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WEIGHTED SOBOLEV-POINCARÉ INEQUALITIES AND POINTWISE ESTIMATES FOR A CLASS OF DEGENERATE ELLIPTIC EQUATIONS

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ABSTRACT. In this paper we prove a Sobolev-Poincaré inequality for a class of function spaces associated with some degenerate elliptic equations. These estimates provide us with the basic tool to prove an invariant Harnack inequality for weak positive solutions. In addition, Hölder regularity of the weak solutions follows in a standard way.

1

Let $\mathscr{L} = \sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j)$ be a second-order degenerate elliptic operator in divergence form with measurable coefficients. In this paper we shall obtain pointwise estimates for the weak solutions of $\mathscr{L}u = 0$ (Hölder continuity of the weak solutions and Harnack inequality for nonnegative solutions).

Let us recall that the original results for elliptic operators were obtained by De Giorgi, Nash, and Moser. An extensive bibliography about the degenerate case can be found in [FL1, FL2, FS].

To introduce the results of the present paper, let us recall some recent results. In [FL1, FL2] a suitable metric d is associated with the differential operator $\mathscr{L}$ in such a way that we obtain a new geometry which is natural for the degenerate operator as the Euclidean geometry is natural for the Laplace operator (or, more precisely, as a suitable Riemannian geometry is natural for a second-order elliptic operator). In the smooth case, this idea is contained in many papers: we refer to [FP, NSW]. The basic results in [FL1, FL2] are obtained via a precise description of this geometry under suitable technical hypotheses on the coefficients whose aim is to give a nonsmooth formulation of the Hörmander hypoellipticity condition for sum-of-squares operators. We note that the same idea is used in [NSW, S, J, V] to obtain pointwise estimates for sum-of-squares operators. On the other hand, a different class of degenerate elliptic operators is considered in [FKS]: instead of a geometrical degeneracy, a measure degeneracy is allowed. A typical example of this class is given by $\mathscr{L}u = \operatorname{div}(\omega(x)\nabla u)$, where ω is a weight function belonging to the A_2-class of Muckenhoupt. Unified results for a class containing both the operators in [FL1] and in [FKS] have

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been recently proved in [FS]. In addition, classes of operators which somehow are intermediate have been considered in [CW1, CW2, CRS].

In the present paper, we assume that the quadratic form of $\mathscr{L}$ is equivalent to the diagonal form $\omega(x)\sum_{j=1}^{n}\lambda_j^2(x)\xi_j^2$, where the λ_j's are Lipschitz continuous nonnegative functions and ω is an A_2-weight function with respect to the balls of the metric d associated with the vector fields $\lambda_1\partial_1, \dots, \lambda_n\partial_n$. For precise definitions, we refer to §2. In addition, we assume that there exists a nice family of integral curves of these vector fields starting from an arbitrary point x. More precisely, we suppose that, if $\xi_1, \dots, \xi_n$ are real parameters bounded away from zero, then the integral curve $t \to \exp_x(\sum_j t\xi_j\lambda_j\partial_j)$ does not approach the coordinate hyperplanes centered at x too fast (i.e., faster than the sides of a family of n-intervals which are equivalent to the d-balls). An analytic formulation of this hypothesis is contained in (H.4): we note that this condition is satisfied in the case considered in [FL1, FL2, FS]. On the other hand, in §6 we shall give some sufficient analytic conditions such that (H.4) holds, showing the class considered here is very large and contains many different kinds of degeneration. In some sense, the present results are related more to the results in [FL3] where a noninvariant Harnack inequality is proved for a large class of degenerate operators. It is possible to prove that if $\omega = 1$, then hypothesis (H.4) implies that a sub-Riemannian structure in the sense of [FL3] is associated with the operator. In the present paper, however, quantitative estimates are obtained in such a way that an invariant Harnack inequality and hence Hölder regularity follows. We note that in some particular cases our results partially overlap with those in [CW1, CW2, CRS]. Moreover, in the case $n = 2$ and $\omega \equiv 1$, recently Xu [X] has proved similar pointwise estimates. Finally, let us note that some related results have been obtained by different techniques by Kusuoka and Stroock (see [KuS] and previous papers quoted therein).

Following the Moser iteration technique, the crucial point of our proof is to obtain a weighted Sobolev-Poincaré inequality. To this end we show that if $\beta \in (0, 1)$ is fixed, it is possible to find a family of deformed quasi-balls such that a large part of these balls is attained by our integral curves. The meaning of 'large part' is that the measure of the region which we can reach by our integral curves is at least β times the measure of the deformed quasi-ball. Successively a careful control of the constants and a geometric result due to Kohn and Jerison enables us to obtain our result. Once the Sovolev-Poincaré inequality is obtained, the proof can be carried out in the same way as in [FS].

2

Let Ω be a bounded open subset of $\mathbb{R}^n$ and let $\mathscr{L}$ be the second-order differential operator in Ω defined in the following way:

$$\mathscr{L} = \sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j),$$

where $a_{ij} = a_{ji}$ are real bounded measurable functions for $i, j = 1, \dots, n$.

We will denote by $A = A(x)$ the matrix of the coefficients a_{ij}.

In the sequel we will assume the following hypotheses are satisfied:

(H.1) There exists $\nu \in (0, 1)$ such that

$$\nu\omega(x)\sum_{j=1}^{n}\lambda_j^2(x)\xi_j^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \frac{1}{\nu}\omega(x)\sum_{j=1}^{n}\lambda_j^2(x)\xi_j^2$$

for any $x, \xi \in \Omega \times \mathbb{R}^n$, where

(H.2) $\lambda_1, \ldots, \lambda_n$ are bounded nonnegative Lipschitz continuous functions in a neighbourhood of Ω.

(H.3) The distance d associated with the vector fields $\lambda_1\partial_1, \ldots, \lambda_n\partial_n$ in the sense of Fefferman and Phong (see [FP, FL1], and Definition 2.1 below) is finite in Ω and the function ω is an A_2-weight function with respect to the distance d, i.e.

$$\frac{1}{|B(x,r)|}\int_{B(x,r)}\omega(y)\,dy\cdot\left(\frac{1}{|B(x,r)|}\int_{B(x,r)}\omega(y)^{-1}\,dy\right) \leq c_{\omega,2}$$

for any $x \in \Omega$ and $r \in (0, \rho_0]$, where ρ_0 is a fixed suitable positive number, and $B(x, r)$ denotes the ball of center x and radius r with respect to the metric d. Here $|E|$ denotes the Lebesgue measure of the subset E of $\mathbb{R}^n$.

In what follows, if E is an L-measurable subset of $\mathbb{R}^n$, we will denote by $\omega(E)$ the ω-measure of E, i.e., $\omega(E) = \int_E \omega(y)\,dy$.

A precise definition of the above distance d will be given in the sequel, together with a basic property (the so-called *doubling property*) giving a precise sense to (H.3).

A further hypothesis on the vector fields $\lambda_1\partial_1, \ldots, \lambda_n\partial_n$ will be specified later (H.4): roughly speaking, we will require that some nice family of optimal curve exists.

Let us now recall some definitions.

Definition 2.1. We will say that an absolutely continuous curve $\gamma\colon [0, T] \to \Omega$ is a subunit curve (with respect to $\lambda_1\partial_1, \ldots, \lambda_n\partial_n$) if

$$(\gamma'(t), \xi)^2 \leq \sum_{j=1}^{n}\lambda_j^2(\gamma(t))\xi_j^2$$

for any $\xi \in \mathbb{R}^n$ and for a.e. $t \in [0, T]$.

If $x, y \in \Omega$, we will put $d(x, y) = \inf\{T > 0$ such that there exists a subunit curve $\gamma\colon [0, T] \to \Omega$ such that $\gamma(0) = x$ and $\gamma(T) = y\}$. We will say that $d(x, y) = \infty$ if the above set is empty.

Clearly, by hypothesis (H.3), d is a metric on Ω; moreover, since we are looking for local properties of the operator $\mathscr{L}$, we may suppose that $\lambda_1, \ldots, \lambda_n$ are defined on all of $\mathbb{R}^n$.

For the sake of simplicity, we will denote by λ the vector-valued function $\lambda = (\lambda_1, \dots, \lambda_n)$, by ∇_λ the vector-valued differential operator $(\lambda_1\partial_1, \dots, \lambda_n\partial_n)$, and by div_λ the differential operator acting on vector-valued functions $f = (f_1, \dots, f_n)$ as

$$\mathrm{div}_\lambda = \sum_{i=1}^n \partial_i f_i.$$

Moreover, without loss of generality, we may suppose $\lambda_1 \equiv 1$, since at least one of the vector fields $\lambda_1\partial_1, \dots, \lambda_n\partial_n$ is different from zero in a neighbourhood of a given point.

In the sequel we will denote by L a positive constant such that

$$(2.1) \qquad |\lambda(x) - \lambda(y)| \le L|x-y| \quad \forall x, y \in \mathbb{R}^n.$$

Definition 2.2. Let $x \in \mathbb{R}^n$ and $r > 0$ be fixed. Put

$$C_j(x, r) = \{u_j(t), 0 \le t \le r, \text{ where } u = (u_1, \dots, u_n) \text{ is any subunit curve such that } u(0) = x\}$$

for $j = 1, \dots, n$.

It is easy to verify that $C_j(x, r)$ is a compact interval containing x_j, the jth component of x, for $j = 1, \dots, n$. Now we can put

$$\Lambda_k(x, r) = \max_{s_j \in C_j(x, r)} \lambda_k(s_1, \dots, s_n).$$

Remark. It follows from Definitions 2.1 and 2.2 that $\Lambda_j(x, r) > 0$ for $r > 0$ if the distance d is continuous with respect to the Euclidean topology. We note also that, if $r \le (2L)^{-1}$,

$$\Lambda_k(x, r) \ge \max_{s_j \in C_j(x, r), j \ne k} \lambda_k(s_1, \dots, x_k, \dots, s_n) \ge \tfrac{1}{2}\Lambda_k(x, r).$$

Indeed, the first inequality is obvious; on the other hand, if $s_j \in C_j(x, r)$, $j = 1, \dots, n$, there exists a subunit curve γ such that $\gamma(0) = x$ and $\gamma_k(t) = s_k$ for a suitable $t \le r$. Hence

$$\begin{aligned} \lambda_k(s_1, \dots, s_n) &\le \lambda_k(s_1, \dots, x_k, \dots, s_n) + L|\gamma_k(t) - x_k| \\ &\le \lambda_k(s_1, \dots, x_k, \dots, s_n) + L\int_0^t |\gamma_k'(s)|\,ds \\ &\le \lambda_k(s_1, \dots, x_k, \dots, s_n) + L\int_0^t \lambda_k(\gamma(s))\,ds \\ &\le \lambda_k(s_1, \dots, x_k, \dots, s_n) + Lt\Lambda_k(x, t), \end{aligned}$$

so that

$$\Lambda_k(x, r) \le \max_{s_j \in C_j(x, r), j \ne k} \lambda_k(s_1, \dots, x_k, \dots, s_n) + Lr\Lambda_k(x, r),$$

and the assertion follows.

If $x \in \mathbb{R}^n$ and $r > 0$, denote by $Q(x, r)$ the n-dimensional open interval

$$Q(x, r) = \prod_{k=1}^{n} (x_k - r\Lambda_k(x, r), x_k + r\Lambda_k(x, r)).$$

We will prove that these intervals are equivalent to the metric balls in the following sense.

Theorem 2.3. *Suppose* $\Lambda_k(x, r) > 0$ *for any* $x \in \mathbb{R}^n$, $r > 0$, *and* $k = 1, \dots, n$. *Then there exists a positive contant* b *such that*

$$Q(x, r) \supseteq B(x, r) \supseteq Q(x, r/b)$$

for every x *and* r, $r \in (0, r_0]$, *where* b *and* r_0 *depend only on* n *and* L.

Proof. Let y belong to $B(x, r)$. If k is fixed, there exists a subunit curve $\gamma: [0, T] \mapsto \mathbb{R}^n$ such that $\gamma(0) = x$, $\gamma(T) = y$, $T < r$, so that

$$|x_k - y_k| \le \int_0^T |\gamma_k'(t)|\, dt \le \int_0^T \lambda_k(\gamma(t))\, dt \le T\Lambda_k(x, T) \le r\Lambda_k(x, r),$$

and the first inclusion is proved. The proof of the second inclusion is a little bit more complicated. To prove it, let us prove some preliminary results (which in turn will be helpful in what follows).

(2.3.1) For fixed x and $r \in (0, r_0]$ it is possible to renumber the variables (in general the permutation depends on x and r) in such a way that

(i) $\Lambda_1(x, r) \ge \Lambda_2(x, r) \ge \cdots \ge \Lambda_n(x, r)$,

(ii) $\Lambda_k(x, r) \le 2\Lambda_k^*(x, r) = \max_{s_j \in C_j(x, r), j<k} \lambda_k(s_1, \dots, x_k, \dots, x_n)$.

Renumber the variables and the indices in such a way that (i) is satisfied. If $s_j \in C_j(x, r)$ for $j = 1, \dots, n$, we have

$$\lambda_k(s_1, \dots, s_n) \le \lambda_k(s_1, \dots, x_k, \dots, x_n) + L \sum_{i=k}^{n} |s_i - x_i|.$$

On the other hand, if $i \ge k$ is fixed, there exists a subunit curve γ starting from x and such that $\gamma_i(t) = s_i$ for a suitable $t \le r$,

$$|\gamma_i(t) - x_i| \le \int_0^t \lambda_i(\gamma(s))\, ds \le t\Lambda_i(x, t)$$

so that

$$\begin{aligned} \lambda_k(s_1, \dots, s_n) &\le \lambda_k(s_1, \dots, x_k, \dots, x_n) + L \sum_{i=k}^{n} r\Lambda_i(x, r) \\ &\le \lambda_k(s_1, \dots, x_k, \dots, x_n) + L(n-k+1)r\Lambda_k(x, r) \end{aligned}$$

and hence

$$\Lambda_k(x, r) \le \Lambda_k^*(x, r) + L(n-k+1)r\Lambda_k(x, r).$$

If $r < 1/(2Ln)$, (ii) follows.

We are now able to prove our second inclusion. Let x and r be fixed and let us renumber the variables as in (2.3.1). For the sake of simplicity the proof will be carried out in the case $n = 3$.

Let $\overline{s_1}$, $\overline{\sigma_1} \in C_1(x, r)$ and $\overline{\sigma_2} \in C_2(x, r)$ such that $\lambda_2(\overline{s_1}, x_2, x_3) = \Lambda_2^*(x, r)$ and $\lambda_3(\overline{\sigma_1}, \overline{\sigma_2}, x_3) = \Lambda_3^*(x, r)$. By definition there exists a subunit curve h such that $h(0) = x$, $h_1(t_0) = \overline{s_1}$, with $0 \le t_0 \le r$. An analogous assertion holds for $\overline{\sigma}_1$. Thus

$$|\overline{s_1} - x_1| \le r \quad \text{and} \quad |\overline{\sigma_1} - x_1| \le r. \tag{2.3.2}$$

In addition, there exists a suitable subunit curve h such that $h(0) = x$, $h_2(t_0) = \overline{\sigma_2}$ with $0 \le t_0 \le r$. We have:

$$\begin{aligned} |\overline{\sigma}_2 - x_2| = |h_2(t_0) - x_2| &\le \int_0^{t_0} \lambda_2(h(s))\, ds \le t_0 \Lambda_2(x, t_0) \\ &\le r\Lambda_2(x, r) \le 2r\Lambda_2^*(x, r). \end{aligned} \tag{2.3.3}$$

Now let y belong to $Q(x, r)$; suppose $y_j > x_j$ for $j = 1, 2, 3$ (otherwise the proof will be modified in an obvious way). In order to prove that $d(x, y) < br$, we will use a technique employed in previous works (see, e.g., [FL1, FL2]). We will construct a piecewise linear curve from x to y by using integral curves of the vector fields $\pm X_j = \pm\lambda_j\partial_j$, $j = 1, 2, 3$, in the following way:

(1) from (x_1, x_2, x_3) to $(\overline{s_1}, x_2, x_3)$ along $\pm X_1$,
(2) from $(\overline{s_1}, x_2, x_3)$ to $(\overline{s_1}, \overline{\sigma_2}, x_3)$ along $\pm X_2$,
(3) from $(\overline{s_1}, \overline{\sigma_2}, x_3)$ to $(\overline{\sigma_1}, \overline{\sigma_2}, x_3)$ along $\pm X_1$,
(4) from $(\overline{\sigma_1}, \overline{\sigma_2}, x_3)$ to $(\overline{\sigma_1}, \overline{\sigma_2}, y_3)$ along X_3,
(5) from $(\overline{\sigma_1}, \overline{\sigma_2}, y_3)$ to $(\overline{s_1}, \overline{\sigma_2}, y_3)$ along $\pm X_1$,
(6) from $(\overline{s_1}, \overline{\sigma_2}, y_3)$ to $(\overline{s_1}, y_2, y_3)$ along $\pm X_2$,
(7) from $(\overline{s_1}, y_2, y_3)$ to (y_1, y_2, y_3) along $\pm X_1$.

Now we must prove that the length of each of the above arcs (i.e., the time required along integral curves) can be estimated by an absolute constant times the radius r. By (2.3.2) and (2.3.3), the length of the arc (1) is less than or equal to r, whereas the lengths of the arcs (3), (5), and (7) are less than or equal to $2r$.

Let us now estimate the length of the arcs (2) and (6). We will estimate the length of an integral curve of $\pm X_2$ from $(\overline{s_1}, z_2, z_3)$ to $(\overline{s_1}, \overline{\sigma_2}, z_3)$ where $|z_2 - x_2| < r\Lambda_2(x, r)$ and $|z_3 - x_3| < r\Lambda_3(x, r)$. Suppose, e.g., $\overline{\sigma_2} > z_2$. Let φ be the solution of the Cauchy problem

$$\begin{cases} \varphi' = \lambda_2(\overline{s_1}, \varphi, z_3), \\ \varphi(0) = z_2. \end{cases}$$

Clearly the curve $t \to (\overline{s_1}, \varphi(t), z_3)$ is a subunit curve. In addition, let us note that $\lambda_2(\overline{s_1}, z_2, z_3) > 0$. Indeed, by (i) and (ii),

$$\begin{aligned} \lambda_2(\overline{s_1}, z_2, z_3) &\ge \lambda_2(\overline{s_1}, x_2, x_3) - L|z_2 - x_2| - L|z_3 - x_3| \\ &\ge \Lambda_2^*(x, r) - Lr\Lambda_2(x, r) - Lr\Lambda_3(x, r) \\ &\ge \tfrac{1}{2}\Lambda_2(x, r) - 2Lr\Lambda_2(x, r) \ge \tfrac{1}{4}\Lambda_2(x, r) \end{aligned}$$

if $r < 1/8L$. Hence, since $\varphi \geq 0$,

$$\varphi(t) - z_2 = \int_0^t \lambda_2(\overline{s_1}, \varphi(s), z_3)\, ds \geq \int_0^t \lambda_2(\overline{s_1}, z_2, z_3)\, ds - Lt(\varphi(t) - z_2)$$
$$\geq \frac{t}{4}\Lambda_2(x, r) - Lt(\varphi(t) - z_2).$$

Then, if $t \leq 1$,

$$\varphi(t) - z_2 \geq \frac{t}{4(1+L)}\Lambda_2(x, r)$$

and hence $\varphi([0, 1]) \supseteq [z_2, z_2 + \Lambda_2(x, r)/4(1+L)]$. Since $0 \leq \overline{\sigma_2} - z_2 \leq |\overline{\sigma_2} - x_2| + |x_2 - z_2| \leq 2r\Lambda_2(x, r)$, provided $r_0 < 1/8(1+L)$ we get $\overline{\sigma_2} \in \varphi([0, 1])$ and hence $\overline{\sigma_2} = \varphi(t_0)$ for a suitable $t_0 \in [0, 1]$. On the other hand

$$2r\Lambda_2(x, r) \geq \overline{\sigma_2} - z_2 = \varphi(t_0) - z_2 \geq \frac{t_0}{4(1+L)}\Lambda_2(x, r)$$

and hence $t_0 \leq 8(1+L)r$. This remark enables us to estimate the length of arcs (2) and (6).

Finally, the estimate of the length of arc (4) is easier. In fact, let us consider the solution φ of the Cauchy problem

$$\begin{cases} \varphi' = \lambda_2(\overline{\sigma_1}, \overline{\sigma_2}, \varphi), \\ \varphi(0) = x_3. \end{cases}$$

Clearly the curve $t \to (\overline{\sigma_1}, \overline{\sigma_2}, \varphi(t))$ is a subunit curve. Moreover, let us note that $\lambda_2(\overline{\sigma_1}, \overline{\sigma_2}, \varphi(t)) > 0$. Arguing as above, if $t \leq 1$, we get

$$\varphi(t) - x_3 \geq \frac{t}{1+L}\Lambda_3(x, r)$$

and hence $\varphi([0, 1]) \supseteq [x_3, x_3 + \Lambda_3(x, r)/(1+L)]$. Thus there exists $t_0 \in (0, 1)$ such that $y_3 = \varphi(t_0)$ with $t_0 \leq r(1+L)$. Thus we have obtained an estimate of the length of arc (4) and hence we have proved our assertion with $b = 17(1+L) + 7$. □

Remark. Let x, y be given and put $r = d(x, y)$. By the above result, $y \in Q(x, 2r)$ and $y \notin Q(x, r/2b)$. Hence there exists $k \in \{1, \dots, n\}$ such that $|y_k - x_k| \geq F_k(x, r/2b)$, where $F_k(x, t)$ denotes the function $t \mapsto t\Lambda_k(x, t)$. Thus,

$$r \leq 2bF_k^{-1}(x, |y_k - x_k|) \leq 2b\sum_{j=1}^{n} F_j^{-1}(x, |y_j - x_j|).$$

On the other hand, $|y_j - x_j| \leq F_j(x, 2r)$ for $j = 1, \dots, n$ and hence

$$r \geq \frac{1}{2n}\sum_{j=1}^{n} F_j^{-1}(x, |y_j - x_j|).$$

Hence we have proved that

$$\frac{1}{2n}\sum_{j=1}^{n} F_j^{-1}(x, |y_j - x_j|) \leq d(x, y) \leq 2b\sum_{j=1}^{n} F_j^{-1}(x, |y_j - x_j|).$$

We are now able to formulate our main hypothesis. We will suppose that:

(H.4) For any $x_0 \in \mathbb{R}^n$ there exists a neighbourhood U of x_0 such that, if $0 < \varepsilon_j \leq |\xi_j| \leq 1$ for $j = 1, \dots, n$ and if we denote by $H(\cdot, x, \xi) = (H_1(\cdots), \dots, H_n(\cdots))$ the integral curve of the vector field $\xi_1\lambda_1\partial_1 + \cdots + \xi_n\lambda_n\partial_n$ starting from x, we have

$$\int_0^t \lambda_j(H(s, x, \xi))\, ds \geq C_j(\varepsilon_1, \dots, \varepsilon_n) t \Lambda_j(x, t)$$

for $j = 1, \dots, n$, where C_j is independent of $t \in (0, t_0]$, $x \in U$, and $\xi \in \prod_{j=1}^n [\varepsilon_j, 1]$.

In what follows we shall denote by ε the vector $(\varepsilon_1, \dots, \varepsilon_n)$, by Δ_ε the n-interval $\prod_{j=1}^n [\varepsilon_j, 1]$, and we shall put $\xi^* = (|\xi_1|, \dots, |\xi_n|)$.

In the sequel, we will explain more explicitly the meaning of the above hypothesis and we will give some examples, but let us first prove some consequences of our hypotheses.

Proposition 2.4. *The functions $t \to \lambda_j(H(t, x, \xi))$ are locally uniformly A_∞-weight functions for $j = 1, \dots, n$, i.e., for any $j = 1, \dots, n$ there exists $p_j \geq 1$ and $c_j(\varepsilon) > 0$ such that*

$$\left(\frac{1}{t}\int_0^t \lambda_j(H(s, x, \xi))\, ds\right) \cdot \left(\frac{1}{t}\int_0^t \lambda_j(H(s, x, \xi))^{-1/(p_j - 1)}\, ds\right)^{p_j - 1} \leq c_j(\varepsilon)$$

for $0 \leq t \leq t_0$, $x \in U$, $(|\xi_1|, \dots, |\xi_n|) \in \Delta_\varepsilon$.

Proof. Note first that $t \to H(t/\sqrt{n}, x, \xi)$ is a subunit curve starting from x. If $\eta > 0$ is fixed we have

$$\begin{aligned}
\frac{1}{t}\int_0^t \lambda_j(H(s, x, \xi))^{1+\eta}\, ds &= \frac{1}{t\sqrt{n}}\int_0^{t\sqrt{n}} \lambda_j(H(s/\sqrt{n}, x, \xi))^{1+\eta}\, ds \\
&\leq \Lambda_j(x, t)^\eta \frac{1}{t\sqrt{n}}\int_0^{t\sqrt{n}} \lambda_j(H(s/\sqrt{n}, x, \xi))\, ds \\
&\leq C_j(\varepsilon)^{-\eta} \frac{1}{t\sqrt{n}}\int_0^{t\sqrt{n}} \lambda_j(H(s/\sqrt{n}, x, \xi))\, ds \\
&\quad \cdot \left(\frac{1}{t}\int_0^t \lambda_j(H(s, x, \xi))\, ds\right)^\eta \\
&= C_j(\varepsilon)^{-\eta}\left(\frac{1}{t}\int_0^t \lambda_j(H(s, x, \xi))\, ds\right)^{1+\eta},
\end{aligned}$$

and hence $t \to \lambda_j(H(t, x, \xi))$ satisfies a reverse Hölder inequality with constants depending only on ε. Then the assertion follows from standard results about A_p-classes [GR, IV, Corollary 2.13]. □

Now, since A_p-weights give doubling measures [GR, IV, Corollary 2.13], we get

Proposition 2.5. *There exists a positive absolute constant* c *such that, if* $t \in (0, t_0]$ *and* $x \in U$, *then*

$$\Lambda_k(x, 2r) \le c\Lambda_k(x, r)$$

for $k = 1, \dots, n$. *In particular,* $\Lambda_k(x, r) > 0$ *if* $r > 0$, $k = 1, \dots, n$.

Proof. Let ξ be fixed such that $|\xi| < 1$. Then $t \mapsto \lambda(H(t, x, \xi))$ is a subunit curve and hence, putting $C_k(\xi) = C_k(|\xi_1|, \dots, |\xi_n|)$, we have

$$\begin{aligned}\Lambda_k(x, 2r) &\le \frac{1}{2C_k(\xi)r}\int_0^{2r} \lambda_k(H(s, x, \xi))\, ds \le \frac{C'(\xi)}{r}\int_0^r \lambda_k(H(s, x, \xi))\, ds \\ &\le C'(\xi)\Lambda_k(x, r). \quad \square\end{aligned}$$

The doubling property of functions Λ_k combined with Theorem 2.3 gives us the basic property of the metric d: the metric space $(\mathbb{R}^n, d, dx)$ is a space of homogeneous type. Here dx denotes the Lebesgue measure.

Theorem 2.6. *There exist two absolute constants* $A, B > 0$ *such that*

$$|B(x, 2r)| \le A|B(x, r)| \quad \text{and} \quad |Q(x, 2r)| \le B|Q(x, r)|$$

for every x *belonging to a compact subset of* $\mathbb{R}^n$ *and for* $r \in (0, r_0]$.

The proof is straightforward.

Remark 1. A theory of A_p-weights in spaces of homogeneous type was developed by A. P. Calderon in [C]. Thus the above theorem gives a precise sense to hypothesis (H.3).

Remark 2. An easy consequence of Proposition 2.5 and Theorem 2.6 is that there exist positive absolute constants $\alpha_1, \dots, \alpha_n$, $k_1, \dots, k_n$, k and α such that

$$\Lambda_j(x, tr) \ge k_j t^{\alpha_j}\Lambda_j(x, r) \quad \text{for } j = 1, \dots, n \quad \text{and} \quad |B(x, tr)| \ge kt^{\alpha}|B(x, r)|$$

for x belonging to a compact subset of $\mathbb{R}^n$, $t \in (0, 1)$, and $r \in (0, r_0]$. In particular, there exist suitable absolute positive constants $k_1, \dots, k_n$ such that, in a compact neighbourhood of Ω, $\Lambda_j(x, t) \ge k_j t^{\alpha_j}$ for $j = 1, \dots, n$ and $t \in (0, 1)$.

Remark 3. Since $t \to H(t/\sqrt{n}, x, \xi)$ is a subunit curve starting from x for any ξ such that $|\xi_j| \le 1$ for $j = 1, \dots, n$, by the doubling property of Λ_k there exists an absolute constant $c_* > 0$ such that, for this choice of ξ, $\lambda_j(H(t, x, \xi)) \le c_*\Lambda_j(x, t)$ for $j = 1, \dots, n$.

Remark 4. From Remark 2, $F_j^{-1}(x, t) \le K_j t^{1/(\alpha_j+1)}$ for t small. Hence, taking into account the Remark after Theorem 2.3, we obtain

the distance d *is Hölder continuous with respect to the euclidean metric.*

In particular, the topology of d is equivalent to the euclidean topology. Hence when quantitative estimates are not involved, we can talk about 'close points' without further specifications.

3

In this section, we will obtain basic properties of the integral curves $H(t, x, \xi)$ which we will use in the following. First, let us note that the function $\xi \to H(t, x, \xi)$ gives nice 'polar coordinates' in a part of the ball $B(x, t)$.

Proposition 3.1. *Let x_0 be fixed. There exists a neighbourhood U of x_0 such that, if $\varepsilon_1, \dots, \varepsilon_n$ are real positive constants and $\varepsilon_j \leq 1$ for $j = 1, \dots, n$, then, putting $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, there exist positive constants $c^*(\varepsilon)$, $c^{**}(\varepsilon)$, $t_0(\varepsilon)$ depending on ε and $c^{***} > 0$, such that, if $x \in U$, $\xi^* = (|\xi_1|, \dots, |\xi_n|) \in \Delta_\varepsilon$, and $0 < t < t_0(\varepsilon_0)$,*

(i) *the map $\xi \to H(t, x, \xi)$ is injective;*
(ii) $c^{***}|Q(x, t)| \geq |\det \frac{\partial H}{\partial \xi}(t, x, \xi)| \geq c^*(\varepsilon)|Q(x, t)|$;
(iii) $c_* \geq |H(t, x, \Delta_\varepsilon)|/|Q(x, t)| \geq c^{**}(\varepsilon)$, *where c_* has been defined in Remark 3 after Theorem 2.6.*

Proof. For the sake of simplicity we will carry out the proof supposing λ_j is a continuously differentiable function for $j = 1, \dots, n$. The result in the general case will be obtained in the same way when derivatives are replaced by finite differences. We note that H is Lipschitz continuous with respect to ξ, so that the derivatives in (ii) exist a.e.

Thus suppose $L \geq \sup|\partial_k \lambda_j|$ for all j, k. It is well known that

$$\frac{\partial H}{\partial \xi_k}(t, x, \xi) = \int_0^t J(H(s, x, \xi), \xi)\frac{\partial H}{\partial \xi_k}\, ds + \int_0^t \lambda_k(H(s, x, \xi))\, ds\, e_k,$$

where $e_k = (0, \dots, 1, \dots, 0)$ and $J(p, \xi)_{ij} = \partial_j \lambda_i(p)\xi_i$. In particular, if $t_0 \leq 1$, there exists $c_1(L)$ such that

$$\left|\frac{\partial H}{\partial \xi_k}(t, x, \xi)\right| \leq c_1(L) \int_0^t \lambda_k(H(s, x, \xi))\, ds.$$

Moreover, without loss of generality, we may suppose $\xi_j > 0$ for $j = 1, \dots, n$.

Let us now consider the matrix

$$D = D(t, x, \xi, \xi') = t\frac{\partial H}{\partial \xi}(t, x, \xi')\frac{\partial H}{\partial \xi}(t, x, \xi),$$

where ξ^*, ξ'^* belong to Δ_ε. We have

$$\begin{aligned}
D_{ij} &= \left\langle \frac{\partial H}{\partial \xi_i}(t, x, \xi'), \frac{\partial H}{\partial \xi_j}(t, x, \xi) \right\rangle \\
&= \left\langle \int_0^t J(H(s, x, \xi'), \xi') \frac{\partial H}{\partial \xi_i} ds, \int_0^t J(H(s, x, \xi), \xi) \frac{\partial H}{\partial \xi_j} ds \right\rangle \\
&\quad + \int_0^t \lambda_j(H(s, x, \xi))\, ds \cdot \int_0^t \left\langle J(H(s, x, \xi'), \xi') \frac{\partial H}{\partial \xi_i}, e_j \right\rangle ds \\
&\quad + \int_0^t \lambda_i(H(s, x, \xi'))\, ds \cdot \int_0^t \left\langle J(H(s, x, \xi), \xi) \frac{\partial H}{\partial \xi_j}, e_i \right\rangle ds \\
&\quad + \delta_{ij} \int_0^t \lambda_i(H(s, x, \xi'))\, ds \cdot \int_0^t \lambda_j(H(s, x, \xi))\, ds \\
&= \alpha_{ij}(t, x, \xi, \xi') + \delta_{ij} \int_0^t \lambda_i(H(s, x, \xi'))\, ds \cdot \int_0^t \lambda_j(H(s, x, \xi))\, ds.
\end{aligned}$$

Now we note that

$$\begin{aligned}
&\left| \left\langle \int_0^t J(H(s, x, \xi'), \xi') \frac{\partial H}{\partial \xi_i} ds, \int_0^t J(H(s, x, \xi), \xi) \frac{\partial H}{\partial \xi_j} ds \right\rangle \right| \\
&\quad \le \left| \int_0^t J(H(s, x, \xi'), \xi') \frac{\partial H}{\partial \xi_i} ds \right| \left| \int_0^t J(H(s, x, \xi), \xi) \frac{\partial H}{\partial \xi_j} ds \right| \\
&\quad \le L^2 \int_0^t \left| \frac{\partial H}{\partial \xi_i} \right| ds \cdot \int_0^t \left| \frac{\partial H}{\partial \xi_j} \right| ds \\
&\quad \le c_2(L) t^2 \int_0^t \lambda_i(H(s, x, \xi))\, ds \cdot \int_0^t \lambda_j(H(s, x, \xi))\, ds \\
&\quad \le c_3(L) t^4 \Lambda_i(x, t) \Lambda_j(x, t),
\end{aligned}$$

by Remark 3 after Theorem 2.6. An analogous argument shows that the second (and the third) term in α_{ij} can be estimated by $c_4(L) t^3 \Lambda_i(x, t) \Lambda_j(x, t)$.

Now, if $\eta \in \mathbb{R}^n$ and $x \in U$, then

$$\begin{aligned}
\sum_{i,j} D_{ij} \eta_i \eta_j &= \sum_{i,j} \alpha_{ij}(\cdots) \eta_i \eta_j + \sum_i \eta_i^2 \int_0^t \lambda_i(H(s, x, \xi'))\, ds \\
&\qquad \cdot \int_0^t \lambda_i(H(s, x, \xi))\, ds \\
&\ge -c_5(L) t^3 \sum_{i,j} \Lambda_i(x, t) \Lambda_j(x, t) \eta_i \eta_j + t^2 \sum_i C_i(\varepsilon) \Lambda_i^2(x, t) \eta_i^2 \\
&= -c_5(L) t^3 \left(\sum_i \Lambda_i(x, t) \eta_i \right)^2 + c_6(\varepsilon) t^2 \sum_i \Lambda_i^2(x, t) \eta_i^2 \\
&\ge (c_6(\varepsilon) - (-c_5(L)) n t) t^2 \sum_i \Lambda_i^2(x, t) \eta_i^2 \ge \frac{c_6(\varepsilon)}{2} t^2 \sum_i \Lambda_i^2(x, t) \eta_i^2
\end{aligned}$$

if $t_0(\varepsilon)$ is sufficiently small.

Suppose now $H(t, x, \xi_1) = H(t, x, \xi_2)$; obviously the coordinates of ξ_1 and ξ_2 have the same sign and hence belong to the same convex component of the domain of the points ξ. Hence we have

$$\begin{aligned} 0 &= \left\langle H(t, x, \xi_1) - H(t, x, \xi_2), \frac{\partial H}{\partial \xi}(t, x, \xi_1)(\xi_1 - \xi_2) \right\rangle \\ &= \int_0^1 \left\langle \frac{\partial H}{\partial \xi}(t, x, \xi_2 + s(\xi_1 - \xi_2))(\xi_1 - \xi_2), \frac{\partial H}{\partial \xi}(t, x, \xi_1)(\xi_1 - \xi_2) \right\rangle ds \\ &= \int_0^1 \langle D(t, x, \xi_2 + s(\xi_1 - \xi_2), \xi_1)(\xi_1 - \xi_2), (\xi_1 - \xi_2) \rangle \, ds \\ &\geq \frac{c_6(\varepsilon)}{2} t^2 \sum_i \Lambda_i^2(x, t)(\xi_{1i} - \xi_{2i})^2 \end{aligned}$$

and hence $\xi_1 = \xi_2$ since $\Lambda_i^2(x, t) > 0$ for any $t > 0$. Thus (i) is proved.

In addition, the same argument shows that, if we denote by $\Lambda(x, t)$ the matrix whose entries are given by $\delta_{ij}\Lambda_i(x, t)$, then

$$D(t, x, \xi, \xi) \geq c_6(\varepsilon) t^2 \Lambda^2(x, t)/2,$$

such that a min-max argument shows that

$$\left| \det \frac{\partial H}{\partial \xi}(t, x, \xi) \right| = \sqrt{\det D} \geq c_7(\varepsilon) t^n \det \Lambda(x, t) = c_8(\varepsilon) |Q(x, t)|.$$

Hence it is enough to choose $c^*(\varepsilon) = c_8(\varepsilon)$. Since the upper estimate for the quadratic form of D is quite obvious, assertion (ii) is proved. □

4

We are now able to prove our main results. To this end, we need to show that it is possible to cover a sufficiently large part of a given ball by integral curves starting from the center. In fact, we are able to prove this result when the ball is replaced by a deformed one, where the deformation parameters depend on how large the part we will attain is. The proof requires a careful control of the constants. Let us begin with some technical preliminaries.

Lemma 4.1. *Denote by $F_j = F_j(x, t)$ the functions $F_j(x, t) = t\Lambda_j(x, t)$, $j = 1, \dots, n$. Let x_0 and $\sigma = (\sigma_1, \dots, \sigma_n) \in \{-1, 1\}^n$ be fixed. For any fixed $\theta \in (0, 1)$ there exist $\varepsilon(\theta) > 0$, $c(\theta) > 0$, and $t(\theta) > 0$ such that for any $t \in (0, t(\theta))$ and x close to x_0, there exist $2n$ positive constants $c_1(x, t, \theta), \dots, c_n(x, t, \theta)$, $\varepsilon_1(x, t, \theta), \dots, \varepsilon_n(x, t, \theta)$ such that:*

(i) *If $\sigma_j p_j \in [\theta F_j(x, c_j(x, t, \theta)t), F_j(x, c_j(x, t, \theta)t)]$, $j = 1, \dots, n$, and $\sigma_j \xi_j \in [\varepsilon_j(x, t, \theta), 1]$, then*

$$\varepsilon_j(x, t, \theta) \leq \frac{\sigma_j p_j}{\int_0^t \lambda_j(H(s, x, \xi)) \, ds} \leq 1 \quad \textit{for } j = 1, \dots, n.$$

(ii) $\varepsilon_j(x, t, \theta) \geq \varepsilon(\theta)$ *for* $j = 1, \dots, n$.

(iii) $1 \geq c_j(x, t, \theta) \geq c(\theta)$ *for* $j = 1, \ldots, n$.

(iv) *Put* $\sigma y = (\sigma_1 y_1, \ldots, \sigma_n y_n)$; *if*

$$\sigma y \in \prod_{j=1}^{n} [x_j + \theta F_j(x, c_j(x, t, \theta))t, x_j + F_j(x, c_j(x, t, \theta))],$$

then there exists ξ *such that* $\varepsilon_j(x, t, \theta) \leq \sigma_j \xi_j \leq 1$ *for* $j = 1, \ldots, n$ *and* $H(t, x, \xi) = y$. *The point* ξ *is unique by Proposition* 3.1.

Proof. First, we note that we can reduce ourselves to the case $\sigma = (1, \ldots, 1)$. In fact if x is fixed, denote by $T_{x,\sigma}$ the mapping $y \to (x_1 + \sigma_1(y_1 - x_1), \ldots, x_n + \sigma_n(y_n - x_n))$ and put $\lambda_j^*(y) = \lambda_j(T_{x,\sigma}(y))$. If we still denote by Q^*, $H^*, \ldots$ the new objects we obtain by replacing λ_j by λ_j^* in the definition of $Q, H, \ldots,$ we get

$$Q(x, t) = Q^*(x, t), \qquad H(t, x, \xi) = x_j + \sigma_j(H_j^*(t, x, \sigma\xi) - \xi_j),$$

and

$$\lambda_j(H_j(s, x, \xi)) = \lambda_j(H_j^*(s, x, \sigma x)).$$

Hence, in what follows, we may suppose $\sigma = (1, \ldots, 1)$.

Let $I = (i_1, \ldots, i_n)$ be a permutation of $\{1, \ldots, n\}$ such that $i_1 = 1$. Let us consider the positive real numbers $\varepsilon(I, 1, \theta), \ldots, \varepsilon(I, n, \theta)$ defined as follows:

$$\varepsilon(I, 1, \theta) = \theta, \qquad \varepsilon(I, k+1, \theta) = \theta \frac{k_{i_{k+1}}}{c^*} \left(\frac{1}{2} C_{i_{k+1}}(\varepsilon^*(I, k, \theta)) \right)^{\alpha_{i_{k+1}}+1}, \quad k = 1, \ldots, n-1,$$

where $C_1, \ldots, C_n$ are the constants of hypothesis (H.4) relative to the neighbourhood U of x_0, c_* has been defined in Remark 3 after Theorem 2.6, and $k_1, \ldots, k_n$, $\alpha_1, \ldots, \alpha_n$ are the constants of Remark 2 after Theorem 2.6. Finally *the vector* $\varepsilon^*(I, k, \theta) = (\varepsilon_1^*(I, k, \theta), \ldots, \varepsilon_n^*(I, k, \theta))$ is defined as follows:

$$\varepsilon_i^*(I, k, \theta) = \begin{cases} \varepsilon(I, j, \theta) & \text{if } i = i_j \text{ for } j \leq k, \\ 1/n & \text{otherwise.} \end{cases}$$

Now we put

$$k(\theta) = \min_{I=(i_1, \ldots, i_n), i_1 = 1} \min_k C_{i_k}(\varepsilon^*(I, k-1, \theta)).$$

The first choice of $t(\theta)$ is the following one: put

$$\Lambda = \max_k \sup_{x, t \leq 1} \Lambda_k(x, t) < \infty;$$

then we will assume that

$$2c_* t(\theta) n L \frac{(t(\theta)Lj)^{[\alpha_j]+1} - 1}{t(\theta)Lj - 1} + \frac{2t(\theta)\Lambda}{k_{i_j}} \max\{L^{[\alpha_j]+1}\} < \frac{1}{2} k(\theta)$$

for $j = 1, \ldots, n$. Here $[x]$ denotes the integral part of x. Without loss of generality, we may suppose $c_* \geq 1$.

Now let $t < t(\theta)$ be fixed and let $I = I(t, x) = (i_1, \dots, i_n)$ be a permutation of $\{1, \dots, n\}$ such that $i_1 = 1$ and

$$\Lambda_{i_1}(x, t) \geq \Lambda_{i_2}(x, t) \geq \cdots \geq \Lambda_{i_n}(x, t).$$

Put

$$\varepsilon_{i_k}(x, t, \theta) = \varepsilon(I(x, t), k, \theta) \quad \text{for } k = 1, \dots, n,$$
$$c_1(x, t, \theta) = 1, \qquad c_{i_{k+1}}(x, t, \theta) = \tfrac{1}{2} C_{i_{k+1}}(\varepsilon^*(I(x, t), k, \theta)) \quad \text{for } k = 1, \dots, n-1.$$

Without loss of generality, we may suppose $c_k(x, t, \theta) \leq 1$, $k = 1, \dots, n$.

In order to prove (i), by an induction argument let us prove first that for any positive integer m, for $j = 2, \dots, n$, and for $\varepsilon_r(x, t, \theta) \leq \xi_r \leq 1$ for $r = 1, \dots, n$ we have

$$\begin{aligned} (4.1.1) \qquad & \int_0^t \lambda_{i_j}(H(s, x, \xi))\, ds \\ & \geq [C_{i_j}(\varepsilon^*(I(x, t), j-1, \theta)) \\ & \qquad - 2tnc_* L(1 + tLi_j + \cdots + (tLi_j)^{m-1})] t \Lambda_{i_j}(x, t) \\ & \quad - L^m j^{m-1} t^{m-1} \sum_{k<j} \int_0^t ds \int_0^s d\sigma |\lambda_{i_k}(H(\sigma, x, \xi)) \\ & \qquad\qquad - \lambda_{i_k}(H(\sigma, x, \xi^*(x, t, j)))|, \end{aligned}$$

where

$$\xi^* = \xi^*(x, t, j) = (\xi_1^*(x, t, j), \dots, \xi_n^*(x, t, j))$$

is defined as follows:

$$\xi_r^*(x, t, j) = \begin{cases} \xi_{i_p} & \text{if } r = i_p \text{ for } p < j, \\ 1/n & \text{otherwise.} \end{cases}$$

We note that, if $r = i_p$ for some $p < j$, then $r = i_p$ for some $p \leq j-1$, so that $\xi_r^*(x, t, j) = \xi_{i_p}$ and $\varepsilon_r^*(I(x, t), j-1, \theta) = \varepsilon(I(t, x), p, \theta) = \varepsilon_{i_p}(t, x, \theta) = \varepsilon_r(x, t, \theta)$ and hence $\xi_r^*(x, t, j) \geq \varepsilon_r^*(I(x, t), j-1, \theta)$. Otherwise, $\xi_r^*(x, t, j) = 1/n = \varepsilon_r^*(I(x, t), j-1, \theta)$, so that

$$\xi_r^*(x, t, j) \geq \varepsilon_r^*(I(x, t), j-1, \theta)$$

for $r = 1, \dots, n$.

Case $m = 1$.

$$\begin{aligned}
&\int_0^t \lambda_{i_j}(H(s,x,\xi))\,ds \\
&= \int_0^t \lambda_{i_j}(H(s,x,\xi^*))\,ds + \int_0^t [\lambda_{i_j}(H(s,x,\xi)) - \lambda_{i_j}(H(s,x,\xi^*))]\,ds \\
&\geq C_{i_j}(\varepsilon^*(I(x,t), j-1, \theta))t\Lambda_{i_j}(x,t) \\
&\quad - L\sum_{k=1}^{n}\int_0^t |H_{i_k}(s,x,\xi) - H_{i_k}(s,x,\xi^*)|\,ds \\
&= C_{i_j}(\varepsilon^*(I(x,t), j-1, \theta))t\Lambda_{i_j}(x,t) \\
&\quad - L\sum_{k<j}\int_0^t |H_{i_k}(s,x,\xi) - H_{i_k}(s,x,\xi^*)|\,ds \\
&\quad - L\sum_{k\geq j}\int_0^t (|H_{i_k}(s,x,\xi) - x_{i_k}| + |H_{i_k}(s,x,\xi^*) - x_{i_k}|)\,ds \\
&\geq C_{i_j}(\varepsilon^*(I(x,t), j-1, \theta))t\Lambda_{i_j}(x,t) \\
&\quad - L\sum_{k<j}\int_0^t ds\int_0^s d\sigma|\lambda_{i_k}(H(\sigma,x,\xi)) - \lambda_{i_k}(H(\sigma,x,\xi))|\xi_{i_k} \\
&\quad - L\sum_{k\geq j}\int_0^t ds\left(\int_0^s d\sigma\lambda_{i_k}(H(\sigma,x,\xi))\xi_{i_k} + \int_0^s d\sigma\lambda_{i_k}(H(\sigma,x,\xi))\xi^*_{i_k}\right).
\end{aligned}$$

Now the assertion follows since

$$\begin{aligned}
&L\sum_{k\geq j}\int_0^t ds\int_0^s d\sigma\,\lambda_{i_k}(H(\sigma,x,\xi))\xi_{i_k} \\
&\qquad \leq c_* Lt^2\sum_{k\geq j}\Lambda_{i_k}(x,t) \leq c_* Lt^2(n-j+1)\Lambda_{i_j}(x,t),
\end{aligned}$$

and the same estimate holds for the other term corresponding to ξ^*.

Case $m+1$. Suppose now (4.1.1) holds for m. We have

$$\begin{aligned}
&\int_0^t \lambda_{i_j}(H(s,x,\xi))\,ds \\
&\geq [C_{i_j}(\varepsilon^*(I(x,t), j-1, \theta)) - 2tnc_*L(1 + tLi_j + \cdots + (tLi_j)^{m-1})]t\Lambda_{i_j}(x,t) \\
&\quad - L^m j^{m-1}t^{m-1}\sum_{h<j}\int_0^t ds\int_0^s d\sigma|\lambda_{i_h}(H(\sigma,x,\xi)) \\
&\quad - \lambda_{i_h}(H(\sigma,x,\xi^*(x,t,j)))| \\
&\geq [C_{i_j}(\varepsilon^*(I(x,t), j-1, \theta)) - 2tnc_*L(1 + tLi_j + \cdots + (tLi_j)^{m-1})]t\Lambda_{i_j}(x,t) \\
&\quad - L^{m+1}j^m t^{m-1}\sum_{k<j}\int_0^t ds\int_0^s d\sigma|H_{i_k}(\sigma,x,\xi) - H_{i_k}(\sigma,x,\xi^*(x,t,j))| \\
&\quad - L^{m+1}j^m t^{m-1}\sum_{k\geq j}\int_0^t ds\int_0^s d\sigma|H_{i_k}(\sigma,x,\xi) - H_{i_k}(\sigma,x,\xi^*(x,t,j))|.
\end{aligned}$$

Now, arguing as above, the last sum can be estimated as follows:

$$
\begin{aligned}
L^{m+1} j^m t^{m-1} \sum_{k \geq j} \int_0^t ds \int_0^s d\sigma |H_{i_k}(\sigma, x, \xi) - H_{i_k}(\sigma, x, \xi^*(x, t, j))| \\
\leq L^{m+1} j^m t^m \sum_{k \geq j} \int_0^t (|H_{i_k}(s, x, \xi) - x_{i_k}| + |H_{i_k}(s, x, \xi^*) - x_{i_k}|)\, ds \\
\leq L^{m+1} j^m t^m \sum_{k \geq j} \int_0^t ds \left(\int_0^s d\sigma \lambda_{i_k}(H(\sigma, x, \xi))\xi_{i_k} \right. \\
\left. + \int_0^s d\sigma \lambda_{i_k}(H(\sigma, x, \xi))\xi^*_{i_k} \right) \\
\leq 2nL^{m+1} j^m t^{m+2} c_* \Lambda_{i_j}(x, t).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
L^{m+1} j^m t^{m-1} \sum_{k<j} \int_0^t ds \int_0^s d\sigma |H_{i_k}(\sigma, x, \xi) - H_{i_k}(\sigma, x, \xi^*(x, t, j))| \\
\leq L^{m+1} j^m t^{m-1} \sum_{k<j} \int_0^t ds \int_0^s d\sigma \int_0^\sigma d\sigma' |\lambda_{i_k}(H(\sigma', x, \xi)) \\
- \lambda_{i_k}(H(\sigma', x, \xi^*(x, t, j)))| \xi_{i_k} \\
\leq L^{m+1} j^m t^m \sum_{k<j} \int_0^t ds \int_0^s d\sigma |\lambda_{i_k}(H(\sigma, x, \xi)) - \lambda_{i_k}(H(\sigma, x, \xi^*(x, t, j)))|.
\end{aligned}
$$

Thus assertion (4.1.1) is proved.

Now, let us choose $m = \max\{[\alpha_i] + 1\}$. Without loss of generality we may suppose $tLi_j < 1$, so that $(tLi_j)^m < (tLi_j)^{[\alpha_j]+1}$. By our choice of $t(\theta) > t$, we have (k_j are the constants of Remark 2 after Theorem 2.6)

$$
\begin{aligned}
(4.1.2) \quad & \int_0^t \lambda_{i_j}(H(s, x, \xi))\, ds \\
& \geq \left[C_{i_j}(\varepsilon^*(I(x, t), j-1, \theta)) - 2tnc_* \frac{(tLi_j)^{[\alpha_j]+1} - 1}{tLi_j - 1} \right] t\Lambda_{i_j}(x, t) \\
& \quad - 2L^m t^{m+1} j^m \Lambda \\
& \geq \left[C_{i_j}(\varepsilon^*(I(x, t), j-1, \theta)) - 2tnc_* \frac{(tLi_j)^{[\alpha_j]+1} - 1}{tLi_j - 1} \right. \\
& \quad \left. - \frac{2t\Lambda}{k_{i_j}} \max\{L^{[\alpha_i]+1}\} \right] t\Lambda_{i_j}(x, t) \\
& \geq \left[C_{i_j}(\varepsilon^*(I(x, t), j-1, \theta)) - \frac{1}{2}k(\theta) \right] t\Lambda_{i_j}(x, t) \\
& \geq \frac{1}{2} C_{i_j}(\varepsilon^*(I(x, t), j-1, \theta)) t\Lambda_{i_j}(x, t) = c_{i_j}(x, t, \theta) t\Lambda_{i_j}(x, t).
\end{aligned}
$$

Obviously, the same estimate holds for $j = 1$.

We are now able to prove (i). We have

$$\frac{p_{i_j}}{\int_0^t \lambda_{i_j}(H(s, x, \xi))\, ds} \le \frac{F_j(x, c_j(x, t, \theta)t)}{c_{i_j}(x, t, \theta)t\Lambda_{i_j}(x, t)} \le 1.$$

On the other hand, by Remark 3 after Theorem 2.6,

$$\begin{aligned}\frac{p_{i_j}}{\int_0^t \lambda_{i_j}(H(s, x, \xi))\, ds} &\ge \frac{\theta c_{i_j}(x, t, \theta)\Lambda_{i_j}(x, c_{i_j}(x, t, \theta)t)}{c_*\Lambda_{i_j}(x, t)} \\ &\ge (\theta/c_*)k_{i_j}c_{i_j}(x, t, \theta)^{\alpha_{i_j+1}} \\ &= (\theta/c_*)k_{i_j}(\tfrac{1}{2}C_{i_j}(\varepsilon^*(I(x, t), j-1, \theta)))^{\alpha_{i_j+1}} \\ &= (\text{by definition})\ \varepsilon(I(x, t), j, \theta) = \varepsilon_{i_j}(x, t, \theta).\end{aligned}$$

Thus (i) is proved for i_j. On the other hand, $\{i_1, \dots, i_n\}$ is a permutation of $\{1, \dots, n\}$ and hence the assertion is completely proved.

Assertion (ii) follows by definition, since, if $j = i_k$ for k suitable, we have:

$$\begin{aligned}\varepsilon_j(x, t, \theta) &= \varepsilon_{i_k}(x, t, \theta) = \varepsilon(I(x, t), k, \theta) \\ &= \theta(k_{i_k}/c^*)(\tfrac{1}{2}C_{i_k}(\varepsilon^*(I(x, t), k-1, \theta)))^{\alpha_{i_k+1}} \\ &\ge \theta(k_{i_k}/c^*)(k(\theta)/2)^{\alpha_{i_k+1}} \\ &\ge (\theta/c^*)\max\{k_{i_r}\}(k(\theta)/2)^{\max\{\alpha_{i_r+1}\}} = \varepsilon(\theta).\end{aligned}$$

Analogously,

$$c_j(x, t, \theta) \ge \tfrac{1}{2}k(\theta) = c(\theta) \quad \text{for } j = 1, \dots, n,$$

and (iii) follows.

Finally, we note that our assumptions imply that

$$\int_0^t \lambda_j(H(s, x, \xi))\, ds > 0 \quad \text{if } t > 0, \xi_j > 0, j = 1, \dots, n.$$

Thus, by (i), and by usual continuous dependence results for ordinary differential equations, the mapping

$$\xi \mapsto T(\xi) = \left(\frac{y_1 - x_1}{\int_0^t \lambda_1(H(s, x, \xi))\, ds}, \dots, \frac{y_n - x_n}{\int_0^t \lambda_n(H(s, x, \xi))\, ds}\right)$$

is continuous from $\prod_{j=1}^n [\varepsilon_j(x, t, \theta), 1] = \Delta(x, t, \theta)$ to itself. A fixed point argument shows that there exists $\xi \in \Delta(x, t, \theta)$ such that

$$y_j = x_j + \int_0^t \lambda_j(H(s, x, \xi))\, ds\xi_j = H_j(t, x, \xi).$$

Thus, assertion (iv) is proved. □

Now put

$$Q^{\theta}(x, t) = \prod_{j=1}^{n}[x_j - F_j(x, c_j(x, t, \theta)t), x_j + F_j(x, c_j(x, t, \theta)t)],$$

$$Q_{+}^{\theta}(x, t) = Q^{\theta}(x, t) \cap \{y_j \geq x_j, j = 1, \ldots, n\},$$

$$S_{+}^{\theta}(x, t) = \prod_{j=1}^{n}[x_j + \theta F_j(x, c_j(x, t, \theta)t), x_j + F_j(x, c_j(x, t, \theta)t)].$$

In what follows, if there is no way of misunderstanding, we will write only Q^{θ}, Q_{+}^{θ}, S_{+}^{θ}.

The aim of the next result is to obtain a suitable representation formula for a function u vanishing on a sufficiently large subset of a ball.

Lemma 4.2. *Let u be a Lipschitz continuous function and let $\beta \in (0, 1)$ be a fixed positive constant. Then there exists $r(\beta) > 0$ such that, if $r < r(\beta)$ the following assertion holds: suppose*

$$|E| = |\{y \in Q^{\theta}(x, r); u(y) = 0\}| \geq \beta |Q^{\theta}(x, r)|,$$

where $\theta = \theta(\beta) = 1 - \sqrt[n]{1 - \beta/2}$. Then there exists $Q \in (0, 1)$ such that

$$|u(x)| \leq c_1(\beta) r |B(x, r)|^{Q-1} M_Q(|\nabla_\lambda| \chi_{c_2 B})(x),$$

where $M_Q(f)(x) = \sup_{t>0} |B(x, t)|^{-Q} \int_{B(x, t)} |f(y)|\, dy$ is the fractional maximal function of order Q, $c_2 B = B(x, c_2 r)$, χ_A is the characteristic function of A, and c_2 is an absolute constant. In particular, these constants are independent of $r < r(\beta)$ and $x \in U$, U being a suitable neighbourhood of a fixed point.

Proof. Without loss of generality we may suppose $|E_+| = |E \cap Q_+^{\theta}| \geq \beta |Q^{\theta}|/2^n$. We have

$$\begin{aligned} 2^{-n}|Q^{\theta}| &= |Q_+^{\theta}| \geq |(E \cup S_+^{\theta}) \cap Q_+^{\theta}| = |(E \cap Q_+^{\theta}) \cup S_+^{\theta}| \\ &= |(E \cap Q_+^{\theta})| + |S_+^{\theta}| - |E \cap S_+^{\theta}| \\ &= |E_+| + |S_+^{\theta}| - |E \cap S_+^{\theta}| \geq \beta 2^{-n}|Q^{\theta}| + (1-\theta)^n 2^{-n}|Q^{\theta}| - |E \cap S_+^{\theta}| \\ &= 2^{-n}(1 + \beta/2)|Q^{\theta}| - |E \cap S_+^{\theta}| \end{aligned}$$

and hence $|E \cap S_+^{\theta}| \geq 2^{-n-1}\beta |Q^{\theta}|$.

Now choose $r(\beta) < t(\theta(\beta))$, where $t(\theta(\beta))$ is defined in Lemma 4.1 and put $\Sigma = \{\xi \in \Delta_{\varepsilon(x, r, \theta)}; H(r, x, \xi) \in E_+ \cap S_+^{\theta}\}$. The first step consists of estimating $|\Sigma|$. By Lemma 4.1 $H(r, x, \Delta_{\varepsilon(x, r, \theta)}) \supseteq S_+^{\theta}(x, r)$; hence, putting

$H(r, x, \xi) = y$, by Proposition 3.1 we get

$$\begin{aligned}|\Sigma| &= \int_\Sigma d\xi \geq \int_{E_+\cap S_+^\theta} dy \left|\det \frac{\partial H}{\partial \xi}(r, x, \xi(y))\right|^{-1} \\ &\geq \frac{|E\cap S_+^\theta|}{c^{***}|Q(x, r)|} \geq \frac{\beta}{2^{n+1}c^{***}} \frac{|Q^\theta(x, r)|}{|Q(x, r)|} \\ &\geq \frac{\beta}{2^{n+1}c^{***}} \prod_{j=1}^n c_j(x, r, \theta)^{\alpha_j+1} \geq \frac{\beta}{2^{n+1}c^{***}} c(\theta)^{n+\sum\alpha_j} = c_3(\beta),\end{aligned}$$

where $c(\theta)$ is the constant of Lemma 4.1.

Now let K be a smooth nonnegative function such that $K \equiv 1$ on $\Delta_{\varepsilon(\theta)}$, where $\varepsilon(\theta)$ is the constant defined in Lemma 4.1(ii). We note explicitly that $\Delta_{\varepsilon(\theta)} \supseteq \Delta_{\varepsilon(x, r, \theta)}$. Assume that u is continuously differentiable. We have

$$|u(x)| = |u(x) - u(H(r, x, \xi))|K(\xi) \quad \text{for } \xi \in \Sigma.$$

Hence

$$\begin{aligned}|\Sigma|\,|u(x)| &= \int_\Sigma |u(x) - u(H(r, x, \xi))|K(\xi)\,d\xi \\ &\leq \int_0^r dt \int_{\operatorname{supp} K} d\xi \left|\frac{d}{dt}u(H(t, x, \xi))\right| \\ &= \int_0^r dt \int_{\operatorname{supp} K} d\xi |\langle \nabla u(H(t, x, \xi)), H'(t, x, \xi)\rangle| \\ &\leq \sqrt{n} \int_0^r dt \int_{\operatorname{supp} K} d\xi |\nabla_\lambda u(H(t, x, \xi))|.\end{aligned}$$

Now put $H(t, x, \xi) = y$; keeping in mind Proposition 3.1, we obtain

$$|\Sigma|\,|u(x)| \leq \frac{\sqrt{n}}{c^*(\varepsilon(\beta))} \int_0^r dt \frac{1}{|Q(x, t)|} \int_{H(t, x, \operatorname{supp} K)} dy |\nabla_\lambda u(y)|,$$

where, for the sake of simplicity, we have denoted by $\varepsilon(\theta)$ the vector $(\varepsilon(\theta), \ldots, \varepsilon(\theta))$.

On the other hand, $t \mapsto H(t/\sqrt{n}, x, \xi)$ is a subunit curve starting from x for any ξ such that $\xi_j \leq 1$ for $j = 1, \ldots, n$, so that $H(t, x, \operatorname{supp} K)$ is contained in $B(x, c_2 t)$ which is in turn contained in $B(x, c_2 r) = c_2 B$. Thus, keeping in mind the estimate of $|\Sigma|$ obtained above and the equivalence between

$|Q(x, t)|$ and $|B(x, t)|$ (Theorems 2.3 and 2.6), for $Q \in (0, 1)$ we get

$$\begin{aligned}|u(x)| &\le c_4(\beta)\int_0^r dt|B(x, t)|^{Q-1}\sup_{\tau>0}|B(x, \tau)|^{-Q}\int_{B(x, \tau)} dy|\nabla_\lambda u(y)|\chi_{c_2B} \\ &= c_4(\beta)M_Q(|\nabla_\lambda|\chi_{c_2B})(x)\int_0^r dt|B(x, t)|^{Q-1} \\ &= c_4(\beta)rM_Q(|\nabla_\lambda|\chi_{c_2B})(x)\int_0^1 d\tau|B(x, \tau r)|^{Q-1} \\ &\le c_4(\beta)r|B(x, r)|^{Q-1}M_Q(|\nabla_\lambda|\chi_{c_2B})(x)\int_0^1 d\tau\, \tau^{\alpha(Q-1)} \\ &= c_5(\beta)r|B(x, r)|^{Q-1}M_Q(|\nabla_\lambda|\chi_{c_2B})(x), \quad \text{if } \alpha(Q-1) > -1.\end{aligned}$$

Now, a standard regularization argument enables us to extend the above inequality to Lipschitz continuous functions, completing the proof of the lemma. □

Lemma 4.3. *Let x_0 belong to Ω and let $\beta \in (0, 1)$ be fixed. Let w be an A_q-weight for a given $q > 1$ with respect to $(\mathbb{R}^n, d, dx)$. In addition, let u be a Lipschitz continuous function such that*

$$|\{y \in Q^\theta(x_0, r)\,;\, u(y) = 0\}| \ge \beta|Q^\theta(x_0, r)|,$$

where $\theta = \theta(\beta) = 1 - \sqrt[n]{1-\beta/2}$. If Q is chosen as in the previous lemma, then for any $p \ge q$ and for any $\kappa \in [1, (1-(1-Q)p/q)^{-1})$, there exist $c_6(\beta) > 0$ (depending only on β, the doubling constant A, p, and κ), a positive constant $r(\beta)$, and an absolute constant c_7 such that, if $r < r(\beta)$,

$$\begin{aligned}&\left(\frac{1}{w(Q^\theta(x_0, r))}\int_{Q^\theta(x_0, r)}|u(y)|^{\kappa p}w(y)\,dy\right)^{1/\kappa p} \\ &\qquad \le c_6(\beta)r\left(\frac{1}{w(B(x_0, c_7 r))}\int_{B(x_0, c_7 r)}|\nabla_\lambda u(y)|^p w(y)\,dy\right)^{1/p}.\end{aligned}$$

Here $c_7B = B(x_0, c_7r)$. We note explicitly that the constant can be chosen uniformly with respect to x_0 belonging to a small neighbourhood V of a fixed point.

Proof. Let x belong to $Q^\theta(x_0, r)$; then there exist two positive constants $c_8(\beta)$ and $c_9(\beta)$ depending only on β such that

$$Q^\theta(x, c_8(\beta)r) \supseteq Q^\theta(x_0, r) \quad \text{and} \quad Q^\theta(x_0, c_9(\beta)r) \supseteq Q^\theta(x, c_8(\beta)r).$$

Let us prove the first assertion. We have $Q^\theta(x_0, r) \subseteq Q(x_0, r) \subseteq B(x_0, br)$. Hence

$$\begin{aligned}Q^\theta(x_0, r) &\subseteq B(x, 2br) \subseteq Q(x, 2br) \\ &= (\text{putting } c_8(\beta) = 2b/c(\theta) > 1)Q(x, c_8(\beta)c(\theta)r) \\ &\subseteq Q^\theta(x, c_8(\beta)r)\end{aligned}$$

since $c(\theta) \le c_j(x, t, \theta)$. Analogously

$$\begin{aligned} Q^\theta(x, c_8(\beta)r) &\subseteq Q(x, c_8(\beta)r) \subseteq B(x, c_8(\beta)br) \\ &\subseteq B(x_0, 2c_8(\beta)br) \subseteq Q(x_0, 2c_8(\beta)br) \\ &= (\text{putting } c_9(\beta) = 2bc_8(\beta)/c(\theta))\ Q(x_0, c_9(\beta)c(\theta)r) \\ &\subseteq Q^\theta(x_0, c_8(\beta)r) \end{aligned}$$

since $c(\theta) \le c_j(x_0, t, \theta)$. We note explicitly that the constant $c(\theta)$ can be chosen locally independent of x and r.

Now

$$\begin{aligned} |\{y \in Q^\theta(x, c_8(\beta)r);\, u(y) = 0\}| &\ge |\{y \in Q^\theta(x_0, r);\, u(y) = 0\}| \\ &\ge \beta|Q^\theta(x_0, r)| \ge c_{10}(\beta)|Q^\theta(x, c_8(\beta)r)|. \end{aligned}$$

Then if r is small enough (depending on θ and hence on β) we can apply Lemma 4.2 to get

$$\begin{aligned} |u(x)| &\le c_{11}(\beta)r|B(x, r)|^{Q-1}M_Q(|\nabla_\lambda u|\chi_{B(x, c_2r)})(x) \\ &\le c_{11}(\beta)r|B(x, r)|^{Q-1}M_Q(|\nabla_\lambda u|\chi_{c_7B})(x), \end{aligned}$$

since, arguing as above, $B(x, c_2r) \subseteq B(x_0, (b + c_2)r) = B(x_0, c_7r)$. Moreover, without loss of generality, we may suppose $Q^\theta(x_0, r) \subseteq Q(x_0, r) \subseteq B(x_0, br) \subseteq B(x_0, c_7r)$. Thus, we have

$$\begin{aligned} &\left(\frac{1}{w(Q^\theta(x_0, r))}\int_{Q^\theta(x_0, r)} |u(y)|^{\kappa p} w(y)\,dy\right)^{1/\kappa p} \\ &\quad\le c_{11}(\beta)r|B(x, r)|^{Q-1} \\ &\qquad\cdot\left(\frac{1}{w(Q^\theta(x_0, r))}\int_{Q^\theta(x_0, r)} M_Q(|\nabla_\lambda u|\chi_{c_7B})(y)^{1/\kappa p} w(y)\,dy\right)^{1/\kappa p} \\ &\quad\le c_{11}(\beta)r|B(x, r)|^{Q-1} \\ &\qquad\cdot\left(\frac{1}{w(Q^\theta(x_0, r))}\int_{B(x_0, c_7r)} M_Q(|\nabla_\lambda u|\chi_{c_7B})(y)^{1/\kappa p} w(y)\,dy\right)^{1/\kappa p} \\ &\quad\le c_{12}(\beta)r|B(x_0, c_7r)|^{Q-1} \\ &\qquad\cdot\left(\frac{1}{w(Q^\theta(x_0, r))}\int_{B(x_0, c_7r)} M_Q(|\nabla_\lambda u|\chi_{c_7B})(y)^{1/\kappa p} w(y)\,dy\right)^{1/\kappa p}, \end{aligned}$$

by the doubling property of the measure of d-balls. Now we note that

$$1 \ge \frac{|Q^\theta(x_0, r)|}{|B(x_0, c_7r)|} \ge \frac{|Q(x_0, r)|}{|B(x_0, c_7r)|} c(\theta)^{\sum \alpha_j} \ge c_{13}(\beta)$$

and hence, taking into account that w is an A_p-weight function, $w(Q^\theta(x_0, r)) \geq c_{14}(\beta)w(B(x_0, c_7 r))$. Thus, we have

$$r|B(x_0, c_7 r)|^{Q-1}\left(\frac{1}{w(Q^\theta(x_0, r))}\int_{B(x_0, c_7 r)} M_Q(|\nabla_\lambda u|\chi_{c_7 B})(y)^{1/\kappa p} w(y)\,dy\right)^{1/\kappa p}$$
$$\leq c_{15}(\beta) r|B(x_0, c_7 r)|^{Q-1} \cdot\left(\frac{1}{w(B(x_0, c_7 r))}\int_{B(x_0, c_7 r)} M_Q(|\nabla_\lambda u|\chi_{c_7 B})(y)^{1/\kappa p} w(y)\,dy\right)^{1/\kappa p}.$$

We can now apply a continuity property for fractional maximal functions in spaces of homogeneous type (with a precise estimate of the continuity constants: see [FS, Lemma 4.4]) in order to complete the proof. □

Lemma 4.4 (Weak Sobolev-Poincaré inequality). *Let u be a Lipschitz continuous function. Let U be a neighbourhood of a fixed point as in the previous lemmas. Then there exists $r_0 > 0$ such that, if w is an A_q-weight function for a given $q > 1$, $p \geq q$, κ is fixed as in Lemma 4.3, $x \in U$, and $r < r_0$, then*

$$\left(\frac{1}{w(B(x_0, r))}\int_{B(x_0, r)} |u(y) - u_B|^{\kappa p} w(y)\,dy\right)^{1/\kappa p}$$
$$\leq c_{16} r\left(\frac{1}{w(B(x_0, c_{17} r))}\int_{B(x_0, c_{17} r)} |\nabla_\lambda u(y)|^p w(y)\,dy\right)^{1/p},$$

where the constants c_{16} and c_{17} depend only on p, q, κ, and on doubling constants. The constant u_B can be chosen to be either the Lebesgue average of u, i.e.,

$$u_B = \frac{1}{|B(x_0, r)|}\int_{B(x_0, r)} u(z)\,dz,$$

or the weighted average

$$u_B = \frac{1}{w(B(x_0, r))}\int_{B(x_0, r)} u(z)w(z)\,dz.$$

Proof. We chose in Lemma 4.3 $\beta = \frac{1}{2}$ and we put $\theta_0 = \theta(\frac{1}{2})$. Now all the constants are fixed. In addition, we note that $B(x_0, r) \subseteq Q(x_0, r) \subseteq Q^{\theta_0}(x_0, r/\theta_0) = Q^{\theta_0}(x_0, \gamma r)$. Let us choose r_0 such that $\gamma r_0 < r(\frac{1}{2})$ in Lemma 4.3.

A standard argument (see e.g., [FS, Theorem 4.5]) shows that there exists $\mu \in \mathbb{R}$ such that

$$|\{y \in Q^{\theta_0}(x_0, \gamma r);\, u(y) \geq \mu\}| \geq \tfrac{1}{2}|Q^{\theta_0}(x_0, \gamma r)|$$

and

$$|\{y \in Q^{\theta_0}(x_0, \gamma r);\, u(y) \leq \mu\}| \geq \tfrac{1}{2}|Q^{\theta_0}(x_0, \gamma r)|.$$

We can apply Lemma 4.3 to $u^+ = \max\{u-\mu, 0\}$ and $u^- = \max\{\mu - u, 0\}$ in $Q^{\theta_0}(x_0, \gamma r)$ to obtain

$$\left(\frac{1}{w(Q^{\theta_0}(x_0, \gamma r))}\int_{Q^{\theta_0}(x_0, \gamma r)} |u(y)-\mu|^{\kappa p} w(y)\, dy\right)^{1/\kappa p} \leq \gamma c_6\left(\frac{1}{2}\right) r\left(\frac{1}{w(B(x_0, \gamma c_7 r))}\int_{Q(x_0, \gamma c_7 r)} |\nabla_\lambda u(y)|^p w(y)\, dy\right)^{1/p}.$$

On the other hand, by the doubling property the Lebesgue measures of $Q^{\theta_0}(x_0, \gamma r)$ and $B(x_0, r)$ are equivalent and hence the w-measures are equivalent also. Thus, keeping in mind that $B(x_0, r) \subseteq Q_0(x_0, \gamma r)$ and putting $\gamma c_7 = c_{17}$, we get

$$\left(\frac{1}{w(B(x_0, r))}\int_{B(x_0, r)} |u(y)-\mu|^{\kappa p} w(y)\, dy\right)^{1/\kappa p} \leq c_{18} r\left(\frac{1}{w(B(x_0, c_{17} r))}\int_{B(x_0, c_{17} r)} |\nabla_\lambda u(y)|^p w(y)\, dy\right)^{1/p}.$$

Finally, the complete proof of our lemma can be obtained from the above inequality in a standard way (see [FS, Theorem 4.5]). □

In fact, up to a new choice of the constant c_{16}, we can put $c_{17} = 1$, by means of a technique due to R. V. Kohn and D. Jerison [K and J]. Arguing as in the proof of Theorem 4.5 in [FS], we get

Theorem 4.5 (Sobolev-Poincaré inequality). *Assume $w \in A_q(\mathbb{R}^n, d, dx)$ and let $B = B(x_0, r)$ be a given ball of radius $r < r_0$. Then, if $p \geq q$, κ is fixed as in Lemma 4.3, and u is Lipschitz continuous in a neighbourhood of B, then*

$$\left(\frac{1}{w(B(x_0, r))}\int_{B(x_0, r)} |u(y)-u_B|^{\kappa p} w(y)\, dy\right)^{1/\kappa p} \leq cr\left(\frac{1}{w(B(x_0, c_{17} r))}\int_{B(x_0, r)} |\nabla_\lambda u(y)|^p w(y)\, dy\right)^{1/p},$$

where c and r_0 can be chosen locally uniform with respect to x_0. In addition, c depends only on p, q, κ, and on the doubling constants. The constant u_B can be chosen to be either the Lebesgue average of u, i.e.,

$$u_B = \frac{1}{|B(x_0, r)|}\int_{B(x_0, r)} u(z)\, dz,$$

or the weighted average

$$u_B = \frac{1}{w(B(x_0, r))}\int_{B(x_0, r)} u(z) w(z)\, dz.$$

We are now able to prove some important inequalities.

Theorem 4.6. *Assume $w \in A_q(\mathbb{R}^n, d, dx)$ for some $q > 1$ and let $B = B(x_0, r)$ be a given ball of radius $r < r_0$. Let $\beta \in (0, 1)$ be fixed. Then, there exists $c(\beta) > 0$ such that, if $p \geq q$, κ is fixed as in Lemma 4.3, and u is a Lipschitz continuous function in a neighbourhood of B such that*

$$|E| = |\{x \in B \,;\, u(x) = 0\}| \geq \beta |B|,$$

then

$$\left(\frac{1}{w(B(x_0, r))} \int_{B(x_0, r)} |u(y)|^{\kappa p} w(y)\, dy\right)^{1/\kappa p} \leq c(\beta) r \left(\frac{1}{w(B(x_0, r))} \int_{B(x_0, r)} |\nabla_\lambda u(y)|^p w(y)\, dy\right)^{1/p},$$

where $c(\beta)$ and r_0 can be chosen locally uniform with respect to x_0. In addition, $c(\beta)$ depends only on β, p, q, κ, and on the doubling constants.

Proof. We have

$$\left(\frac{1}{w(B(x_0, r))} \int_{B(x_0, r)} |u(y)|^{\kappa p} w(y)\, dy\right)^{1/\kappa p} \leq \left(\frac{1}{w(B(x_0, r))} \int_{B(x_0, r)} |u(y) - u_B|^{\kappa p} w(y)\, dy\right)^{1/\kappa p} + |u_B|.$$

Let us now estimate the second term in the right-hand side of the above inequality. Keeping in mind that w is an A_q-weight, we have

$$|u_B| = \frac{1}{|E|} \int_E |u(y) - u_B|\, dy \leq \frac{1}{\beta} \frac{1}{|B|} \int_B |u(y) - u_B|\, dy \leq \frac{c_1}{\beta} \left(\frac{1}{w(B(x_0, r))} \int_{B(x_0, r)} |u(y) - u_B|^{\kappa p} w(y)\, dy\right)^{1/\kappa p}.$$

Thus,

$$\left(\frac{1}{w(B(x_0, r))} \int_{B(x_0, r)} |u(y)|^{\kappa p} w(y)\, dy\right)^{1/\kappa p} \leq c(\beta) \left(\frac{1}{w(B(x_0, r))} \int_{B(x_0, r)} |u(y) - u_B|^{\kappa p} w(y)\, dy\right)^{1/\kappa p}$$

and the assertion follows from the Sobolev-Poincaré inequality. □

Keeping in mind the doubling property of the measure of d-balls, a straightforward consequence of the preceding inequality is the following Sobolev theorem for compactly supported functions.

Theorem 4.7. *Assume* $w \in A_q(\mathbb{R}^n, d, dx)$ *for some* $q > 1$. *If* $p \geq q$, κ *is fixed as in Lemma* 4.3, *and* u *is a compactly supported Lipschitz continuous function such that* $\operatorname{supp} u$ *is contained in a given ball* $B = B(x_0, r)$ *of radius* $r < r_0$, *then*

$$\left(\frac{1}{w(B(x_0, r))}\int_{B(x_0, r)} |u(y)|^{\kappa p} w(y)\,dy\right)^{1/\kappa p} \leq c(\beta) r \left(\frac{1}{w(B(x_0, r))}\int_{B(x_0, r)} |\nabla_\lambda u(y)|^p w(y)\,dy\right)^{1/p},$$

where c *and* r_0 *can be chosen locally uniform with respect to* x_0. *In addition,* c *depends only on* p, q, κ, *and on the doubling constants.*

5

In this section we shall apply the inequalities of §4 to prove pointwise estimates for the weak solutions of the class of degenerate elliptic equations defined at the beginning of §2. In the sequel we shall use the notations introduced therein. In addition,

In the sequel we will suppose hypotheses (H.1)–(H.4) *are satisfied.*

Let us now recall some standard definitions [FL1, FS].

Given a measurable set $E \subseteq \mathbb{R}^n$ we denote by $L^p(E)$, $1 \leq p \leq \infty$, the usual Lebesgue spaces, while, if w is an A_p-weight function, we denote by $L^p(E, w)$, $1 \leq p \leq \infty$, the Banach space of the measurable functions f such that $\|f; L^p(E, w)\| = (\int_E |f|^p w(x)\,dx)^{1/p} < \infty$. Observe that since $w^{-1/(p-1)} \in L^1_{\text{loc}}(\mathbb{R}^n)$, then $L^1(E) \supseteq L^p(E, w)$ for any bounded measurable set.

We use the notations $H^{1,p}(\Omega)$ and $\overset{\circ}{H}{}^{1,p}(\Omega)$ for the usual Sobolev spaces, while we indicate by $H^{1,p}_\lambda(\Omega, w)$ (respectively by $\overset{\circ}{H}{}^{1,p}_\lambda(\Omega, w)$) the closure of the space $\operatorname{Lip}(\Omega)$ of Lipschitz continuous functions on Ω (respectively of $\operatorname{Lip}(\Omega) \cap \mathscr{E}'(\Omega)$) with respect to the norm

$$\|f; H^{1,p}_\lambda(\Omega, w)\| = \|f; L^p(\Omega, w)\| + \|\nabla_\lambda f; L^p(\Omega, w)\|.$$

Moreover, the spaces $L^p_{\text{loc}}(E, w)$ and $H^{1,p}_{\lambda,\text{loc}}(\Omega, w)$ are defined in the usual way.

The following assertion is straightforward.

Proposition 5.1. *The bilinear form* $\mathscr{B}$ *on* $\operatorname{Lip}(\Omega) \cap H^{1,2}_\lambda(\Omega, \omega)$ *defined as*

$$\mathscr{B}(u, v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij} \partial_i u \partial_j v\,dx$$

can be extended to all of $H^{1,2}_\lambda(\Omega, \omega)$.

Definition 5.2. Let $f = (f_1, \dots, f_n)$ be a vector-valued function such that $|f|/\omega \in L^2(\Omega, \omega)$. We say that u is a solution of the Dirichlet problem

$$\text{(DP)} \qquad \begin{cases} \mathscr{L}u = \operatorname{div}_\lambda f & \text{in } \Omega, \\ u = g & \text{in } \partial\Omega \end{cases}$$

if $u - g \in \overset{\circ}{H}{}_\lambda^{1,2}(\Omega, \omega)$ and

$$\mathscr{B}(u, \varphi) = \sum_{i=1}^{n} \int_\Omega \lambda_i f_i \partial_i \varphi \, dx \quad \forall \varphi \in \overset{\circ}{H}{}_\lambda^{1,2}(\Omega, \omega).$$

In addition, we say that $v \in H^{1,2}_{\lambda,\mathrm{loc}}(\Omega, \omega)$ is a local subsolution (local supersolution) for $\mathscr{L}$ if for any open set $\Omega' \Subset \Omega$ and for any $\varphi \in \overset{\circ}{H}{}_\lambda^{1,2}(\Omega, \omega)$, $\varphi \geq 0$, we have $\mathscr{B}(u, \varphi) \leq 0$ $(\mathscr{B}(u, \varphi) \geq 0)$. Moreover we say that $u \in H^{1,2}_{\lambda,\mathrm{loc}}(\Omega, \omega)$ is a local solution for $\mathscr{L}$ if it is both a local subsolution and a local supersolution.

Theorem 4.7 says that if $w \in A_2(R^n, d, dx)$ and Ω is bounded, then $\overset{\circ}{H}{}_1^{1,2}(\Omega, w)$ is continuously imbedded in $L^{2\kappa}(\Omega, w)$. In fact, $\overset{\circ}{H}{}_\lambda^{1,2}(\Omega, w)$ is compactly imbedded in $L^{2\kappa}(\Omega, w)$. The proof can be carried out by using the Sobolev-Poincaré inequality by the same arguments of Theorem 4.6 in [FS]. We note explicitly that the proof therein relies only on the Poincaré inequality and on the fact that $(\mathbb{R}^n, d, dx)$ is a metric space of homogeneous type. Then, by (H.3) we have:

Theorem 5.3. *$\overset{\circ}{H}{}_\lambda^{1,2}(\Omega, \omega)$ is compactly imbedded in $L^{2\kappa}(\Omega, \omega)$ for $1 \leq \kappa < \kappa_0$.*

Moreover, by the Lax-Milgram theorem we have

Proposition 5.4. *Let $f = (f_1, \dots, f_n)$ be a vector-valued function such that $|f|/\omega \in L^2(\Omega, \omega)$ and let g belong to $H_\lambda^{1,2}(\Omega, \omega)$. Then there exists a unique solution $u \in H_\lambda^{1,2}(\Omega, \omega)$ of the Dirichlet problem* (DP) *of Definition* 5.2.

We can now repeat the arguments in §5 of [FS] in order to obtain the basic pointwise estimates for weak solutions of the equation $\mathscr{L}u = \operatorname{div}_\lambda f$.

Theorem 5.5. *If $u \in H^{1,2}_{\lambda,\mathrm{loc}}(\Omega, \omega)$ is a local supersolution for $\mathscr{L}$, then*

$$u(x) \geq \operatorname*{Inf}_{\partial\Omega} u \quad \text{a.e. in } \Omega,$$

where $\operatorname{Inf}_{\partial\Omega} u$ is taken in the $H_\lambda^{1,2}(\Omega, \omega)$ sense (*the definition is the same as in the elliptic case: see* [KS, *Definition* 5.1]).

Theorem 5.6. *Let $\kappa > 1$ be fixed as in the Sobolev-Poincaré inequality. Let $f = (f_1, \dots, f_n)$ be a vector-valued function such that $|f|/\omega \in L^p(\Omega, \omega)$ for*

$p > 2\kappa/(\kappa - 1)$. *Then if* $u \in \overset{\circ}{H}{}^{1,2}_{\lambda}(\Omega, \omega)$ *is a solution in* Ω *of* $\mathscr{L}u = \operatorname{div}_{\lambda} f$, *we have*

$$\operatorname*{Sup}_{\Omega} |u(x)| \le C(\Omega) w(\Omega)^{1/2 - 1/p - 1/2\kappa} \|f/\omega ; L^p(\Omega, \omega)\|,$$

where $C(\Omega)$ *depends only on the diameter of* Ω.

Theorem 5.7. *Let* $u \in H^{1,2}_{\lambda, \mathrm{loc}}(\Omega, \omega)$ *be a local subsolution of* $\mathscr{L}u = 0$. *Then there exists* $M > 0$, M *independent of* $r < r_0$ *and* u, *such that, if* $Q(x, r) \subset \Omega$, *then*

$$\operatorname*{Sup}_{Q(x, r/2)} u \le M \left(\frac{1}{\omega(Q(x, r))} \int_{Q(x, r)} u^2 \omega \, dy \right)^{1/2}.$$

Theorem 5.8. *Let* $u \in H^{1,2}_{\lambda, \mathrm{loc}}(\Omega, \omega)$ *be a local solution of* $\mathscr{L}u = 0$. *Then there exists* $M > 0$, M *independent of* $r < r_0$ *and* u, *such that, if* $Q(x, r) \subset \Omega$, *then*

$$\operatorname*{Sup}_{Q(x, r/2)} |u| \le M \left(\frac{1}{\omega(Q(x, r))} \int_{Q(x, r)} u^2 \omega \, dy \right)^{1/2}.$$

Theorem 5.9 (Harnack inequality). *Let* $u \in H^{1,2}_{\lambda, \mathrm{loc}}(\Omega, \omega)$ *be a local solution of* $\mathscr{L}u = 0$. *Then there exists* $M > 0$ *and* $a > 0$, M *and* a *independent of* x, r *and* u, *such that, if* $r < a \cdot d(x, \partial\Omega)$, *then*

$$\operatorname*{Sup}_{Q(x, r)} u \le M \operatorname*{Inf}_{Q(x, r)} u.$$

The proofs of the above results can be obtained by repeating verbatim the proofs of the corresponding results in [FS]: in fact the techniques we used in [FS] require only the following tools:

(i) $(\mathbb{R}^n, d, dx)$ is a space of homogeneous type (here proved in Theorem 2.6);
(ii) Sobolev-Poincaré inequality (here proved in Theorem 4.5);
(iii) existence of cut-off functions on the d-balls (or, equivalently, on the quasi-balls $Q(x, r)$).

The following proposition provides us with the next tool.

Proposition 5.10. *Let* x, r_1, *and* r_2 *be given,* $0 < r_1 < r_2 < r_0$. *Then there exists a cut-off function* $\varphi \in C_0^{\infty}(\mathbb{R}^n)$ *such that*:

(i) $\operatorname{supp} \varphi \subset Q(x, r_1)$, $\varphi \equiv 1$ *on* $Q(x, r_2)$, $0 \le \varphi \le 1$;
(ii) $|\nabla_{\lambda} \varphi| \le C(r_2 - r_1)^{-1}$, *where* C *is an absolute constant.*

Proof. Let φ be a smooth function on $[0, \infty)$ such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ on $[0, r_1/r_2]$, $\varphi \equiv 0$ on $[1, \infty)$, and $|\varphi'(t)| \le 2(1 - r_1/r_2)^{-1}$. Put

$$\varphi(y) = \prod_{j=1}^{n} \varphi \left(\frac{|y_j - x_j|}{r_2 \Lambda_j(x, r_2)} \right).$$

Obviously φ is a smooth function and $0 \leq \varphi \leq 1$. If $y \notin Q(x, r_2)$, there exists $j \in \{1, \dots, n\}$ such that $|y_j - x_j| \geq r_2\Lambda_j(x, r_2)$ and hence

$$\varphi(|y_j - x_j|/r_2\Lambda_j(x, r_2)) = 0$$

so that $\varphi(y) = 0$. Let now y belong to $Q(x, r_1)$; then $|y_j - x_j| \leq r_1\Lambda_j(x, r_1)$ for $j = 1, \dots, n$ and hence

$$|y_j - x_j|/r_2\Lambda_j(x, r_2) \leq r_1\Lambda_j(x, r_1)/r_2\Lambda_j(x, r_2) \leq r_1/r_2,$$

since $\Lambda_j(x, r_1) \leq \Lambda_j(x, r_2)$, so that $\varphi(|y_j - x_j|/r_2\Lambda_j(x, r_2)) = 1$ for $j = 1, \dots, n$. Thus $\varphi(y) = 1$ and part (i) is proved. On the other hand,

$$|\lambda_k(y)\partial_k\varphi(y)| = \prod_{j\neq k} \varphi\left(\frac{|y_j - x_j|}{r_2\Lambda_j(x, r_2)}\right)\left|\varphi'\left(\frac{|y_k - x_k|}{r_2\Lambda_k(x, r_2)}\right)\right| \frac{\lambda_k(y)}{r_2\Lambda_k(x, r_2)} = I.$$

Obviously, if $I = 0$, then (ii) holds. Hence we may suppose $y \in Q(x, r_2)$. By Theorem 2.3, $y \in B(x, br_2)$ and hence there exists a subunit curve $h\colon [0, T] \mapsto \mathbb{R}^n$ such that $h(0) = x$, $h(T) = y$, and $T < br_2$. In particular, $\lambda_k(y) = \lambda_k(h(T)) \leq \Lambda_k(x, br_2) \leq c\Lambda_k(x, r_2)$, where c is an absolute constant. Then

$$I \leq 2c\frac{\Lambda_k(x, r_2)}{(1 - r_1/r_2)r_2\Lambda_k(x, r_2)} = \frac{2c}{r_2 - r_1},$$

and the assertion is completely proved. □

In a standard way it follows from the above results that weak solutions are locally Hölder continuous (De Giorgi-Nash-Moser theorem).

Theorem 5.11. *Let $\kappa > 1$ be fixed as in the Sobolev-Poincaré inequality. Let $f = (f_1, \dots, f_n)$ be a vector-valued function such that $|f|/\omega \in L^p(\Omega, \omega)$, where $p > 2\kappa/(\kappa - 1)$ and $\omega \in A_{p/\delta}$, $\delta = \sum_j \alpha_j$. Then if $u \in \overset{\circ}{H}{}^{1,2}_\lambda(\Omega, \omega)$ is a solution in Ω of $\mathscr{L}u = \operatorname{div}_\lambda f$, u is locally Hölder continuous in Ω. More precisely, there exist $\sigma \in (0, 1)$ and $a > 0$ such that, if $x_0 \in \Omega$ and $0 < r < R < a \cdot d(x_0, \partial\Omega)$, then*

$$\sup_{|x - x_0| < r} |u(x) - u(y)| \leq c(R)\left(\left(\frac{1}{\omega(B(x_0, R))}\int_{B(x_0, R)} u^2\omega\, dx\right)^{1/2} + \||f|/\omega; L^p(B(x_0, r), \omega)\|\right) r^\sigma.$$

6

In this section, we will exhibit simple examples showing that a large class of operators satisfy our hypotheses. In particular, we are concerned with hypothesis (H.4). In what follows, we will deal mainly with the case $\omega \equiv 1$, since the two different degenerations (the metric degeneration and the measure degeneration) can be treated separately. A discussion of some examples of A_2-weights with respect to degenerate metrics is given in [FS].

Example 1. Previous results obtained in [FL1, FS] are particular cases of the present ones. In fact, in these papers pointwise estimates for the weak solutions are obtained under the following assumptions:

(i) $\lambda_j(x) = \lambda_j(x_1, \dots, x_{j-1})$, $j = 1, \dots, n$;
(ii) λ_j is continuously differentiable when $x_1 \cdot \dots \cdot x_{j-1} \neq 0$, $j = 1, \dots, n$;
(iii) $\lambda_j(x) = \lambda_j(x_1, \dots, |x_k|, \dots, x_{j-1})$, and $0 \le x_k \partial_k \lambda_j(x) \le \alpha_{jk} \lambda_j(x)$ for any x and for $k = 1, \dots, j-1$.

Now, if (i)–(iii) are verified, hypothesis (H.4) is an immediate consequence of Proposition 7.3 in [FS], keeping in mind Proposition 2.7 in [FS] and Theorem 2.3 in the present paper.

Moreover, we note that if hypothesis (H.4) is satisfied, by Proposition 3.1 a sub-Riemannian structure is associated with the operator $\mathscr{L}$ in the sense of [FL3]. Thus, the regularity results for the weak solutions of $\mathscr{L}$ are, in a suitable sense, an improvement of the estimates in [FL3] in the diagonal case.

Example 2. Let us consider in detail the case $n = 2$, where hypothesis (H.4) assumes a very simple form. As we pointed out in §1, in the particular case $n = 2$ and $\omega \equiv 1$ similar Sobolev-Poincaré estimates have been recently obtained in [X] by different techniques; nevertheless, an explicit form of (H.4) in this case can suggest the meaning of the condition in higher dimension. Without loss of generality, we may suppose $\lambda_1 \equiv 1$ so that a straightforward calculation shows that (H.4) is satisfied if

$$\text{(H.4}'\text{)} \qquad \int_0^t \lambda_2(x_1 \pm s, x_2)\, ds \ge ct \max_{0 \le s \le t} \lambda_2(x_1 \pm s, x_2).$$

We will say that a function g belongs to RH_∞ (i.e., satisfies an infinite order reverse Hölder inequality) if $\int_I g(s)\, ds \ge c|I| \max_I g$ for every compact interval I. Thus, (H.4$'$) can be formulated in the following way:

(H.4$''$) $\quad s \to \lambda_2(s, x_2)$ belongs to RH_∞, uniformly with respect to x_2.

It is easy to show that (H.4$''$) is satisfied by a very large class of Lipschitz continuous functions satisfying no monotonicity condition as in Example 1. Consider, e.g., the function $g = g(|x_1|)$ defined in the following way: if $\alpha > 0$ and n is a positive integer, put

$$g\left(\frac{1}{n^\alpha}\right) = 0, \qquad g\left(\frac{1}{2n^\alpha} + \frac{1}{2(n+1)^\alpha}\right) = \frac{1}{2n^\alpha} - \frac{1}{2(n+1)^\alpha}$$

and let us linearly interpolate between these points. Clearly, g is a Lipschitz continuous, bounded nonnegative function. Now a straightforward direct calculation shows that g belongs to RH_∞. We note explicitly that g is essentially nonsmooth.

Example 3. Suppose $n > 2$ and $\lambda_j(x) = \lambda_{j,1}(x_1) \cdot \dots \cdot \lambda_{j,n}(x_n)$, where $\lambda_{j,k}$ is a positive Lipschitz continuous real function for $j, k = 1, \dots, n$. If $d(x, y) < \infty$ for any x, y and the distance d is continuous, it was shown in [FL1, Theorem 2.4] that without loss of generality we may suppose $\lambda_{j,j} \equiv \cdots \equiv \lambda_{j,n} \equiv 1$ for $j = 1, \ldots, n$. Suppose now $\lambda_{j,k}$ belongs to RH_∞ for $k < j$, $j = 2, \ldots, n$. We will show that (H.4) holds. Preliminary, we note explicitly that in particular the $\lambda_{j,k}$'s are A_∞-weights. Hence, if J and I are intervals, $J \subseteq I$, then

$$\text{(6.3.a)} \quad \int_I \lambda_{j,k}(s)\,ds \cdot (|J|/|I|)^{b_{j,k}} \geq \int_J \lambda_{j,k}(s)\,ds \geq \int_I \lambda_{j,k}(s)\,ds \cdot (|J|/|I|)^{a_{j,k}}$$

for some positive constants $a_{j,k}$ and $b_{j,k}$.

Let us now prove that (H.4) holds. For the sake of simplicity, let us restrict ourselves to the case $n = 3$. In this case

$$\begin{aligned} H_1(t, x, \xi) &= x_1 + t\xi_1, \\ H_2(t, x, \xi) &= x_2 + \int_0^t \lambda_{2,1}(x_1 + s\xi_1)\,ds\xi_2, \\ H_3(t, x, \xi) &= x_2 + \int_0^t \lambda_{3,1}(x_1 + s\xi_1)\lambda_{3,2}(H_2(s, x, \xi))\,ds\xi_3. \end{aligned}$$

Obviously (with the notations in (H.4)),

$$\int_0^t \lambda_1(H(s, x, \xi))\,ds = 1 = t\Lambda_1(x, t).$$

In addition, if we denote by $I(a, b)$ the interval whose endpoints are the real numbers a and b, we get

$$\int_0^t \lambda_2(H(s, x, \xi))\,ds = \int_0^t \lambda_{2,1}(x_1 + s\xi_1)\,ds = \frac{1}{|\xi_1|} \int_{I(x_1, x_1 + t\xi_1)} \lambda_{2,1}(s)\,ds.$$

Hence, keeping in mind (6.3.a) and the RH_∞ hypothesis,

$$\begin{aligned} \int_0^t \lambda_2(H(s, x, \xi))\,ds &\geq |\xi_1|^{a_{2,1}-1} \int_{I(x_1, x_1+t)} \lambda_{2,1}(s)\,ds \\ &\geq c_1 t |\xi_1|^{a_{2,1}-1} \int_{x_1-t}^{x_1+t} \lambda_{2,1}(s)\,ds \\ &\qquad \text{(by the doubling property of } \lambda_{2,1}(s)\,ds) \\ &\geq c_2 \varepsilon_1^{a_{2,1}-1} \Lambda_2(x, t), \end{aligned}$$

since $\lambda_{2,1}$ belongs to RH_∞ and $C_1(x, t) = [x_1 - t, x_1 + t]$.

Let us now prove the corresponding estimate for λ_3. Keeping in mind that

$\lambda_{3,1}$ belongs to A_p for some $p > 1$, by Theorem 2.8 in [GR] we have

$$\int_0^t \lambda_3(H(s, x, \xi))\, ds = \int_0^t \lambda_{3,1}(x_1 + s\xi_1)\lambda_{3,2}(H_2(s, x, \xi))\, ds$$

$$= \frac{1}{|\xi_1|} \int_{I(x_1, x_1+t\xi_1)} \lambda_{3,1}(s')\lambda_{3,2}(H_2((s' - x_1)/\xi_1, x, \xi))\, ds'$$

$$\text{(putting } x_1 + s\xi_1 = s')$$

$$\geq c_2 \frac{1}{|\xi_1|} \int_{I(x_1, x_1+t\xi_1)} \lambda_{3,1}(s)\, ds$$

$$\cdot \left(\frac{1}{t|\xi_1|} \int_{I(x_1, x_1+t\xi_1)} \lambda_{3,2}^{1/p}(H_2((s - x_1)/\xi_1, x, \xi))\, ds \right)^p.$$

On the other hand,

$$t^{-p} \left(\int_{I(x_1, x_1+t\xi_1)} \lambda_{3,2}^{1/p}(H_2((s - x_1)/\xi_1, x, \xi))\, ds \right)^p$$

$$= t^{-p}|\xi_1|^p \left(\int_0^t \lambda_{3,2}^{1/p}(H_2(s, x, \xi))\, ds \right)^p$$

$$= t^{-p} \frac{|\xi_1|^p}{|\xi_2|^p} \left(\int_0^t \lambda_{3,2}^{1/p}(H_2(s, x, \xi)) \frac{|dH_2(s, x, \xi)|}{\lambda_{2,1}(H_1(s, x, \xi))} \right)^p$$

$$\geq t^{-p} \frac{|\xi_1|^p}{|\xi_2|^p} \left(\max_{I(x_1, x_1+t\xi_1)} \lambda_{2,1} \right)^{-p} \left(\int_{I(x_2, H_2(t, x, \xi))} \lambda_{3,2}^{1/p}(s)\, ds \right)^p$$

$$\geq t^{-p} \frac{|\xi_1|^p}{|\xi_2|^p} \left(\max_{I(x_1, x_1+t\xi_1)} \lambda_{2,1} \right)^{-p} \left(\max_{I(x_2, H_2(t, x, \xi))} \lambda_{3,2} \right)^{1-p}$$

$$\cdot \left(\int_{I(x_2, H_2(t, x, \xi))} \lambda_{3,2}(s)\, ds \right)^p$$

$$\geq c_3 t^{-p} \frac{|\xi_1|^p}{|\xi_2|^p} \left(\max_{I(x_1, x_1+t\xi_1)} \lambda_{2,1} \right)^{-p} |H_2(t, x, \xi) - x_2|^p \max_{I(x_2, H_2(t, x, \xi))} \lambda_{3,2}$$

$$= c_3 t^{-p} |\xi_1|^p \left(\max_{I(x_1, x_1+t\xi_1)} \lambda_{2,1} \right)^{-p} \left(\int_0^t \lambda_2(H(s, x, \xi))\, ds \right)^p \max_{I(x_2, H_2(t, x, \xi))} \lambda_{3,2}$$

$$\geq \text{(arguing as above) } c_3 \xi_1^p \max_{I(x_2, H_2(t, x, \xi))} \lambda_{3,2},$$

so that

$$\int_0^t \lambda_3(H(s, x, \xi))\, ds \geq c_4 \xi_1^{p-1} \int_{I(x_1, x_1+t\xi_1)} \lambda_{3,2}(s)\, ds \cdot \max_{I(x_2, H_2(t, x, \xi))} \lambda_{3,2}.$$

Now we note that a straightforward calculation shows that $C_1(x, t) = [x_1 - t, x_1 + t]$ and

$$C_2(x, t) \subseteq \left[x_1 - t \max_{[x_1 - t, x_1 + t]} \lambda_{2,1}, x_1 + t \max_{[x_1 - t, x_1 + t]} \lambda_{2,1} \right].$$

Hence, if we suppose for the sake of simplicity that $\xi_j > 0$ for $j = 1, 2$, we get

$$\begin{aligned} t\Lambda_3(x, t) &\le t \max_{[x_1-t, x_1+t]} \lambda_{3,1} \cdot \max_{[x_2 - t \max_{[x_1-t,x_1+t]} \lambda_{2,1}, \dots]} \lambda_{3,2} \\ &\le c_5 \int_{x_1-t}^{x_1+t} \lambda_{3,1}(s)\, ds \int_{x_2-\cdots}^{x_2+\cdots} \lambda_{3,2}(s)\, ds \frac{1}{t \max_{[x_1-t,x_1+t]} \lambda_{2,1}} \\ &\le c_6 \int_{x_1}^{x_1+t} \lambda_{3,1}(s)\, ds \int_{x_2}^{x_2+\cdots} \lambda_{3,2}(s)\, ds \frac{1}{t \max_{[x_1-t,x_1+t]} \lambda_{2,1}} \end{aligned}$$

(by the doubling property of $\lambda_{3,i}(s)\, ds$, $i = 1, 2$).

In order to estimate the second integral, let us note that

$$\begin{aligned} t \max_{[x_1-t, x_1+t]} \lambda_{2,1} &\ge H_2(t, x, \xi) - x_2 = \xi_2 \int_0^t \lambda_2(x_1 + s\xi_1)\, ds \\ &\ge \xi_2 \xi_1^{a_{2,1}-1} \int_{x_1}^{x_1+t} \lambda_{2,1}(s)\, ds \ge c_7 \xi_2 \xi_1^{a_{2,1}-1} t \max_{[x_1-t,x_1+t]} \lambda_{2,1}. \end{aligned}$$

Hence

$$\begin{aligned} &\frac{1}{t \max_{[x_1-t,x_1+t]} \lambda_{2,1}} \int_{x_2}^{x_2+\cdots} \lambda_{3,2}(s)\, ds \\ &\le \int_{[x_2, H_2(t,x,\xi)]} \lambda_{3,2}(s)\, ds \cdot \left(\frac{t \max_{[x_1-t,x_1+t]} \lambda_{2,1}}{H_2(t, x, \xi) - x_2} \right)^{a_{3,2}} \frac{1}{t \max_{[x_1-t,x_1+t]} \lambda_{2,1}} \\ &\le \max_{[x_2, H_2(t,x,\xi)]} \lambda_{3,2} \cdot \left(\frac{t \max_{[x_1-t,x_1+t]} \lambda_{2,1}}{H_2(t, x, \xi) - x_2} \right)^{a_{3,2}-1} \\ &\le \max_{[x_2, H_2(t,x,\xi)]} \lambda_{3,2} \cdot \left(\frac{1}{c_7 \xi_2 \xi_1^{a_{2,1}-1}} \right)^{a_{3,2}-1}. \end{aligned}$$

Thus (without loss of generality we may suppose $a_{j,k} \ge 1$),

$$t\Lambda_3(x, t) \le c_8 \left(\frac{1}{\xi_2 \xi_1^{a_{2,1}-1}} \right)^{a_{3,2}-1} \int_{x_1}^{x_1+t} \lambda_{3,1}(s)\, ds \cdot \max_{[x_2, H_2(t,x,\xi)]} \lambda_{3,2}.$$

Hence,

$$\begin{aligned} \int_0^t \lambda_3(H(s, x, \xi))\, ds &\ge c_9 \xi_1^{p-1+(a_{2,1}-1)(a_{3,2}-1)} \xi_2^{a_{3,2}-1} t\Lambda_3(x, t) \\ &\ge c_9 \varepsilon_1^{\alpha_1} \varepsilon_2^{\alpha_2} t\Lambda_3(x, t), \end{aligned}$$

for suitable $\alpha_1, \alpha_2 > 0$.

Thus, the assertion is completely proved.

Remark. A final remark about RH_∞ is now in order. As is easy to see from our proof, the doubling property of $|B(x, t)|$ plays a key role in the Harnack inequality for degenerate elliptic operators. On the other hand, the doubling property of the measure of the d-balls is equivalent to the same inequality for the function $t \to \Lambda_k(x, t)$, $k = 1, \dots, n$. Now we can prove the following result:

Let φ be a real smooth function and put $\Phi(x, t) = \max_{[x-t, x+t]} |\varphi|$. If

$$(*) \qquad \Phi(x, 2t) \le C\Phi(x, t) \quad \text{and} \quad \Phi(x, t) > 0 \quad \text{for any real } x \text{ and any } t > 0,$$

then $|\varphi|$ belongs to RH_∞ on any compact subset K of the real line.

Suppose $(*)$ is satisfied. A standard argument shows that there exists $\alpha > 0$ such that $\Phi(x, t) \ge t^\alpha \Phi(x, 1) \ge C_K t^\alpha$ for $x \in K$ and $t \in (0, 1)$. Now we note that if $p_m = p_m(t) = a_0 + \cdots + a_m t^m$ is a real polynomial of degree m, then there exist two positive constants $c_{1,m}$, $c_{2,m}$ *depending only on* m such that

$$(**) \qquad t \max_{[-t, t]} |p_m| \le c_{1,m} \sum_{j=0}^{m} |a_j| t^j \le c_{2,m} \int_{-t}^{t} |p_m(s)|\, ds.$$

Indeed, up to rescaling the variables, we can reduce ourselves to the interval $[-1, 1]$ where the uniform norm, the L^1-norm, and the sum of the absolute values of the coefficients give equivalent norms (the space of polynomials of degree $\le m$ is finite-dimensional).

Now let $x \in K$ be fixed and let us write the Taylor polynomial of φ up to order $[\alpha]$. If $\max_{[x-t, x+t]} |\varphi| = \varphi(x + s^*)$ with $|s^*| \le t$, we have

$$\begin{aligned}
\Phi(x, t) = \varphi(x + s^*) &\le \left| \sum_{i=0}^{[\alpha]} \varphi^{(i)}(x) s^{*i} \right| + C(K) t^{[\alpha]+1} \\
&\le \max_{|s| \le t} \left| \sum_{i=0}^{[\alpha]} \varphi^{(i)}(x) s^i \right| + C(K) t^{[\alpha]+1} \\
&\le C_\alpha \frac{1}{t} \int_0^t \left| \sum_{i=0}^{[\alpha]} \varphi^{(i)}(x) s^i \right| ds + C(K) t^{[\alpha]+1} \\
&\le C_\alpha \frac{1}{t} \int_0^t |\varphi(s)|\, ds + C'(K) t^{[\alpha]+1} \\
&\le C_\alpha \frac{1}{t} \int_0^t |\varphi(s)|\, ds + C''(K) t^\varepsilon \Phi(x, t),
\end{aligned}$$

where $\varepsilon = [\alpha] - \alpha + 1 > 0$. Thus, the assertion is proved if t is small enough.

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