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Weighted Sum-rate Functional Dependence Bound for Network Coding Capacity

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***Abstract*—Explicit characterization of network coding capacity for multi-source multi-sink networks is an extremely hard problem. The linear programming bound is an explicit outer bound on network coding capacity but it is computationally very intensive. An edge-cut bound called functional dependence bound is an easily computable relaxation of the linear programming bound. However, the functional dependence bound is still very loose, even for two source unicast networks. In this paper, we characterize a set of Shannon-type inequalities for a given network that leads to new weighted bounds providing strict improvement over the functional dependence bound.**

I. Introduction

By allowing intermediate nodes to combine the input data, network coding can achieve higher throughput than pure routing. For single-source networks, the capacity region is given by the max-flow bound. However, the max-flow bound is not tight for multi-source networks. An exact expression of capacity region for multi-source networks is given in terms of entropy region $\Gamma_n^\star$ in [1]. Unfortunately, characterization of $\Gamma_n^\star$ is still open for $n \geq 3$. Characterizing the network capacity is proved to be as difficult as finding the entropy region, $\Gamma_n^\star$, which is equivalent to the determination of the set of all possible information inequalities [2]. By replacing the set of entropy function $\Gamma_n^\star$ with the set of polymatroidal functions Γ_n (which is characterized by elemental Shannon-type inequalities), a computable outer bound for the network, called linear programming (LP) bound, is obtained. However, the number of variables and constraints for this linear programming problem grows exponentially with the network size. An easily computable relaxation of the LP bound is proposed in [3] called functional dependence (FD) bound.

Besides the FD bound, the cut-set bound, the P*d*E bound [4], the network sharing bound [5] and the bounds based on information dominance [6] are well known edge-cut bounds in the literature. Theoretical comparisons of various edge-cut bounds are given in [7]. All these edge-cut bounds are for the sum of source rates for some subset of sources in a network (see e.g., (10)) also known as sum-rate bounds. It is shown in [8] that sum-rate bounds are not sufficient for characterizing the capacity for general multi-source multi-sink networks, even for simple two-unicast networks. Therefore, *weighted* bounds are required for improving the existing sum-rate edge-cut bounds. A simple way for finding weighted bound is introduced in [9] for multi-source *multicast* networks. The weighted bound of [9] uses "multicast" argument and hence it is not applicable for multiple-unicast networks.

In this paper, we give weighted sum-rate functional dependence bound for multi-source multi-sink networks using Shannon-type inequalities. For a given multi-source multi-sink network, we characterize a set of Shannon-type inequalities that renders weighted sum-rate functional dependence bound. An algorithm for finding the weighted bound for networks is also given. It is shown that the weighted bound strictly improve the FD bound. Therefore, the gap between the FD bound and the "computationally infeasible" LP bound is further reduced.

In section II, a network model and two explicit outer bounds are presented: the linear programming bound and the functional dependence bound. The notion of irreducible sets introduced in [3] will also be discussed. The main contribution of the paper is presented in Section III. In particular, a utility of Shannon-type (submodular) inequality to derive a weighted bound is described by example. Then a class of shannon-type inequalities is characterized for a given network to derive weighted bounds and an algorithm to find weighted bounds is given. Finally, the paper is concluded in section IV.

In the sequel, set elements are denoted by numbers or capital letters e.g., $1, A$, sets are denoted by font e.g., $\mathcal{A}$ and sets of sets are denoted by bold font e.g., $\boldsymbol{\mathcal{A}}$. The power set of a set $\mathcal{A}$ is denoted by $\boldsymbol{\mathcal{P}}(\mathcal{A})$. The notation $A_\mathcal{B}$ means the set $\{A_B : B \in \mathcal{B}\}$.

II. Background

A noise-free point-to-point network is represented by a directed acyclic graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes and $\mathcal{E}$ is the set of edges. For an edge $E = (A, B) \in \mathcal{E}$, define $\mathrm{Head}(E) = B$ (means B is the head node of E) and $\mathrm{Tail}(E) = A$. For $A \in V$, the set of edges entering into A and leaving A are denoted by $\mathrm{In}(A)$ and $\mathrm{Out}(A)$ respectively, i.e. $\mathrm{In}(A) = \{E \in \mathcal{E} : \mathrm{Head}(E) = A\}$, $\mathrm{Out}(A) = \{E \in \mathcal{E} : \mathrm{Tail}(E) = A\}$.

Let $\mathcal{S} = \{1, \ldots, |\mathcal{S}|\}$ denote the set of independent information sources available at some nodes (called source nodes) in a network via mapping $a : \mathcal{S} \mapsto \mathcal{V}$. The sources are demanded by some nodes in the network called sink nodes. A set of sources demanded by a given sink node is described by mapping $b : \mathcal{V} \mapsto \boldsymbol{\mathcal{P}}(\mathcal{S})$, e.g., the set of sources demanded by the node A is $b(A)$. If each source is demanded by exactly one sink, the network is called multiple-unicast network. For multiple-unicast network, imaginary sink nodes can be added to make the mapping b become bijective and the mapping from $\mathcal{S} \mapsto \mathcal{V}$ is denoted as b^{-1}.

Following the definition of a network code for graph $\mathcal{G}$ in [10, Section 21.4], an information rate tuple $\mathbf{R} = (R_S : S \in \mathcal{S})$ is achievable if there exists network codes such that every source S can be transmitted at rate R_S from its source node and the corresponding sink nodes can decode the source with arbitrarily small probability of error. The capacity region of a network is the set of all achievable rate tuples.

A. Linear Programming Bound

Given a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with sets of random variables $\{Y_S : S \in \mathcal{S}\}$ and $\{U_E : E \in \mathcal{E}\}$ representing the information generated by sources and carried by edges, a valid network code for the network must satisfy the following constraints:

$$R_S \leq H(Y_S) \quad (1)$$

$$H(Y_{\mathcal{S}}) = \sum_{S \in \mathcal{S}} H(Y_S) \quad (2)$$

$$H(U_{\mathrm{Out}(V)} | Y_{\{S:a(S)=V\}}, U_{\mathrm{In}(V)}) = 0, V \in \mathcal{V} \quad (3)$$

$$H(Y_{b(V)} | U_{\mathrm{In}}(V)) = 0, V \in \mathcal{V} \quad (4)$$

$$H(U_E) \leq C_E, E \in \mathcal{E} \quad (5)$$

where C_E denotes the capacity constraint for edge $E \in \mathcal{E}$. The constraint (2) specifies independence of source random variables. (3) and (4) correspond to encoding and decoding requirements respectively; (5) means that the entropy of the random variable carried by any link cannot exceed the link capacity.

The LP bound is the set of all rate tuples satisfying (2)-(5) together with the basic inequalities. The basic inequalities are non-negativity of entropy, conditional entropy and conditional mutual information. The set of the elemental basic inequalities

$$H(A | \{Y_{\mathcal{S}}, U_{\mathcal{E}}\} \setminus A) \geq 0, A \in \{Y_{\mathcal{S}}, U_{\mathcal{E}}\} \quad (6)$$

$$I(A; B | \mathcal{C}) \geq 0, A \neq B \neq \emptyset, A, B \in \{Y_{\mathcal{S}}, U_{\mathcal{E}}\}, \mathcal{C} \subseteq \{Y_{\mathcal{S}}, U_{\mathcal{E}}\} \setminus \{A, B\} \quad (7)$$

represents (implies) all Shannon-type inequalities for the random variables $Y_{\mathcal{S}}, U_{\mathcal{E}}$ and is minimal [10]. Note that constraints (1)-(7) are linear and hence weighted sum-rate LP bound can be computed by solving the linear programm

$$\text{maximize} \sum_{S \in \mathcal{S}} w_S H(Y_S) \text{ subject to } (2) - (7) \quad (8)$$

where w_S is any non-negative constant for source S called weight coefficient.

The simple and elegant formulation of the weighted sum-rate LP bound in (8) is deceptive since it is computationally very expensive. In fact, the number of variables (all possible joint entropies) and the constraints (7) for the LP bound computation increase exponentially with the number of random variables. Hence the LP bound is not useful for most practical network scenarios.

B. Functional Dependence Bound

An easily computable relaxation of the LP bound is introduced in [3], called functional dependence (FD) bound. The notion of irreducible sets in the functional dependence graphs (FDG) was developed to characterize the functional dependence bound. FDG is defined in [3] as:

Definition 1 (Functional dependence graph [3]): Let $\bar{\mathcal{V}}$ be a set of random variables. A directed graph $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ is called a functional dependence graph for $\bar{\mathcal{V}}$ if and only if for all $V \in \bar{\mathcal{V}}$

$$H(V | \{U : (U, V) \in \bar{\mathcal{E}}\}) = 0. \quad (9)$$

Note that, using the encoding and decoding constraints (3)-(4) for a given network $\mathcal{G}$ (i.e., local functional dependence constraints at nodes), an FDG can be constructed for the set of source and edge random variables. For two disjoint sets $\mathcal{A}, \mathcal{B} \subset \bar{\mathcal{V}}$ in an FDG $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$, we say $\mathcal{A}$ *determines* $\mathcal{B}$, denoted as $\mathcal{A} \to \mathcal{B}$, if no elements in $\mathcal{B}$ are left in the FDG after deleting all outgoing edges from $\mathcal{A}$ and subsequently the nodes and edges with no incoming edges or nodes respectively. The largest set of nodes that is determined by $\mathcal{A}$ is denoted by $\phi(\mathcal{A})$. It is shown in [3] that, if $\mathcal{A} \to \mathcal{B}$, we have $H(\mathcal{B}|\mathcal{A}) = 0$.

A set of nodes $\mathcal{A} \subset \bar{\mathcal{V}}$ is called *irreducible* if there is no proper subset of $\mathcal{A}$ dominates $\mathcal{A}$, i.e., $\nexists \mathcal{B} \subsetneq \mathcal{A} : \mathcal{B} \to \mathcal{A}$. An irreducible set $\mathcal{A}$ is *maximal* if $\mathcal{A} \to \bar{\mathcal{V}}$, which is equivalent to $\bar{\mathcal{V}} = \phi(\mathcal{A})$ or $Y_{\mathcal{S}} \subset \phi(\mathcal{A})$ [3]. Following the same notations in [7], we use $\boldsymbol{\mathcal{K}}$ to represent the set of all irreducible sets for $\bar{\mathcal{G}}$, $\boldsymbol{\mathcal{M}}$ to denote the set of all maximal irreducible sets and thus $\boldsymbol{\mathcal{M}}^c \triangleq \boldsymbol{\mathcal{K}} \setminus \boldsymbol{\mathcal{M}}$ is the set of all non-maximal irreducible sets. The functional dependence bound [3] is obtained by invoking the sub-additivity of entropy of elements in maximal irreducible sets.

Theorem 1: [3] Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a given network with source and edge random variables $Y_{\mathcal{S}}, U_{\mathcal{E}}$, mappings a, b and constraints (1)-(5). Let $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ be a functional dependence graph for the given network and let $\boldsymbol{\mathcal{M}}$ be the collection of all maximal irreducible sets. Then

$$\sum_{S \in \mathcal{W}} R_S \leq C(\mathcal{W}) \triangleq \min \sum_{E \in \mathcal{A}, \{U_{\mathcal{A}}, Y_{\mathcal{W}^c}\} \in \boldsymbol{\mathcal{M}}} C_E \quad (10)$$

where $\mathcal{W} \subseteq \mathcal{S}, \mathcal{W}^c \triangleq \mathcal{S} \setminus \mathcal{W}$ and C_E is the capacity of edge $E \in \mathcal{E}$.

III. Main Result

In this section, given a network, we characterize a class of Shannon-type inequalities that are useful to obtain a weighted sum-rate bound and thus improve the FD bound. We first give an example to elaborate the main idea.

Example 1: For the network shown in Fig.1(a), assume that each edge has unit capacity. The maximal irreducible sets obtained based on its reduced FDG [11] shown in Fig.1(b) are $\{Y_1, Y_2\}$, $\{Y_2, U_6, U_{12}\}$, $\{Y_2, U_9, U_{12}\}$, $\{Y_2, U_6, U_{15}\}$, $\{Y_2, U_6, U_{15}\}$, $\{U_6, U_9, U_{15}\}$ and $\{U_6, U_{12}, U_{15}\}$. Thus, the FD bound of this network is formed by the following inequalities:

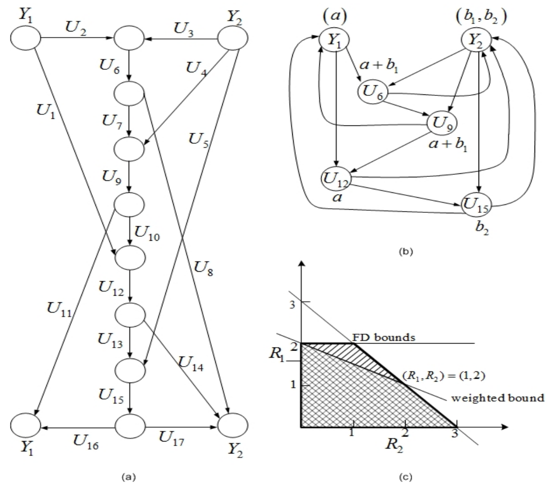

Fig. 1. A 2-unicast network counter-example for FD bound

$$H(Y_1, Y_2) = H(Y_2, U_6, U_{12}) \quad (11)$$
$$H(Y_1) + H(Y_2) \leq H(Y_2) + H(U_6) + H(U_{12}) \quad (12)$$
$$R_1 \leq 2 \quad (13)$$
$$H(Y_1, Y_2) = H(U_6, U_{12}, U_{15}) \quad (14)$$
$$H(Y_1) + H(Y_2) \leq H(U_6) + H(U_{12}) + H(U_{15}) \quad (15)$$
$$R_1 + R_2 \leq 3 \quad (16)$$

However, the FD bound for the network is not tight since the rate tuple $(R_1, R_2) = (2, 1)$ satisfying (13) and (16) is not achievable. We prove this using a submodular inequality as follows.

$$\begin{aligned} & H(Y_1) + H(Y_2) = H(Y_1, Y_2) \\ =& H(U_6, U_{12}, U_{15}) \\ \leq& H(U_6) + H(U_{12}, U_{15}) \\ =& H(U_6) + H(U_{12}, U_{15}) + H(Y_1, Y_2) - H(Y_1, Y_2) \\ \stackrel{(a)}{=}& H(U_6) + H(U_{12}, U_{15}) + H(U_{12}, U_{15}, Y_1, Y_2) - H(Y_1, Y_2) \\ \stackrel{(b)}{\leq}& H(U_6) + H(U_{12}, U_{15}, Y_1) + H(U_{12}, U_{15}, Y_2) - H(Y_1, Y_2) \\ \stackrel{(c)}{\leq}& H(U_6) + H(U_9, U_{15}) + H(U_{12}, Y_2) - H(Y_1, Y_2) \\ \leq& 4 - H(Y_1) \end{aligned}$$

Hence

$$2R_1 + R_2 = 2H(Y_1) + H(Y_2) \leq 4 \quad (17)$$

where (a) follows since in the FDG $\{U_{12}, U_{15}\} \subset \phi(Y_1, Y_2)$ and hence $H(U_{12}, U_{15}, Y_1, Y_2) = H(Y_1, Y_2)$, (b) follows from the submodularity of entropy

$$\begin{aligned} & H(U_{12}, U_{15}) + H(Y_1, Y_2, U_{12}, U_{15}) \\ & \leq H(U_{12}, U_{15}, Y_1) + H(U_{12}, U_{15}, Y_2) \end{aligned} \quad (18)$$

and (c) follows since $\{U_{12}, Y_1\} \subset \phi(U_9, U_{15})$ and $U_{15} \in \phi(U_{12}, Y_2)$, which imply $H(U_{12}, U_{15}, Y_1) \leq H(U_9, U_{15})$ and $H(U_{12}, U_{15}, Y_2) = H(U_{12}, Y_2)$ respectively.

Note that, the inequality (17) is not implied by (13) and (16) and hence its inclusion strictly improves the functional dependence bound as shown in Fig.1(c). The region defined by (13), (16) and (17) together with non-negativity of information rates is in fact the capacity region of this network. To prove this, we only need to show the achievement of rate pair $(R_1, R_2) = (1, 2)$ and other points are achieved simply by pure routing and time sharing. Assume that source 1 want to transmit the packet a and source 2 wants to transmit two packets (b_1, b_2) to their corresponding receivers. This can be achieved by letting head nodes of the edges 6, 9, 12, 15 perform network coding as indicated in the FDG (Fig.1(b)) while letting head nodes of remaining edges perform routing.

A. A Class of Shannon-type Inequalities

Now, we characterize a class of Shannon-type inequalities that renders, together with sub-additivity of entropy, weighted sum-rate bounds and is guaranteed to improve the functional dependence bound. Let $\boldsymbol{\mathcal{L}} \triangleq \{\mathcal{A} : \mathcal{A} \in \boldsymbol{\mathcal{M}}, Y_S \notin \mathcal{A}, \forall S \in \mathcal{S}\}$ be the set of all maximal irreducible sets that contain edge variables only. Consider $\mathcal{A} \in \boldsymbol{\mathcal{L}}$ and $\mathcal{B} \subset U_{\mathcal{E}}$, for any $\mathcal{W} \subset \mathcal{S}$.

$$\begin{aligned} & H(Y_{\mathcal{W}}) + H(Y_{\mathcal{W}^c}) = H(Y_{\mathcal{W}}, Y_{\mathcal{W}^c}) = H(\mathcal{A}) = H(\mathcal{A}, \mathcal{B}) \\ \leq& H(\mathcal{B}) + H(\mathcal{A} \setminus \mathcal{B}) + H(Y_{\mathcal{S}}) - H(Y_{\mathcal{S}}) \\ =& H(\mathcal{B}) + H(\mathcal{A} \setminus \mathcal{B}) + H(Y_{\mathcal{W}}, Y_{\mathcal{W}^c}) - H(Y_{\mathcal{S}}) \\ \stackrel{(a)}{\leq}& H(\mathcal{A} \setminus \mathcal{B}) + H(Y_{\mathcal{W}}, \mathcal{B}) + H(Y_{\mathcal{W}^c}, \mathcal{B}) - H(Y_{\mathcal{W}}, Y_{\mathcal{W}^c}) \end{aligned}$$

where (a) follows from the submodular inequality

$$H(Y_{\mathcal{W}}, Y_{\mathcal{W}^c}) + H(\mathcal{B}) \leq H(Y_{\mathcal{W}}, \mathcal{B}) + H(Y_{\mathcal{W}^c}, \mathcal{B}). \quad (19)$$

Proposition 1: For a maximal irreducible set $\mathcal{A} \in \boldsymbol{\mathcal{L}}$ and some $\mathcal{B} \subset U_{\mathcal{E}}$, Shannon-type inequalities of the form (19) render weighted sum-rate bound and improve the functional dependence bound if following three conditions are satisfied.

1) There exists some set of edge random variables $U_{\mathcal{X}}$ such that $(Y_{\mathcal{W}}, \mathcal{B}) \subseteq \phi(U_{\mathcal{X}})$,
2) There exists some set of edge random variables $U_{\mathcal{Z}}$ such that $\mathcal{B} \subseteq \phi(Y_{\mathcal{W}^c}, U_{\mathcal{Z}})$ and
3) $\displaystyle\sum_{E \in \mathcal{A} \setminus \mathcal{B}} C_E + \sum_{E \in \mathcal{X}} C_E + \sum_{E \in \mathcal{Z}} C_E < C(\mathcal{S}) + C(\mathcal{W})$.

Proof: Condition 1, $(Y_{\mathcal{W}}, \mathcal{B}) \subseteq \phi(U_{\mathcal{X}})$, is equivalent to $U_{\mathcal{X}} \to (Y_{\mathcal{W}}, \mathcal{B})$. Hence, by [3, Theorem 1], $H(Y_{\mathcal{W}}, \mathcal{B} | U_{\mathcal{X}}) = 0$. Expressing this conditional entropy in the elemental form (6) leads to

$$H(U_{\mathcal{X}}) = H(U_{\mathcal{X}}, Y_{\mathcal{W}}, \mathcal{B}) \geq H(Y_{\mathcal{W}}, \mathcal{B}). \quad (20)$$

Similarly, Condition 2 implies

$$H(Y_{\mathcal{W}^c}, U_{\mathcal{Z}}) \geq H(Y_{\mathcal{W}^c}, U_{\mathcal{Z}}, \mathcal{B}) \geq H(Y_{\mathcal{W}^c}, \mathcal{B}). \quad (21)$$

A weighted sum-rate bound can be obtained by utilizing (20), (21) and (19) as follows.

$$\begin{aligned} & H(Y_{\mathcal{W}}) + H(Y_{\mathcal{W}^c}) = H(Y_{\mathcal{W}}, Y_{\mathcal{W}^c}) = H(\mathcal{A}) = H(\mathcal{A}, \mathcal{B}) \\ & \leq H(\mathcal{A} \setminus \mathcal{B}) + H(Y_{\mathcal{W}}, \mathcal{B}) + H(Y_{\mathcal{W}^c}, \mathcal{B}) - H(Y_{\mathcal{S}}) \\ & \stackrel{(a)}{\leq} H(\mathcal{A} \setminus \mathcal{B}) + H(U_{\mathcal{X}}) + H(Y_{\mathcal{W}^c}, U_{\mathcal{Z}}) - H(Y_{\mathcal{W}}) - H(Y_{\mathcal{W}^c}) \\ & \leq H(\mathcal{A} \setminus \mathcal{B}) + H(U_{\mathcal{X}}) + H(U_{\mathcal{Z}}) - H(Y_{\mathcal{W}}) \end{aligned} \quad (22)$$

where (a) follows from (20) and (21). Hence, by (1) and (5)

$$\begin{aligned} 2\sum_{S\in\mathcal{W}} R_S + \sum_{S\in\mathcal{W}^c} R_S &\le 2H(Y_{\mathcal{W}}) + H(Y_{\mathcal{W}^c}) \\ &\le H(\mathcal{A}\setminus\mathcal{B}) + H(U_{\mathcal{X}}) + H(U_{\mathcal{Z}}) \\ &\le \sum_{E\in\mathcal{A}\setminus\mathcal{B}} C_E + \sum_{E\in\mathcal{X}} C_E + \sum_{E\in\mathcal{Z}} C_E. \end{aligned} \tag{23}$$

The FD bound states $\sum_{S\in\mathcal{W}} R_S \le C(\mathcal{W})$; $\sum_{S\in\mathcal{W}^c} R_S \le C(\mathcal{W}^c)$; $\sum_{S\in\mathcal{S}} R_S \le C(\mathcal{S})$. The trivial weighted bounded implied by these FD bounds is $2\sum_{S\in\mathcal{W}} R_S + \sum_{S\in\mathcal{W}^c} R_S \le C(\mathcal{S}) + C(\mathcal{W})$. Thus, (23) provides improvement over the FD bound if Condition 3 is satisfied. ■

B. An Algorithm to Find the Weighted Bound

Proposition 1 provides sufficient conditions to obtain the weighted bound. However, it does not tell how to find the sets of variables $\mathcal{B}$, $U_{\mathcal{X}}$ and $U_{\mathcal{Z}}$. In this subsection, we examine the properties of these variables and describe the steps for finding $U_{\mathcal{X}}$, $\mathcal{A}\setminus\mathcal{B}$, $U_{\mathcal{Z}}$ and $\mathcal{B}$ sequentially. Finally, an algorithm is constructed to obtain the weighted bounds.

Lemma 1: For $\mathcal{W}_1, \mathcal{W}_2 \subseteq \mathcal{S}$, let $U_{\mathcal{C}} \subseteq U_{\mathcal{E}}$ be a subset of edge variables. If $Y_{\mathcal{W}_2} \subseteq \phi(Y_{\mathcal{S}\setminus(\mathcal{W}_1\cup\mathcal{W}_2)}, U_{\mathcal{C}})$ in the FDG $\bar{\mathcal{G}}$, then $\mathcal{C}$ is an edge-cut separating $a(\mathcal{W}_1)$ and $b^{-1}(\mathcal{W}_2)$ in $\mathcal{G}$.

Proof: Suppose the edge cut $\mathcal{C}$ does not separate $a(\mathcal{W}_1)$ and $b^{-1}(\mathcal{W}_2)$ in $\mathcal{G}$ and hence there exists a path from $a(\mathcal{W}_1)$ to $b^{-1}(\mathcal{W}_2)$ in $\mathcal{G}$ which does not contain any edge from the set $\mathcal{C}$. Then there must exist a directed path from $Y_{\mathcal{W}_1}$ to $Y_{\mathcal{W}_2}$ in $\bar{\mathcal{G}}$ which does not contain any node from the set $U_{\mathcal{C}}$. But, this contradicts the assumption $Y_{\mathcal{W}_2} \subseteq \phi(Y_{\mathcal{S}\setminus(\mathcal{W}_1\cup\mathcal{W}_2)}, U_{\mathcal{C}})$. ■

To simplify the description, let $\mathbf{\Gamma}_{\mathcal{W}_1-\mathcal{W}_2}(\mathcal{G})$ denote the collection of all minimal edge-cuts[1] separating $b^{-1}(\mathcal{W}_2)$ and $a(\mathcal{W}_1)$ in $\mathcal{G}$.

As $Y_{\mathcal{W}} \in \phi(U_{\mathcal{X}})$, according to Lemma 1, $U_{\mathcal{X}}$ must be a cut separating $a(\mathcal{S})$ from $b^{-1}(\mathcal{W})$ in $\mathcal{G}$. To tighten the resulted weighted bound, we search for suitable $U_{\mathcal{X}}$ from $\mathbf{\Gamma}_{\mathcal{S}-\mathcal{W}}(\mathcal{G})$.

Lemma 2: $\sum_{E\in\mathcal{X}} C_E < C(\mathcal{S})$

Proof: Assume that $\sum_{E\in\mathcal{X}} C_E \ge C(\mathcal{S})$. This assumption, together with Condition 3 in Proposition 1, gives

$$\sum_{E\in\mathcal{A}\setminus\mathcal{B}} C_E + \sum_{E\in\mathcal{Z}} C_E < C(\mathcal{W}). \tag{24}$$

However, Condition 2 in Proposition 1 states $(Y_{\mathcal{W}^c}, \mathcal{B}) \subseteq \phi(Y_{\mathcal{W}^c}, U_{\mathcal{Z}})$. Thus, $\phi(Y_{\mathcal{W}^c}, U_{\mathcal{Z}}, \mathcal{A}\setminus\mathcal{B}) \supseteq \phi(Y_{\mathcal{W}^c}, \mathcal{B}, \mathcal{A}\setminus\mathcal{B}) \supseteq \phi(\mathcal{A}) = \bar{\mathcal{V}}$. Therefore, according to the definition of $C(\mathcal{W})$ in (10), we have $\sum_{E\in\mathcal{A}\setminus\mathcal{B}} C_E + \sum_{E\in\mathcal{Z}} C_E \ge C(\mathcal{W})$ which contradicts (24). ■

Denote $\mathcal{D} = \mathcal{A}\setminus\mathcal{B}$. Since $\phi(\mathcal{D}, U_{\mathcal{X}}) \supseteq \phi(\mathcal{D}, \mathcal{B}) = \bar{\mathcal{V}}$, $\mathcal{D}$ must be an edge-cut separating $b^{-1}(\mathcal{W}^c)$ from $a(\mathcal{W}^c)$ in $\mathcal{G}[\mathcal{V}\setminus U_{\mathcal{X}}]$[2]. Thus, we search for $\mathcal{D}$ from $\mathbf{\Gamma}_{\mathcal{W}^c-\mathcal{W}^c}(\mathcal{G}[\mathcal{V}\setminus U_{\mathcal{X}}])$.

[1] If the cut contains any edge that is not present in the reduced FDG, it is not included in $\mathbf{\Gamma}_{\mathcal{W}_1-\mathcal{W}_2}(\mathcal{G})$.

[2] For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{K} \subset \mathcal{V}$, $\mathcal{G}[\mathcal{V}\setminus\mathcal{K}]$ denotes the subgraph of $\mathcal{G}$ induced by $\mathcal{V}\setminus\mathcal{K}$ and it is defined by the node set $\mathcal{V}\setminus\mathcal{K}$ and the edge set $\{E = (A, B) \in \mathcal{E} : A, B \in V\setminus\mathcal{K}\}$.

Since $\phi(\mathcal{D}, Y_{\mathcal{W}^c}, U_{\mathcal{Z}}) \supseteq \phi(\mathcal{D}, \mathcal{B}) = \bar{\mathcal{V}}$, $U_{\mathcal{Z}}$ must separate $b^{-1}(\mathcal{W})$ from $a(\mathcal{W})$ in $\mathcal{G}[\mathcal{V}\setminus\mathcal{D}]$. Therefore, we search $U_{\mathcal{Z}}$ from $\mathbf{\Gamma}_{\mathcal{W}-\mathcal{W}}(\mathcal{G}[\mathcal{V}\setminus\mathcal{D}])$.

Lemma 3: $C(\mathcal{W}) - \sum_{E\in\mathcal{D}} C_E \le \sum_{E\in\mathcal{Z}} C_E < C(\mathcal{S}) + C(\mathcal{W}) - \sum_{E\in\mathcal{X}} C_E - \sum_{E\in\mathcal{D}} C_E$

Proof: The lower bound on $\sum_{E\in\mathcal{Z}} C_E$ follows from the definition of $C(\mathcal{W})$ and the upper bound results from Condition 3 in Proposition 1. ■

To sum up, the search space for $U_{\mathcal{Z}}$ is reduced to

$$\begin{aligned} \boldsymbol{\mathcal{O}}_{\mathcal{Z}} = \{U_{\mathcal{Z}} : U_{\mathcal{Z}} \in \mathbf{\Gamma}_{\mathcal{W}-\mathcal{W}}(\mathcal{G}[\mathcal{V}\setminus\mathcal{D}]), C(\mathcal{W}) - \sum_{E\in\mathcal{D}} C_E \\ \le \sum_{E\in\mathcal{Z}} C_E < C(\mathcal{S}) + C(\mathcal{W}) - \sum_{E\in\mathcal{X}} C_E - \sum_{E\in\mathcal{D}} C_E\}. \end{aligned} \tag{25}$$

Finally, we need to determine the set $\mathcal{B}$. According to Condition 1 and Condition 2 in Proposition 1, $\mathcal{B} \subseteq \phi(U_{\mathcal{X}}) \cap \phi(Y_{\mathcal{W}^c}, U_{\mathcal{Z}}) \setminus Y_{\mathcal{S}}$. The weighted bound in (22) is valid if $\phi(\mathcal{D}, \mathcal{B}) = \bar{\mathcal{V}}$. Therefore, we maximize the set $\mathcal{B}$ by setting $\mathcal{B} = \phi(U_{\mathcal{X}}) \cap \phi(Y_{\mathcal{W}^c}, U_{\mathcal{Z}}) \setminus Y_{\mathcal{S}}$.

The procedures of computing weighted bounds based on above lemmas are summarized in Algorithm 1.

Algorithm 1 WeightedBd($\mathcal{G} = (\mathcal{V}, \mathcal{E}), \{C_E, E\in\mathcal{E}\}, \bar{\mathcal{G}}, \boldsymbol{\mathcal{M}}$)

for $\mathcal{W} \subsetneq \mathcal{S}$ **do**
  Find $\mathbf{\Gamma}_{\mathcal{S}-\mathcal{W}}(\mathcal{G})$
  $\boldsymbol{\mathcal{O}}_{\mathcal{X}} = \{U_{\mathcal{X}} : U_{\mathcal{X}} \in \mathbf{\Gamma}_{\mathcal{S}-\mathcal{W}}, \sum_{E\in\mathcal{X}} C_E < C(\mathcal{S})\}$
  for $U_{\mathcal{X}} \in \boldsymbol{\mathcal{O}}_{\mathcal{X}}$ **do**
    Find $\mathbf{\Gamma}_{\mathcal{W}^c-\mathcal{W}^c}(\mathcal{G}[\mathcal{V}\setminus U_{\mathcal{X}}])$
    for $\mathcal{D} \in \mathbf{\Gamma}_{\mathcal{W}^c-\mathcal{W}^c}(\mathcal{G}[\mathcal{V}\setminus U_{\mathcal{X}}])$ **do**
      Find $\mathbf{\Gamma}_{\mathcal{W}-\mathcal{W}}(\mathcal{G}[\mathcal{V}\setminus\mathcal{D}])$
      Find $\boldsymbol{\mathcal{O}}_{\mathcal{Z}}$ (refer (25))
      for $U_{\mathcal{Z}} \in \boldsymbol{\mathcal{O}}_{\mathcal{Z}}$ **do**
        $\mathcal{B} = \phi(U_{\mathcal{X}}) \cap \phi(Y_{\mathcal{W}^c}, U_{\mathcal{Z}}) \setminus Y_{\mathcal{S}}$
        if $\phi(\mathcal{D}, \mathcal{B}) = \bar{\mathcal{V}}$ **then**
          Output (23)
        end if
      end for
    end for
  end for
end for

Example 2: Consider the 2-unicast network shown in Fig.2(a) with its reduced FDG shown in Fig.2(b). Assume unit edge capacity. We can obtain:

$$\begin{aligned} \boldsymbol{\mathcal{M}} = \{&\{Y_1, U_{17}\}, \{Y_1, U_6, U_{13}\}, \{Y_2, U_6, U_7\}, \{Y_2, U_6, U_{13}\}, \\ &\{Y_2, U_{12}, U_7\}, \{Y_2, U_{12}, U_7\}, \{Y_2, U_{12}, U_{13}\}, \\ &\{U_6, U_7, U_{13}\}, \{U_6, U_{12}, U_{13}\}, \{U_{17}, U_6, U_7\}, \\ &\{U_{17}, U_6, U_{13}\}, \{U_{17}, U_{12}, U_7\}, \{U_{17}, U_{12}, U_{13}\}\} \end{aligned}$$

Consider $Y_{\mathcal{W}} = Y_2$, we have $C(\mathcal{W}) = 2, C(\mathcal{S}) = 3$ and $\mathbf{\Gamma}_{\mathcal{S}-\mathcal{W}} = \{\{U_{12}, U_{13}\}, \{U_6, U_7, U_{13}\}\}$. Combining with the capacity constraint, the search space of $U_{\mathcal{X}}$ is further reduced to $\mathcal{O}_{\mathcal{X}} = \{\{U_{12}, U_{13}\}\}$. Given $U_{\mathcal{X}} = \{U_{12}, U_{13}\}$, $\mathbf{\Gamma}_{\mathcal{W}^c - \mathcal{W}^c}(\mathcal{G}[\mathcal{V} \setminus U_{\mathcal{X}}]) = \{\{U_6\}, \{U_{17}\}\}$.

When $\mathcal{D} = \{U_6\}$, $\mathbf{\Gamma}_{\mathcal{W}-\mathcal{W}}(\mathcal{G}[\mathcal{V} \setminus \mathcal{D}]) = \{\{U_7\}, \{U_{13}\}\}$, $\mathcal{O}_{\mathcal{Z}} = \{\{U_7\}, \{U_{13}\}\}$. Note that $\phi(U_{\mathcal{X}}) = \{Y_2, U_{12}, U_{13}, U_7\}$. When $U_{\mathcal{Z}} = \{U_7\}$, $\phi(Y_{\mathcal{W}^c}, U_{\mathcal{Z}}) = \{Y_1, U_7, U_{13}\}$. Therefore, $\mathcal{B} = \{U_7, U_{13}\}$ which forms a maximal set with $\mathcal{D}$. We can obtain the nontrivial weighted bound: $R_1 + 2R_2 \leq |\mathcal{D}| + |\mathcal{X}| + |\mathcal{Z}| = 4$. However, when $U_{\mathcal{Z}} = \{U_{13}\}$, $\phi(Y_{\mathcal{W}^c}, U_{\mathcal{Z}}) = \{Y_1, U_{13}\}$ which leads to $\mathcal{B} = \{U_{13}\}$ and no weighted bound can be obtained as $\phi(\mathcal{B}, \mathcal{D}) \neq \bar{\mathcal{V}}$.

When $\mathcal{D} = \{U_{17}\}$, $\mathbf{\Gamma}_{\mathcal{W}-\mathcal{W}}(\mathcal{G}[\mathcal{V} \setminus \mathcal{D}]) = \{\{U_6, U_7\}, \{U_6, U_{13}\}, \{U_{12}, U_{13}\}\}$ and $\mathcal{O}_{\mathcal{Z}} = \emptyset$. It does not render any weighted bound.

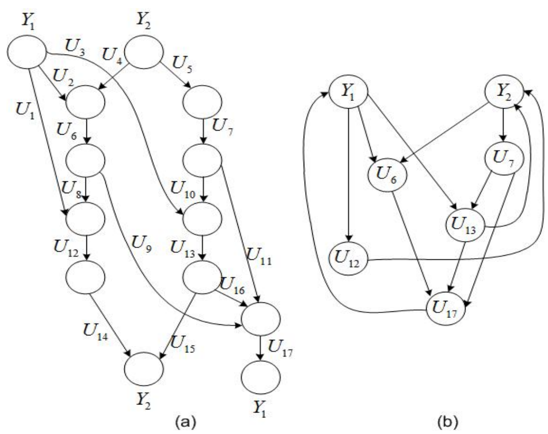

Fig. 2. A 2-unicast network example

C. Discussion

The weighted bound obtained in Algorithm 1 can be interpreted as follows: when the sources $\mathcal{W}$ are transmitting at their maximal possible sum rate $C(\mathcal{W})$, data carried by edge set $\mathcal{B}$ can no longer be independent and $H(\mathcal{B}) \leq \sum_{E \in \mathcal{X}} C_E + \sum_{E \in \mathcal{Z}} C_E - H(Y_{\mathcal{W}}) < \sum_{E \in \mathcal{B}} C_E$. If Condition 3 in Proposition 1 is satisfied, it strictly improves the FD bound together with $\sum_{S \in \mathcal{S}} R_S \leq H(\mathcal{D}) + H(\mathcal{B})$.

For computing this weighted bound, proposed algorithm can be much more efficient than the LP bound for small to medium size networks. Consider Example 2, to find the bound of $(R_1 + 2R_2)$ using linear programming method requires solving the problem of dimension 31 with 691 constraints. Although proposed algorithm contains 3 for-loops, the number of iterations is actually very small, as $|\mathcal{O}_{\mathcal{X}}| = 1$ and with the unique element in $\mathcal{O}_{\mathcal{X}}$, $|\mathbf{\Gamma}_{\mathcal{W}^c - \mathcal{W}^c}(\mathcal{G}[\mathcal{V} \setminus U_{\mathcal{X}}])| = 2$. For the two choices of $\mathcal{D}$, we have $|\mathcal{O}_{\mathcal{Z}}| = 2$ and 0 respectively. Moreover, there exists efficient polynomial-time algorithm for finding the minimal cuts between a pair of nodes in a directed acyclic graph [12].

IV. Conclusion

While LP bound is one of the tightest explicit upper bound, it is "computationally infeasible". On the other hand, the FD bound is an easily computable upper bound but it is quite loose. In this paper, we develop an algorithm for computing weighted bounds that strictly improves the FD bound. The weighted bound with weight coefficients $w_S \in \{0, 1, 2\}, S \in \mathcal{S}$ is obtained using a set of submodular inequalities of entropy by examining the maximal irreducible sets of a given network. Along this line of research, the following problems are interesting to investigate: (1) Characterize Shannon-type inequalities leading to weighted bounds with broader values of w_S. (2) Characterize non-Shannon-type inequalities leading to weighted bounds for a given network.

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