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WEIGHTED SUMS OF CERTAIN DEPENDENT RANDOM VARIABLES

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1. Let $(\Omega, \mathfrak{A}, \mathrm{P})$ be a probability space and $(\mathfrak{A}_n)_{n=1,2,\dots}$ be an increasing family of sub σ-fields of $\mathfrak{A}$ (we put $\mathfrak{A}_0 = (\phi, \Omega)$). Let $(x_n)_{n=1,2,\dots}$ be a sequence of bounded martingale differences on $(\Omega, \mathfrak{A}, \mathrm{P})$, that is, $x_n(\omega)$ is bounded almost surely (a.s.) and $\mathrm{E}\{x_n | \mathfrak{A}_{n-1}\} = 0$ a.s. for $n = 1, 2, \cdots$. It is easily seen that this sequence has the following properties [G] and [M], which have been introduced by Y. S. Chow ([1]) in an analogous form and by G. Alexits ([4]), respectively, and may be of independent interest.

[G] (x_n) is a sequence of martingale differences and there exist non negative constants c_n such that for every real number t

$$\mathrm{E}\{\exp(tx_n) | \mathfrak{A}_{n-1}\} \leqq \exp(c_n^2 t^2/2) \text{ a.s. } (n = 1, 2, \cdots).$$

For each n, the minimum of those c_n is denoted by $\tau(x_n)$.

[M] $$|x_n(\omega)| \leqq K_n \quad \text{a.s. for } n = 1, 2, \cdots$$

and $\mathrm{E}\{x_{i_1} x_{i_2} \cdots x_{i_k}\} = 0$ for $i_1 < i_2 < \cdots < i_k\,; k = 1, 2, \cdots$.

In this note we investigate the asymptotic behavior of the weighted sums of those random variables. In §3 we will deal with the class [M] and in §4 with the class [G] and the uniformly bounded case of martingale differences.

2. Preliminary Lemmas.

LEMMA 1. *If (x_n) is a sequence of random variables for which* [M] *holds with $K_n = 1$ for all n, then for every real number t*

$$\mathrm{E}\left\{\exp\left(t \sum_{k=1}^{n} b_{nk} x_k\right)\right\} \leqq \exp\left(\frac{t^2}{2} \sum_{k=1}^{n} b_{nk}^2\right), \tag{2. 1}$$

where $(b_{nk})_{k=1,2,\cdots n;\, n=1,2,\cdots}$ is an arbitrary sequence of real numbers.

PROOF. We may assume that $|b_{nk}| \neq 0$ for $k = 1, 2, \cdots$. Since $|b_{nk} x_k| \leqq |b_{nk}|$ a.s. and the exponential function $\exp(t b_{nk} x_k)$ is convex, we have

(2. 2) $$\exp(tb_{nk}x_k) \leqq \cosh(t|b_{nk}|) + (x_k/|b_{nk}|)\sinh(t|b_{nk}|) \quad \text{a.s.}$$

Then, using the property [M], we have

$$\begin{aligned} E\left\{\exp\left(t\sum_{k=1}^{n} b_{nk}x_k\right)\right\} &\leqq \prod_{k=1}^{n} \cosh(t|b_{nk}|) \\ &= \prod_{k=1}^{n}\sum_{m=0}^{\infty} \frac{t^{2m}b_{nk}^{2m}}{(2m)!} \\ &\leqq \prod_{k=1}^{n}\sum_{m=0}^{\infty} \frac{t^{2m}b_{nk}^{2m}}{2^m m!} \\ &= \exp\left(\frac{t^2}{2}\sum_{k=1}^{n} b_{nk}^2\right), \end{aligned}$$ q.e.d.

REMARK 1. If (x_n) is a sequence of martingale differences such that $|x_n| \leqq K_n$ a.s. for all n, then from (2.2), we obtain

$$\begin{aligned} E\{\exp(tx_n)\,|\mathfrak{A}_{n-1}\} &\leqq \cosh(tK_n) \quad \text{a.s.} \\ &\leqq \exp(t^2K_n^2/2) \quad \text{a.s. for } n = 1, 2, \cdots. \end{aligned}$$

Therefore (x_n) has the property [G] with $\tau(x_n) \leqq K_n$, $n = 1, 2, \cdots$.

LEMMA 2. *If (x_n) is a sequence of random variables for which* [G] *holds with $\tau(x_n) \leqq 1$, $n = 1, 2, \cdots$,* then

(2. 3) $$E\{\exp(tS_n^*)\} \leqq 8 \exp\left(\frac{t^2}{2}\sum_{k=1}^{n} b_k^2\right),$$

where $S_n^*(\omega) = \max_{1\leqq m\leqq n}\left|\sum_{k=1}^{m} b_kx_k\right|$ and (b_k) is an arbitrary sequence of real numbers.

PROOF. Noting that $\tau(b_nx_n) = |b_n|\tau(x_n) \leqq |b_n|$, we have

(2. 4) $$\begin{aligned} E\left\{\exp\left(t\sum_{k=1}^{n} b_kx_k\right)\right\} &= E\left\{\exp\left(t\sum_{k=1}^{n-1} b_kx_k\right) E\left\{\exp\left(tb_nx_n\right)\middle|\mathfrak{A}_{n-1}\right\}\right\} \\ &\leqq \exp\left(\frac{t^2b_n^2}{2}\right) E\left\{\exp\left(t\sum_{k=1}^{n-1} b_kx_k\right)\right\} \\ &\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

$$\leqq \exp\left(\frac{t^2}{2}\sum_{k=1}^{n} b_k^2\right).$$

On the other hand we have ([2], p. 317) for $\alpha > 1$

$$\mathrm{E}\{(S_n^*)^\alpha\} \leqq \left(\frac{\alpha}{\alpha-1}\right)^\alpha \mathrm{E}\{|S_n|^\alpha\},$$

where $S_n = \sum_{k=1}^{n} b_k x_k$, since $(|S_n|)$ is a sequence of non negative submartingale. Then

$$\begin{aligned}
\mathrm{E}\{\exp(tS_n^*)\} &\leqq \mathrm{E}\{\exp(tS_n^*)\} + \mathrm{E}\{\exp(-tS_n^*)\}\\
&= 2\,\mathrm{E}\left\{\sum_{j=0}^{\infty} \frac{t^{2j}(S_n^*)^{2j}}{(2j)!}\right\}\\
&\leqq 8\,\mathrm{E}\left\{\sum_{j=0}^{\infty} \frac{t^{2j}S_n^{2j}}{(2j)!}\right\}\\
&\leqq 8\exp\left(\frac{t^2}{2}\sum_{k=1}^{n} b_k^2\right), \qquad \text{q.e.d.}
\end{aligned}$$

3. Let $(a_{nk})_{k=1,2,\cdots n;\ n=1,2\cdots}$ be a sequence of real numbers and put $T_n = \sum_{k=1}^{n} a_{nk}x_k$ and $B_n = \left(\sum_{k=1}^{n} a_{nk}^2\right)^{\frac{1}{2}}$.

THEOREM 1. *If (x_n) is a sequence of random variables for which* [M] *holds with $K_n = 1$ for all n, then*

$$\limsup_{n\to\infty} \frac{|T_n|}{\sqrt{2B_n^2 \log n}} \leqq 1 \quad \text{a.s.} \tag{3. 1}$$

PROOF. Supposing in Lemma 1 that $t = (2(\log n)/\mathrm{B}_n^2)^{\frac{1}{2}}$ and $b_{nk} = a_{nk}$ and multiplying both sides by $\exp(-(2+\varepsilon)\log n)$, where $\varepsilon > 0$, we obtain

$$\mathrm{E}\left\{\exp\left(\left(\frac{2\log n}{B_n^2}\right)^{\frac{1}{2}} |T_n| - (2+\varepsilon)\log n\right)\right\} \leqq 2\left(\frac{1}{n}\right)^{1+\varepsilon}$$

so that by the Beppo-Levi theorem

$$\sum_{n=1}^{\infty} \exp\left(\left(\frac{2\log n}{B_n^2}\right)^{\frac{1}{2}} |T_n| - (2+\varepsilon)\log n\right) < \infty \quad \text{a.s.}$$

Hence,

$$\limsup_{n\to\infty}\left(\left(\frac{2\log n}{B_n^2}\right)^{\frac{1}{2}} |T_n| - (2+\varepsilon)\log n\right) < 0 \quad \text{a.s.}$$

that is,

$$\limsup_{n\to\infty} \frac{|T_n|}{\sqrt{2B_n^2 \log n}} \leqq \frac{2+\varepsilon}{2} \quad \text{a.s.}$$

Since ε is an arbitrary positive number, letting $\varepsilon \to 0$, we obtain (3.1), q.e.d.

COROLLARY 1. *Let* (x_n) *be the same as in Theorem* 1 *and* (a_j) *be a sequence of positive numbers. Put* $A_n = \sum_{j=1}^{n} a_j$. *If*

(3. 2) $$a_n/A_n = o(1/\log n),\ A_n/\log n \uparrow \infty \ \textit{as}\ n \to \infty,$$

then

(3. 3) $$(a_n x_1 + a_{n-1}x_2 + \cdots + a_1 x_n)/A_n \to 0 \ \textit{as}\ n \to \infty,\ \textit{a.s.}$$

PROOF. Write $a_n^* = a_{k(n)} = \max_{1\leqq k\leqq n} a_k$, then $1 \leqq k(n) \leqq n$ and $k(n)$ is increasing. If $k(n) = O(1)$, we have by (3.2) $a_{k(n)}/A_n = o(1/\log n)$ and if $k(n) \to \infty$, then we have again

$$\frac{a_{k(n)}\log n}{A_n} = \frac{a_{k(n)}\log k(n)}{A_{k(n)}} \frac{A_{k(n)}\log n}{A_n \log k(n)} = o(1).$$

Hence in any case we get $a_n^*/A_n = o(1/\log n)$. Therefore by Theorem 1,

$$\frac{|a_n x_1 + \cdots + a_1 x_n|}{A_n} \leqq \frac{|a_n x_1 + \cdots + a_1 x_n|}{\sqrt{2\sum_{i=1}^{n} a_i^2 \log n}} \sqrt{\frac{2a_n^* \log n}{A_n}}$$

which is $o(1)$, and we get (3.3).

REMARK 2. In (3.3), if we consider $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ instead of $a_nx_1 + a_{n-1}x_2 + \cdots + a_1x_n$, we may replace the condition (3.2) by

$$(3.2)' \qquad a_n/A_n = o(1/\log\log A_n),\ A_n \to \infty \quad \text{as } n \to \infty .$$

This can be established by the same way as in the proof of Theorem 3 and we omit the proof.

REMARK 3. From Theorem 1 we find that $S_n = a_1x_1 + a_2x_2 + \cdots + a_nx_n = o(\log n)^{\frac{1}{2}}$, when $\sum_{j=1}^{\infty} a_j^2 < \infty$. For we may take an integer m such that $\sum_{j=m+1}^{\infty} a_j^2$ is sufficiently small and apply Theorem 1 to $S_n - S_m$ for $n > m$.

COROLLARY 2. *Let* (x_n) *be the same as in Theorem* 1 *and put*

$$(3.4) \qquad \sigma_n^\alpha = \frac{1}{E_n^\alpha}\sum_{k=0}^{n-1} E_{n-k}^\alpha x_{k+1} \qquad \text{for } \alpha > -\frac{1}{2}\ ,\ E_n^\alpha = \binom{n+\alpha}{n}.$$

Then

$$(3.5) \qquad \limsup_{n\to\infty} \frac{|\sigma_n^\alpha|}{\sqrt{(2/(2\alpha+1))n \log n}} \leqq 1 \quad a.s.$$

PROOF. Since

$$\frac{\sum_{k=0}^{n-1}(n-k)^{2\alpha}}{n^{2\alpha}(n/(2\alpha+1))} = \frac{\sum_{k=1}^{n} k^{2\alpha}}{n^{2\alpha}(n/(2\alpha+1))} \to 1 \text{ as } n \to \infty ,$$

we have, taking $a_{nk} = (n-k)^\alpha$ in Theorem 1

$$(3.6) \qquad \limsup_{n\to\infty} \frac{\left|\sum_{k=0}^{n-1}(1-(k/n))^\alpha x_{k+1}\right|}{\sqrt{(2/(2\alpha+1))n \log n}} = \limsup_{n\to\infty} \frac{\left|\sum_{k=0}^{n-1}(n-k)^\alpha x_{k+1}\right|}{\sqrt{n^{2\alpha}(n/(2\alpha+1))\log n}} \leqq 1 \text{ a.s.}$$

On the other hand ([3])

$$E_{n-k}^\alpha / E_n^\alpha = (1-(k/n))^\alpha + O(1/n)$$

and therefore, (3.5) follows from (3.6), q.e.d.

4. Let (a_n) be a sequence of real numbers and put $D_n = \left(\sum_{j=1}^{n} a_j^2\right)^{\frac{1}{2}}$, $S_n = a_1x_1 + \cdots + a_nx_n$ and $\overline{S}_n = a_nx_1 + a_{n-1}x_2 + \cdots + a_1x_n$.

THEOREM 2. *Let (x_n) be a sequence of random variables for which* [G] *holds with $\tau(x_n) \leqq 1$ for all n. If*

$$(4.1) \qquad a_n^2/D_n^2 \to 0\,,\ D_n^2 \to \infty \ \text{as}\ n \to \infty$$

then

$$(4.2) \qquad \limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2D_n^2 \log\log D_n^2}} \leqq 1 \ \ a.s.$$

PROOF. Take an arbitrary positive number ε and fix it. Next we define the sequence (n_j) of positive integers as follows: We may choose n_1 by (4.1) such that

$$(4.3) \qquad D_{n_1}^2 > \frac{4}{3}$$

$$(4.4) \qquad a_n^2/D_n^2 < \frac{1}{3} \quad \text{for } n > n_1\,.$$

And generally, after $n_1, n_2, \cdots, n_{k-1}$ are defined we may choose n_k such that

$$(4.5) \qquad D_{n_{k-1}}^2 < D_{n_k}^2 \leqq 2D_{n_{k-1}}^2 < D_{n_k+1}^2\,.$$

From (4.4) we get $D_{n_{k-1}+1}^2/D_{n_{k-1}}^2 < 2$ and therefore n_k is well defined. Then from (4.4), (4.5) and (4.3),

$$D_{n_k}^2 = D_{n_k+1}^2 - a_{n_k+1}^2 \geqq \frac{2}{3} D_{n_k+1}^2 \geqq \frac{4}{3} D_{n_{k-1}}^2\,,$$

so that

$$(4.6) \qquad D_{n_k}^2 > (4/3)^k \quad k = 1, 2, \cdots.$$

Further, using the Tchebycheff inequality and (2.3),

$$\sum_{k=1}^{\infty} \mathrm{P}\left\{ \bigcup_{m=n_k+1}^{n_{k+1}} (|S_m| > (1+\varepsilon)\sqrt{2D_m^2 \ \mathrm{loglog}\ D_m^2)}\right\}$$

$$\leqq \sum_{k=1}^{\infty} \mathrm{P}\{\max_{n_k < m \leqq n_{k+1}} |S_m| > (1+\varepsilon)\sqrt{2D_{n_k}^2 \log\log D_{n_k}^2}\}$$

$$\leqq 8 \sum_{k=1}^{\infty} \exp\left(- \frac{2(1+\varepsilon)^2 D_{n_k}^2 \log\log D_{n_k}^2}{2(D_{n_{k+1}}^2 - D_{n_k}^2)} \right)$$

$$\leqq 8 \sum_{k=1}^{\infty} \exp\left(- \frac{(1+2\varepsilon) \log\log \mathrm{D}_{n_k}^2}{(D_{n_{k+1}}^2 / D_{n_k}^2) - 1} \right).$$

By (4.5) and (4.6), the last series is dominated by

$$8 \sum_{k=1}^{\infty} \exp(-(1+2\varepsilon) \log\log D_{n_k}^2) \leqq K \sum_{k=1}^{\infty} (1/k)^{1+2\varepsilon} < \infty$$

Where K is a positive constant. Therefore (4.2) follows immediately in virtue of the Borel-Cantelli lemma, q.e.d.

COROLLARY 3. *Let* (x_n) *be the same as in Theorem* 2 *and* (a_n) *be a sequence of positive numbers. Put* $A_n = a_1 + a_2 + \cdots + a_n$. *If*

(4. 7) $$a_n / A_n = o(1/\log\log A_n),\ A_n \to \infty \ \textit{as}\ n \to \infty,$$

then

(4. 8) $$S_n / A_n \to 0 \ \textit{as}\ n \to \infty,\ \textit{a.s.}$$

This result may be proved along the same line as the proof of Theorem 2, and we omit the detail. If the condition (4.1) is satisfied, the proof may be done as that of Corollary 1. But in general, (4.1) need not follow from (4.7). In fact, we give an example (due to T. Tsuchikura) of sequence (a_n) which satisfies the condition (4.7) but does not (4.1).

Put $p_n = n!$, $n = 1, 2, \cdots$. Then, since

$$1p_1^2 + 2p_2^2 + \cdots + np_n^2 = (n+1)! - 1,$$

we have

$$\frac{p_{n+1}^2}{1p_1^2 + 2p_2^2 + \cdots + np_n^2 + p_{n+1}^2} > \frac{1}{2} \ (n = 1, 2 \cdots),$$

and

$$\left[\frac{p_n \log\log(1p_1 + 2p_2 + \cdots + (n-1)p_{n-1})}{1p_1 + 2p_2 + \cdots + (n-1)p_{n-1}}\right]^2 \leqq \frac{n}{(n-1)^2} (\log n)^2 = o(1).$$

Therefore, if we define (a_j) by $a_1 = p_1, a_2 = a_3 = p_2, a_4 = a_5 = a_6 = p_3, \cdots$, and generally $a_m = p_n$, for $1 + 2 + \cdots + (n-1) + 1 \leqq m \leqq 1 + 2 + \cdots + n$, we have

$$\frac{a_m}{A_m} \log\log A_m$$

$$\leqq \frac{p_n}{1p_1 + 2p_2 + \cdots + (n-1)p_{n-1}} \log\log(1p_1 + \cdots + (n-1)p_{n-1}) = o(1)$$

but

$$\frac{a_m^2}{D_m^2} \geqq \frac{p_{n+1}^2}{1p_1^2 + 2p_2^2 + \cdots + np_n^2 + p_{n+1}^2} > \frac{1}{2}.$$

REMARK 4. In [3] V.F. Gaposhkin showed that the law of iterated logarithm of uniformly bounded independent random variables holds for the Cesàro's summation method. In our case we follow his proof, word by word, starting from Theorem 2 and (2.4) and then we can obtain the following result.

If (x_n) is the same as in Theorem 2 and σ_n^α $(\alpha > 0)$ is in (3.4), then

$$\limsup_{n\to\infty} \frac{|\sigma_n^\alpha|}{\sqrt{(2/(2\alpha+1))n \log\log n}} \leqq 1 \ a.s.$$

THEOREM 3. *Let (x_n) be a sequence of martingale differences such that $|x_n| \leqq 1$ a.s. and (a_n) be a sequence of positive increasing numbers. If*

(4. 9) $$a_n/A_n = o(1/\log\log A_n), \text{ as } n \to \infty,$$

then

(4.10) $$\bar{S}_n/A_n \to 0 \quad \text{as } n \to \infty, \quad a.s.$$

PROOF. Give $\varepsilon > 0$ and define (n_j) in the previous manner, that is,

(4.11) $$A_{n_1} > 2(3+\varepsilon)/(6+\varepsilon) \ (>1)$$

$$(4.12) \qquad a_n/A_n < \varepsilon/(6+\varepsilon) \qquad \text{for } n > n_1$$

$$(4.13) \qquad a_n(\log\log A_n)/A_n < \varepsilon^2/64 \qquad \text{for } n > n_1,$$

and

$$(4.14) \qquad A_{n_{k-1}} < A_{n_k} \leqq (1+(\varepsilon/3))A_{n_{k-1}} < A_{n_k+1}.$$

We will show that

$$(4.15) \qquad 2\mathrm{P}\{\overline{S}_{n_{k+1}} > (\varepsilon/2)A_{n_k}\} \geqq \mathrm{P}\{\max_{n_k<n\leqq n_{k+1}} \overline{S}_n > \varepsilon\, A_{n_k}\}.$$

For this purpose we put the event

$$F_l = \{\overline{S}_{n_k+1} < x, \cdots, \overline{S}_{l-1} < x, \overline{S}_l \geqq x\},\ l = n_k+1,\ n_k+2, \cdots, n_{k+1}$$

and denote the conditional probability (expectation) with respect the event F_l by $\mathrm{P}\{\cdot\,|F_l\}$ $(\mathrm{E}\{\cdot\,|F_l\})$. We may suppose that $\mathrm{P}\{F_l\} > 0$, and then

$$\begin{aligned}
\mathrm{E}\{(\overline{S}_{n_{k+1}} - \overline{S}_l)^2\,|\,F_l\} &= \mathrm{E}\left\{\left(\sum_{j=1}^{n_{k+1}} a_{n_{k+1}-j+1}x_j - \sum_{j=1}^{l} a_{l-j+1}x_j\right)^2 \middle|\, F_l\right\} \\
&= \mathrm{E}\left\{\left(\sum_{j=1}^{l} (a_{n_{k+1}-j+1} - a_{l-j+1})x_j\right)^2 \middle|\, F_l\right\} + \mathrm{E}\left\{\left(\sum_{j=l+1}^{n_{k+1}} a_{n_{k+1}-j+1}x_j\right)^2 \middle|\, F_l\right\} \\
&\leqq \left(\sum_{j=1}^{l} (a_{n_{k+1}-j+1} - a_{l-j+1})\right)^2 + \left(\sum_{j=l+1}^{n_{k+1}} a_{n_{k+1}-j+1}\right)^2 \\
&\leqq \left(\sum_{j=l+1}^{n_{k+1}} a_j\right)^2 \\
&\leqq (A_{n_{k+1}} - A_{n_k})^2,
\end{aligned}$$

where we used the fact that if $1 \leqq i \leqq l < j \leqq n_{k+1}$,

$$\begin{aligned}
\mathrm{E}\{x_i x_j\,|\,F_l\} &= (1/\mathrm{P}\{F_l\})\mathrm{E}\{x_i x_j I(F_l)\} \\
&= (1/\mathrm{P}\{F_l\})\mathrm{E}\{x_i I(F_l)\mathrm{E}\{x_j\,|\,\mathfrak{A}_{j-1}\}\} = 0,
\end{aligned}$$

where $I(F_l)$ is the indicator of F_l, and that

$$a_{n_{k+1}-j+1} \geqq a_{l-j+1}, \qquad j = 1, 2, \cdots, l;\, n_{k+1} \geqq l.$$

Therefore

$$\begin{aligned}
\mathrm{P}\{|\overline{S}_{n_{k+1}}-\overline{S}_l| \geqq (\varepsilon/2)A_{n_k}|F_l\} &\leqq (4/(\varepsilon^2 A_{n_k}^2))\mathrm{E}\{(\overline{S}_{n_{k+1}} - \overline{S}_l)^2|F_l\} \\
&\leqq (4/(\varepsilon^2 A_{n_k}^2))(A_{n_{k+1}} - A_{n_k})^2 \\
&= (4/\varepsilon^2)((A_{n_{k+1}}/A_{n_k}) - 1)^2 \\
&\leqq (4/\varepsilon^2)(\varepsilon^2/9) < 1/2,
\end{aligned}$$

hence

$$\mathrm{P}\{\overline{S}_{n_{k+1}} > x - (\varepsilon/2)A_{n_k}|F_l\} \geqq 1/2,$$

and consequently

$$\begin{aligned}
\mathrm{P}\{\overline{S}_{n_{k+1}} > (\varepsilon/2)A_{n_k}\} &= \mathrm{P}\{\overline{S}_{n_{k+1}} > \varepsilon A_{n_k} - (\varepsilon/2)A_{n_k}\} \\
&\geqq \sum_{l=n_k+1}^{n_{k+1}} \mathrm{P}\{F_l\}\ \mathrm{P}\{\overline{S}_{n_{k+1}} > \varepsilon A_{n_k} - (\varepsilon/2)A_{n_k}|F_l\} \\
&\geqq (1/2)\mathrm{P}\{\max_{n_k<n\leqq n_{k+1}} \overline{S}_n > \varepsilon A_{n_k}\}.
\end{aligned}$$

Thus, (4.15) has been shown. From (4.15), (2.4), (4.14) and (4.13),

$$\begin{aligned}
(4.16) \qquad \sum_{k=1}^{\infty} \mathrm{P}\{\max_{n_k<n\leqq n_{k+1}} \overline{S}_n > \varepsilon A_{n_k}\} &\leqq 2\sum_{k=1}^{\infty} \mathrm{P}\{\overline{S}_{n_{k+1}} > (\varepsilon/2)A_{n_k}\} \\
&\leqq 2\sum_{k=1}^{\infty} \exp\left(-\frac{\varepsilon^2 A_{n_k}^2}{8\sum_{j=1}^{n_{k+1}} a_j^2}\right) \\
&\leqq 2\sum_{k=1}^{\infty} \exp\left(-\frac{\varepsilon^2 A_{n_{k+1}}}{32 a_{n_{k+1}}}\right) \\
&\leqq 2\sum_{k=1}^{\infty} \exp(-2\log\log A_{n_{k+1}}).
\end{aligned}$$

On the other hand, from (4.11), (4.12) and (4.14) we obtain

$$A_{n_k} > (2(3+\varepsilon)/(6+\varepsilon))^{k-1} \quad k = 1, 2, \cdots.$$

In conjunction with (4.16) this gives

$$\sum_{k=1}^{\infty} \mathrm{P}\{\max_{n_k < n \leqq n_{k+1}} \overline{S}_n > \varepsilon A_{n_k}\} \leqq C \sum_{k=1}^{\infty} (1/k)^2 < \infty,$$

where C is a positive constant. By the same way we can obtain

$$\sum_{k=1}^{\infty} \mathrm{P}\{\min_{n_k < n \leqq n_{k+1}} \overline{S}_n < -\varepsilon A_{n_k}\} < \infty,$$

and therefore (4.10) holds in virtue of the Borel-Cantelli lemma, q.e.d.

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