega):\ \int_\Omega \mathbf{grad}\,\varphi\cdot\mathbf{grad}\,\psi = -\int_\Omega q\,\psi.$$

2. As Y is contained in $\mathrm{H}^{-1}(\Omega)$, and $\mathrm{D}(\Delta^{\mathrm{Dir}}[\mathrm{Y}])$ in $\mathring{\mathrm{H}}^1(\Omega)$, for any $p \in \mathrm{Y}$ and $\varphi \in \mathrm{D}(\Delta^{\mathrm{Dir}}[\mathrm{Y}])$, the scalar product (p, φ) makes sense in the duality $\mathrm{H}^{-1}(\Omega)$ - $\mathring{\mathrm{H}}^1(\Omega)$ and is an extension of the $\mathrm{L}^2(\Omega)$ product. Thus by the Riesz representation theorem there exists a bounded operator

$$K: \begin{array}{rcl} \mathrm{D}(\Delta^{\mathrm{Dir}}[\mathrm{Y}]) & \longrightarrow & \mathrm{Y} \\ \varphi & \longmapsto & K\varphi \quad \text{such that } \forall p \in \mathrm{Y},\ \langle p, K\varphi\rangle_{\mathrm{Y}} = (p, \varphi). \end{array} \tag{1.5}$$

Theorem 1.1 *Let* $\mathbf{J} \in \mathrm{L}^2(\Omega)^3$ *be divergence-free: div* $\mathbf{J} = 0$. *We assume* $\omega \neq 0$.
(i) *If* $(\mathbf{E}, \mathbf{H})$ *solves* (0.1)*, then* $\boldsymbol{u} = \mathbf{E}$ *solves* (1.3).
(ii) *If* $\boldsymbol{u}$ *solves* (1.3) *and if the range of the operator* $\Delta^{\mathrm{Dir}}[\mathrm{Y}] + \omega^2 K$ *is dense in* Y*, then* $(\mathbf{E}, \mathbf{H}) = \big(\boldsymbol{u}, (i\omega)^{-1}\,\mathbf{curl}\,\boldsymbol{u}\big)$ *solves* (0.1).

PROOF.
(i) It is standard that $\boldsymbol{u} = \mathbf{E}$ solves problem (0.4). Since $\operatorname{div} \boldsymbol{u} = 0$, $\boldsymbol{u}$ belongs to $\mathrm{X}_N[\mathrm{Y}]$ too and is solution of (1.3).
(ii) The only thing to be proved is that $\operatorname{div} \boldsymbol{u} = 0$. We take as test function any $\mathbf{grad}\, \varphi$ with $\varphi \in \mathrm{D}(\Delta^{\mathrm{Dir}}[\mathrm{Y}])$. We obtain

$$\forall \varphi \in \mathrm{D}(\Delta^{\mathrm{Dir}}[\mathrm{Y}]), \quad -\omega^2 \int_\Omega \boldsymbol{u} \cdot \mathbf{grad}\, \varphi \, \mathrm{d}\boldsymbol{x} + \langle \operatorname{div} \boldsymbol{u}, \Delta\varphi \rangle_{\mathrm{Y}} = \int_\Omega \boldsymbol{f} \cdot \mathbf{grad}\, \varphi \, \mathrm{d}\boldsymbol{x}.$$

As there holds $\int_\Omega \boldsymbol{f} \cdot \mathbf{grad}\, \varphi = -\int_\Omega \operatorname{div} \boldsymbol{f}\, \varphi$ (and similarly for $\boldsymbol{u}$), the assumption $\operatorname{div} \boldsymbol{f} = 0$ implies

$$\forall \varphi \in \mathrm{D}(\Delta^{\mathrm{Dir}}[\mathrm{Y}]), \quad \omega^2(\operatorname{div} \boldsymbol{u}\,, \varphi) + \langle \operatorname{div} \boldsymbol{u}\,, \Delta\varphi \rangle_{\mathrm{Y}} = 0.$$

This means that

$$\forall \varphi \in \mathrm{D}(\Delta^{\mathrm{Dir}}[\mathrm{Y}]), \quad \big\langle \operatorname{div} \boldsymbol{u}\,, (\Delta + \omega^2 K)\varphi \big\rangle_{\mathrm{Y}} = 0.$$

The assumption about the density of the range of $\Delta^{\mathrm{Dir}}[\mathrm{Y}] + \omega^2 K$ implies that $\operatorname{div} \boldsymbol{u} = 0$.
■

We will see that when Y is realized as a weighted L^2 space, the condition about the density of the range $\Delta^{\mathrm{Dir}}[\mathrm{Y}] + \omega^2 K$ is equivalent to requiring that ω^2 is not an eigenvalue of a certain self-adjoint partial differential operator. Moreover, in this situation we can prove that if the range of the operator $\Delta^{\mathrm{Dir}}[\mathrm{Y}] + \omega^2 K$ is not dense, then there are spurious solutions of problem (1.3), see §4.

1.b Density of smooth functions

Our criterion for the density of smooth functions in the space $\mathrm{X}_N[\mathrm{Y}]$ relies on the following decomposition theorem.

Theorem 1.2 *Let* Y *satisfy* (1.1). *Then any element* $\boldsymbol{u} \in \mathrm{X}_N[\mathrm{Y}]$ *can be decomposed into a sum*

$$\boldsymbol{u} = \boldsymbol{u}_0 + \mathbf{grad}\, \varphi, \quad \textit{with } \boldsymbol{u}_0 \in \mathrm{H}_N \textit{ and } \varphi \in \mathrm{D}(\Delta^{\mathrm{Dir}}[\mathrm{Y}])\,, \tag{1.6}$$

with the estimate

$$\|\boldsymbol{u}_0\|_{\mathrm{H}^1(\Omega)^3} + \|\varphi\|_{\mathrm{H}^1(\Omega)} + \|\Delta\varphi\|_{\mathrm{Y}} \le C\|\boldsymbol{u}\|_{\mathrm{X}_N[\mathrm{Y}]}\,.$$

Conversely, any element of the form (1.6) *belongs to* $\mathrm{X}_N[\mathrm{Y}]$.

PROOF. Let $\boldsymbol{u}$ belong to $\mathrm{X}_N[\mathrm{Y}]$. The divergence $\operatorname{div} \boldsymbol{u}$ belongs to Y. Let ψ be the element of $\mathrm{D}(\Delta^{\mathrm{Dir}}[\mathrm{Y}])$ such that $\Delta^{\mathrm{Dir}}[\mathrm{Y}]\, \psi = \operatorname{div} \boldsymbol{u}$. We have

$$\|\psi\|_{\mathrm{H}^1(\Omega)} + \|\Delta\psi\|_{\mathrm{Y}} \le C\| \operatorname{div} \boldsymbol{u}\|_{\mathrm{Y}}\,.$$

The field $\boldsymbol{v} := \boldsymbol{u} - \mathbf{grad}\,\psi$ satisfies $\mathbf{curl}\,\boldsymbol{v} = \mathbf{curl}\,\boldsymbol{u}$ and $\operatorname{div}\boldsymbol{v} = 0$. Therefore $\boldsymbol{v}$ belongs to the "standard" space $\mathrm{X}_N = \mathrm{X}_N[\mathrm{L}^2(\Omega)]$ with the estimate

$$\|\boldsymbol{v}\|_{\mathrm{X}_N[\mathrm{L}^2(\Omega)]} \le C\|\boldsymbol{u}\|_{\mathrm{X}_N[\mathrm{Y}]}\,.$$

Then, since Ω is supposed to have a Lipschitz boundary, we may use the decomposition result of [3] for $\boldsymbol{v}$: There exists $\boldsymbol{u}_0 \in \mathrm{H}_N$ and $\chi \in \Delta^{\mathrm{Dir}}[\mathrm{L}^2(\Omega)]$ such that $\boldsymbol{v} = \boldsymbol{u}_0 + \mathbf{grad}\,\chi$, with the estimate

$$\|\boldsymbol{u}_0\|_{\mathrm{H}^1(\Omega)^3} + \|\chi\|_{\mathrm{H}^1(\Omega)} + \|\Delta\chi\|_{\mathrm{L}^2(\Omega)} \le C\|\boldsymbol{v}\|_{\mathrm{X}_N[\mathrm{L}^2(\Omega)]}\,.$$

Setting $\varphi := \chi + \psi$, we obtain an element of Y (here we use that $\mathrm{L}^2(\Omega) \subset \mathrm{Y}$), and there holds the estimate of the Theorem. ∎

Corollary 1.3 *If the embedding of* Y *into* $\mathrm{H}^{-1}(\Omega)$ *is compact, then the space* $\mathrm{X}_N[\mathrm{Y}]$ *is compactly embedded in* $\mathrm{L}^2(\Omega)^3$.

This is a consequence of the decomposition (1.6): Since $\mathrm{Y} \subset\subset \mathrm{H}^{-1}(\Omega)$, we also have $\mathrm{D}(\Delta^{\mathrm{Dir}}[\mathrm{Y}]) \subset\subset \mathrm{H}^1(\Omega)$, hence $\mathbf{grad}\,\mathrm{D}(\Delta^{\mathrm{Dir}}[\mathrm{Y}]) \subset\subset \mathrm{L}^2(\Omega)$.

Concerning the density or the non-density of smooth functions, we obtain the fundamental result as an immediate corollary of the decomposition theorem:

Theorem 1.4 (i) *If* $\mathrm{H}^2 \cap \mathring{\mathrm{H}}^1(\Omega)$ *is dense in* $\mathrm{D}(\Delta^{\mathrm{Dir}}[\mathrm{Y}])$ *for the graph norm, then* H_N *is dense in* $\mathrm{X}_N[\mathrm{Y}]$.

(ii) $\mathrm{H}^2 \cap \mathring{\mathrm{H}}^1(\Omega)$ *is closed in* $\mathrm{D}(\Delta^{\mathrm{Dir}}[\mathrm{Y}])$ *if and only if* H_N *is closed in* $\mathrm{X}_N[\mathrm{Y}]$.

(iii) $\mathrm{D}(\Delta^{\mathrm{Dir}}[\mathrm{Y}])$ *is contained in* $\mathrm{H}^2(\Omega)$ *if and only if* $\mathrm{X}_N[\mathrm{Y}] = \mathrm{H}_N$.

In the rest of the paper, we realize the spaces Y as weighted L^2 spaces on Ω, where the weight $\underline{w}$ is a product of distances to corners and edges with different exponents.

2 Laplacian in weighted spaces on polygons

We begin with the discussion of 2-dimensional domains, because their geometry is simpler and the corresponding weighted spaces are better known. The 2-dimensional situation also constitutes the first step for the 3-dimensional case.

2.a Domains

We denote by $\boldsymbol{x} = (x, y)$ the cartesian coordinates $\mathbb{R}^2$. Let $B(\boldsymbol{x}, r)$ denote the ball of center $\boldsymbol{x}$ and radius r. In order to include curvilinear polygonal domains, we define the *Lipschitz 2D corner domains* as follows:

These are the bounded domains Ω in $\mathbb{R}^2$ or $\mathbb{S}^2$ such that in each point $\boldsymbol{a}$ of the boundary there exists $r_{\boldsymbol{a}} > 0$ and a diffeomorphism $\chi_{\boldsymbol{a}}$ transforming the neighborhood

$\mathscr{V}_{\boldsymbol{a}} := \Omega \cap B(\boldsymbol{a}, r_{\boldsymbol{a}})$ into a neighborhood of the corner 0 of a plane sector $\Gamma_{\boldsymbol{a}}$ of opening $\omega_{\boldsymbol{a}} \in (0, 2\pi)$, $\boldsymbol{a}$ being sent into 0. We assume without restriction that, at point $\boldsymbol{a}$, the diffeomorphism $\chi_{\boldsymbol{a}}$ is an *isometric transformation*. Therefore the opening $\omega_{\boldsymbol{a}}$ is an intrinsic parameter of the domain Ω.

The set $\mathscr{A}$ of corners of Ω is the set of points $\boldsymbol{a} \in \partial\Omega$ such that the corresponding sector $\Gamma_{\boldsymbol{a}}$ is non-trivial (opening $\omega_{\boldsymbol{a}} \neq \pi$). With each corner $\boldsymbol{a}$, we associate local polar coordinates such that

$$\Gamma_{\boldsymbol{a}} = \big\{ (r_{\boldsymbol{a}}, \theta_{\boldsymbol{a}}) \mid r_{\boldsymbol{a}} > 0,\ 0 < \theta_{\boldsymbol{a}} < \omega_{\boldsymbol{a}} \big\}.$$

2.b Weighted Sobolev spaces

Let m be a non-negative integer and for any $\boldsymbol{a} \in \mathscr{A}$ let $\gamma_{\boldsymbol{a}}$ be a real number. We denote the collection of $(\gamma_{\boldsymbol{a}})_{\boldsymbol{a}\in\mathscr{A}}$ by $\boldsymbol{\gamma}$. We recall that $\mathscr{V}_{\boldsymbol{a}}$ is a neighborhood of the vertex $\boldsymbol{a}$ which does not contain any other vertex, and we introduce a complementary open set $\mathscr{V}^0 \subset \Omega$ such that no vertex $\boldsymbol{a}$ belongs to $\overline{\mathscr{V}^0}$ and that

$$\Omega = \mathscr{V}^0 \cup \big(\textstyle\bigcup_{\boldsymbol{a}\in\mathscr{A}} \mathscr{V}_{\boldsymbol{a}} \big).$$

The space Y associated with the multi-exponent $\boldsymbol{\gamma}$ is the space $\mathrm{V}^0_{\boldsymbol{\gamma}}(\Omega)$ in the scale of the spaces $\mathrm{V}^m_{\boldsymbol{\gamma}}(\Omega)$ defined as, *cf* [22]

$$\begin{aligned}\mathrm{V}^m_{\boldsymbol{\gamma}}(\Omega) = \Big\{ \varphi \in \mathscr{D}'(\Omega) \mid \quad & \varphi \in \mathrm{H}^m(\mathscr{V}^0) \quad \text{and} \quad \forall \alpha \in \mathbb{N}^2,\ |\alpha| \leq m, \\ & \forall \boldsymbol{a} \in \mathscr{A}, \quad r_{\boldsymbol{a}}^{\gamma_{\boldsymbol{a}}+|\alpha|-m} \partial^\alpha_{\boldsymbol{x}} \varphi \in \mathrm{L}^2(\mathscr{V}_{\boldsymbol{a}}) \Big\}.\end{aligned} \tag{2.1}$$

Here are a few standard and useful properties of these spaces. In local polar coordinates $(r_{\boldsymbol{a}}, \theta_{\boldsymbol{a}})$, the condition $r_{\boldsymbol{a}}^{\gamma_{\boldsymbol{a}}+|\alpha|-m} \partial^\alpha_{\boldsymbol{x}} \varphi \in \mathrm{L}^2(\Omega)$ becomes

$$r_{\boldsymbol{a}}^{\gamma_{\boldsymbol{a}}-m} (r_{\boldsymbol{a}} \partial_{r_{\boldsymbol{a}}})^k \partial^\ell_{\theta_{\boldsymbol{a}}} \varphi_{\boldsymbol{a}} \in \mathrm{L}^2(\Gamma_{\boldsymbol{a}}), \quad k + \ell \leq m,$$

where $\varphi_{\boldsymbol{a}}$ is transformed from φ by the diffeomorphism $\chi_{\boldsymbol{a}}$ (after localization). It is understood that $\varphi_{\boldsymbol{a}}$ is written in polar coordinates: $\varphi_{\boldsymbol{a}} = \varphi_{\boldsymbol{a}}(r_{\boldsymbol{a}}, \theta_{\boldsymbol{a}})$. In Euler coordinates $t_{\boldsymbol{a}} = \log r_{\boldsymbol{a}}$ and $\theta_{\boldsymbol{a}}$ and with $\widetilde{\varphi}_{\boldsymbol{a}}(t_{\boldsymbol{a}}, \theta_{\boldsymbol{a}}) := \varphi_{\boldsymbol{a}}(r_{\boldsymbol{a}}, \theta_{\boldsymbol{a}})$, the above condition becomes

$$e^{(\gamma_{\boldsymbol{a}}-m+1)t_{\boldsymbol{a}}}\, \partial^k_{t_{\boldsymbol{a}}} \widetilde{\varphi}_{\boldsymbol{a}} \in \mathrm{L}^2\big(\mathbb{R}\,, \mathrm{H}^{m-k}(0, \pi/\omega_{\boldsymbol{a}})\big), \quad k \leq m. \tag{2.2}$$

The Mellin transform of $\varphi_{\boldsymbol{a}}$ with respect to $r_{\boldsymbol{a}}$

$$\lambda \longmapsto \int_0^\infty r_{\boldsymbol{a}}^{-\lambda}\, \varphi_{\boldsymbol{a}}(r_{\boldsymbol{a}}, \theta_{\boldsymbol{a}})\, \frac{\mathrm{d}r_{\boldsymbol{a}}}{r_{\boldsymbol{a}}}$$

is therefore well-defined for $\operatorname{Re}\lambda = -\gamma_{\boldsymbol{a}} + m - 1$.

The natural inclusions are, for $m \geq 1$

$$\mathrm{V}^m_{\boldsymbol{\gamma}}(\Omega) \subset \mathrm{V}^{m-1}_{\boldsymbol{\gamma}-\boldsymbol{1}}(\Omega) \tag{2.3}$$

where $\boldsymbol{\gamma}-1$ means the collection $(\gamma_{\boldsymbol{a}}-1)_{\boldsymbol{a}\in\mathscr{A}}$, and for any $\boldsymbol{\gamma}'\le\boldsymbol{\gamma}$ (which means that for any $\boldsymbol{a}\in\mathscr{A}$, $\gamma'_{\boldsymbol{a}}\le\gamma_{\boldsymbol{a}}$) and any m:

$$\mathrm{V}^m_{\boldsymbol{\gamma}'}(\Omega)\subset\mathrm{V}^m_{\boldsymbol{\gamma}}(\Omega). \tag{2.4}$$

Poincaré's inequality allows to prove that

$$\mathring{\mathrm{H}}^1(\Omega)\subset\mathrm{V}^1_{\mathbf{0}}(\Omega).$$

For any weight multi-exponent $\boldsymbol{\gamma}$, the subspace of $\mathrm{V}^1_{\boldsymbol{\gamma}}(\Omega)$ with null traces on the boundary $\partial\Omega$ is the closure of $\mathscr{C}^\infty_0(\Omega)$ functions in $\mathrm{V}^1_{\boldsymbol{\gamma}}(\Omega)$ and is denoted by $\mathring{\mathrm{V}}^1_{\boldsymbol{\gamma}}(\Omega)$. We thus have

$$\mathring{\mathrm{H}}^1(\Omega)=\mathring{\mathrm{V}}^1_{\mathbf{0}}(\Omega). \tag{2.5}$$

Finally, the by-product inclusion $\mathring{\mathrm{H}}^1(\Omega)\subset\mathrm{V}^0_{-\mathbf{1}}(\Omega)$ yields by duality

$$\mathrm{V}^0_{\mathbf{1}}(\Omega)\subset\mathrm{H}^{-1}(\Omega). \tag{2.6}$$

2.c Regularity results

Let us first assume that for a fixed weight multi-exponent $\boldsymbol{\gamma}'$ the function φ belongs to $\mathring{\mathrm{V}}^1_{\boldsymbol{\gamma}'-\mathbf{1}}(\Omega)$ and is such that $\Delta\varphi$ belongs to $\mathrm{V}^0_{\boldsymbol{\gamma}'}(\Omega)$. Then elliptic estimates on dyadic partitions of the sectors $\Gamma_{\boldsymbol{a}}$ for each $\boldsymbol{a}\in\mathscr{A}$ allow to prove in a standard way that φ belongs to $\mathrm{V}^2_{\boldsymbol{\gamma}'}(\Omega)$. We note that the Mellin transform of $\varphi_{\boldsymbol{a}}$ for each $\boldsymbol{a}\in\mathscr{A}$ is defined for $\operatorname{Re}\lambda=-\gamma'_{\boldsymbol{a}}+1$.

Let us assume that, additionally, $\Delta\varphi$ belongs to $\mathrm{V}^0_{\boldsymbol{\gamma}}(\Omega)$, with $\boldsymbol{\gamma}<\boldsymbol{\gamma}'$. We deduce then directly from KONDRAT'EV [22] that the Mellin transform of the localized function $\varphi_{\boldsymbol{a}}$ defines a meromorphic function on the strip $\operatorname{Re}\lambda\in[-\gamma'_{\boldsymbol{a}}+1,-\gamma_{\boldsymbol{a}}+1]$ and that, if for any $\boldsymbol{a}\in\mathscr{A}$ the intervals $[-\gamma'_{\boldsymbol{a}}+1,-\gamma_{\boldsymbol{a}}+1]$ do not contain any number of the form $k\pi/\omega_{\boldsymbol{a}}$, $k\in\mathbb{Z}$, $k\neq0$, then there holds

$$\varphi\in\mathrm{V}^2_{\boldsymbol{\gamma}}\cap\mathring{\mathrm{V}}^1_{\boldsymbol{\gamma}-\mathbf{1}}(\Omega).$$

Applying this result with $\boldsymbol{\gamma}'=\mathbf{1}$ and $\gamma_{\boldsymbol{a}}$ in the interval $(1-\pi/\omega_{\boldsymbol{a}},1]$ and using the equality (2.5), the inclusion (2.6), and the uniqueness of the solution of the Dirichlet problem with data in $\mathrm{H}^{-1}(\Omega)$, we obtain the following result.

Theorem 2.1 *For any weight multi-exponent $\boldsymbol{\gamma}$ such that for every $\boldsymbol{a}\in\mathscr{A}$ the inequality*

$$1-\pi/\omega_{\boldsymbol{a}}<\gamma_{\boldsymbol{a}}\le1 \tag{2.7}$$

holds, the Laplace operator is an isomorphism from $\mathrm{V}^2_{\boldsymbol{\gamma}}\cap\mathring{\mathrm{V}}^1_{\boldsymbol{\gamma}-\mathbf{1}}(\Omega)$ onto $\mathrm{V}^0_{\boldsymbol{\gamma}}(\Omega)$. The solution space can equivalently be written as $\mathrm{V}^2_{\boldsymbol{\gamma}}\cap\mathring{\mathrm{H}}^1(\Omega)$.

For the intended applications, it is useful to note the following.

Proposition 2.2 *For any weight multi-exponent* $\boldsymbol{\gamma}$ *, the* $\mathscr{C}^\infty(\overline{\Omega})$ *functions with null trace on* $\partial\Omega$ *are dense in* $\mathrm{V}^2_{\boldsymbol{\gamma}} \cap \mathring{\mathrm{V}}^1_{\boldsymbol{\gamma}-\mathbf{1}}(\Omega)$.

PROOF. Let us take $\varphi \in \mathrm{V}^2_{\boldsymbol{\gamma}} \cap \mathring{\mathrm{V}}^1_{\boldsymbol{\gamma}-\mathbf{1}}(\Omega)$. In the neighborhood of any smooth point of the boundary of Ω, the approximability property is a consequence of the density of smooth functions in standard Sobolev spaces. It remains to prove that each function $\varphi_{\boldsymbol{a}}$ can be approximated in the sector $\Gamma_{\boldsymbol{a}}$ by $\mathscr{C}^\infty(\overline{\Omega})$ functions with null trace on $\partial\Gamma_{\boldsymbol{a}}$. By the change of coordinates $t_{\boldsymbol{a}} = \log r_{\boldsymbol{a}}$, this is equivalent to prove that $\mathscr{C}^\infty(\overline{\Omega})$ functions with null traces on $\theta_{\boldsymbol{a}} = 0$ and $\theta_{\boldsymbol{a}} = \pi/\omega_{\boldsymbol{a}}$ are dense in the space (2.2) with null traces ($m = 2$).

Multiplying by the exponential weight $e^{(\gamma_{\boldsymbol{a}} - m+1)t_{\boldsymbol{a}}}$, this is equivalent to prove that $\mathscr{C}^\infty(\overline{\Omega})$ functions with null traces on $\theta_{\boldsymbol{a}} = 0$ and $\theta_{\boldsymbol{a}} = \pi/\omega_{\boldsymbol{a}}$ are dense in

$$\mathrm{H}^2 \cap \mathring{\mathrm{H}}^1\big(\mathbb{R} \times (0, \pi/\omega_{\boldsymbol{a}})\big).$$

As in the regular case, this fact is a consequence of the standard density of smooth functions and the use of a lifting operator for the first traces on $\theta_{\boldsymbol{a}} = 0$ and $\theta_{\boldsymbol{a}} = \pi/\omega_{\boldsymbol{a}}$. ∎

2.d Simplified weighted spaces

We will now provide equivalent expressions for the weighted spaces $\mathrm{V}^m_{\boldsymbol{\gamma}}(\Omega)$ based on global weights instead of a partition of unity. Then we define a subclass of these weighted spaces which we shall use in practice for our regularization method.

Let $\boldsymbol{x} \mapsto \underline{d}(\boldsymbol{x})$ denote the distance function to the set of corners of Ω :

$$\underline{d}(\boldsymbol{x}) = \mathrm{dist}(\boldsymbol{x}, \textstyle\bigcup_{\mathscr{A}} \{\boldsymbol{a}\}). \tag{2.8}$$

Since the corners $\boldsymbol{a}$ are isolated from each other, the function

$$\underline{w} := \prod_{\boldsymbol{a} \in \mathscr{A}} r_{\boldsymbol{a}}^{\gamma_{\boldsymbol{a}}} \tag{2.9}$$

is equivalent to $r_{\boldsymbol{a}}^{\gamma_{\boldsymbol{a}}}$ in each neighborhood $\mathscr{V}_{\boldsymbol{a}}$ and to 1 in $\mathscr{V}^0$. We may also interpret $\underline{w}$ as the distance function $\underline{d}$ raised to a certain (variable) power $\underline{\gamma} = \underline{\gamma}(\boldsymbol{x})$. Since $\underline{d}$ is equivalent to the product $\prod_{\boldsymbol{a} \in \mathscr{A}} r_{\boldsymbol{a}}$, the weight $\underline{w}$ in (2.9) is equivalent to

$$\underline{w}(\boldsymbol{x}) \simeq \underline{d}^{\underline{\gamma}(\boldsymbol{x})} \quad \text{with} \quad \gamma(\boldsymbol{x}) \equiv \gamma_{\boldsymbol{a}} \ \text{ for } \ \boldsymbol{x} \in \mathscr{V}_{\boldsymbol{a}}. \tag{2.10}$$

It is then clear that the space $\mathrm{V}^m_{\boldsymbol{\gamma}}(\Omega)$ in (2.1) can be equivalently defined as

$$\mathrm{V}^m_{\boldsymbol{\gamma}}(\Omega) = \{\varphi \in \mathrm{L}^2_{\mathrm{loc}}(\Omega) | \quad \underline{w}\, \underline{d}^{|\alpha|-m} \partial^\alpha \varphi \in \mathrm{L}^2(\Omega), \ \ \forall \alpha, \ |\alpha| \le m\}. \tag{2.11}$$

A simpler class of weights $\underline{w}$ is defined if we take a unique constant value $\gamma \in \mathbb{R}$ of the exponent for a certain subset $\mathscr{A}_0$ of corners and 0 for the others. This simplification will be useful for the implementation of the weighted regularization and for the convergence proof of the finite element method. Let $\mathscr{A}_0$ be a subset of $\mathscr{A}$. We define the global weight multi-exponent $\boldsymbol{\gamma}$ by

$$\forall \boldsymbol{a} \in \mathscr{A}_0, \;\; \gamma_{\boldsymbol{a}} = \gamma, \quad \text{and} \quad \forall \boldsymbol{a} \in \mathscr{A} \setminus \mathscr{A}_0, \;\; \gamma_{\boldsymbol{a}} = 0. \tag{2.12}$$

Then instead of (2.9), we can choose the equivalent weight

$$\underline{w} = \underline{d}_0^{\gamma} \quad \text{with} \quad \underline{d}_0(\boldsymbol{x}) = \text{dist}(\boldsymbol{x}, \textstyle\bigcup_{\mathscr{A}_0} \{\boldsymbol{a}\}), \tag{2.13}$$

and we still have the description (2.11) of the space $\mathrm{V}^m_{\boldsymbol{\gamma}}(\Omega)$.

3 Laplacian in weighted spaces on polyhedra

We shall see now how the previous results in two dimensions can be extended to three-dimensional domains. Although general definitions are more complicated due to the corner-edge interaction, we will see at the end of this section that a subclass of weighted spaces can be described as simply as in (2.11) and (2.13) above.

3.a Domains and distance functions

We denote by $\boldsymbol{x} = (x, y, z)$ the cartesian coordinates in $\mathbb{R}^3$. As in [18], the definition of the classes of domains is recursive. We define the *Lipschitz 3D corner domains* as bounded Lipschitz domains Ω in $\mathbb{R}^3$ such that in each point $\boldsymbol{x}$ of the boundary there exists $r_{\boldsymbol{x}} > 0$ and a diffeomorphism $\chi_{\boldsymbol{x}}$ transforming $\Omega \cap B(\boldsymbol{x}, r_{\boldsymbol{x}})$ into a neighborhood of the corner 0 of a cone $\Gamma_{\boldsymbol{x}}$ of the form $\{\boldsymbol{x} \in \mathbb{R}^3, \; \boldsymbol{x}/|\boldsymbol{x}| \in G_{\boldsymbol{x}}\}$ with $G_{\boldsymbol{x}}$ a Lipschitz 2D corner domain of $\mathbb{S}^2$, $\boldsymbol{x}$ being sent into 0. Like in the two-dimensional case, we assume that at the point $\boldsymbol{x}$, the diffeomorphism $\chi_{\boldsymbol{x}}$ is an *isometric transformation*.

Let $\mathscr{C}$ be the set of the corners $\boldsymbol{c}$ of the 3D corner domain $\Omega \subset \mathbb{R}^3$ which we define by the requirement that for any $\boldsymbol{c} \in \mathscr{C}$, the corresponding cone $\Gamma_{\boldsymbol{c}}$ is a non-trivial cone (i.e. it is neither a half space nor a wedge). The corresponding neighborhood $\Omega \cap B(\boldsymbol{x}, r_{\boldsymbol{c}})$ is denoted by $\mathscr{V}_{\boldsymbol{c}}$. In local spherical coordinates $\rho_{\boldsymbol{c}} \in \mathbb{R}_+$, $\vartheta_{\boldsymbol{c}} \in \mathbb{S}^2$, the cone $\Gamma_{\boldsymbol{c}}$ is:

$$\Gamma_{\boldsymbol{c}} = \big\{(\rho_{\boldsymbol{c}}, \vartheta_{\boldsymbol{c}}) \mid r_{\boldsymbol{c}} > 0, \; \vartheta_{\boldsymbol{c}} \in G_{\boldsymbol{c}}\big\}$$

with the spherical polygonal domain $G_{\boldsymbol{c}} = \Gamma_{\boldsymbol{c}} \cap \mathbb{S}^2$. Let $r_{\boldsymbol{c}}$ be defined as

$$r_{\boldsymbol{c}}(\boldsymbol{x}) := \text{dist}(\boldsymbol{c}, \boldsymbol{x}), \quad \boldsymbol{x} \in \Omega. \tag{3.1}$$

It is clear that $r_{\boldsymbol{c}}$ is equivalent to $\rho_{\boldsymbol{c}}$ in the neighborhood $\mathscr{V}_{\boldsymbol{c}}$.

Let $\mathscr{E}$ be the set of the (open) edges $\boldsymbol{e}$ of Ω: for each point $\boldsymbol{x} \in \boldsymbol{e}$ the local cone $\Gamma_{\boldsymbol{x}}$ is a wedge $\Gamma_{\boldsymbol{e}}(\boldsymbol{x}) \times \mathbb{R}$ and Ω is diffeomorphic to this wedge in the neighborhood

$\Omega \cap B(\boldsymbol{x}, r_{\boldsymbol{x}}) =: \mathscr{V}_{\boldsymbol{e}}(\boldsymbol{x})$. Here $\Gamma_{\boldsymbol{e}}(\boldsymbol{x})$ is a plane sector whose opening we denote by $\omega_{\boldsymbol{e}}(\boldsymbol{x})$. Like for 2D domains, this opening is intrinsic. Let $r_{\boldsymbol{e}}$ be defined as

$$r_{\boldsymbol{e}}(\boldsymbol{x}) := \operatorname{dist}(\overline{\boldsymbol{e}}, \boldsymbol{x}), \quad \boldsymbol{x} \in \Omega. \tag{3.2}$$

It is clear that for each $\boldsymbol{x} \in \boldsymbol{e}$, in the neighborhood $\mathscr{V}_{\boldsymbol{e}}(\boldsymbol{x})$ the function $r_{\boldsymbol{e}}$ is equivalent to the radial coordinate in $\Gamma_{\boldsymbol{e}}(\boldsymbol{x})$.

In order to define our weighted spaces, we need another "distance" function $\rho_{\boldsymbol{e}}$ to the edges. Let $\boldsymbol{e}$ be an edge. Typically, one out of two situations is valid (The situation of an edge containing exactly one corner, which is possible according to our definitions, is left to the reader):

1. $\overline{\boldsymbol{e}}$ contains no corner, then it is a closed curve ($\boldsymbol{e} = \overline{\boldsymbol{e}}$) and we define $\rho_{\boldsymbol{e}} := r_{\boldsymbol{e}}$.
2. $\overline{\boldsymbol{e}}$ contains exactly two corners $\boldsymbol{c}, \boldsymbol{c}' \in \mathscr{C}$. Then we define $\rho_{\boldsymbol{e}}$ such that there holds

$$r_{\boldsymbol{e}} = \rho_{\boldsymbol{e}}\, r_{\boldsymbol{c}}\, r_{\boldsymbol{c}'} \tag{3.3}$$

It is clear that $\rho_{\boldsymbol{e}}$ is equivalent to $r_{\boldsymbol{e}}/r_{\boldsymbol{c}}$ in $\mathscr{V}_{\boldsymbol{c}}$, and to $r_{\boldsymbol{e}}$ outside $\mathscr{V}_{\boldsymbol{c}} \cup \mathscr{V}_{\boldsymbol{c}'}$.

Conversely, for each $\boldsymbol{c} \in \mathscr{C}$, the corners $\boldsymbol{a}$ of the spherical domain $G_{\boldsymbol{c}}$ correspond bijectively to the subset $\mathscr{E}_{\boldsymbol{c}}$ of edges $\boldsymbol{e} \in \mathscr{E}$ such that $\boldsymbol{c}$ belongs to $\overline{\boldsymbol{e}}$. We set $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{e})$. The function $\rho_{\boldsymbol{e}}$ is equivalent to the radial coordinate $r_{\boldsymbol{a}}$ at the corner $\boldsymbol{a}(\boldsymbol{e})$ in $G_{\boldsymbol{c}}$. We denote by $\vartheta_{\boldsymbol{e}}$ the corresponding angular coordinate $\theta_{\boldsymbol{a}}$ in $G_{\boldsymbol{c}}$. Then, for each $\boldsymbol{c} \in \mathscr{C}$ and $\boldsymbol{e} \in \mathscr{E}_{\boldsymbol{c}}$ the three coordinates $(r_{\boldsymbol{c}}, \rho_{\boldsymbol{e}}, \vartheta_{\boldsymbol{e}})$ are local spherical "sectorial" coordinates in a neighborhood $\mathscr{V}_{\boldsymbol{e}}(\boldsymbol{c})$ which has the form $r_{\boldsymbol{c}} < \varepsilon$ and $\vartheta_{\boldsymbol{c}} \in \mathscr{V}_{\boldsymbol{a}}$, where according to the 2D definitions, $\mathscr{V}_{\boldsymbol{a}}$ is a neighborhood of the corner $\boldsymbol{a}$. Let $\mathscr{V}^0_{\boldsymbol{c}}$ be an open set such that $\boldsymbol{e} \cap \overline{\mathscr{V}^0_{\boldsymbol{c}}} = \emptyset$ for any edge $\boldsymbol{e} \in \mathscr{E}_{\boldsymbol{c}}$ and such that $\mathscr{V}_{\boldsymbol{c}} = \mathscr{V}^0_{\boldsymbol{c}} \cup \big(\bigcup_{\boldsymbol{e} \in \mathscr{E}_{\boldsymbol{c}}} \mathscr{V}_{\boldsymbol{e}}(\boldsymbol{c})\big)$.

Besides the neighborhoods $\mathscr{V}_{\boldsymbol{e}}(\boldsymbol{c})$ and $\mathscr{V}^0_{\boldsymbol{c}}$, we introduce $\mathscr{V}^0_{\boldsymbol{e}}$ such that $\overline{\mathscr{V}^0_{\boldsymbol{e}}}$ does not contain any other edge than $\boldsymbol{e}$, nor any corner and such that $\overline{\boldsymbol{e}}$ is contained in $\overline{\mathscr{V}^0_{\boldsymbol{e}}} \cup \big(\bigcup_{\boldsymbol{c} \in \overline{\boldsymbol{e}}} \overline{\mathscr{V}}_{\boldsymbol{e}}(\boldsymbol{c})\big)$. And finally let $\mathscr{V}^0$ such that $\overline{\mathscr{V}^0}$ contains no edge and no corner and such that

$$\Omega = \mathscr{V}^0 \cup \Big(\bigcup_{\boldsymbol{e} \in \mathscr{E}} \mathscr{V}^0_{\boldsymbol{e}}\Big) \cup \Big(\bigcup_{\boldsymbol{c} \in \mathscr{C}} \mathscr{V}^0_{\boldsymbol{c}} \cup \big(\bigcup_{\boldsymbol{e} \in \mathscr{E}_{\boldsymbol{c}}} \mathscr{V}_{\boldsymbol{e}}(\boldsymbol{c})\big)\Big).$$

The opening angle $\omega_{\boldsymbol{e}}(\boldsymbol{x})$ is till now defined for any interior point $\boldsymbol{x}$ in the edge $\boldsymbol{e}$. If a corner $\boldsymbol{c}$ belongs to $\overline{\boldsymbol{e}}$, we define $\omega_{\boldsymbol{e}}(\boldsymbol{c})$ as the opening of $G_{\boldsymbol{c}}$ in $\boldsymbol{a}(\boldsymbol{e})$. Equivalently, $\omega_{\boldsymbol{e}}(\boldsymbol{x})$ is the limit as $\boldsymbol{x} \to \boldsymbol{c}$ of $\omega_{\boldsymbol{e}}(\boldsymbol{x})$.

3.b Weighted Sobolev spaces

Let m be a non-negative integer and, for any $\boldsymbol{e} \in \mathscr{E}$ and any $\boldsymbol{c} \in \mathscr{C}$, let $\gamma_{\boldsymbol{e}}$ and $\gamma_{\boldsymbol{c}}$ be real numbers. We denote by $\boldsymbol{\gamma}$ the collection of $(\gamma_{\boldsymbol{e}})_{\boldsymbol{e} \in \mathscr{E}} \cup (\gamma_{\boldsymbol{c}})_{\boldsymbol{c} \in \mathscr{C}}$. The space Y

associated with the multi-exponent $\boldsymbol{\gamma}$ is the space $\mathrm{V}^0_{\boldsymbol{\gamma}}(\Omega)$ in the scale of spaces $\mathrm{V}^m_{\boldsymbol{\gamma}}(\Omega)$ defined as, *cf* [26]

$$\begin{aligned}\mathrm{V}^m_{\boldsymbol{\gamma}}(\Omega) = \Big\{\varphi \in \mathscr{D}'(\Omega) \mid \varphi \in \mathrm{H}^m(\mathscr{V}^0) \quad &\text{and} \quad \forall\alpha \in \mathbb{N}^2,\ |\alpha| \le m, \\ \forall \boldsymbol{e} \in \mathscr{E}, \qquad & r_{\boldsymbol{e}}^{\gamma_e+|\alpha|-m}\partial^\alpha_{\boldsymbol{x}}\varphi \in \mathrm{L}^2(\mathscr{V}^0_{\boldsymbol{e}}), \\ \forall \boldsymbol{c} \in \mathscr{C}, \qquad & r_{\boldsymbol{c}}^{\gamma_c+|\alpha|-m}\partial^\alpha_{\boldsymbol{x}}\varphi \in \mathrm{L}^2(\mathscr{V}^0_{\boldsymbol{c}}), \\ \forall \boldsymbol{c} \in \mathscr{C},\ \forall \boldsymbol{e} \in \mathscr{E}_{\boldsymbol{c}}, \quad & r_{\boldsymbol{c}}^{\gamma_c+|\alpha|-m}\rho_{\boldsymbol{e}}^{\gamma_e+|\alpha|-m}\partial^\alpha_{\boldsymbol{x}}\varphi \in \mathrm{L}^2(\mathscr{V}_{\boldsymbol{e}}(\boldsymbol{c}))\Big\}.\end{aligned} \tag{3.4}$$

For any corner $\boldsymbol{c}$, in the neighborhood $\mathscr{V}^0_{\boldsymbol{c}}$ the condition in spherical coordinates is

$$r_{\boldsymbol{c}}^{\gamma_c-m}(r_{\boldsymbol{c}}\partial_{r_c})^k\partial^\beta_{\vartheta_c}\varphi_{\boldsymbol{c}} \in \mathrm{L}^2(\mathscr{V}^0_{\boldsymbol{c}}), \quad k+|\beta| \le m,$$

where $\varphi_{\boldsymbol{c}}$ denotes the localized function. Moreover for any $\boldsymbol{e} \in \mathscr{E}_{\boldsymbol{c}}$, in the sectorial coordinates $(r_{\boldsymbol{c}}, \rho_{\boldsymbol{e}}, \vartheta_{\boldsymbol{e}})$ the function φ is written locally as $\varphi_{\boldsymbol{c},\boldsymbol{e}}$ and the condition of integrability becomes

$$r_{\boldsymbol{c}}^{\gamma_c-m}\rho_{\boldsymbol{e}}^{\gamma_e+k-m}(r_{\boldsymbol{c}}\partial_{r_c})^k(\rho_{\boldsymbol{e}}\partial_{\rho_e})^j\partial^\ell_{\vartheta_e}\varphi_{\boldsymbol{c},\boldsymbol{e}} \in \mathrm{L}^2(\mathscr{V}_{\boldsymbol{e}}(\boldsymbol{c})), \quad k+j+\ell \le m.$$

In Euler coordinates $t_{\boldsymbol{c}} = \log r_{\boldsymbol{c}}$ and $\vartheta_{\boldsymbol{c}}$, the above condition for any $\boldsymbol{e} \in \mathscr{E}_{\boldsymbol{c}}$ becomes

$$e^{(\gamma_c-m+\frac{3}{2})t_c}\,\partial^k_{t_c}\widetilde{\varphi}_{\boldsymbol{c},\boldsymbol{e}} \in \mathrm{L}^2\big(\mathbb{R}\,, \mathrm{V}^{m-k}_{\boldsymbol{\gamma}(\boldsymbol{c})}(G_{\boldsymbol{c}})\big), \quad k \le m,$$

where $\boldsymbol{\gamma}(\boldsymbol{c})$ is the collection of weight exponents $(\gamma_{\boldsymbol{e}})_{\,\boldsymbol{e}\in\mathscr{E}_{\boldsymbol{c}}}$. The Mellin transform with respect to the variable $r_{\boldsymbol{c}}$ is then well defined for $\operatorname{Re}\lambda = -\gamma_{\boldsymbol{c}} + m - \frac{3}{2}$. The same inclusions (2.3) and (2.4) hold in polyhedra. And we still have the embeddings (2.5) and (2.6) too.

3.c Regularity results

Let us assume that φ belongs to $\mathring{\mathrm{V}}^1_{\boldsymbol{\gamma}'-1}(\Omega)$ and is such that $\Delta\varphi \in \mathrm{V}^0_{\boldsymbol{\gamma}'}(\Omega)$. As in 2D corner domains, there holds the elliptic regularity result $\varphi \in \mathrm{V}^2_{\boldsymbol{\gamma}'}(\Omega)$.

Let assume that, additionally, $\Delta\varphi \in \mathrm{V}^0_{\boldsymbol{\gamma}}(\Omega)$ with $\boldsymbol{\gamma} < \boldsymbol{\gamma}'$. From the two dimensional result, we obtain as a condition for the regularity of φ along edges that for any $\boldsymbol{e} \in \mathscr{E}$ and any $\boldsymbol{x} \in \overline{\boldsymbol{e}}$ the intervals $[-\gamma'_{\boldsymbol{e}}+1, -\gamma_{\boldsymbol{e}}+1]$ do not contain any number of the form $k\pi/\omega_{\boldsymbol{e}}(\boldsymbol{x})$, $k \in \mathbb{Z}$, $k \neq 0$.

The condition for the regularity at corners involves the eigenvalues μ^{Dir}_k, $0 < \mu^{\mathrm{Dir}}_1 < \mu^{\mathrm{Dir}}_2 \le \ldots$, of the Laplace-Beltrami operator on $G_{\boldsymbol{c}}$ with Dirichlet condition. We set for any $k \in \mathbb{N}$, $k \neq 0$

$$\lambda^{\mathrm{Dir}}_{\boldsymbol{c},\pm k} = -\tfrac{1}{2} \pm \sqrt{\mu^{\mathrm{Dir}}_k + \tfrac{1}{4}}. \tag{3.5}$$

The Mellin transform of the localized function $\varphi_{\boldsymbol{c}}$ defines a meromorphic function on the strip $\operatorname{Re}\lambda \in [-\gamma'_{\boldsymbol{c}} + \frac{1}{2}, -\gamma_{\boldsymbol{c}} + \frac{1}{2}]$ and the associated regularity condition is that the interval $[-\gamma'_{\boldsymbol{c}} + \frac{1}{2}, -\gamma_{\boldsymbol{c}} + \frac{1}{2}]$ does not contain any number $\lambda^{\mathrm{Dir}}_{\boldsymbol{c},k}$, $k \in \mathbb{Z}$, $k \neq 0$.

If the above conditions are satisfied, there holds

$$\varphi \in \mathrm{V}^2_{\boldsymbol{\gamma}} \cap \mathring{\mathrm{V}}^1_{\boldsymbol{\gamma}-\mathbf{1}}(\Omega).$$

Like in the case of 2D corner domains, we deduce the following

Theorem 3.1 *For any weight multi-exponent* $\boldsymbol{\gamma}$ *satisfying*

$$\forall \boldsymbol{e} \in \mathscr{E}, \forall \boldsymbol{x} \in \overline{\boldsymbol{e}}, \ \ 1 - \pi/\omega_{\boldsymbol{e}}(\boldsymbol{x}) < \gamma_{\boldsymbol{e}} \leq 1 \quad \textit{and} \quad \forall \boldsymbol{c} \in \mathscr{C}, \ \ \tfrac{1}{2} - \lambda^{\mathrm{Dir}}_{\boldsymbol{c},1} < \gamma_{\boldsymbol{c}} \leq 1, \tag{3.6}$$

the Laplace operator is an isomorphism from $\mathrm{V}^2_{\boldsymbol{\gamma}} \cap \mathring{\mathrm{V}}^1_{\boldsymbol{\gamma}-\mathbf{1}}(\Omega)$ *onto* $\mathrm{V}^0_{\boldsymbol{\gamma}}(\Omega)$. *The space of solutions can be equivalently written as* $\mathrm{V}^2_{\boldsymbol{\gamma}} \cap \mathring{\mathrm{H}}^1(\Omega)$. *Moreover* $\mathrm{H}^2 \cap \mathring{\mathrm{H}}^1(\Omega)$ *is* dense *in* $\mathrm{V}^2_{\boldsymbol{\gamma}} \cap \mathring{\mathrm{H}}^1(\Omega)$.

3.d Global weights

Like for two dimensional domains, we provide global expressions for the weights defining the spaces $\mathrm{V}^m_{\boldsymbol{\gamma}}(\Omega)$.

In 3D domains, as the corners $\boldsymbol{c}$ are isolated from each other, the function $\prod_{\boldsymbol{c}} r_{\boldsymbol{c}}^{\gamma_{\boldsymbol{c}}}$ is equivalent to $r_{\boldsymbol{c}}^{\gamma_{\boldsymbol{c}}}$ in each neighborhood $\mathscr{V}_{\boldsymbol{c}}$, and as $\rho_{\boldsymbol{e}}$ is the distance to edges blown up at corners which are also isolated from each other, the function $\prod_{\boldsymbol{e}} \rho_{\boldsymbol{e}}^{\gamma_{\boldsymbol{e}}}$ is equivalent to $\rho_{\boldsymbol{e}}^{\gamma_{\boldsymbol{e}}}$ in each neighborhood $\mathscr{V}_{\boldsymbol{e}}(\boldsymbol{c})$. We may therefore take as a global weight

$$\underline{w} = \Big(\prod_{\boldsymbol{c}\in\mathscr{C}} r_{\boldsymbol{c}}^{\gamma_{\boldsymbol{c}}}\Big) \ \Big(\prod_{\boldsymbol{e}\in\mathscr{E}} \rho_{\boldsymbol{e}}^{\gamma_{\boldsymbol{e}}}\Big). \tag{3.7}$$

The space $\mathrm{V}^0_{\boldsymbol{\gamma}}(\Omega)$ can be defined equivalently as

$$\mathrm{V}^0_{\boldsymbol{\gamma}}(\Omega) = \{\varphi \in \mathrm{L}^2_{\mathrm{loc}}(\Omega) | \quad \underline{w}\,\varphi \in \mathrm{L}^2(\Omega)\}.$$

We may also obtain an expression analogous to (2.10), but it is more involved. We need to introduce two distance functions

$$\underline{d}_{\mathscr{C}}(\boldsymbol{x}) = \mathrm{dist}(\boldsymbol{x}, \textstyle\bigcup_{\mathscr{C}}\{\boldsymbol{c}\}) \quad \text{and} \quad \underline{d}_{\mathscr{E}}(\boldsymbol{x}) = \mathrm{dist}(\boldsymbol{x}, \textstyle\bigcup_{\mathscr{E}} \overline{\boldsymbol{e}}) \tag{3.8}$$

and two exponent functions $\underline{\gamma}[\mathscr{C}]$ and $\underline{\gamma}[\mathscr{E}]$ on Ω such that

$$\begin{aligned} &\underline{\gamma}[\mathscr{C}](\boldsymbol{x}) \equiv \gamma_{\boldsymbol{c}} \ \text{ for } \ \boldsymbol{x} \in \mathscr{V}_{\boldsymbol{c}} \\ &\underline{\gamma}[\mathscr{E}](\boldsymbol{x}) \equiv \gamma_{\boldsymbol{e}} \ \text{ for } \ \boldsymbol{x} \in \mathscr{V}^0_{\boldsymbol{e}} \cup \textstyle\bigcup_{\boldsymbol{c}\in\overline{\boldsymbol{e}}} \mathscr{V}_{\boldsymbol{e}}(\boldsymbol{c}), \quad \underline{\gamma}[\mathscr{E}](\boldsymbol{x}) \equiv 0 \ \text{ for } \ \boldsymbol{x} \in \mathscr{V}_{\boldsymbol{c}}, \ \forall \boldsymbol{c} \in \mathscr{C}', \end{aligned} \tag{3.9}$$

where $\mathscr{C}'$ is the subset of corners $\boldsymbol{c} \in \mathscr{C}$ which do not belong no any edge.

The product $\prod_{\boldsymbol{c}\in\mathscr{C}} r_{\boldsymbol{c}}^{\gamma_{\boldsymbol{c}}}$ is, of course, equivalent to $\underline{d}_{\mathscr{C}}^{\underline{\gamma}[\mathscr{C}]}$, and for each $\boldsymbol{c}$ the following equivalence holds

$$r_{\boldsymbol{c}} \prod_{\boldsymbol{e}\in\mathscr{E}_{\boldsymbol{c}}} \rho_{\boldsymbol{e}} \ \simeq \ \underline{d}_{\mathscr{E}} \quad \text{in} \quad \mathscr{V}_{\boldsymbol{c}}\,.$$

We find that the weight $\underline{w}$ in (3.7) is equivalent to

$$\underline{w} \;\simeq\; \underline{d}_{\mathscr{C}}^{\underline{\gamma}[\mathscr{C}]-\underline{\gamma}[\mathscr{E}]}\,\underline{d}_{\mathscr{E}}^{\underline{\gamma}[\mathscr{E}]}. \tag{3.10}$$

We see that if for all corners and edges the exponents are equal to a fixed number δ, then we can take $\underline{\gamma}[\mathscr{C}] \equiv \delta$, moreover $\underline{\gamma}[\mathscr{E}] = \delta$ in a neighborhood of the edges, and $\underline{\gamma}[\mathscr{E}] = 0$ in a neighborhood of the subset $\mathscr{C}'$ of corners which do not belong to any edge.

Therefore the weight $\underline{d}_{\mathscr{C}}^{\underline{\gamma}[\mathscr{C}]-\underline{\gamma}[\mathscr{E}]}\,\underline{d}_{\mathscr{E}}^{\underline{\gamma}[\mathscr{E}]}$ is simply equal to $\underline{d}_{\mathscr{C}'}^{\delta}\,\underline{d}_{\mathscr{E}}^{\delta}$ where $\underline{d}_{\mathscr{C}'}$ is the distance function to the subset $\mathscr{C}'$. It is then clear that the distance function $\underline{d}$

$$\underline{d}(\boldsymbol{x}) = \mathrm{dist}\Big(\boldsymbol{x}, \textstyle\bigcup_{\mathscr{C}}\{\boldsymbol{c}\} \cup \bigcup_{\mathscr{E}} \overline{\boldsymbol{e}}\Big). \tag{3.11}$$

is equivalent to $\underline{d}_{\mathscr{C}'}\,\underline{d}_{\mathscr{E}}$. As a consequence the space $\mathrm{V}^m_{\boldsymbol{\gamma}}(\Omega)$ in (3.4) is equivalently defined as

$$\mathrm{V}^m_{\boldsymbol{\gamma}}(\Omega) = \{\varphi \in \mathrm{L}^2_{\mathrm{loc}}(\Omega) | \quad \underline{w}\,\underline{d}^{|\alpha|-m}\partial^\alpha\varphi \in \mathrm{L}^2(\Omega),\ \ \forall\alpha,\ |\alpha| \le m\}. \tag{3.12}$$

3.e Simple weights

For a 3D corner domain Ω, let $\mathscr{E}_0$ be a subset of edges and $\mathscr{C}_0$ be a subset of corners with the following compatibility condition:

$$\textit{If } \boldsymbol{e} \in \mathscr{E}_0 \textit{ is an open curve, then its end points belong to } \mathscr{C}_0\,. \tag{3.13}$$

We define the global weight multi-exponent $\boldsymbol{\gamma}$ by

$$\begin{cases} \forall \boldsymbol{c} \in \mathscr{C}_0, \ \ \gamma_{\boldsymbol{c}} = \gamma, \quad \text{and} \quad \forall \boldsymbol{c} \in \mathscr{C}\setminus\mathscr{C}_0, \ \ \gamma_{\boldsymbol{c}} = 0 \\ \forall \boldsymbol{e} \in \mathscr{E}_0, \ \ \gamma_{\boldsymbol{e}} = \gamma, \quad \text{and} \quad \forall \boldsymbol{e} \in \mathscr{E}\setminus\mathscr{E}_0, \ \ \gamma_{\boldsymbol{e}} = 0. \end{cases} \tag{3.14}$$

With this choice of multi-exponent, using (3.10), we can prove that

$$\underline{w} = \underline{d}_0^{\gamma} \quad \text{with} \quad \underline{d}_0(\boldsymbol{x}) = \mathrm{dist}\Big(\boldsymbol{x}, \textstyle\bigcup_{\mathscr{C}_0}\{\boldsymbol{c}\} \cup \bigcup_{\mathscr{E}_0} \overline{\boldsymbol{e}}\Big). \tag{3.15}$$

Indeed, the only situation where (3.15) is not trivial is when $\boldsymbol{x}$ belongs to the neighborhood $\mathscr{V}_{\boldsymbol{c}}$ of a corner $\boldsymbol{c} \in \mathscr{C}_0$:

- If $\boldsymbol{c}$ does not belong to any edge $\boldsymbol{e} \in \mathscr{E}_0$, then $\underline{\gamma}[\mathscr{E}] \equiv 0$ in $\mathscr{V}_{\boldsymbol{c}}$ and the equivalence of $\underline{d}_0^{\gamma}$ with $\underline{d}_{\mathscr{C}}^{\underline{\gamma}[\mathscr{C}]-\underline{\gamma}[\mathscr{E}]}\,\underline{d}_{\mathscr{E}}^{\underline{\gamma}[\mathscr{E}]}$ in $\mathscr{V}_{\boldsymbol{c}}$ is clear;
- If $\boldsymbol{c}$ belongs to edge(s) $\boldsymbol{e} \in \mathscr{E}_0$, then in the conical neighborhoods $\mathscr{V}_{\boldsymbol{e}}(\boldsymbol{c})$ of these edges $\underline{\gamma}[\mathscr{E}] \equiv \gamma$, therefore $\underline{d}_{\mathscr{C}}^{\underline{\gamma}[\mathscr{C}]-\underline{\gamma}[\mathscr{E}]}\,\underline{d}_{\mathscr{E}}^{\underline{\gamma}[\mathscr{E}]}$ is equivalent to $\underline{d}_{\mathscr{E}}^{\gamma}$ which is itself equivalent to $\underline{d}_0^{\gamma}$; outside the conical neighborhoods of these edges (but still in $\mathscr{V}_{\boldsymbol{c}}$), $\underline{\gamma}[\mathscr{E}] \equiv 0$, therefore $\underline{d}_{\mathscr{C}}^{\underline{\gamma}[\mathscr{C}]-\underline{\gamma}[\mathscr{E}]}\,\underline{d}_{\mathscr{E}}^{\underline{\gamma}[\mathscr{E}]}$ is equivalent to $\underline{d}_{\mathscr{C}}^{\gamma}$ which is itself equivalent to $\underline{d}_0^{\gamma}$.

4 Regularization with weight

We are going to realize our space Y as a space $\mathrm{V}^0_{\boldsymbol{\gamma}}(\Omega)$. For suitable choices of the weight multi-exponent $\boldsymbol{\gamma}$, the condition of Theorem 1.4 *(i)* will be satisfied.

4.a Density of smooth functions for weighted regularizations

In order that the embedding (1.1) hold we must have

$$\mathbf{0} \le \boldsymbol{\gamma} \le \mathbf{1}, \tag{4.1}$$

that is $0 \le \gamma_{\boldsymbol{a}} \le 1$ for all $\boldsymbol{a}$ in 2D, and $0 \le \gamma_{\boldsymbol{c}} \le 1$ for all $\boldsymbol{c}$, $0 \le \gamma_{\boldsymbol{e}} \le 1$ for all $\boldsymbol{e}$ in 3D. If moreover condition (2.7) in 2D or condition (3.6) in 3D is satisfied for the weight multi-exponent, then there holds for the operator $\Delta^{\mathrm{Dir}}[\mathrm{Y}]$:

$$\mathrm{Y} = \mathrm{V}^0_{\boldsymbol{\gamma}}(\Omega) \quad \Longrightarrow \quad \mathrm{D}(\Delta^{\mathrm{Dir}}[\mathrm{Y}]) = \mathrm{V}^2_{\boldsymbol{\gamma}} \cap \mathring{\mathrm{V}}^1_{\boldsymbol{\gamma}-\mathbf{1}}(\Omega).$$

In the sequel we will write conditions (2.7) and (3.6) in a unified way by an inequality between two weights. Let us define the multi exponent $\boldsymbol{\delta}^{\mathrm{Dir}}$ by

$$\begin{cases} \boldsymbol{\delta}^{\mathrm{Dir}} = (\delta_{\boldsymbol{a}})_{\boldsymbol{a}\in\mathscr{A}} & : \quad \delta_{\boldsymbol{a}} = 1 - \frac{\pi}{\omega_{\boldsymbol{a}}} & \text{in 2D,} \\ \boldsymbol{\delta}^{\mathrm{Dir}} = \big((\delta_{\boldsymbol{e}})_{\boldsymbol{e}\in\mathscr{E}}\,,\,(\delta^{\mathrm{Dir}}_{\boldsymbol{c}})_{\boldsymbol{c}\in\mathscr{C}}\big) & : \quad \delta_{\boldsymbol{e}} = 1 - \min\limits_{x\in\overline{\boldsymbol{e}}} \frac{\pi}{\omega_{\boldsymbol{e}}(\boldsymbol{x})}, \quad \delta^{\mathrm{Dir}}_{\boldsymbol{c}} = \frac12 - \lambda^{\mathrm{Dir}}_{\boldsymbol{c},1} & \text{in 3D.} \end{cases} \tag{4.2}$$

As a corollary of Theorems 2.1, 3.1 and 1.4 *(i)* we have

Theorem 4.1 *Let $\boldsymbol{\gamma}$ be a weight multi-exponent satisfying $\boldsymbol{\delta}^{\mathrm{Dir}} < \boldsymbol{\gamma}$ and* (4.1)*. Then for the choice $\mathrm{Y} = \mathrm{V}^0_{\boldsymbol{\gamma}}(\Omega)$, the space H_N is dense in $\mathrm{X}_N[\mathrm{Y}]$.*

4.b The operator K

Let Y be given by $\{\varphi \in \mathrm{L}^2_{\mathrm{loc}}(\Omega) |\ \underline{w}\,\varphi \in \mathrm{L}^2(\Omega)\}$, with $\underline{w}$ defined in (2.9) or (3.7) with $\boldsymbol{\gamma}$ satisfying (4.1). It is obvious that the operator K in (1.5) is simply given by

$$K\varphi = \underline{w}^{-2}\varphi.$$

Moreover it is possible to exhibit a self-adjoint operator $A_{\underline{w}}$ the spectrum of which characterizes the set of ω such that $\Delta^{\mathrm{Dir}}[\mathrm{Y}] + \omega^2 K$ has no dense range: it suffices to define

$$A_{\underline{w}} := -K^{-\frac12}\,\Delta^{\mathrm{Dir}}[\mathrm{Y}]\,K^{-\frac12} = -\underline{w}\,\Delta^{\mathrm{Dir}}[\mathrm{Y}]\,\underline{w}.$$

The operator $A_{\underline{w}}$ is the self-adjoint realization on $\mathrm{L}^2(\Omega)$ of the operator from $\mathring{\mathrm{V}}^1_{\boldsymbol{\gamma}}(\Omega)$ onto its dual defined by the symmetric positive bilinear form

$$a_{\underline{w}}(p,q) = \int_\Omega \mathbf{grad}(\underline{w}\,p)\cdot\mathbf{grad}(\underline{w}\,q)\;\mathrm{d}\boldsymbol{x}, \quad p,q \in \mathring{\mathrm{V}}^1_{\boldsymbol{\gamma}}(\Omega).$$

Under condition $\boldsymbol{\gamma} > \boldsymbol{\delta}^{\rm Dir}$ with $\boldsymbol{\delta}^{\rm Dir}$ defined in (4.2), the domain of $A_{\underline{w}}$ is $\mathrm{V}^2_{2\boldsymbol{\gamma}} \cap \mathring{\mathrm{V}}^1_{\boldsymbol{\gamma}}(\Omega)$.

Then the range of $\Delta^{\rm Dir}[\mathrm{Y}] + \omega^2 K$ is not dense in Y if and only if ω^2 is an eigenvalue of $A_{\underline{w}}$. In this situation, problem (1.3) admits spurious solutions: It suffices to take an eigenvector $p \neq 0$ of $A_{\underline{w}}$ and to define $\boldsymbol{u}$ as $\mathbf{grad}(\underline{w}\, p)$ in order to obtain a non-zero solution of the homogeneous problem (1.3).

The operator $A_{\underline{w}}$ has a compact inverse if and only if $\boldsymbol{\gamma} < 1$. In this situation, the spectrum of $A_{\underline{w}}$ is discrete, formed of positive eigenvalues which accumulate at infinity. When some of the exponents belonging to $\boldsymbol{\gamma}$ are equal to 1, the operator $A_{\underline{w}}$ has no more a discrete spectrum, but an essential spectrum. But still, $A_{\underline{w}}$ is > 0 . Thus, for any choice of multi-exponent $\boldsymbol{\gamma} > \boldsymbol{\delta}^{\rm Dir}$, multiplying the term $\langle\, \mathrm{div}\, \boldsymbol{u}, \mathrm{div}\, \boldsymbol{v}\rangle_{\mathrm{Y}}$ in problem (1.3) by a large enough positive factor s , we can guarantee that the spurious eigenvalues are avoided. In the forthcoming paper [15], we are studying this and other aspects of the Maxwell eigenvalue problem in more detail.

5 Regularity and singularities of Maxwell solutions

We have proved in [14] that the electric parts $\mathbf{E}$ of solutions of problem (0.1) with $\mathbf{J} \in \mathrm{L}^2(\Omega)^3$ have two sorts of singularities along edges and at corners:

Type 1 which are the gradients of Dirichlet singularities of the Laplace operator,

Type 2 which have the same singularity exponents as the Neumann singularities of the Laplace operator.

Concerning the magnetic part $\mathbf{H}$, the situations of Dirichlet and Neumann are inverted. The Type 3, also present in [14], does not appear here, because $\mathrm{div}\, \mathbf{E} = \mathrm{div}\, \mathbf{H} = 0$.

This means that, if the condition of Theorem 1.1 *(ii)* is fulfilled, we have a decomposition of $\boldsymbol{u}$ in three parts

$$\boldsymbol{u} = \boldsymbol{u}_0^{\rm reg} + \boldsymbol{u}_0^{\rm sing} + \mathbf{grad}\, \varphi. \tag{5.1}$$

In order to describe the Sobolev regularity of the last two terms, we introduce the notation

$$\underline{\mathrm{H}}^{\sigma}(\Omega) := \bigcap_{s<\sigma} \mathrm{H}^s(\Omega). \tag{5.2}$$

We also need the minimum singularity exponents for the Dirichlet and Neumann Laplace operators: for 2D domains

$$\lambda^{\rm Dir} = \lambda^{\rm Neu} := \min_{\boldsymbol{a}\in\mathscr{A}} \frac{\pi}{\omega_{\boldsymbol{a}}} \tag{5.3}$$

and for 3D domains

$$\lambda^{\rm Dir} := \min\left(\left(\min_{\boldsymbol{e}\in\mathscr{E}} \min_{x\in\overline{\boldsymbol{e}}} \frac{\pi}{\omega_{\boldsymbol{e}}(\boldsymbol{x})} \right) ,\ \left(\min_{\boldsymbol{c}\in\mathscr{C}} \lambda^{\rm Dir}_{\boldsymbol{c},1} + \frac{1}{2} \right) \right) \tag{5.4}$$

and

$$\lambda^{\mathrm{Neu}} := \min\left(\left(\min_{\boldsymbol{e}\in\mathscr{E}}\min_{x\in\overline{\boldsymbol{e}}}\frac{\pi}{\omega_{\boldsymbol{e}}(\boldsymbol{x})}\right),\ \left(\min_{\boldsymbol{c}\in\mathscr{C}}\lambda^{\mathrm{Neu}}_{\boldsymbol{c},1}+\frac{1}{2}\right)\right) \tag{5.5}$$

where $\lambda^{\mathrm{Neu}}_{\boldsymbol{c},1}$ is defined as in (3.5) with μ^{Dir}_1 replaced by μ^{Neu}_1, the first non-zero eigenvalue of the Laplace-Beltrami operator on $G_{\boldsymbol{c}}$ with Neumann condition.

Then in (5.1) there holds

$$\boldsymbol{u}_0^{\mathrm{reg}} \in \mathrm{H}^2(\Omega)^n, \quad \boldsymbol{u}_0^{\mathrm{sing}} \in \underline{\mathrm{H}}^{1+\lambda^{\mathrm{Neu}}}(\Omega)^n, \quad \varphi \in \underline{\mathrm{H}}^{1+\lambda^{\mathrm{Dir}}}(\Omega), \tag{5.6}$$

hence $\mathbf{grad}\,\varphi \in \underline{\mathrm{H}}^{\lambda^{\mathrm{Dir}}}(\Omega)^n$ with $n=2$ or 3.

Here, of course, the regularity of $\boldsymbol{u}_0^{\mathrm{reg}}$ is optimal, as opposed to the regularity of the two other parts in (5.6). Indeed, the singular parts are $\mathscr{C}^\infty$ inside Ω and are better described using the following limits of weighted spaces: we first define

$$\mathrm{K}^\infty_{\boldsymbol{\gamma}}(\Omega) := \bigcap_{m\in\mathbb{N}} \mathrm{V}^m_{\boldsymbol{\gamma}+m}(\Omega)$$

and then

$$\underline{\mathrm{K}}^\infty_{\boldsymbol{\beta}}(\Omega) := \bigcap_{\boldsymbol{\gamma}>\boldsymbol{\beta}} \mathrm{K}^\infty_{\boldsymbol{\gamma}}(\Omega).$$

In 2D, the singularities have the simple structure $r_{\boldsymbol{a}}^\lambda\,\psi(\theta_{\boldsymbol{a}})$ with smooth angular functions $\psi_{\boldsymbol{a}}$ and $\lambda = k\pi/\omega_{\boldsymbol{a}}$. We obtain immediately

Theorem 5.1 *If Ω is a 2D corner domain, there holds for the splitting* (5.1)*, with the weight multi-exponents $\boldsymbol{\beta}_{\mathrm{Dir}} = \boldsymbol{\beta}_{\mathrm{Neu}} := (\beta_{\boldsymbol{a}})_{\boldsymbol{a}\in\mathscr{A}}$ with $\beta_{\boldsymbol{a}} = -\frac{\pi}{\omega_{\boldsymbol{a}}} - 1$:*

$$u_0^{\mathrm{sing}} \in \underline{\mathrm{K}}^\infty_{\boldsymbol{\beta}_{\mathrm{Neu}}}(\Omega)^2 \quad \textit{and} \quad \varphi \in \underline{\mathrm{K}}^\infty_{\boldsymbol{\beta}_{\mathrm{Dir}}}(\Omega).$$

In 3D, the structure of the singularities is much more involved. Using the splitting into edge and vertex singularities of [18] and estimates along edges like in [11] we can prove

Theorem 5.2 *If Ω is a 3D corner domain, with the weight multi-exponents $\boldsymbol{\beta}_{\mathrm{Dir}} = \left((\beta_{\boldsymbol{e}})_{\boldsymbol{e}\in\mathscr{E}},\ (\beta^{\mathrm{Dir}}_{\boldsymbol{c}})_{\boldsymbol{c}\in\mathscr{C}}\right)$ and $\boldsymbol{\beta}_{\mathrm{Neu}} = \left((\beta_{\boldsymbol{e}})_{\boldsymbol{e}\in\mathscr{E}},\ (\beta^{\mathrm{Neu}}_{\boldsymbol{c}})_{\boldsymbol{c}\in\mathscr{C}}\right)$ where*

$$\beta_{\boldsymbol{e}} = -\min_{x\in\overline{\boldsymbol{e}}}\frac{\pi}{\omega_{\boldsymbol{e}}(\boldsymbol{x})} - 1, \qquad \beta^{\mathrm{Dir}}_{\boldsymbol{c}} = -\lambda^{\mathrm{Dir}}_{\boldsymbol{c},1} - \frac{3}{2} \qquad \textit{and} \qquad \beta^{\mathrm{Neu}}_{\boldsymbol{c}} = -\lambda^{\mathrm{Neu}}_{\boldsymbol{c},1} - \frac{3}{2}$$

there holds for the splitting (5.1)*,*

$$u_0^{\mathrm{sing}} \in \underline{\mathrm{K}}^\infty_{\boldsymbol{\beta}_{\mathrm{Neu}}}(\Omega)^3 \quad \textit{and} \quad \varphi \in \underline{\mathrm{K}}^\infty_{\boldsymbol{\beta}_{\mathrm{Dir}}}(\Omega).$$

Remark 5.3 The multi-exponents $\boldsymbol{\beta}_{\mathrm{Dir}}$ in Theorems 5.1 and 5.2 satisfy with the multi-exponent $\boldsymbol{\delta}^{\mathrm{Dir}}$ defined in (4.2),

$$\boldsymbol{\beta}_{\mathrm{Dir}} = \boldsymbol{\delta}^{\mathrm{Dir}} - \mathbf{2}\,. \qquad \blacksquare$$

The above theorems give optimal results in the scale of spaces which we have defined. In the sequel, we will only use the following corollary:

Corollary 5.4 *If the condition of Theorem* 1.1 (ii) *is fulfilled, there holds for* $\boldsymbol{u}$

$$\boldsymbol{u} = \boldsymbol{u}_0 + \mathbf{grad}\,\varphi \quad \textit{with} \quad \boldsymbol{u}_0 \in \underline{\mathrm{H}}^{1+\lambda^{\mathrm{Neu}}}(\Omega)^n \quad \textit{and} \quad \varphi \in \underline{\mathrm{K}}^{\infty}_{\boldsymbol{\beta}_{\mathrm{Dir}}} \tag{5.7}$$

with the multi-exponent $\boldsymbol{\beta}_{\mathrm{Dir}}$ *equal to* $\boldsymbol{\delta}^{\mathrm{Dir}} - \mathbf{2}$. *We recall that* $\boldsymbol{\delta}^{\mathrm{Dir}}$ *is defined in* (4.2) *and* λ^{Neu} *in* (5.5).

6 Approximation by finite elements

6.a The principles

From Theorem 1.4 we know that H_N is dense in $\mathrm{X}_N[\mathrm{Y}]$ for suitable choices of Y. From the density of standard finite element function spaces in H_N it follows then that the solution $\boldsymbol{u}$ of the variational problem can be approximated by a convergent sequence of finite element Galerkin approximations $\boldsymbol{u}_h$. In this section we obtain error estimates and convergence rates for such finite element approximations.

The convergence rates we obtain are limited by the choice of the weight multi-exponent $\boldsymbol{\gamma}$ and by geometry dependent parameters like λ^{Dir} and λ^{Neu}, *cf* (5.4) and (5.5). These convergence rates could be improved by standard methods of mesh refinement. But even with uniform meshes, they show – and this is confirmed by the results of numerical computations described below – that the present method provides efficient finite element methods for the approximation of Maxwell boundary value problems on non smooth domains.

We compare this to the previously studied situations, namely

- The regularization without weight [14, 13], where the finite element solution would converge, but to the wrong solution, so the error would not tend to zero,
- The boundary penalization [12, 16], where a theoretical density (and hence convergence) result is available, too, but without explicit error estimates, and numerical experiments show very poor approximation results, to the extent that this method is practically unusable.

We have found, however, that our present weighted regularization method can very well be combined with the boundary penalization method. This might give, in some situations, a more efficient way of implementing the boundary conditions, and it gives good numerical results.

For the finite element approximation, we use a space X_N^h of H^1 -conforming vector-valued finite elements with the following assumptions (see $(\mathfrak{A}1)$ - $(\mathfrak{A}3)$ below)

1. X_N^h has standard good approximation properties in the H^1 norm,
2. There is a space Φ^h of scalar $\mathscr{C}^1$ finite elements whose gradients belong to X_N^h .
3. Φ^h has good approximation properties in the weighted H^2 -space $\mathrm{V}^2_{\boldsymbol{\gamma}}(\Omega)$.

We shall show that (at least in 2D domains) these conditions are satisfied if X_N^h is one of the following standard ($\mathscr{C}^0$) finite element spaces

- $\mathbb{Q}_p$ elements on rectangular grids for $p \geq 3$,
- $\mathbb{Q}_p$ elements on rectangles or trapezoidal quadrilaterals for $p \geq 5$,
- $\mathbb{P}_p$ elements on triangular grids for $p \geq 4$ (and for $p \geq 2$ on some special triangular grids).

From the numerical experiments it seems that any standard H^1 -conforming finite elements should be usable, and that for quadratic or cubic elements the results are quite good for 2D and 3D domains.

6.b Galerkin methods

Let us fix the weight multi-exponent $\boldsymbol{\gamma}$ satisfying the conditions of Theorem 4.1, i.e. $\mathbf{0} \leq \boldsymbol{\gamma} \leq \mathbf{1}$ and $\boldsymbol{\delta}^{\mathrm{Dir}} < \boldsymbol{\gamma}$, and take $\mathrm{Y} = \mathrm{V}^0_{\boldsymbol{\gamma}}(\Omega)$. For any finite dimensional subspace X_N^h (realized as a finite element space) of $\mathrm{X}_N[\mathrm{Y}]$, the Galerkin method associated with the variational problem (1.3) is

$$\left\{ \begin{array}{l} \boldsymbol{u}_h \in \mathrm{X}_N^h, \quad \forall \boldsymbol{v}_h \in \mathrm{X}_N^h, \\ \displaystyle\int_\Omega (\mathbf{curl}\, \boldsymbol{u}_h \cdot \mathbf{curl}\, \boldsymbol{v}_h - \omega^2\, \boldsymbol{u}_h \cdot \boldsymbol{v}_h)\, \mathrm{d}\boldsymbol{x} + \langle\, \mathrm{div}\, \boldsymbol{u}_h, \mathrm{div}\, \boldsymbol{v}_h \rangle_{\mathrm{Y}} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h\, \mathrm{d}\boldsymbol{x}. \end{array} \right. \tag{6.1}$$

By Céa's Lemma we have the error estimate

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{X}_N[\mathrm{Y}]} \leq C \min_{\boldsymbol{v}_h \in \mathrm{X}_N^h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{X}_N[\mathrm{Y}]}\,, \tag{6.2}$$

where the constant C does not depend on the subspace X_N^h . Thus, as usual, we have to evaluate the approximation error of the solution of problem (1.3) by the space X_N^h .

The idea of the error estimate is simple: From the splitting (5.7), and recalling that φ is a Dirichlet singularity, hence satisfies zero boundary conditions, we obtain the decomposition of our solution $\boldsymbol{u}$ in a regular part and a gradient

$$\boldsymbol{u} = \boldsymbol{u}_0 + \mathbf{grad}\, \varphi \quad \text{with} \quad \boldsymbol{u}_0 \in \underline{\mathrm{H}}^{1+\lambda^{\mathrm{Neu}}} \cap \mathrm{H}_N(\Omega) \text{ and } \varphi \in \underline{\mathrm{K}}^\infty_{\boldsymbol{\beta}_{\mathrm{Dir}}} \cap \overset{\circ}{\mathrm{H}}{}^1(\Omega)\,. \tag{6.3}$$

We analyze the approximation error separately for $\boldsymbol{u}_0$ and for $\mathbf{grad}\, \varphi$ as follows:

The energy norm of any element $\boldsymbol{v} \in \mathrm{X}_N[\mathrm{Y}]$ can be estimated, given any splitting of $\boldsymbol{v}$ into $\boldsymbol{v}_0 + \mathbf{grad}\,\psi$ with $\boldsymbol{v}_0 \in \mathrm{H}_N$ and $\psi \in \mathrm{D}(\Delta^{\mathrm{Dir}}[\mathrm{Y}])$, *cf* (1.6), by

$$\|\boldsymbol{v}\|_{\mathrm{X}_N[\mathrm{Y}]} \leq C\big(\|\boldsymbol{v}_0\|_{\mathrm{H}^1} + \|\Delta\psi\|_{\mathrm{Y}} + \|\psi\|_{\mathrm{L}^2}\big). \tag{6.4}$$

Moreover, since $\boldsymbol{\gamma}$ satisfies the conditions of Theorem 4.1, the domain $\mathrm{D}(\Delta^{\mathrm{Dir}}[\mathrm{Y}])$ coincides with the weighted space $\mathrm{V}^2_{\boldsymbol{\gamma}} \cap \mathring{\mathrm{H}}^1(\Omega)$. Therefore, the energy norm (6.4) is bounded by

$$\|\boldsymbol{v}\|_{\mathrm{X}_N[\mathrm{Y}]} \leq C\big(\|\boldsymbol{v}_0\|_{\mathrm{H}^1} + \|\psi\|_{\mathrm{V}^2_{\boldsymbol{\gamma}}}\big). \tag{6.5}$$

Now we require the following assumptions on the family of finite element spaces $(\mathrm{X}^h_N)_{h\in\mathscr{H}}$. We assume that there exists $\tau \in (0,1)$ such that the three following assumptions $(\mathfrak{A}1)$ - $(\mathfrak{A}3)$ hold. First a (standard) global approximation property:

$(\mathfrak{A}1)$ $\forall \boldsymbol{w} \in \underline{\mathrm{H}}^{1+\lambda^{\mathrm{Neu}}} \cap \mathrm{H}_N(\Omega),\ \exists c(\boldsymbol{w}),\ \forall h \in \mathscr{H},\ \exists \boldsymbol{w}_h \in \mathrm{X}^h_N, \quad \|\boldsymbol{w} - \boldsymbol{w}_h\|_{\mathrm{H}^1} \leq c(\boldsymbol{w})\,h^\tau.$

Second, that the spaces X^h_N contain gradients

$(\mathfrak{A}2)$ $\quad \forall h \in \mathscr{H},$ there exists a non-zero space Φ^h such that $\mathbf{grad}\,\Phi^h \subset \mathrm{X}^h_N.$

And, third, that Φ^h has good approximation properties in the $\mathrm{V}^2_{\boldsymbol{\gamma}}$-norm for the elements $\varphi \in \underline{\mathrm{K}}^\infty_{\boldsymbol{\beta}_{\mathrm{Dir}}} \cap \mathring{\mathrm{H}}^1(\Omega)$

$(\mathfrak{A}3)$ $\quad \forall \varphi \in \underline{\mathrm{K}}^\infty_{\boldsymbol{\beta}_{\mathrm{Dir}}} \cap \mathring{\mathrm{H}}^1(\Omega),\ \exists c(\varphi),\ \forall h \in \mathscr{H},\ \exists \varphi_h \in \Phi^h, \quad \|\varphi - \varphi_h\|_{\mathrm{V}^2_{\boldsymbol{\gamma}}} \leq c(\varphi)\,h^\tau.$

With these assumptions, we realize an approximation of the solution $\boldsymbol{u}$ of problem (1.3) using the decomposition (6.3), by the element $\boldsymbol{w}_h + \mathbf{grad}\,\varphi_h$ of X^h_N where $\boldsymbol{w}_h$ is the "interpolant" of $\boldsymbol{u}_0$ according to $(\mathfrak{A}1)$ and φ_h is the "interpolant" of φ according to $(\mathfrak{A}3)$. We have

$$\|\boldsymbol{u} - (\boldsymbol{w}_h + \mathbf{grad}\,\varphi_h)\|_{\mathrm{X}_N[\mathrm{Y}]} = \|(\boldsymbol{u}_0 - \boldsymbol{w}_h) + \mathbf{grad}(\varphi - \varphi_h)\|_{\mathrm{X}_N[\mathrm{Y}]},$$

and with the expression of the energy (6.5)

$$\|\boldsymbol{u} - (\boldsymbol{w}_h + \mathbf{grad}\,\varphi_h)\|_{\mathrm{X}_N[\mathrm{Y}]} \leq C\|\boldsymbol{u}_0 - \boldsymbol{w}_h\|_{\mathrm{H}^1} + \|\varphi - \varphi_h\|_{\mathrm{V}^2_{\boldsymbol{\gamma}}}.$$

Assumptions $(\mathfrak{A}1)$ - $(\mathfrak{A}3)$ yield that

$$\|\boldsymbol{u} - (\boldsymbol{w}_h + \mathbf{grad}\,\varphi_h)\|_{\mathrm{X}_N[\mathrm{Y}]} \leq C\,h^\tau.$$

We have obtained

Theorem 6.1 *Let the multi-exponent* $\boldsymbol{\gamma}$ *satisfy* $\mathbf{0} \leq \boldsymbol{\gamma} \leq \mathbf{1}$ *and* $\boldsymbol{\delta}^{\mathrm{Dir}} < \boldsymbol{\gamma}$ *and set* $\mathrm{Y} = \mathrm{V}^0_{\boldsymbol{\gamma}}(\Omega)$, cf *Theorem* 4.1. *Let* ω^2 *satisfy the uniqueness hypothesis of Theorem* 1.1 (ii) *and for* $\boldsymbol{f} \in \mathrm{L}^2(\Omega)^n$ *with* $\operatorname{div}\boldsymbol{f} = 0$ *let* $\boldsymbol{u}$ *be the solution of problem* (1.3). *If the family* $(\mathrm{X}^h_N)_{h\in\mathscr{H}}$ *satisfy the approximation properties* $(\mathfrak{A}1)$ - $(\mathfrak{A}3)$, *then for the solutions* $\boldsymbol{u}_h$ *of the corresponding Galerkin problems* (6.1) *there holds the estimate*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{X}_N[\mathrm{Y}]} \leq c(\boldsymbol{u})\,h^\tau \tag{6.6}$$

for any $h \in \mathscr{H}$, *where the constant* $c(\boldsymbol{u})$ *does not depend on* h.

7 Performances of the FEM with the weighted regularization

7.a Convergence rates

We will show that the hypotheses of Theorem 6.1 are satisfied for some standard classes of finite elements and with simplified weights $\boldsymbol{\gamma}$ according to section 3.e.

Consider a family of finite element spaces X_N^h, $h \in \mathscr{H}$, on the domain Ω. For simplicity, we assume that Ω is a polygon in $\mathbb{R}^2$ or a polyhedron in $\mathbb{R}^3$ that is discretized into elements $\mathscr{T}_h$:

$$\overline{\Omega} = \bigcup \big\{ K \mid \ K \in \mathscr{T}_h \big\} \quad \text{and} \quad h = \max \big\{ \operatorname{diam}(K) \mid \ K \in \mathscr{T}_h \big\}.$$

We denote $\operatorname{diam}(K)$ by h_K.

For the first approximation property $(\mathfrak{A}1)$, one can take any almost-affine family of $\mathscr{C}^0$ elements. Here we use the definition of almost-affine in the sense of [9]. For our case, this means that the local interpolation operator Π_K on the space P_K of polynomials of the element K has the approximation property (with a constant C independent of K, and $|\cdot|_{\mathrm{H}^{k+1}(K)}$ denoting the $\mathrm{H}^{k+1}(K)$-seminorm)

$$\forall v \in \mathrm{H}^{k+1}(K): \quad \|v - \Pi_K v\|_{\mathrm{H}^1(K)} \le C\, h_K^k |v|_{\mathrm{H}^{k+1}(K)}. \tag{7.1}$$

Here $k \ge 1$ is such that the space P_K contains the space $\mathbb{P}_k\big|_K$ of all polynomials of degree $\le k$ and such that $\Pi_K : \mathrm{H}^{k+1}(K) \to \mathrm{P}_K$ is continuous (this means that $\mathrm{H}^{k+1}(K) \subset \mathscr{C}^s$, where s is the maximal order of derivatives defining the interpolation operator Π_K).

The approximation property $(\mathfrak{A}1)$ is a standard consequence of these assumptions:

Proposition 7.1 *Let X_N^h, $h \in \mathscr{H}$, be a family of vector-valued finite element spaces whose vector components are $\mathscr{C}^0(\overline{\Omega})$ and are defined by almost-affine elements satisfying the estimates* (7.1) *with $k \ge 1$. In addition we assume that $\mathrm{X}_N^h \subset \mathrm{H}_N$:*

$$\forall \boldsymbol{v}_h \in \mathrm{X}_N^h, \ \ \boldsymbol{v}_h \times \boldsymbol{n} = 0 \ \ \textit{on} \ \ \partial\Omega \quad \textit{and} \quad \forall \boldsymbol{v} \in \mathrm{H}^{k+1} \cap \mathrm{H}_N(\Omega), \ \Pi_h \boldsymbol{v} \in \mathrm{H}_N, \tag{7.2}$$

where Π_h is the global interpolation operator which matches the individual Π_K on each element K and each component v_i. Then for any $\tau \in [0,k]$ there is a constant C_τ independent of h such that

$$\forall \boldsymbol{v} \in \mathrm{H}^{1+\tau} \cap \mathrm{H}_N(\Omega), \quad \exists \boldsymbol{v}_h \in \mathrm{X}_N^h : \quad \|\boldsymbol{v} - \boldsymbol{v}_h\|_{\mathrm{H}^1(\Omega)} \le C_\tau\, h^\tau \|\boldsymbol{v}\|_{\mathrm{H}^{\tau+1}(\Omega)}.$$

Remark. Alternatively, we could just have taken any $(\ell, m)-$ system $S_h^{\ell,m}(\Omega)$ in the sense of [2] with $m \ge 1$ and $\ell = k+1 \ge 2$. Since we need the element-wise error estimate in the following less standard estimate, we preferred to write it even for this first well-known case. In particular the assumptions (7.1)-(7.2) are satisfied for the standard $\mathbb{P}_k$ or $\mathbb{Q}_k$ elements on triangles or rectangles for any $k \geq 1$. ■

For the spaces Φ^h, we use the same discretization $\mathscr{T}_h$ of Ω, but with different polynomial spaces $\overline{\mathrm{P}}_K$ and interpolation operators $\overline{\Pi}_K$ on each element. In fact, since we need **grad** $\Phi^h \subset \mathrm{X}_N^h$ to satisfy assumption $(\mathfrak{A}2)$, we require

$$\begin{cases} \Phi^h \subset \mathscr{C}^1(\overline{\Omega}) \cap \mathring{\mathrm{H}}^1(\Omega) \quad \text{and} \quad \forall\varphi \in \mathrm{H}^{\ell+1} \cap \mathring{\mathrm{H}}^1(\Omega),\ \overline{\Pi}_h\varphi \in \mathring{\mathrm{H}}^1(\Omega), \\ \partial_i \overline{\mathrm{P}}_K \subset \mathrm{P}_K \quad (K \in \mathscr{T}_h\ ;\ \ i = 1, \ldots, n). \end{cases} \tag{7.3}$$

We also assume the element-wise approximation property (Φ^h is an almost-affine family of $\mathscr{C}^1$ elements) for all $\varphi \in \mathrm{H}^{\ell+1}(K)$ and for $j = 0, 1, 2$:

$$|\varphi - \overline{\Pi}_K\varphi|_{\mathrm{H}^j(K)} \leq C\, h_K^{\ell+1-j} |\varphi|_{\mathrm{H}^{\ell+1}(K)}, \tag{7.4}$$

for some ℓ such that $\mathbb{P}_\ell\big|_K \subset \overline{\mathrm{P}}_K$ and $\overline{\Pi}_K : \mathrm{H}^{\ell+1}(K) \to \overline{\mathrm{P}}_K$ is continuous.

With these assumptions, we obtain an approximation property in weighted Sobolev spaces which will yield the validity of assumption $(\mathfrak{A}3)$:

Proposition 7.2 *Let the multi-exponent* $\boldsymbol{\gamma}$ *be associated with subsets* $\mathscr{A}_0$*, or* $\mathscr{C}_0$ *and* $\mathscr{E}_0$*, of selected corners and edges with condition* (3.13) *according to* § 3.e. *Let* $\boldsymbol{\beta}$ *be a multi-exponent such that*

$$\boldsymbol{\gamma} - \mathbf{1} - \boldsymbol{\ell} \leq \boldsymbol{\beta} \leq \boldsymbol{\gamma} - \mathbf{2}$$

with the same integer ℓ *as in* (7.4)*. Let the family* Φ^h*,* $h \in \mathscr{H}$*, of* $\mathscr{C}^1-$ *finite element spaces satisfy the compatibility condition* (7.3) *and the almost-affine estimate* (7.4)*. Then there is a constant* $C_{\boldsymbol{\gamma},\boldsymbol{\beta}}$ *independent of* h *such that*

$$\forall\varphi \in \mathrm{V}^{\ell+1}_{\boldsymbol{\beta}+\boldsymbol{\ell}+\mathbf{1}} \cap \mathring{\mathrm{H}}^1(\Omega), \quad \exists\varphi_h \in \Phi^h : \quad \|\varphi - \varphi_h\|_{\mathrm{V}^2_{\boldsymbol{\gamma}}(\Omega)} \leq C_{\boldsymbol{\gamma},\boldsymbol{\beta}}\, h^\tau \|\varphi\|_{\mathrm{V}^{\ell+1}_{\boldsymbol{\beta}+\boldsymbol{\ell}+\mathbf{1}}(\Omega)} \tag{7.5}$$

where the real number τ *is defined as* $\min(\boldsymbol{\gamma} - \boldsymbol{\beta}) - 2$*,*

$$\tau = \begin{cases} \min_{\boldsymbol{a}\in\mathscr{A}}(\gamma_{\boldsymbol{a}} - \beta_{\boldsymbol{a}}) - 2 & \textit{in 2D}, \\ \min\big(\min_{\boldsymbol{c}\in\mathscr{C}}(\gamma_{\boldsymbol{c}} - \beta_{\boldsymbol{c}}),\, \min_{\boldsymbol{e}\in\mathscr{E}}(\gamma_{\boldsymbol{e}} - \beta_{\boldsymbol{e}})\big) - 2 & \textit{in 3D}. \end{cases} \tag{7.6}$$

Remark 7.3 The assumption $\boldsymbol{\gamma} - \mathbf{1} - \boldsymbol{\ell} \leq \boldsymbol{\beta} \leq \boldsymbol{\gamma} - \mathbf{2}$ implies $\tau \in [0, \ell - 1]$. ■

PROOF. Let us recall that according to (2.11) and (2.13) the space $\mathrm{V}^2_{\boldsymbol{\gamma}}(\Omega)$ satisfies

$$\mathrm{V}^2_{\boldsymbol{\gamma}}(\Omega) = \{\varphi \in \mathrm{L}^2_{\mathrm{loc}}(\Omega) | \quad \underline{d}_0^{\gamma}\, \underline{d}^{|\alpha|-2} \partial^\alpha\varphi \in \mathrm{L}^2(\Omega), \ \ \forall\alpha,\ |\alpha| \leq 2\} \tag{7.7}$$

where $\underline{d}$ is the distance to the whole set $\mathscr{S}$ of corners and edges, and $\underline{d}_0$ the distance to the set $\mathscr{S}_0$ of selected corners and edges. Let us define a new multi-exponent $\boldsymbol{\beta}'$ by

$$\boldsymbol{\beta}' := \boldsymbol{\gamma} - \boldsymbol{\tau} - \mathbf{2},$$

where $\boldsymbol{\tau}$ is the repetition of τ over all corners and edges. As a consequence of the definition (7.6) of τ there holds

$$\boldsymbol{\beta} \leq \boldsymbol{\beta}',$$

which shows that the estimate (7.5) for $\boldsymbol{\beta}'$ instead of $\boldsymbol{\beta}$ implies that the estimate (7.5) holds for $\boldsymbol{\beta}$, too. Therefore we assume from now on that $\boldsymbol{\beta} = \boldsymbol{\beta}'$. With $\boldsymbol{\beta} = \boldsymbol{\gamma} - \boldsymbol{\tau} - \mathbf{2}$, we also have a simple expression for the spaces $\mathrm{V}^{\ell+1}_{\boldsymbol{\beta}+\boldsymbol{\ell}+\mathbf{1}}(\Omega)$. The shift of the multi-exponent $\boldsymbol{\gamma}$ by a *constant*, viz $-\tau - 2 + \ell + 1$, corresponds to a factor $\underline{d}^{-\tau-2}$ in the weight for each derivative. Hence

$$\mathrm{V}^{\ell+1}_{\boldsymbol{\beta}+\boldsymbol{\ell}+\mathbf{1}}(\Omega) = \{\varphi \in \mathrm{L}^2_{\mathrm{loc}}(\Omega) | \quad \underline{d}_0^{\gamma}\, \underline{d}^{|\alpha|-\tau-2} \partial^\alpha \varphi \in \mathrm{L}^2(\Omega), \ \ \forall \alpha, \ |\alpha| \leq m\}. \tag{7.8}$$

We choose a cutoff function $\chi_h \in \mathscr{C}^\infty(\overline{\Omega})$ with the properties

$$0 \leq \chi_h \leq 1, \quad \chi_h(\boldsymbol{x}) \equiv 1 \ \text{ if } \ \underline{d}(\boldsymbol{x}) \leq 2h\,, \quad \chi_h(\boldsymbol{x}) \equiv 0 \ \text{ if } \ \underline{d}(\boldsymbol{x}) \geq 4h\,,$$

together with the estimates on its derivatives

$$\|\partial^\alpha \chi_h\|_{\mathrm{L}^\infty(\Omega)} \leq C_\alpha h^{-|\alpha|}, \quad \forall \alpha \in \mathbb{N}^n. \tag{7.9}$$

Let φ belong to $\mathrm{V}^{\ell+1}_{\boldsymbol{\beta}+\boldsymbol{\ell}+\mathbf{1}} \cap \mathring{\mathrm{H}}^1(\Omega)$. Then $\varphi = \chi_h\varphi + (1-\chi_h)\varphi$ and we will choose φ_h as the Φ^h- interpolant of $(1-\chi_h)\varphi$:

$$\varphi_h = \overline{\Pi}_h\big((1-\chi_h)\varphi\big).$$

As $(1-\chi_h)\varphi \in \mathring{\mathrm{H}}^1(\Omega)$, according to condition (7.3), φ_h belongs to $\mathring{\mathrm{H}}^1(\Omega)$.
There holds

$$\|\varphi - \varphi_h\|_{\mathrm{V}^2_{\boldsymbol{\gamma}}(\Omega)} \leq \|\chi_h\varphi\|_{\mathrm{V}^2_{\boldsymbol{\gamma}}(\Omega)} + \|(1-\chi_h)\varphi - \varphi_h\|_{\mathrm{V}^2_{\boldsymbol{\gamma}}(\Omega)}$$

and we estimate the two terms on the right hand side separately.
(i) $\|\chi_h\varphi\|$
On the support of $\chi_h\varphi$, we have $\underline{d} \leq 4h$, hence

$$\begin{aligned}
\|\chi_h\varphi\|^2_{\mathrm{V}^2_{\boldsymbol{\gamma}}(\Omega)} &= \sum_{|\alpha|\leq 2} \int_\Omega \underline{d}^{2(|\alpha|-2)}\, \underline{d}_0^{2\gamma}\, |\partial^\alpha(\chi_h\varphi)|^2 \,\mathrm{d}\boldsymbol{x} \\
&\leq \sum_{|\alpha|\leq 2} \int_\Omega (4h)^{2\tau} \underline{d}^{2(|\alpha|-\tau-2)}\, \underline{d}_0^{2\gamma}\, |\partial^\alpha(\chi_h\varphi)|^2 \,\mathrm{d}\boldsymbol{x} \\
&\leq C\, h^{2\tau} \|\chi_h\varphi\|^2_{\mathrm{V}^2_{\boldsymbol{\gamma}-\boldsymbol{\tau}}(\Omega)} \ \leq \ C\, h^{2\tau} \|\chi_h\varphi\|^2_{\mathrm{V}^{\ell+1}_{\boldsymbol{\beta}+\boldsymbol{\ell}+\mathbf{1}}(\Omega)}
\end{aligned}$$

Due to (7.8) and (7.9), we have for any m

$$
\begin{aligned}
\|\chi_h\varphi\|^2_{\mathrm{V}^m_{\boldsymbol{\beta}+\boldsymbol{m}}(\Omega)} &\leq C\sum_{|\alpha|\leq m}\sum_{\kappa\leq\alpha}\int_\Omega \underline{d}^{2(|\alpha|-\tau-2)}\,\underline{d}_0^{2\gamma}\,|\partial^{\alpha-\kappa}\chi_h|^2\,|\partial^\kappa\varphi|^2\,\mathrm{d}\boldsymbol{x} \\
&\leq C\sum_{|\alpha|\leq m}\sum_{\kappa\leq\alpha}\int_{\Omega,\ \underline{d}\leq 4h} \underline{d}^{2(|\alpha|-\tau-2)}\,\underline{d}_0^{2\gamma}\,h^{2(|\kappa|-|\alpha|)}\,|\partial^\kappa\varphi|^2\,\mathrm{d}\boldsymbol{x} \\
&\leq C\sum_{|\kappa|\leq m}\int_{\Omega,\ \underline{d}\leq 4h} \underline{d}^{2(|\kappa|-\tau-2)}\,\underline{d}_0^{2\gamma}\,|\partial^\kappa\varphi|^2\,\mathrm{d}\boldsymbol{x} \;\leq\; C\|\varphi\|^2_{\mathrm{V}^m_{\boldsymbol{\beta}+\boldsymbol{m}}(\Omega)}\,.
\end{aligned}
$$

From the last two series of inequalities we obtain

$$
\|\chi_h\varphi\|_{\mathrm{V}^2_{\boldsymbol{\gamma}}(\Omega)} \leq C\,h^\tau\,\|\varphi\|_{\mathrm{V}^{\ell+1}_{\boldsymbol{\beta}+\boldsymbol{\ell}+\mathbf{1}}(\Omega)}\,. \tag{7.10}
$$

(ii) $\|(1-\chi_h)\varphi-\varphi_h\|$
Since $\varphi_h = \overline{\Pi}_h\big((1-\chi_h)\varphi\big)$ and since $\overline{\Pi}_h$ is a local interpolant, the support of φ_h contains only the elements K such that $\operatorname{supp}\big((1-\chi_h)\varphi\big)\cap K$ is not empty. As the diameter of any K is less than h and as the distance function $\underline{d}$ is $\geq 2h$ on $\operatorname{supp}(1-\chi_h)$, there holds $\underline{d}\geq h$ on the support of φ_h. As $\underline{d}$ and $\underline{d}_0$ are the distance functions to the sets $\mathscr{S}$ and $\mathscr{S}_0$ respectively, with $\mathscr{S}\supset\mathscr{S}_0$, there holds $\underline{d}_0\geq\underline{d}$, hence $\underline{d}_0\geq h$, too, on the support of φ_h. Set for each element K

$$
\underline{d}_K = \inf\{\,\underline{d}(\boldsymbol{x})\mid\quad \boldsymbol{x}\in K\}\quad\text{and}\quad \underline{d}_{0,K} = \inf\{\,\underline{d}_0(\boldsymbol{x})\mid\quad \boldsymbol{x}\in K\}.
$$

As $\operatorname{diam}(K)\leq h$, on K the function $\underline{d}(\boldsymbol{x})$ takes its values in $[\underline{d}_K,\underline{d}_K+h]$ and as $\underline{d}\geq h$ on the support of φ_h, we obtain

$$
\forall K,\ \operatorname{supp}(\varphi_h)\cap K\neq\emptyset\ :\qquad \underline{d}_K\geq h\quad\text{and}\quad \underline{d}(\boldsymbol{x})\simeq\underline{d}_K\ \text{ on }\ K\,.
$$

There holds similarly

$$
\forall K,\ \operatorname{supp}(\varphi_h)\cap K\neq\emptyset\ :\qquad \underline{d}_0(\boldsymbol{x})\simeq\underline{d}_{0,K}\ \text{ on }\ K\,.
$$

Therefore we have the equivalence (uniformly in h)

$$
\|(1-\chi_h)\varphi-\varphi_h\|^2_{\mathrm{V}^2_{\boldsymbol{\gamma}}(\Omega)} \;\simeq\; \sum_{j=0}^{2}\sum_{\underline{d}_K\geq h}\underline{d}_{0,K}^{2\gamma}\,\underline{d}_K^{2(j-2)}\,|(1-\chi_h)\varphi-\varphi_h|^2_{\mathrm{H}^j(K)}\,.
$$

On each element K, taking advantage of $\varphi_h\big|_K = \overline{\Pi}_K\big((1-\chi_h)\varphi\big)$, we use the almost-affine estimate (7.4) and obtain for $j=0,1,2$

$$
\begin{aligned}
|(1-\chi_h)\varphi-\varphi_h|_{\mathrm{H}^j(K)} &\leq C\,h_K^{\ell+1-j}|(1-\chi_h)\varphi|_{\mathrm{H}^{\ell+1}(K)} \\
&\leq C\,h^{\ell+1-j}|(1-\chi_h)\varphi|_{\mathrm{H}^{\ell+1}(K)}
\end{aligned}
$$

$$\leq \; C\Big(\frac{h}{\underline{d}_K}\Big)^{\ell+1-j-\tau} \underline{d}_K^{\ell+1-j-\tau}\, h^\tau |(1-\chi_h)\varphi|_{\mathrm{H}^{\ell+1}(K)}\,.$$

As $\underline{d}_K \geq h$ and $\ell+1-j-\tau \geq \ell-1-\tau \geq 0$, we deduce from the previous inequality that

$$|(1-\chi_h)\varphi - \varphi_h|_{\mathrm{H}^j(K)} \leq C\,\underline{d}_K^{\ell+1-j-\tau}\, h^\tau |(1-\chi_h)\varphi|_{\mathrm{H}^{\ell+1}(K)}\,.$$

Therefore on each element K there holds

$$\underline{d}_{0,K}^{2\gamma}\;\underline{d}_K^{2(j-2)}\,|(1-\chi_h)\varphi - \varphi_h|^2_{\mathrm{H}^j(K)} \leq C\,h^{2\tau}\underline{d}_{0,K}^{2\gamma}\;\underline{d}_K^{2(\ell+1-\tau-2)}\,|(1-\chi_h)\varphi|^2_{\mathrm{H}^{\ell+1}(K)}\,.$$

Taking the sum of these inequalities over all $K \in \mathscr{T}_h$ with $\underline{d}_K \geq h$ and using (7.8), we obtain finally

$$\|(1-\chi_h)\varphi - \varphi_h\|_{\mathrm{V}^2_\gamma(\Omega)} \leq C\,h^\tau\,|(1-\chi_h)\varphi|_{\mathrm{V}^{\ell+1}_{\beta+\ell+1}(\Omega)}$$

and therefore

$$\|(1-\chi_h)\varphi - \varphi_h\|_{\mathrm{V}^2_\gamma(\Omega)} \leq C\,h^\tau\,\|\varphi\|_{\mathrm{V}^{\ell+1}_{\beta+\ell+1}(\Omega)}\,. \qquad (7.11)$$

In the last inequality we used again (7.9) as before.

From (7.10) and (7.11) we obtain

$$\|\varphi - \varphi_h\|_{\mathrm{V}^2_\gamma(\Omega)} \leq C\,h^\tau\,\|\varphi\|_{\mathrm{V}^{\ell+1}_{\beta+\ell+1}(\Omega)}\,,$$

which concludes the proof of the proposition. ∎

Now we are in a position to use Theorem 6.1 in order to get estimates for the convergence rates of our finite element method.

Theorem 7.4 *Let the multi-exponent $\boldsymbol{\gamma}$ be associated with subsets $\mathscr{A}_0$, or $\mathscr{C}_0$ and $\mathscr{E}_0$, of selected corners and edges with condition* (3.13) *according to* § 3.e. *We assume that,* cf *Theorem* 4.1*,*

$$\mathbf{0} \leq \boldsymbol{\gamma} \leq \mathbf{1} \qquad \textit{and} \qquad \boldsymbol{\delta}^{\mathrm{Dir}} < \boldsymbol{\gamma}.$$

Let ω^2 satisfy the uniqueness hypothesis of Theorem 1.1 (ii) *and for $\boldsymbol{f} \in \mathrm{L}^2(\Omega)^n$ with* div $\boldsymbol{f} = 0$ *let $\boldsymbol{u}$ be the solution of problem* (1.3). *Let the finite element family X_N^h satisfy conditions* (7.1)-(7.2) *and let there exist another family Φ^h satisfying **grad** $\Phi^h \subset \mathrm{X}_N^h$ and conditions* (7.3)-(7.4). *Then for the solutions $\boldsymbol{u}_h$ of the corresponding Galerkin problems* (6.1) *there holds the error estimate*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{X}_N[\mathrm{Y}]} \leq C_\varepsilon\, h^{\min\left\{k,\,\ell-1,\,\lambda^{\mathrm{Neu}}-\varepsilon,\,\min(\boldsymbol{\gamma}-\boldsymbol{\delta}^{\mathrm{Dir}})-\varepsilon\right\}}\,\|\boldsymbol{f}\|_{\mathrm{L}^2(\Omega)^n}\,, \quad \forall\varepsilon > 0. \quad (7.12)$$

We recall that $\boldsymbol{\delta}^{\mathrm{Dir}}$ is defined in (4.2), *λ^{Neu} in* (5.5) *and that $\min(\boldsymbol{\gamma}-\boldsymbol{\delta}^{\mathrm{Dir}})$ is the least component of $\boldsymbol{\gamma}-\boldsymbol{\delta}^{\mathrm{Dir}}$, which is positive by assumption.*

PROOF. Let us fix $\varepsilon > 0$. We have to check that the approximation properties $(\mathfrak{A}1)$ - $(\mathfrak{A}3)$ hold with $\tau = \min(k, \ell - 1, \tau')$ for

$$\tau' = \min\left(\lambda^{\mathrm{Neu}},\ \min(\boldsymbol{\gamma}\ -\ \boldsymbol{\delta}^{\mathrm{Dir}})\right) - \varepsilon,$$

and the result of Theorem 7.4 will be a consequence of Theorem 6.1.

By assumption, X_N^h satisfies the hypotheses of Proposition 7.1. As a consequence, $(\mathfrak{A}1)$ holds for any τ such that $\tau \leq k$ and $\tau < \lambda^{\mathrm{Neu}}$.

The assumption $(\mathfrak{A}2)$ holds by the hypotheses of Theorem 7.4.

Finally, by assumption, Φ^h satisfies the hypotheses of Proposition 7.2. Let $\varphi \in \underline{\mathrm{K}}^{\infty}_{\boldsymbol{\beta}_{\mathrm{Dir}}} \cap \mathring{\mathrm{H}}^1(\Omega)$. Then $\varphi \in \mathrm{V}^{\ell+1}_{\boldsymbol{\beta}+\boldsymbol{\ell}+\mathbf{1}}(\Omega)$ for all $\boldsymbol{\beta} > \boldsymbol{\beta}_{\mathrm{Dir}}$. As a consequence, the estimate (7.5) holds for $\tau \leq \ell - 1$, $\tau < \min(\boldsymbol{\gamma} - \boldsymbol{\beta}_{\mathrm{Dir}}) - 2$. But $\boldsymbol{\beta}_{\mathrm{Dir}} = \boldsymbol{\delta}^{\mathrm{Dir}} - 2$, *cf* Remark 5.3. Therefore $(\mathfrak{A}3)$ holds for any τ such that $\tau \leq \ell - 1$ and $\tau < \min(\boldsymbol{\gamma} - \boldsymbol{\delta}^{\mathrm{Dir}})$.

The theorem is proved. ∎

7.b Concrete applications

For concrete applications, we finally exhibit families of finite element spaces X_N^h that satisfy the conditions of Theorem 7.4, together with relevant choices for the multi-exponent $\boldsymbol{\gamma}$.

(*i*) FINITE ELEMENT SPACES. The leading principle is first to choose a family of spaces Φ^h satisfying conditions (7.3)-(7.4) and then to determine X_N^h as a standard finite element space containing **grad** Φ^h and satisfying (7.1)-(7.2). Thus for the space Φ^h, one can take any almost-affine $\mathscr{C}^1$ finite element as described in [9]. Then X_N^h is any space of almost-affine $\mathscr{C}^0$ elements containing all the gradients of elements of Φ^h. Let us give examples.

We consider only the 2D case.

1. *The Argyris triangle.* Here $\mathscr{T}_h$ can be any triangle, Φ^h consists of polynomials of degree ≤ 5 on each element, so that for X_N^h we can take the standard $\mathbb{P}_4$ elements (or $\mathbb{P}_p$ with $p \geq 4$). In (7.4) we have $\ell = 5$.
2. *The Bogner-Fox-Schmit rectangle.* In this case, $\mathscr{T}_h$ consists of rectangles and $\mathrm{P}_K = \mathbb{Q}_3(K)$, the space of polynomials of partial degree ≤ 3 in each variable. In order to contain all their gradients, X_N^h can be a space of $\mathscr{C}^0$ $\mathbb{Q}_p$ elements with $p \geq 3$.
3. *The Hsieh-Clough-Tocher triangles.* Here $\mathscr{T}_h$ is a triangulation consisting of "supertriangles" (triangular macroelements), each of which is subdivided into 3 triangles by one interior node. Since the HCT functions are $\mathbb{P}_3$ on each subtriangle, its gradients are just $\mathbb{P}_2$, and therefore for X_N^h we can take standard $\mathbb{P}_p$ elements for any $p \geq 2$ on such a triangulation.

(ii) CHOICE OF THE WEIGHT MULTI-EXPONENT. We discuss first the choice of $\boldsymbol{\gamma}$ for a 2D polygon Ω. Let us denote by $\omega_0 > \pi$ its largest non-convex opening and by $\omega_1 < \pi$ its largest convex opening.

A. If Ω is non-convex, we are obliged to use a weight and to take as set of selected corners $\mathscr{A}_0$ at least the set of *non-convex corners* and define γ so that

$$1 - \frac{\pi}{\omega_0} < \gamma.$$

Then the convergence rate in (7.12) is

$$\min\Big(\frac{\pi}{\omega_0},\ \gamma - 1 + \frac{\pi}{\omega_0},\ \frac{\pi}{\omega_1} - 1\Big) - \varepsilon,$$

the contribution $\frac{\pi}{\omega_1} - 1$ coming from the convex corners where there is no weight.
By a more clever choice of $\mathscr{A}_0$ and γ we can obtain the (optimal) convergence rate

$$\frac{\pi}{\omega_0} - \varepsilon.$$

For this we take $\gamma = 1$ and $\mathscr{A}_0$ the set of corners $\boldsymbol{a}$ such that $\frac{\pi}{\omega_{\boldsymbol{a}}} - 1 < \frac{\pi}{\omega_0}$. For example if ω_0 is close to π, we obtain a convergence rate close to 1 if we put into the set of selected corners any corner of opening $> \frac{\pi}{2}$.

B. If Ω is convex, we are not obliged to put a weight and obtain the convergence rate without weight

$$\min\big(k,\ \ell - 1,\ \frac{\pi}{\omega_1} - \varepsilon,\ \frac{\pi}{\omega_1} - 1 - \varepsilon\big)$$

which may be very small ($\frac{\pi}{\omega_1} - 1$) if there are angles close to π. The introduction of a weight allows for restoring the (optimal) convergence rate

$$\min\big(k,\ \ell - 1,\ \frac{\pi}{\omega_1} - \varepsilon\big).$$

For this we take again $\gamma = 1$ and define the set $\mathscr{A}_0$ of selected corners as the set of corners $\boldsymbol{a}$ such that $\frac{\pi}{\omega_{\boldsymbol{a}}} - 1 < \frac{\pi}{\omega_1}$.

For a 3D polyhedron Ω, the principles are the same. We have to choose as set $\mathscr{S}_0$ of selected edges and corners, at least the set of non-convex edges and corners. As Ω is a polyhedron, any edge is a segment the ends of which are corners. And any non-convex edge ends in non-convex corners. Therefore the compatibility condition (3.13) is automatically satisfied for such a choice of $\mathscr{S}_0$. Let ω_0 be the largest non-convex edge opening, λ_0^{Dir} the smallest corner exponent $\lambda_{\boldsymbol{c},1}^{\mathrm{Dir}}$ for non-convex corners, and ω_1 be the largest convex edge opening. If Ω is non-convex and if we choose as $\mathscr{S}_0$ the set of non-convex edges and corners, we have the convergence rate

$$\min\Big(\lambda^{\mathrm{Neu}},\ \gamma - 1 + \frac{\pi}{\omega_0},\ \gamma - \tfrac{1}{2} + \lambda_0^{\mathrm{Dir}},\ \frac{\pi}{\omega_1} - 1\Big) - \varepsilon,$$

for any $\gamma > 1 - \pi/\omega_0$ and $\gamma > \frac{1}{2} - \lambda_0^{\mathrm{Dir}}$. The exponents $\lambda_{\boldsymbol{c},1}^{\mathrm{Dir}}$ of convex corners $\boldsymbol{c}$ have no influence because they are larger than $\pi/\omega_{\boldsymbol{e}}$ for any edge $\boldsymbol{e}$ such that $\boldsymbol{c} \in \overline{\boldsymbol{e}}$ (this is a consequence of the monotonicity principle for Dirichlet eigenvalues, *cf* [18, Ch.19]).

Thus, for a better choice of $\mathscr{S}_0$ and with $\gamma = 1$, we obtain the rate

$$\min\Big(\lambda^{\mathrm{Neu}},\ \frac{\pi}{\omega_0},\ \tfrac{1}{2} + \lambda_0^{\mathrm{Dir}}\Big) - \varepsilon.$$

8 Numerical results

As an illustration of the error estimates, we present results of some finite element computations on an L-shaped domain

$$\Omega = (0, \tfrac{1}{2})^2 \setminus (\tfrac{1}{4}, \tfrac{1}{2})^2 \subset \mathbb{R}^2 .$$

We use $\mathbb{Q}_p$ elements on rectangular grids, discretizing the variational form

$$\int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} + s\int_\Omega r^\alpha \operatorname{div}\boldsymbol{u}\ \operatorname{div}\boldsymbol{v} - \omega^2 \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v} = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \tag{8.1}$$

and subject to the boundary condition $\boldsymbol{u} \times \boldsymbol{n} = 0$ on $\partial\Omega$. This is the regularized formulation with weight

$$w = s\, r^\alpha, \quad \alpha = 2\gamma,$$

where $s > 0$ is a constant and r is the distance to the reentrant corner $\boldsymbol{a}_0 = (\frac{1}{4}, \frac{1}{4})$ of opening angle $\omega_0 = 3\pi/2$.

We present results for two types of problems:

- The *boundary value problem*: $\omega^2 = 0$, $\boldsymbol{f}$ given, in Tables 1 and 2,
- The *eigenvalue problem*: $\boldsymbol{f} = 0$, $\lambda = \omega^2$ unknown, in Tables 3 , 4 and 5.

In both cases, we first illustrate our error estimates which are asymptotic in h by choosing $\mathbb{Q}_p$ elements on a sequence of uniform grids, starting with a very simple grid containing only 3 squares which are then repeatedly divided in four. For both $p = 2$ and $p = 7$ or 8 one can see that the computed convergence rates τ are at least as high as those predicted by our theoretical results in Section 7.

We then show the performance of the p version of our method on a grid with 3 layers of geometric mesh refinement near the corner $\boldsymbol{a}_0$ and $\mathbb{Q}_p$ elements with $p = 1, \ldots, 10$.

In the examples shown in Tables 1 and 2, we chose the right hand side $\boldsymbol{f}$ in such a way that the exact solution $\boldsymbol{u}$ coincides with a singular function $\mathbf{grad}\, S_k$, where S_k in local polar coordinates is given by

$$S_k = r^{\frac{2k}{3}} \sin \tfrac{2k}{3}\theta .$$

We show the first two singular functions $k = 1, 2$. The obvious difference between the two is that $\mathbf{grad}\, S_1 \notin H^1(\Omega)^2$, so that the non-weighted regularized method ($\alpha = 0$)

does not converge, and a weight with $\alpha \in [\frac{2}{3}, 2]$ is necessary, whereas S_2 is more regular, so that we do have convergence even for $\alpha = 0$. We see, however, that in this case, a weight with $\alpha > 0$ improves the convergence, too.

In Table 1, we choose $s = 2$, and we present the quadratic error $e_2 = \|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{\mathrm{L}^2(\Omega)}$ as well as the computed convergence rate τ as functions of the total number of degrees of freedom N. Note that τ is equivalent to the convergence rate of the L^2 norm $\sqrt{e_2}$ with respect to h.

$\boxed{k = 1,\, p = 2}$		$\alpha = 0$		$\alpha = 1$		$\alpha = 2$	
h	N	e_2	τ	e_2	τ	e_2	τ
$1/4$	42	0.84211741		0.04235700		0.07411075	
$1/8$	130	1.01358422	−0.164	0.04768209	−0.105	0.03777494	0.596
$1/16$	450	1.06201671	−0.038	0.03614337	0.223	0.01707371	0.640
$1/32$	1666	1.06756118	−0.004	0.02437231	0.301	0.00732893	0.646

$\boxed{k = 2,\, p = 2}$		$\alpha = 0$		$\alpha = 1$		$\alpha = 2$	
h	N	e_2	τ	e_2	τ	e_2	τ
$1/4$	42	0.00901950		0.00274676		0.01281581	
$1/8$	130	0.00382858	0.758	0.00140259	0.595	0.00728658	0.500
$1/16$	450	0.00152661	0.740	0.00052674	0.789	0.00274435	0.786
$1/32$	1666	0.00059810	0.716	0.00017189	0.856	0.00065502	1.094

$\boxed{k = 1,\, p = 7}$		$\alpha = 0$		$\alpha = 1$		$\alpha = 2$	
h	N	e_2	τ	e_2	τ	e_2	τ
$1/4$	42	1.01609591		0.01846936		0.00760876	
$1/8$	130	1.03912235	−0.017	0.01316194	0.261	0.00307983	0.696
$1/16$	450	1.04370936	−0.003	0.00875260	0.304	0.00123537	0.681
$1/32$	1666	1.04236404	0.001	0.00562906	0.324	0.00049364	0.673

$\boxed{k = 2,\, p = 7}$		$\alpha = 0$		$\alpha = 1$		$\alpha = 2$	
h	N	e_2	τ	e_2	τ	e_2	τ
$1/4$	42	0.00052035		0.00011192		0.00027287	
$1/8$	130	0.00020966	0.700	0.00002948	1.027	0.00004782	1.341
$1/16$	450	0.00008341	0.688	0.00000648	1.130	0.00000778	1.355
$1/32$	1666	0.00003302	0.680	0.00000125	1.207	0.00000124	1.349

Table 1. Boundary value problem, uniform grid

In Table 2 we give e_2 on a fixed (refined) grid as the degree p varies.

		$k=1$			$k=2$		
p	N	$\alpha=0$	$\alpha=1$	$\alpha=2$	$\alpha=0$	$\alpha=1$	$\alpha=2$
1	112	1.5966	0.47592	0.042220	3.15 e-03	7.57 e-05	1.87 e-04
2	390	1.4460	0.02746	0.000497	1.35 e-04	1.71 e-06	1.20 e-05
3	836	1.4264	0.01811	0.000215	4.93 e-05	3.13 e-07	3.55 e-07
4	1450	1.4213	0.01307	0.000113	2.56 e-05	8.91 e-08	8.53 e-08
5	2232	1.4187	0.01003	0.000067	1.52 e-05	3.10 e-08	2.77 e-08
6	3182	1.4170	0.00803	0.000044	9.88 e-06	1.27 e-08	1.09 e-08
7	4300	1.4158	0.00663	0.000031	6.82 e-06	5.84 e-09	4.87 e-09
8	5586	1.4150	0.00561	0.000023	4.93 e-06	2.96 e-09	2.43 e-09
9	7040	1.4144	0.00483	0.000018	3.70 e-06	1.62 e-09	1.31 e-09
10	8662	1.4139	0.00422	0.000014	2.85 e-06	9.37 e-10	7.47 e-10

Table 2. Boundary value problem, refined grid, $s=8$, quadratic L^2-error

In Tables 3, 4 and 5, we show the results of the computation of the first two Maxwell eigenvalues $\lambda = \omega^2$ on the L-shaped domain Ω. We choose $s = 100$ to avoid the spurious eigenvalues. The first eigenfunction is in $\mathrm{X}_N \setminus H_N$, and therefore the first eigenvalue cannot be approximated without weight. The second eigenfunction belongs to H^1, and we see convergence for λ_2 even for $\alpha = 0$. With weight exponent $\alpha = 2$, we see the expected convergence rates $\tau = 2/3$ for the first eigenvalue and $\tau = 4/3$ for the second eigenvalue.

$\boxed{\lambda_1,\ p=2}$		Maxwell		
N	Neumann	$\alpha = 0$	$\alpha = 1$	$\alpha = 2$
42	24.13701659	120.90488226	110.46906479	49.49443834
130	23.81339665	102.04184498	90.48517831	33.23172507
450	23.69017022	98.25631602	82.71080795	27.28036540
1666	23.64177400	96.98393766	76.65583255	25.01965466
$\boxed{\lambda_2,\ p=2}$				
42	57.38784397	75.99447724	71.75781417	61.32884651
130	56.60883955	62.53207509	60.97123999	57.28067304
450	56.55026143	58.76614173	58.07956603	56.66672908
1666	56.54513337	57.40691268	57.09394908	56.56457358
$\boxed{\lambda_1,\ p=8}$				
42	23.62993021	98.66470749	75.44438016	24.49393351
130	23.61789596	97.15288075	69.23173390	23.95780545
450	23.61310276	96.56438960	63.47164884	23.74642793
1666	23.61120066	96.33263584	58.03793226	23.66368700
$\boxed{\lambda_2,\ p=8}$				
42	56.54462340	59.24751481	57.54028363	56.55419599
130	56.54452112	57.59166691	56.90303442	56.54622903
450	56.54450490	56.95618365	56.66869140	56.54478825
1666	56.54450235	56.70727431	56.58653566	56.54454794

Table 3. Eigenvalues, uniform grids, $s = 100$

In 2 dimensions, the Maxwell eigenvalues are the same as the Laplace-Neumann eigenvalues. Therefore we show for comparison the numerical computation of the first two Neumann eigenvalues with the same finite element method, using the standard H^1 variational formulation. For $\alpha = 0$, we compare in fact the first Neumann eigenvalue with the *second* computed eigenvalue, because considered as an analytic function of α, this is the one that for large enough values of α will give an approximation of the first

	Neumann		Maxwell					
$\boxed{\lambda_1, p=2}$			$\alpha=0$		$\alpha=1$		$\alpha=2$	
N	$\|\lambda-\lambda_h\|$	τ	$\|\lambda-\lambda_h\|$	τ	$\|\lambda-\lambda_h\|$	τ	$\|\lambda-\lambda_h\|$	τ
42	0.02232261		4.1209226		3.6789139		1.0963354	
130	0.00861570	0.843	3.3219792	0.191	2.8324970	0.231	0.4075287	0.876
450	0.00339645	0.750	3.1616433	0.040	2.5032137	0.100	0.1554590	0.776
1666	0.00134663	0.707	3.1077517	0.013	2.2467554	0.083	0.0597067	0.731
$\boxed{\lambda_2, p=2}$								
42	0.01491466		0.3439764		0.2690502		0.0846120	
130	0.00113782	2.277	0.1058913	1.043	0.0782876	1.093	0.0130193	1.657
450	0.00010186	1.944	0.0392901	0.798	0.0271478	0.853	0.0021616	1.446
1666	0.00001117	1.689	0.0152519	0.723	0.0097170	0.785	0.0003549	1.380
$\boxed{\lambda_1, p=8}$								
42	0.00084498		3.1789407		2.1954444		0.0374398	
130	0.00033527	0.706	3.1149073	0.016	1.9323080	0.098	0.0147321	0.713
450	0.00013226	0.691	3.0899818	0.006	1.6883397	0.100	0.0057792	0.695
1666	0.00005169	0.688	3.0801659	0.002	1.4581948	0.107	0.0022747	0.683
$\boxed{\lambda_2, p=8}$								
42	0.00000215		0.0478032		0.0176105		0.0001714	
130	0.00000034	1.408	0.0185193	0.724	0.0063407	0.780	0.0000305	1.318
450	0.00000005	1.372	0.0072806	0.694	0.0021963	0.788	0.0000050	1.335
1666	0.00000001	1.345	0.0028786	0.679	0.0007433	0.793	0.0000008	1.338

Table 4. Eigenvalue problem, uniform grids, $s = 100$, errors

exact eigenvalue.

We see that our weighted regularization method with $\alpha = 2$ gives the same convergence rates as for the Neumann eigenvalues, and this despite the fact that the Maxwell eigenfunctions are one order less regular than the Neumann eigenfunctions (the former are the curls of the latter). In Table 5 we see that the p version on a refined grid performs rather well even with a modest number of unknowns.

$\boxed{\lambda_1}$				Maxwell					
p	N	Neumann		$\alpha = 0$		$\alpha = 1$		$\alpha = 2$	
2	390	23.62073663	5 e-04	96.6129	3.09	56.6139	1.40	23.7110324	4 e-3
3	836	23.61085665	4 e-05	96.1942	3.07	55.2695	1.34	23.6452078	1 e-3
4	1450	23.61036255	2 e-05	96.1850	3.07	51.9190	1.20	23.6280100	8 e-4
5	2232	23.61019233	1 e-05	96.1839	3.07	49.3547	1.09	23.6205595	4 e-4
6	3182	23.61010647	6 e-06	96.1832	3.07	47.3127	1.00	23.6167563	3 e-4
7	4300	23.61005760	4 e-06	96.1828	3.07	45.6409	0.93	23.6146005	2 e-4
8	5586	23.61002750	3 e-06	96.1825	3.07	44.2418	0.87	23.6132818	1 e-4
$\boxed{\lambda_2}$									
2	390	56.57346262	5 e-04	56.6162	1 e-3	59.8606	6 e-2	56.6083112	1 e-3
3	836	56.54474445	4 e-06	56.5503	1 e-4	56.5485	7 e-5	56.5448802	7 e-6
4	1450	56.54450293	2 e-08	56.5486	7 e-5	56.5471	5 e-5	56.5445087	1 e-7
5	2232	56.54450190	5 e-10	56.5476	6 e-5	56.5464	3 e-5	56.5445037	3 e-8
6	3182	56.54450188	1 e-10	56.5470	4 e-5	56.5459	2 e-5	56.5445026	1 e-8
7	4300	56.54450187	6 e-11	56.5466	4 e-5	56.5455	2 e-5	56.5445022	6 e-9
8	5586	56.54450187	3 e-11	56.5462	3 e-5	56.5453	1 e-5	56.5445021	3 e-9

Table 5. Eigenvalue problem, refined grid, $s = 100$, eigenvalues and relative errors

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