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WEIGHTED SHARING AND UNIQUENESS OF MEROMORPHIC FUNCTIONS

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Abstract. Introducing the idea of weighted sharing of values we prove some uniqueness theorems for meromorphic functions which improve some existing results.

§1. Introduction and Definitions

Let f and g be two nonconstant meromorphic functions defined in the open complex plane C. If for some $a \in C \cup \{\infty\}$ the a-points of f and g coincide in locations and multiplicities, we say that f and g share the value a CM (counting multiplicities). On the other hand, if the a-points of f and g coincide in locations only, we say that f and g share the value a IM (ignoring multiplicities).

Though we do not explain the standard notations of the value distribution theory because those are available in [2], we explain some notations which will be needed in the sequel.

Definition 1. If s is a nonnegative integer, we denote by $N(r, a; f \mid= s)$ the counting function of those a-points of f whose multiplicity is s, where each a-point is counted according to its multiplicity.

Definition 2. If s is a positive integer, we denote by $\overline{N}(r, a; f \mid\geq s)$ the counting function of those a-points of f whose multiplicities are greater than or equal to s, where each a-point is counted only once.

Definition 3. If s is a nonnegative integer, we denote by $N_s(r, a; f)$ the counting function of a-points of f where an a-point with multiplicity m is counted m times if $m \leq s$ and s times if $m > s$. We put $N_\infty(r, a; f) = N(r, a; f)$.

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Definition 4. Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the counting function of those a-points of f whose multiplicities are different from multiplicities of the corresponding a-points of g, where each a-point is counted only once.

Clearly $\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f)$.

Definition 5. Let f, g share a value a IM. We denote by $\overline{N}(r,a;f < g)$ $(\overline{N}(r,a;f > g))$ the counting function of those a-points of f whose multiplicities are less (greater) than the multiplicities of the corresponding a-points of g, where each a-point is counted only once.

We denote by I a set of infinite linear measure not necessarily the same in all its occurrences. Also $T(r)$ denotes the maximum of $T(r,f)$ and $T(r,g)$.

M. Ozawa [4] proved the following result.

Theorem A. ([4]) *Let f, g be entire functions of finite order such that f and g share 0, 1 CM. If $\delta(0,f) > 1/2$ then $f.g \equiv 1$ unless $f \equiv g$.*

Removing the order restriction in the above result H. Ueda [6] proved the following theorem.

Theorem B. ([6]) *If f, g share 0, 1, ∞ CM and*

$$\limsup_{r\to\infty} \frac{\overline{N}(r,0;f) + \overline{N}(r,\infty,f)}{T(r,f)} < \frac{1}{2}$$

then either $f \equiv g$ or $f.g \equiv 1$.

In this direction H. X. Yi proved the following two results.

Theorem C. ([7]) *If f, g share $0, 1, \infty$ CM and $\overline{N}(r,0;f)+\overline{N}(r,\infty;f) < \{\lambda + o(1)\}T(r,f)$ for $r \in I$ and $0 < \lambda < 1/2$, then $f \equiv g$ or $f.g \equiv 1$.*

Theorem D. ([9]) *If f, g share 0, 1, ∞ CM and $N(r,0;f \mid= 1) + N(r,\infty;f \mid= 1) < \{\lambda + o(1)\}T(r)$ for $r \in I$ and $0 < \lambda < 1/2$ then either $f \equiv g$ or $f.g \equiv 1$.*

Following examples show that in Theorem D the sharing of 0 can not be relaxed from CM to IM.

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EXAMPLE 1. Let $f(z) = \left(\frac{1+e^z}{1-e^z}\right)^2$ and $g(z) = \frac{1+e^z}{1-e^z}$. Then f, g share 0, ∞ IM and 1 CM. Also $N(r, 0; f \mid= 1) \equiv N(r, \infty; f \mid= 1) \equiv 0$ but neither $f \equiv g$ nor $f.g \equiv 1$.

EXAMPLE 2. Let $f(z) = (e^z - 1)^2$ and $g(z) = e^z - 1$. Then f, g share 0 IM and 1, ∞ CM. Also $N(r, 0; f \mid= 1) \equiv N(r, \infty; f \mid= 1) \equiv \overline{N}(r, \infty; f) \equiv 0$ but neither $f \equiv g$ nor $f.g \equiv 1$.

Now one may ask: *Is it possible to relax the nature of sharing of* 0 *in the above results and if possible how far?*

The purpose of the paper is to discuss this problem. To this end we introduce a gradation of sharing of values which we call the weight of sharing.

DEFINITION 6. Let k be a nonnegative integer or infinity. For $a \in C \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$.

DEFINITION 7. Let k be a nonnegative integer or infinity. If for $a \in C \cup \{\infty\}$, $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_o is a zero of $f - a$ with multiplicity m $(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity m $(\leq k)$ and z_o is a zero of $f - a$ with multiplicity m $(> k)$ if and only if it is a zero of $g - a$ with multiplicity n $(> k)$ where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

§2. Lemmas

In this section we present some lemmas which are necessary in the sequel.

LEMMA 1. *If f, g share $(a, 0)$, $(b, 0)$, $(\infty, 0)$ where $b \neq \infty$ and $a \neq b, \infty$ then $T(r, f) \leq 3T(r, g) + S(r, f)$ and $T(r, g) \leq 3T(r, f) + S(r, g)$.*

Proof. The lemma follows as a direct consequence of the second fundamental theorem.

Lemma 2. *Let $c_1 f + c_2 g \equiv c_3$, where c_1, c_2, c_3 are nonzero constants. Then*

(i) $T(r,f) \le \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;f) + S(r,f)$,

(ii) $T(r,g) \le \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + S(r,g)$.

Proof. By the second fundamental theorem we get

$$\begin{aligned} T(r,f) &\le \overline{N}(r,0;f) + \overline{N}(r,c_3/c_1;f) + \overline{N}(r,\infty;f) + S(r,f) \\ &= \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(f,\infty;f) + S(r,f). \end{aligned}$$

In a similar manner we can prove (ii). This proves the lemma.

Lemma 3. *Let f, g share $(a,0)$ and $\phi = \frac{f'}{f-b} - \frac{g'}{g-b}$ where $a \neq \infty$, $b \neq a, \infty$. If $\overline{N}(r,a;f) \neq S(r,f)$ and $\phi \equiv 0$ then $f \equiv g$.*

Proof. Since $\phi \equiv 0$, we get $f - b = c(g-b)$, where c is a constant. Since f, g share $(a,0)$ and $\overline{N}(r,a;f) \neq S(r,f)$, there exists $z_0 \in C$ such that $f(z_0) = g(z_0) = a$. This shows that $c = 1$ because $a \neq b$. Therefore $f \equiv g$. This proves the lemma.

Lemma 4. *Let $a \neq \infty$, $b \neq a, \infty$ be two complex numbers. If f, g share $(a,1)$, $(\infty,0)$, (b,∞) and $f \not\equiv g$ then*

$$\begin{aligned} \overline{N}(r,a;f \mid\ge 2) &\le \overline{N}_*(r,\infty;f,g) + S(r,f), \\ \overline{N}(r,a;g \mid\ge 2) &\le \overline{N}_*(r,\infty;f,g) + S(r,f). \end{aligned}$$

Proof. Since the lemma is obvious when $\overline{N}(r,a;f) = S(r,f)$, we suppose that $\overline{N}(r,a;f) \neq S(r,f)$.

Let $\phi = \frac{f'}{f-b} - \frac{g'}{g-b}$. Since f, g share $(a,1)$ and $f \not\equiv g$, by Lemma 3 it follows that $\phi \not\equiv 0$. Since f, g share $(a,1)$, every multiple a-point of f is a multiple a-point of g and so it is a zero of ϕ. Hence

$$\begin{aligned} \overline{N}(r,a;f \mid\ge 2) &\le N(r,0;\phi) \le T(r,\phi) + O(1) \\ &= N(r,\phi) + m(r,\phi) + O(1) \\ &\le N(r,\phi) + m\Big(r, \frac{f'}{f-b}\Big) + m\Big(r, \frac{g'}{g-b}\Big) + O(1) \\ &= N(r,\phi) + S(r,f) + S(r,g), \end{aligned}$$

by Milloux theorem [2, p. 55].

So by Lemma 1 we get

$$\overline{N}(r,a;f\mid\geq 2)\leq N(r,\phi)+S(r,f). \tag{1}$$

Since f, g share $(a,1)$, it follows that $\overline{N}(r,a;f\mid\geq 2)=\overline{N}(r,a;g\mid\geq 2)$ and so

$$\overline{N}(r,a;g\mid\geq 2)\leq N(r,\phi)+S(r,f). \tag{2}$$

Clearly the possible poles of ϕ occur at the b-points and poles of f, g.

Let z_0 be a b-point of f with multiplicity m. Then $f-b=(z-z_0)^m\alpha(z)$ in some neighbourhood of z_0, where α is analytic at z_0 and $\alpha(z_0)\neq 0$. So $\frac{f'}{f-b}=\frac{\alpha'}{\alpha}+\frac{m}{z-z_0}$ in some neighbourhood of z_0.

Since f, g share (b,∞), in a similar manner we get $\frac{g'}{g-b}=\frac{\beta'}{\beta}+\frac{m}{z-z_0}$ in some neighbourhood of z_0, where β is analytic at z_0 and $\beta(z_0)\neq 0$.

Hence in some neighbourhood of z_0, $\phi=\frac{\alpha'}{\alpha}-\frac{\beta'}{\beta}$ so that z_0 is not a pole of ϕ.

Let z_1 be a pole of f with multiplicity m and a pole of g with multiplicity n. Then in some neighbourhood of z_1 we get $f-b=\gamma(z)/(z-z_1)^m$ and $g-b=\delta(z)/(z-z_1)^n$, where γ, δ are analytic at z_1 and $\gamma(z_1)\neq 0$, $\delta(z_1)\neq 0$. So

$$f'=\frac{\gamma'}{(z-z_1)^m}-\frac{m\gamma}{(z-z_1)^{m+1}}\quad\text{and}\quad g'=\frac{\delta'}{(z-z_1)^n}-\frac{n\delta}{(z-z_1)^{n+1}}$$

in some neighbourhood of z_1.

Hence $\phi=\frac{\gamma'}{\gamma}-\frac{\delta'}{\delta}-\frac{m-n}{z-z_1}$ in some neighbourhood of z_1. This shows that if $m\neq n$, z_1 is a simple pole of ϕ and if $m=n$, z_1 is not a pole of ϕ. Since all the poles of ϕ are simple, we get

$$N(r,\phi)=\overline{N}(r,\phi)\leq\overline{N}_*(r,\infty;f,g). \tag{3}$$

Now the lemma follows from (1), (2) and (3). This proves the lemma.

LEMMA 5. *Let $a\neq\infty$, $b\neq a,\infty$ be two complex numbers. If f, g share $(a,1)$, (b,∞), $(\infty,0)$ and $f\not\equiv g$ then*

$$N_2(r,a;f)\leq N(r,a;f\mid=1)+2\overline{N}_*(r,\infty;f,g)+S(r,f),$$

and

$$N_2(r,a;g)\leq N(r,a;f\mid=1)+2\overline{N}_*(r,\infty;f,g)+S(r,f).$$

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Proof. By Lemma 4 we get

$$\begin{aligned} N_2(r,a;f) &= N(r,a;f\mid=1) + 2\overline{N}(r,a;f\mid\geq 2) \\ &\leq N(r,a;f\mid=1) + 2\overline{N}_*(r,\infty;f,g) + S(r,f) \end{aligned}$$

and

$$\begin{aligned} N_2(r,a;g) &= N(r,a;g\mid=1) + 2\overline{N}(r,a;g\mid\geq 2) \\ &\leq N(r,a;f\mid=1) + 2\overline{N}_*(r,\infty;f,g) + S(r,f). \end{aligned}$$

This proves the lemma.

Lemma 6. *Let $a \neq \infty$, $b \neq a, \infty$ be two comlpex numbers. If f, g share $(a,1)$, (b,∞), $(\infty,1)$ and $f \not\equiv g$ then*

(i) $\overline{N}(r,\infty;f\mid\geq 2) \leq \overline{N}_*(r,\infty;f,g) + S(r,f)$, *and*

(ii) $\overline{N}(r,\infty;g\mid\geq 2) \leq \overline{N}_*(r,\infty;f,g) + S(r,f)$.

Proof. Let $F = a + \frac{(b-a)^2}{f-a}$ and $G = a + \frac{(b-a)^2}{g-a}$. Then F, G share $(a,1)$, (b,∞), $(\infty,1)$. So by Lemma 4 we get

$$\overline{N}(r,a;F\mid\geq 2) \leq \overline{N}_*(r,\infty;F,G) + S(r,f)$$

i.e.,

$$\begin{aligned} \overline{N}(r,\infty;f\mid\geq 2) &\leq \overline{N}_*(r,a;f,g) + S(r,f) \\ &\leq \overline{N}(r,a;f\mid\geq 2) + S(r,f). \end{aligned} \tag{4}$$

Again by Lemma 4 we get

$$\overline{N}(r,a;f\mid\geq 2) \leq \overline{N}_*(r,\infty;f,g) + S(r,f). \tag{5}$$

Now (i) follows from (4) and (5). Since by Lemma 1 $S(r,G) = S(r,g) = S(r,f)$, we can prove (ii) in a similar manner. This proves the lemma.

Lemma 7. *Let $a \neq \infty$, $b \neq a, \infty$ be two complex numbers. If f, g share $(a,1)$, (b,∞), $(\infty,1)$ and $f \not\equiv g$ then*

(i) $N_2(r,\infty;f) \leq N(r,\infty;f\mid=1) + 2\overline{N}_*(r,\infty;f,g) + S(r,f)$,

(ii) $N_2(r,\infty;g) \leq N(r,\infty;f\mid=1) + 2\overline{N}_*(r,\infty;f,g) + S(r,f)$.

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Proof. By Lemma 6 we get

$$\begin{aligned} N_2(r,\infty;f) &= N(r,\infty;f \mid= 1) + 2\overline{N}(r,\infty;f \mid\geq 2) \\ &\leq N(r,\infty;f \mid= 1) + 2\overline{N}_*(r,\infty;f,g) + S(r,f) \end{aligned}$$

and

$$\begin{aligned} N_2(r,\infty;g) &= N(r,\infty;g \mid= 1) + 2\overline{N}(r,\infty;f \mid\geq 2) \\ &\leq N(r,\infty;f \mid= 1) + 2\overline{N}_*(r,\infty;f,g) + S(r,f). \end{aligned}$$

This proves the lemma.

LEMMA 8. *Let $a \neq \infty$, $b \neq a, \infty$ be two complex numbers. If f, g share $(a,1)$, (b,∞), (∞,∞) and $f \not\equiv g$ then*

(i) $N_2(r,a;f) \leq N(r,a;f \mid= 1) + S(r,f)$,

(ii) $N_2(r,a;g) \leq N(r,a;f \mid= 1) + S(r,f)$,

(iii) $N_2(r,\infty;f) \leq N(r,\infty;f \mid= 1) + S(r,f)$, *and*

(iv) $N_2(r,\infty;g) \leq N(r,\infty;f \mid= 1) + S(r,f)$.

Proof. Since f, g share (∞,∞), $\overline{N}_*(r,\infty;f,g) \equiv 0$ and the lemma follows from Lemma 5 and Lemma 7. This proves the lemma.

LEMMA 9. ([3]) *Let f_1, f_2, f_3 be nonconstant meromorphic functions such that $f_1 + f_2 + f_3 \equiv 1$. If f_1, f_2, f_3 are linearly independent then for $i = 1,2,3$*

$$T(r,f_i) \leq \sum_{j=1}^{3} N_2(r,0;f_j) + \sum_{j=1}^{3} \overline{N}(r,\infty;f_j) + \sum_{j=1}^{3} S(r,f_j).$$

§3. Theorems

In this section we present the main results of the paper.

THEOREM 1. *Let f, g share $(0,1)$, $(\infty,0)$, $(1,\infty)$. If*

$$N(r,0;f \mid= 1) + 4\overline{N}(r,\infty;f) < \{\lambda + o(1)\}T(r) \tag{6}$$

for $r \in I$ and $0 < \lambda < 1/2$ then either $f \equiv g$ or $f.g \equiv 1$.

Theorem 2. *Let f, g share $(0,1)$, (∞,∞), $(1,\infty)$. If*

$$N(r,0;f\mid=1)+N(r,\infty;f\mid=1)<\{\lambda+o(1)\}T(r) \tag{7}$$

for $r\in I$ and $0<\lambda<1/2$ then either $f\equiv g$ or $f.g\equiv 1$.

Example 2 shows that in Theorem 1 and Theorem 2 sharing $(0,1)$ can not be relaxed to sharing $(0,0)$. Also the following example shows that the conditions (6) and (7) are sharp.

Example 3. Let $f(z)=e^z(1-e^z)$, $g(z)=e^{-z}(1-e^{-z})$. Then f, g share $(0,\infty)$, (∞,∞), $(1,\infty)$ and $N(r,0;f\mid=1)\sim\frac{1}{2}T(r)$, $N(r,\infty;f\mid=1)\equiv\overline{N}(r,\infty;f)\equiv 0$. Also neither $f\equiv g$ nor $f.g\equiv 1$.

Proof of Theorem 1. We suppose that $f\not\equiv g$. Without loss of generality, we suppose that there exists a set I of infinite linear measure such that $T(r,g)\le T(r,f)$ for $r\in I$, because otherwise we have only to interchange f and g in our discussion, noting by Lemma 1 that $S(r,f)=S(r,g)$. Let

$$h=\frac{f-1}{g-1}. \tag{8}$$

Since f, g share $(1,\infty)$, $(\infty,0)$ it follows that

$$N_2(r,0;h)\le 2\overline{N}(r,0;h)\le 2\overline{N}(r,\infty;f<g)$$

and

$$N_2(r,\infty;h)\le 2\overline{N}(r,\infty;h)\le 2\overline{N}(r,\infty;f>g).$$

Let $f_1=f$, $f_2=-gh$ and $f_3=h$. Then by (8) it follows that

$$f_1+f_2+f_3\equiv 1. \tag{9}$$

If possible, we suppose that f_1, f_2, f_3 are linearly independnt. It is clear that a zero of h is not a zero of f_2 so that $N_2(r,0;f_2)\le N_2(r,0;g)$. Then by Lemma 9, Lemma 5 and Lemma 1 we get

$$\begin{aligned}T(r,f)&\le\sum_{j=1}^{3}N_2(r,0;f_j)+\sum_{j=1}^{3}\overline{N}(r,\infty;f_j)+S(r,f)\\&\le N_2(r,0;f)+N_2(r,0;g)+N_2(r,0;h)+\overline{N}(r,\infty;f)\\&\quad+\overline{N}(r,\infty;gh)+\overline{N}(r,\infty;h)+S(r,f)\end{aligned}$$

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$$\begin{aligned}
&\le 2N(r,0;f\mid=1)+4\overline{N}_*(r,\infty;f,g)+2\overline{N}(r,0;h)+\overline{N}(r,\infty;f)\\
&\qquad+\overline{N}(r,\infty;h(g-1))+\overline{N}(r,\infty;h)+S(r,f)\\
&\le 2N(r,0;f\mid=1)+4\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;f)\\
&\qquad+\{\overline{N}(r,0;h)+\overline{N}(r,\infty;h)\}+S(r,f)\\
&\le 2N(r,0;f\mid=1)+7\overline{N}(r,\infty;f)\\
&\qquad+\{\overline{N}(r,\infty;f<g)+\overline{N}(r,\infty;f>g)\}+S(r,f)\\
&\le 2N(r,0;f\mid=1)+8\overline{N}(r,\infty;f)+S(r,f)\\
&<\{2\lambda+o(1)\}T(r,f)\quad \text{for } r\in I,
\end{aligned}$$

which is a contradiction.

Therefore f_1, f_2, f_3 are linearly dependent and so there exist constants c_1, c_2, c_3, not all zero, such that

$$c_1f_1+c_2f_2+c_3f_3\equiv 0. \tag{10}$$

If $c_1=0$, we get from (10) $h(c_3-c_2g)\equiv 0$, which is a contradiction because f, g are nonconstant. So $c_1\ne 0$ and eliminating f_1 from (9) and (10) we get

$$cf_2+df_3\equiv 1, \tag{11}$$

where $c=1-(c_2/c_1)$ and $d=1-(c_3/c_1)$ and clearly $|c|+|d|\ne 0$.

Now we consider the following cases.

Case I. Let $c.d\ne 0$. Then from (11) and (8) we get

$$\begin{aligned}
&-cgh+dh\equiv 1,\\
&\text{i.e., } -c\Big(1+\frac{f-1}{h}\Big)h+dh\equiv 1,\\
&\text{i.e., } (d-c)h-cf\equiv 1+c.
\end{aligned}\tag{12}$$

Since f is nonconstant, it follows that $c\ne d$. Let $c\ne -1$. Then by Lemma 2 and Lemma 5 we get

$$\begin{aligned}
T(r,f)&\le \overline{N}(r,0;f)+\overline{N}(r,0;h)+\overline{N}(r,\infty;f)+S(r,f)\\
&\le \overline{N}(r,0;f)+\overline{N}(r,\infty;f<g)+\overline{N}(r,\infty;f)+S(r,f)\\
&\le N(r,0;f\mid=1)+2\overline{N}_*(r,\infty;f,g)+2\overline{N}(r,\infty;f)+S(r,f)\\
&\le N(r,0;f\mid=1)+4\overline{N}(r,\infty;f)+S(r,f)\\
&<\{\lambda+o(1)\}T(r,f)\quad \text{for } r\in I,
\end{aligned}$$

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which is a contradiction.

Let $c = -1$. Then $d \neq -1$ and from (12) we get

$$\begin{aligned}(d+1)h + f &\equiv 0,\\ \text{i.e., } (d+1)\frac{f-1}{g-1} + f &\equiv 0,\\ \text{i.e., } \frac{d+1}{f} - g &\equiv d.\end{aligned}$$

So by Lemma 2, Lemma 5 and the first fundamental theorem we get

$$T\Big(r, \frac{1}{f}\Big) \leq \overline{N}\Big(r, 0; \frac{1}{f}\Big) + \overline{N}(r, 0; g) + \overline{N}\Big(r, \infty; \frac{1}{f}\Big) + S(r, f)$$

i.e.,

$$\begin{aligned}T(r, f) &\leq 2\overline{N}(r, 0; f) + \overline{N}(r, \infty; f) + S(r, f)\\ &\leq 2N(r, 0; f \mid= 1) + 4\overline{N}_*(r, \infty; f, g) + \overline{N}(r, \infty; f) + S(r, f)\\ &\leq 2N(r, 0; f \mid= 1) + 5\overline{N}(r, \infty; f) + S(r, f)\\ &< \{2\lambda + o(1)\}T(r, f) \quad \text{for } r \in I,\end{aligned}$$

which is a contradiction. Therefore the case $c.d \neq 0$ does not arise.

Case II. Let $c.d = 0$.

Let $c = 0$. Then $d \neq 0$ and so from (11) we get $df - g \equiv d - 1$. Since $f \not\equiv g$, $d \neq 1$ and so by Lemma 2 and Lemma 5 we get

$$\begin{aligned}T(r, f) &\leq \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, \infty; f) + S(r, f)\\ &\leq 2N(r, 0; f \mid= 1) + 4\overline{N}_*(r, \infty; f, g) + \overline{N}(r, \infty; f) + S(r, f)\\ &\leq 2N(r, 0; f \mid= 1) + 5\overline{N}(r, \infty; f) + S(r, f)\\ &< \{2\lambda + o(1)\}T(r, f) \quad \text{for } r \in I,\end{aligned}$$

which is a contradiction.

Therefore $c \neq 0$ and so $d = 0$. From (11) we get

$$-cf + \frac{1}{g} \equiv 1 - c. \tag{13}$$

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If $c \neq 1$, by Lemma 2 and Lemma 5 we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,0;f) + \overline{N}(r,0;1/g) + \overline{N}(r,\infty;f) + S(r,f) \\ &\leq N(r,0;f \mid= 1) + 2\overline{N}_*(r,\infty;f,g) + 2\overline{N}(r,\infty;f) + S(r,f) \\ &\leq N(r,0;f \mid= 1) + 4\overline{N}(r,\infty;f) + S(r,f) \\ &< \{\lambda + o(1)\}T(r,f) \quad \text{for } r \in I, \end{aligned}$$

which is a contradiction.

So $c = 1$ and from (13) we get $f.g \equiv 1$. This proves the theorem.

Proof of Theorem 2. Using Lemma 8 the theorem can be proved in a similar manner noting that $\overline{N}(r,0;h) \equiv \overline{N}(r,\infty;h) \equiv 0$ and $N_2(r,0;h) \leq 2\overline{N}(r,0;h)$, $\overline{N}(r,\infty;f) \leq N_2(r,\infty;f)$.

§4. Consequences

In this section we discuss some consequences of Theorem 1 and Theorem 2.

DEFINITION 8. For $S \subset C \cup \{\infty\}$ we denote by $E_f(S)$ the set $E_f(S) = \bigcup_{a \in S}\{z : f(z) - a = 0\}$, where an a-point of multiplicity m is counted m times.

DEFINITION 9. For $S \subset C \cup \{\infty\}$ we define $E_f(S,k)$ as $E_f(S,k) = \bigcup_{a \in S} E_k(a;f)$, where k is a nonnegative integer or infinity.

Clearly $E_f(S) = E_f(S,\infty)$.

Gross and Osgood [1] proved the following theorem.

THEOREM E. ([1]) *Let $S_1 = \{-1,1\}$, $S_2 = \{0\}$. If f and g are entire functions of finite order such that $E_f(S_j) = E_g(S_j)$ for $j = 1,2$ then $f \equiv \pm g$ or $f.g \equiv \pm 1$*

Extending this result Tohge [5] and Yi [8] proved the following two theorems.

THEOREM F. ([5]) *Let $S_1 = \{1,\omega,\omega^2,\ldots,\omega^{n-1}\}$, $S_2 = \{0\}$, $S_3 = \{\infty\}$ where n is an integer (≥ 2) and $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$. If $E_f(S_j) = E_g(S_j)$ for $j = 1,2,3$ then $f^n \equiv g^n$ or $f^n.g^n \equiv 1$.*

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Theorem G. ([8]) *Let* $S_1 = \{a+b, a+b\omega, \dots, a+b\omega^{n-1}\}$, $S_2 = \{a\}$, $S_3 = \{\infty\}$, *where* n *is an integer* (≥ 2), a, b $(b \neq 0)$ *are constants and* $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$. *If* $E_f(S_j) = E_g(S_j)$ *for* $j = 1,2,3$ *then* $f - a \equiv t(g-a)$ *where* $t^n = 1$ *or* $(f-a)(g-a) \equiv s$ *where* $s^n = b^{2n}$.

As an application of Theorem 2 we improve Theorem G.

Theorem 3. *Let* S_1, S_2, S_3 *be defined as in Theorem G. If* $E_f(S_1,\infty) = E_g(S_1,\infty)$, $E_f(S_2,1) = E_g(S_2,1)$ *and* $E_f(S_3,\infty) = E_g(S_3,\infty)$ *then* $f - a \equiv t(g-a)$ *where* $t^n = 1$ *or* $(f-a)(g-a) \equiv s$ *where* $s^n = b^{2n}$.

Proof. Let $F = \left(\frac{f-a}{b}\right)^n$, $G = \left(\frac{g-a}{b}\right)^n$. Then F, G share $(0,1)$, $(1,\infty)$ and (∞,∞). Since $N(r,0;F \mid= 1) \equiv N(r,\infty;F \mid= 1) \equiv 0$, it follows from Theorem 2 that either $F \equiv G$ or $F.G \equiv 1$ from which the theorem follows. This proves the theorem.

Following are two simple consequences of Theorem 1 and Theorem 2.

Theorem 4. *Let* f, g *share* $(0,0)$, $(1,\infty)$ *and* $(\infty,1)$. *If*

$$N(r,\infty;f \mid= 1) + 4\overline{N}(r,0;f) < \{\lambda + o(1)\}T(r) \quad \textit{for } r \in I,$$

where $0 < \lambda < 1/2$, *then either* $f \equiv g$ *or* $f.g \equiv 1$.

Proof. Let $F = 1/f$ and $G = 1/g$. Then F, G satisfy the conditions of Theorem 1. So either $F \equiv G$ or $F.G \equiv 1$, from which the theorem follows.

Theorem 5. *Let* f, g *share* $(0,\infty)$, $(1,\infty)$ *and* $(\infty,1)$. *If*

$$N(r,0;f \mid= 1) + N(r,\infty;f \mid= 1) < \{\lambda + o(1)\}T(r) \quad \textit{for } r \in I,$$

where $0 < \lambda < 1/2$ *then either* $f \equiv g$ *or* $f.g \equiv 1$.

Proof. Let $F = 1/f$, $G = 1/g$. Then F, G satisfy the conditions of Theorem 2. So either $F \equiv G$ or $F.G \equiv 1$, from which the theorem follows.

Remark 1. If f has at least one zero or pole then the possibility $f.g \equiv 1$ does not arise in Theorems 1, 2, 4, 5.

Definition 10. ([6]) Let $\{a_n\}$, $\{b_n\}$ and $\{p_n\}$ be three disjoint sequences with no finite limit point. If it is possible to construct a meromorphic function f in the plain C whose zeros, 1-points and poles are exactly $\{a_n\}$, $\{b_n\}$ and $\{p_n\}$ respectively, where their multiplicities are taken into consideration, then the given triad $(\{a_n\}, \{b_n\}, \{p_n\})$ is called a zero-one-pole set. Further if there exists only one meromorphic function f whose zero-one-pole set is just the given triad then the triad is called unique.

H. Ueda [6] proved the following result.

Theorem H. ([6]) *If* $n(r, 0; f) + n(r, \infty; f) \not\equiv 0$ *and*

$$\limsup_{r\to\infty} \frac{\overline{N}(r, 0; f) + \overline{N}(r, \infty; f)}{T(r, f)} < \frac{1}{2}$$

then the zero-one-pole set of f *is unique.*

As an application of Theorem 2 and Remark 1 we can improve Theorem H.

Theorem 6. *If* $n(r, 0; f) + n(r, \infty; f) \not\equiv 0$ *and*

$$N(r, 0; f \mid = 1) + N(r, \infty; f \mid = 1) < \{\lambda + o(1)\}T(r, f) \quad \text{for } r \in I,$$

where $0 < \lambda < 1/2$ *then the zero-one-pole set of* f *is unique.*

Corollary 1. *If* $n(r, 0; f) + n(r, \infty; f) \not\equiv 0$ *and* f *has at most a finite number of simple zeros and poles then zero-one-pole set of* f *is unique.*

Proof. If f is transcendental, the corollary follows from Theorem 6. Let f be rational and g have the same zero-one-pole set of f. Then g is also rational and $f = cg$, where c is a constant. Since f is rational, there exists a point $z_0 \in C$ such that $f(z_0) = 1$ and so $g(z_0) = 1$. This shows that $c = 1$ and hence $f \equiv g$. This proves the corollary.

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