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نظم التصويت الموزونة من منظور الدوال البولانية الحدية

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الخلاصة

تلعب نظم التصويت الموزونة دورا هاما في استقصاء ونمذجة كثير من البنى الهندسية والظواهر السياسية والاجتماعية الاقتصادية. توجد حاجة ملحة لوصف هذه النظم بطريقة رياضية مبسطة وقوية يمكن تعميمها للنظم الكبيرة. ثمة وصف بارع لنظم التصويت بدلالة الدوال البولانية الحدية. ينتفع هذا الوصف كثيرا من ثراء المعلومات المتوفرة حول هذه الدوال ومن وجود باقة من الطرائق الجبرية والخريطية المعنية بمعالجتها. توضح الورقة أن الضامنات الأولية للدالة الحدية للنظام هي التحالفات الفائزة الصغرى (ح ف ص) فيه. تشرح الورقة المشتقة البولانية (الفرق البولاني) للدالة الحدية لنظام بالنسبة لكل عضو من الأعضاء المكونة له. إن الضامنات الأولية لهذا الفرق البولاني تفيد في استنباط التحالفات الفائزة (ح ف) التي يتعذر فيها الاستغناء عن العضو المعني. يمثل كل حد من الحدود الأصغرية لهذا الفرق البولاني تحالفا فائزا يلعب فيه هذا العضو دورا محوريا بمعنى أن التحالف يفقد الفوز إذا انسحب منه العضو. ومن ثم فإن عدد هذه الحدود الأصغرية هو مؤشر القدرة التصويتية أو النفوذ كما عرفه العالم بانزهاف. يتم شرح المفاهيم المقدمة بأمثلة توضيحية يتبين منها بعض الملغزات الموجية بالتناقض التي تعاني منها نظرية نظم التصويت. تخلص الورقة الى تأكيد فائدة الدوال البولانية الحدية في فهم ودراسة وتحليل وتصميم نظم التصويت أياً كان حجمها.

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Weighted voting systems: A threshold- Boolean perspective

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ABSTRACT

Weighted voting systems play a crucial role in the investigation and modeling of many engineering structures and political and socio-economic phenomena. There is an urgent need to describe these systems in a simplified powerful mathematical way that can be generalized to systems of any size. An elegant description of voting systems is presented in terms of threshold Boolean functions. This description benefits considerably from the wealth of information about these functions, and of the potpourri of algebraic and map techniques for handling them. The paper demonstrates that the prime implicants of the system threshold function are its Minimal Winning Coalitions (MWC). The paper discusses the Boolean derivative (Boolean difference) of the system threshold function with respect to each of its member components. The prime implicants of this Boolean difference can be used to deduce the winning coalitions (WC) in which the pertinent member cannot be dispensed with. Each of the minterms of this Boolean difference is a winning coalition in which this member plays a pivotal role. However, the coalition ceases to be winning if the member defects from it. Hence, the number of these minterms is identified as the Banzhaf index of voting power. The concepts introduced are illustrated with detailed demonstrative examples that also exhibit some of the known paradoxes of voting- system theory. Finally, the paper stresses the utility of threshold Boolean functions in the understanding, study, analysis, and design of weighted voting systems irrespective of size.

Keywords: Banzhaf index; Prime implicants; Threshold Boolean functions; Voting systems; Winning coalitions.

INTRODUCTION

A weighted voting system is a group of entities which have to come to a decision through voting. Each member of the system has a specific weight for its vote, and the decision is passed if it secures a minimum threshold of supporting votes. For simplicity, we shall not consider "abstention" here, i.e., we assume that every member of the system casts a vote of 'yes' or 'no'. There is a wealth of examples of weighted voting systems in a variety of political and socio-economic entities such as (a) a presidential council or parliament of a federal government composed of states of different sizes, (b) a state council with weighted representatives for the participating districts or counties, (c) the European Union (EU), and (d) the board of directors representing stockholders of a company or a corporation (March, 1962; Cross, 1967; Holler, 1982; Hershey, 1973; Steen, 1994; Taylor and Pacelli, 2008).

Our interest in the topic of weighted systems stems from an *engineering* application, namely, the evaluation of system reliability for a threshold system, i.e., a system whose success is a weighted voting function of the successes of its components (Rushdi, 1990; 1993; 2010; Rushdi and Alturki, 2015, Eryilmaz, 2015). Despite the urgent need for an adequate description of weighted voting systems that is scalable or generalizable to large systems, the only current descriptions rely on trial and error or computer simulations for large systems and use of lattice diagrams for very small systems (Steiner, 1967; Steen, 1994; Stewart, 1995; Taylor and Pacelli, 2008). Our study of the reliability of threshold systems revealed the availability of a very powerful tool for the study of weighted voting systems, namely the theory of threshold Boolean functions. There is already a great wealth of information in that theory that we are going to utilize in (and adapt to) the study of weighted voting systems. Moreover, we will benefit much from an associated pictorial tool, viz. the Karnaugh map (Rushdi, 1997; Rushdi & Al-Yahya, 2000; 2001a; 2001b).

The organization of the rest of this paper is as follows. Section 2 reviews the basic concepts of threshold Boolean functions and uses them in interpreting important concepts in the theory of weighted voting systems, including those of a decision, minimal winning coalitions, and voting power. Section 3 demonstrates the findings of section 2 via three illustrative examples. The first example compares the existing method of lattice diagram to the proposed method of a threshold function. The second example discusses three schemes for the same problem, and nicely exposes some of the paradoxes of voting-system theory. Example 3 relates concepts of coherent Boolean threshold functions to common terminology of political coalitions. Section 4 concludes the paper and proposes some future work.

THRESHOLD BOOLEAN FUNCTIONS

By definition, a Boolean function $S(\vec{X}) = S(X_1, X_2, \ldots\ldots, X_n)$ is a threshold function (Muroga, 1971; Lee, 1978; Muroga, 1979; Rushdi, 1990; Crama and Hammer, 2011) if and only if there exists a set of real numbers $W_1, W_2, \ldots\ldots, W_n$, called weights, and T, called a threshold, such that

$$S(\vec{X}) = 1 \ \text{ iff } \ \sum_{i=1}^{n} W_i\, X_i \geq T. \tag{1}$$

A threshold function $S(\vec{X})$ satisfying Equation (1) will be denoted by $H(n; \vec{X}; \vec{W}; T)$. Here, the magnitudes of the weights $|W_i|$ were thought to give the relative importance of the respective values of the elements or components X_i in determining the values of the function (Hurst et al., 1985; Rushdi, 1990). However, Rushdi and Alturki (2015) demonstrated that this was not necessarily the case. In fact, they made a clear distinction between the weight of an element and its voting power. Such a distinction appears to be in agreement with the earlier findings of Banzhaf (1964).

A threshold Boolean function with positive weights and threshold is a natural description for the success of a threshold reliability system, or equivalently, for the decision made by a weighted voting system (Rushdi and Alturki, 2015). This function is a non-decreasing function, and hence it has a minimal sum that is identical to its complete sum, and both are expressible without complemented literals (Lee, 1978; Rushdi, 1986a; 1986b; Rushdi and Alturki, 2015). A prime implicant of this function is a Minimal Winning Coalition (Rushdi and Alturki, 2015), i.e., it is a winning coalition such that any defection from it negates its winning status (Steiner, 1967; Fishburn and Brams, 1996).

Now, we note that the famous Banzhaf index of voting power (Banzhaf, 1964; Dubey and Shapley, 1979; Hammer and Holzman, 1992; Alonso-Meijide and Freixas, 2010; Yamamoto, 2012), is simply the weight of the Boolean derivative (Boolean difference) (Reed, 1973; Lee, 1978; Muroga, 1979; Rushdi, 1986b) of the system function with respect to the pertinent element variable

$$B_i = \text{weight}\ (\partial f/\partial X_i) \tag{2a}$$

$$= \text{weight}\ (f\ (\vec{X}|1_i) \oplus f\ (\vec{X}|0_i)), \tag{2b}$$

where $f(\vec{X}|1_i)$ and $f(\vec{X}|0_i)$ are the subfunctions obtained by restricting the input $\vec{X}$ of f such that f is a 1 or a 0, respectively. In Equation (2), the weight of a Boolean function is the number of its true vectors (Rushdi, 1987a; Rushdi, 1987b), i.e., the number of vectors $\vec{X}$ for which $S(\vec{X}) = 1$. The prime implicants of this Boolean difference can be used to deduce the winning coalitions (WC) in which the pertinent member cannot be dispensed with. Each of the minterms of this Boolean difference is a winning coalition in which this member plays a pivotal role, in the sense that the coalition ceases to be winning if the member defects from it. That is why the number of these minterms is identified as the Banzhaf index of voting power.

ILLUSTRATIVE EXAMPLES

Example 1

Consider the weighted voting system

$$H\ (n; \vec{X}; \vec{W}; T) = H\ (3; A, B, C; 2, 1, 1; 3) \tag{3}$$

taken from Stewart (1995). Here, member A has two votes, each of members, B and C has a single vote, and a majority of three votes upholds a decision. This system can be solved by the lattice diagram in Figure 1(a). The diagram shows all possible coalitions. These are given by the power set of the set S = {A, B, C}, namely

$$2^s = \{\phi, \{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}\}. \tag{4}$$

In Figure 1(a), two possible coalitions are linked by one edge if they differ by just one member, and such an edge is labeled by the member that the two coalitions do not have in common. Figure 1(a) also shows the total weight for every coalition. A winning coalition (one with total weight ≥ 3) is depicted as a black node, while a losing coalition (one with total weight < 3) is characterized as a white node. An edge going from a white node (losing coalition) to a black one (winning coalition) is a pivotal edge and is marked in bold red. These are five such edges. The voting power B_i of member i is the number of pivotal edges bearing its name, and hence $B_A = 3$, $B_B = 1$, and $B_C = 1$. In Figure 1(b), we redraw the lattice diagram of Figure 1(a) using a Karnaugh map layout (Rushdi and Ba-Rukab, 2004; Rushdi and Ba-Rukab, 2007; Rushdi and Albarakati, 2012).

Our alternative approach is to represent the system decision by the threshold Boolean function

$$\{f(A, B, C) = 1\} \iff \{2A + B + C \geq 3\} \tag{5}$$

Figure 1(c) is a Karnaugh-map expression of the pseudo-Boolean function

$$F(A, B, C) = 2A + B + C \tag{6}$$

and Figure 1(d) is a Karnaugh-map representation of the corresponding threshold function ($F \geq 3$), namely

$$F(A, B, C) = AB \vee AC. \tag{7}$$

Equation (7) states that f has two prime implicants AB and AC, which correspond to the minimal winning coalitions {A, B} and {A, C}, respectively. The Karnaugh map in Figure 1(d) is folded with respect to each of its arguments to obtain the Boolean differences ($\partial f/\partial A$), ($\partial f/\partial B$), ($\partial f/\partial C$), according to Equation (2) (Rushdi, 1986b) in Figures 1(e2), 1(e3), 1(e3), respectively. These are given as

$$\partial f/\partial A = B \vee C \tag{8a}$$

$$\partial f/\partial B = A\overline{C} \tag{8b}$$

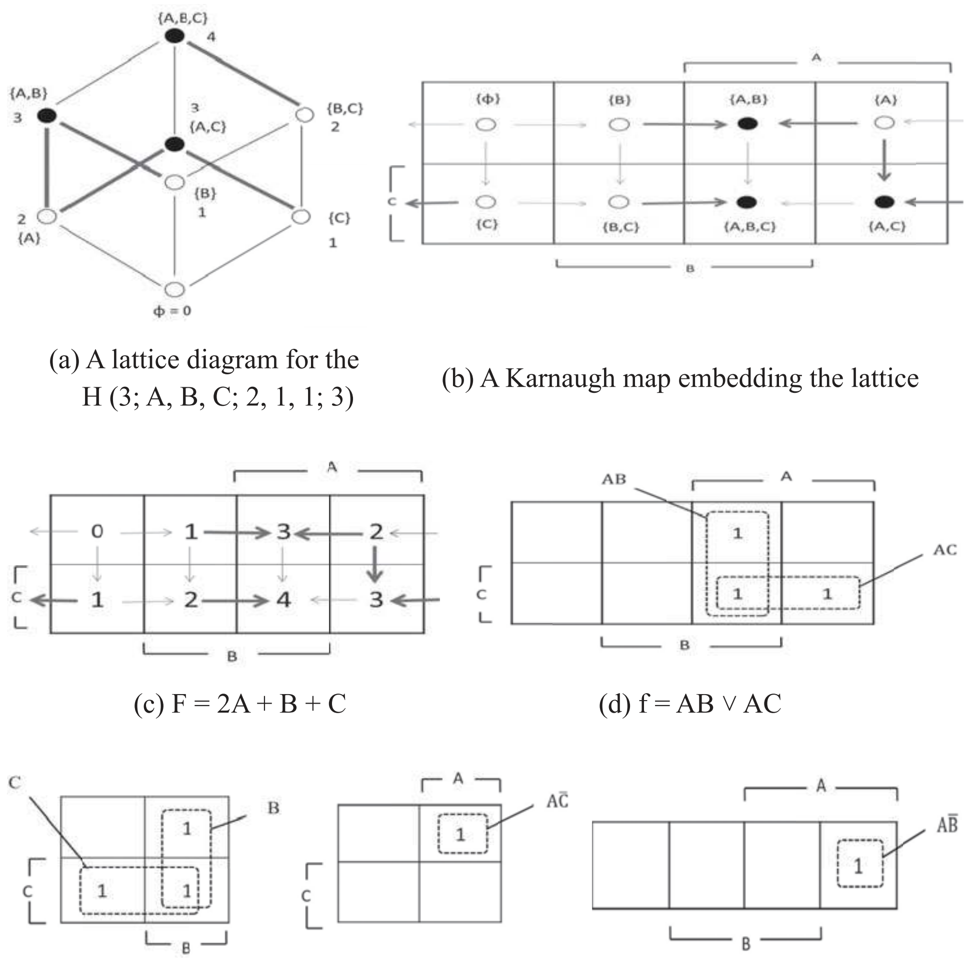
$$\partial f/\partial C = A\overline{B} \tag{8c}$$

and indicate that A cannot be dispensed with in MWCs {A, B} and {A, C}, while B is necessary in MWC {A, B}, and the C must be included in MWC {A, C}. The Banzhaf indices of the three members are:

$$B_A = \text{Weight}\ (\partial f/\partial A) = 3 \quad (9a)$$

$$B_B = \text{Weight}\ (\partial f/\partial B) = 1 \quad (9b)$$

$$B_C = \text{Weight}\ (\partial f/\partial C) = 1 \quad (9c)$$



(a) A lattice diagram for the H (3; A, B, C; 2, 1, 1; 3)

(b) A Karnaugh map embedding the lattice

(c) F = 2A + B + C

(d) f = AB ∨ AC

(e1) ∂f/∂A (e2) ∂f/∂B (e3) ∂f/∂C

Fig. 1. Representation of a 3-member weighting system via (a) a lattice diagram, (b) a Karnaugh map, (c) a pseudo-Boolean function, (d) a threshold function, and (e) Boolean derivatives.

Example 2

Table 1 shows three schemes for the voting weights of the six districts of a fictitious country called Blockvotia (Stewart, 1995). The six districts are Sheepshire (H), Richfolk (R), Candlewick (C), Fiddlesex (F), Slurrey (L) and Porkney Isles (P). For the sake of brevity, assume that the abbreviation of the name of a district, is also the two-valued Boolean indicator variable for its voting position. The voting positions and weights are expressed by the 6-element vectors

$$\vec{X} = [H\ \ R\ \ C\ \ F\ \ L\ \ P]^T \tag{10}$$

$$\vec{W} = [W_H\ \ W_R\ \ W_C\ \ W_F\ \ W_L\ \ W_P]^T \tag{11}$$

Table 1. Voting weights for the districts of Blockvotia.

	First scheme	**Second scheme**	**Third scheme**
W_H	10	10	12
W_R	9	9	9
W_C	7	7	7
W_F	3	3	3
W_L	1	2	1
W_P	1	2	1
Sum = ΣW_i	31	33	33
T = ceiling ($\Sigma W_i / 2$)	16	17	17

Now, introduce the pseudo-Boolean function $\vec{F}(\vec{X})$: $\{0, 1\}^6 \longrightarrow R$ such that

$$F(\vec{X}) = W_H H + W_R R + W_C C + W_F F + W_L L + W_P P, \tag{12}$$

and hence the system is described by a threshold function $f(\vec{X})$: $\{0, 1\}^6 \longrightarrow \{0, 1\}$ such that

$$\{f(\vec{X}) = 1\} \text{ iff } \{F(\vec{X}) \geq T\} \tag{13}$$

where T is the threshold of the voting system, expressed as the ceiling of half the total sum of weights. In the following section, we discuss the three voting schemes presented in Table 1.

Scheme 1

Figures 2(a) and 2(b) are Karnaugh-map representations for $F(\vec{X})$ and $f(\vec{X})$ for the first scheme, herein designated $F_1(\vec{X})$ and $f_1(\vec{X})$, respectively. Since the function $f_1(\vec{X})$ is monotonically non-decreasing or coherent, its prime implicants entail solely un-complemented literals, and its minimal sum is identical to its complete sum (Lee, 1978; Rushdi, 1986a; Rushdi and Alturki, 2015), namely:

$$f_1(\vec{X}) = HR \vee HC \vee RC. \tag{14}$$

The threshold Boolean function $f_1(\vec{X})$ in Equation (14) has three prime implicants, HR, HC, and RC, each of which represents a Minimal Wining Coalition (MWC). The total weights of these MWCs are

$$W_{HR} = W_H + W_R = 10 + 9 = 19, \tag{15a}$$

$$W_{HC} = W_H + W_C = 10 + 7 = 17, \tag{15b}$$

$$W_{RC} = W_R + W_C = 9+ 7 = 16. \tag{15c}$$

Here, the coalition RC is the least MWC and just meets the bare minimum requirement of T = 16. Fishburn and Brans (1996) suggest that this least MWC is the most stable among the class of MWCs. Figure 2(b) indicates that out of the $64 = 2^6$ system states or coalitions, there are 32 primitive winning coalitions (depicted with map cells of entry 1) and also 32 primitive losing coalitions (depicted with map cells of entry 0 (that are actually left blank)). Figure 2(c) is a Karnaugh-map representation of the Boolean derivative (Boolean difference) $\partial f_1/\partial H$. This map is obtained by folding the map shown in Figure 2(b) with respect to the variable H, so that a cell $(\vec{X}|1_H)$ {of the right half of the map} and a cell $(\vec{X}|0_H)$ {of the left half of the map} coincide as a single cell whose entry is obtained by XORing the entries of the two original cells, in accordance with Equation (2). The function $(\partial f/\partial H)$ has two prime implicants $R\overline{C}$ and $\overline{R}C$, which can be used to deduce the wining coalitions $HR\overline{C}$ and $H\overline{R}C$ in which member H cannot be dispensed with.

Figures 2(d) –2(h) express the Boolean difference of f_1 with respect to variables R, C, F, L, and P respectively. The Banzhaf indices are:

$$B_H = \text{Weight } (\partial f_1/\partial H) = 16, \tag{16a}$$

$$B_R = \text{Weight } (\partial f_1/\partial R) = 16, \tag{16b}$$

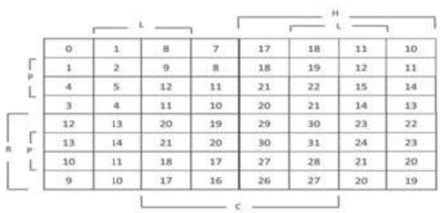
$$B_C = \text{Weight } (\partial f_1/\partial C) = 16, \tag{16c}$$

$$B_F = \text{Weight } (\partial f_1/\partial F) = 0, \tag{16d}$$

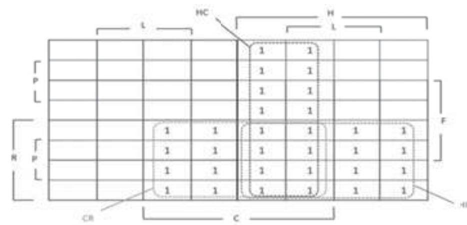
$$B_L = \text{Weight } (\partial f_1/\partial L) = 0, \tag{16e}$$

$$B_P = \text{Weight } (\partial f_1/\partial P) = 0. \tag{16f}$$

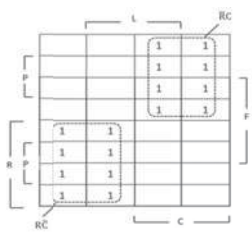
This means that the three largest districts have equal voting power, while the three smallest ones have no power at all. In fact, in any vote, at least two of the three largest districts will vote the same way, securing a MWC and leaving the three smallest districts powerless. In fact, none of the smallest districts can ever play a pivotal role in decision making. Furthermore, none of them can turn a winning coalition to a losing one by defecting from it, and none of them can turn a losing coalition to a winning one by joining it.



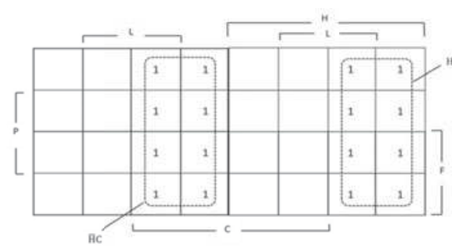
(a) $F_1(\vec{F}) = 10\,H + 9\,R + 7\,C + 3\,F + L + P$



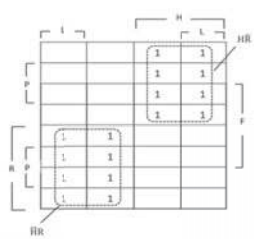
(b) $f_1(X) = HR \vee HC \vee RC$



(c) $\partial f_1/\partial H$

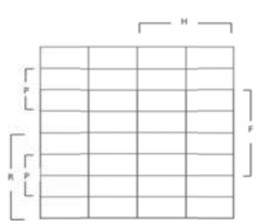


(d) $\partial f_1/\partial R$

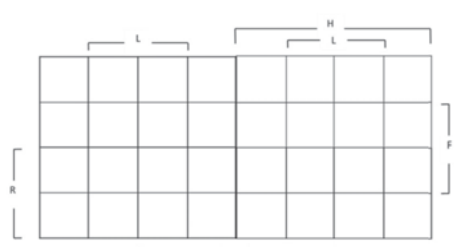


(e) $\partial f_1/\partial C$

(f) $\partial f_1/\partial F$



(g) $\partial f_1/\partial L$



(h) $\partial f_1/\partial P$

Fig. 2. The pseudo Boolean function $F_1(\vec{X})$, the threshold function $f_1(\vec{X})$, and the Boolean derivatives for the first scheme in Table 1.

Scheme 2

Scheme 2 was proposed as a remedy to the unfortunate situation in scheme 1 by adding an extra vote to each of the two smallest districts (see Table 1). Figure 3(a) and 3(b) are Karnaugh-map representations for $F(\vec{X})$ and $f(\vec{X})$ for the second scheme, herein designated as $F_2(\vec{X})$ and $f_2(\vec{X})$, respectively. The functions $F_2(\vec{X})$ and $f_2(\vec{X})$ are given by:

$$F_1(\vec{X}) = 10\ H + 9\ R + 7\ C + 3\ F + 2\ L + 2\ P \tag{17}$$

$$f_2(\vec{X}) = HR \vee HC \vee RCL \vee RCP \vee RCF \vee HFLP \tag{18}$$

The function $f_2(\vec{X})$ has six prime implicants HR, RCF, RCL, RCP, HC and HFLP, each of which represents an MWC. The total weights of these MWCs are

$$W_{HR} = W_H + W_R = 10 + 9 = 19, \tag{19a}$$

$$W_{RCF} = W_R + W_C + W_F = 9 + 7 + 3 = 19, \tag{19b}$$

$$W_{RCL} = W_R + W_C + W_L = 9 + 7 + 2 = 18, \tag{19c}$$

$$W_{RCP} = W_R + W_C + W_P = 9 + 7 + 2 = 18, \tag{19d}$$

$$W_{HC} = W_H + W_C = 10 + 7 = 17, \tag{19e}$$

$$W_{HFLP} = W_H + W_F + W_H + W_F = 10 + 3 + 2 + 2 = 17. \tag{19f}$$

Here, the two coalitions, HC and HFLP are the least MWCs and each of them just meets the bare minimum requirement of T = 17. Figure 3(b) indicates that out of the 64 = 2^6 system states or coalitions, there are still 32 primitive winning coalitions and also 32 primitive losing coalitions. Figures 3(c) – 3(h) are Karnaugh-map representations of the Boolean derivatives, from which the Banzhaf indices are obtained as:

$$B_H = \text{Weight}\ (\partial f_2/\partial H) = 17, \tag{20a}$$

$$B_R = \text{Weight}\ (\partial f_2/\partial R) = 15, \tag{20b}$$

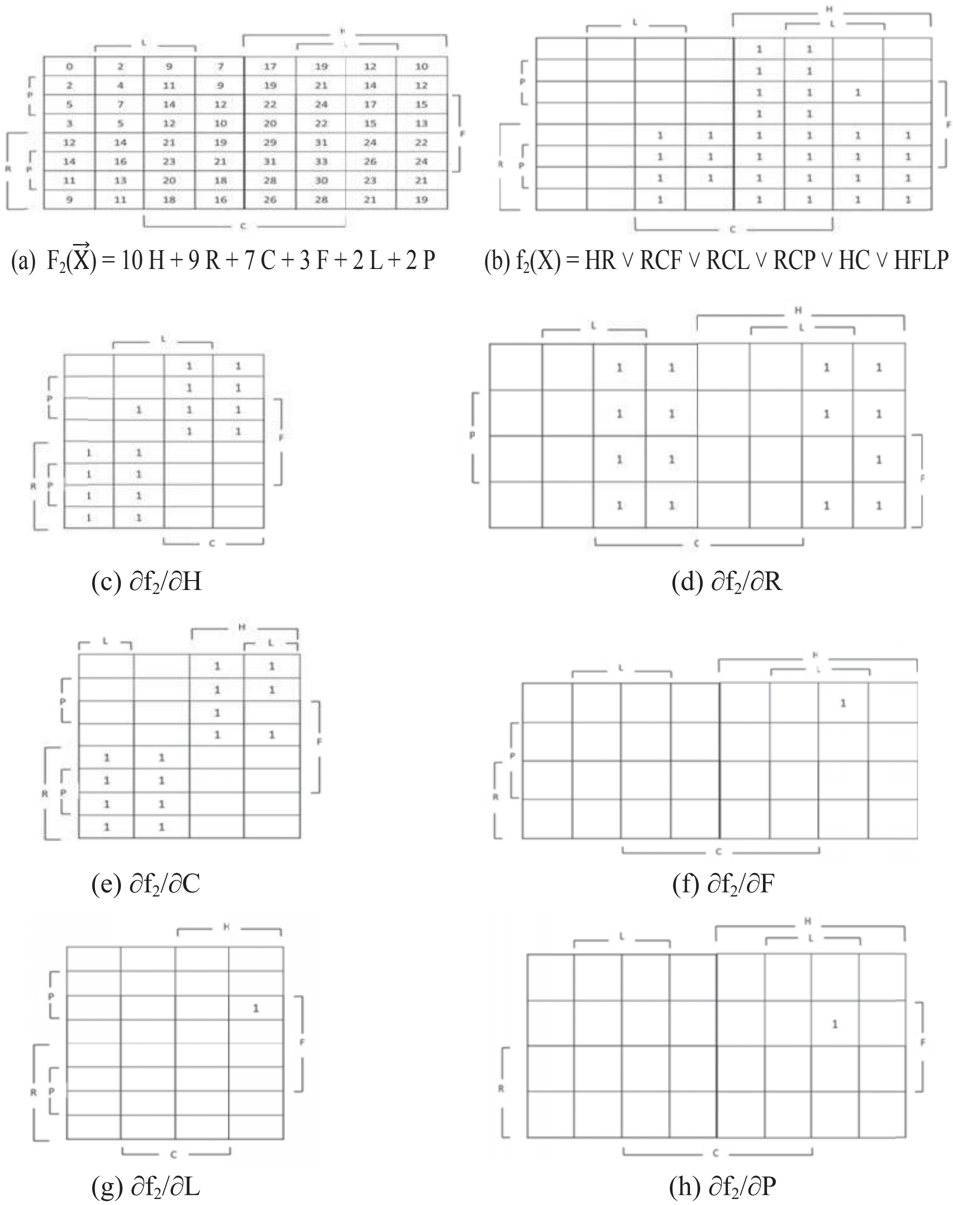
$$B_C = \text{Weight}\ (\partial f_2/\partial C) = 15, \tag{20c}$$

$$B_F = \text{Weight}\ (\partial f_2/\partial F) = 1, \tag{20d}$$

$$B_L = \text{Weight}\ (\partial f_2/\partial L) = 1, \tag{20e}$$

$$B_P = \text{Weight}\ (\partial f_2/\partial P) = 1. \tag{20f}$$

Scheme 2 is better than scheme 1, since each of the three smallest districts has now some power. It might seem paradoxical that district F gained some power, and also district H became slightly more powerful than districts R and C by simply adding votes to districts L and P. Though scheme 2 is better than scheme 1, it is still not entirely fair. For example, district F has more weight than any of districts L and P, but it has just the same power as each of them.



(a) $F_2(\vec{X}) = 10\,H + 9\,R + 7\,C + 3\,F + 2\,L + 2\,P$ (b) $f_2(X) = HR \vee RCF \vee RCL \vee RCP \vee HC \vee HFLP$

(c) $\partial f_2/\partial H$ (d) $\partial f_2/\partial R$

(e) $\partial f_2/\partial C$ (f) $\partial f_2/\partial F$

(g) $\partial f_2/\partial L$ (h) $\partial f_2/\partial P$

Fig. 3. The pseudo Boolean function $F_2(\vec{X})$, the threshold function $f_2(\vec{X})$, and the Boolean differences for the first scheme in Table 1.

Scheme 3

Scheme 3 is an alternative remedy for the unfortunate situation in scheme 1. In scheme 3, the largest district H, is assigned two more votes. Figure 4 is a Karnaugh-map representation of $F(\vec{X})$ for this scheme designated $F_3(\vec{X})$ namely:

$$F_3(\vec{X}) = 12\,H + 9\,R + 7\,C + 3\,F + L + P \tag{21}$$

Now, with a threshold of T = 17, we discover that the governing threshold function $f_3(\vec{X})$ for this scheme is nothing but $f_2(\vec{X})$ of Figure 3(b) and Equation (18). Hence, this scheme has exactly the same set of MWCs and Banzhaf indices as scheme 2. Again, it is paradoxical that by granting more votes to the largest district, the three smallest districts cease to be powerless.

Example 3

We determine the number N_n and the list of all coherent switching functions for n = 0, 1, 2, and 3, and then identify among them those that are threshold with majority voting. A switching function $f(\vec{X})$ is coherent if it satisfies the conditions of (Rushdi, 2010, Rushdi and Hassan, 2015; 2016):

(a) relevancy (causality): $f(\vec{0}) = 0$, $f(\vec{1}) = 1$;

(b) monotonicity: $\{\vec{X} \geq \vec{Y}\} \Rightarrow \{f(\vec{X}) \geq f(\vec{Y})\}$

The case n = 0

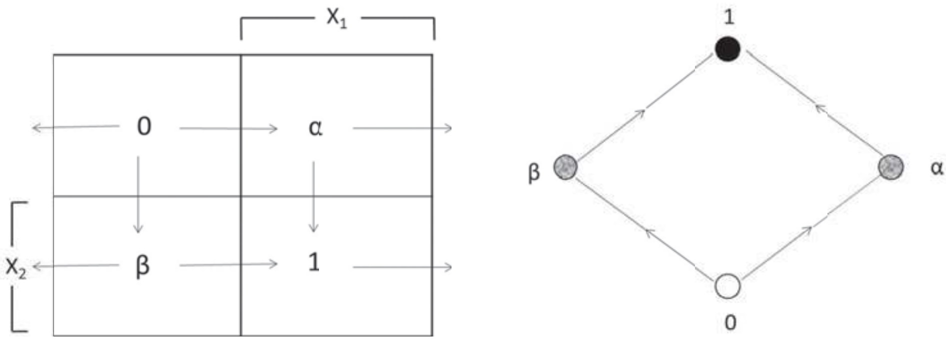
Here f( ) ε {0, 1}, and hence $N_0 = 0$.

The case n = 1

Here f (X) ε $\{0, 1, X, \vec{X}\}$, and hence $N_1 = 1$, i.e., there is a single coherent switching function f (X) of one variable, namely f(X) = X.

The case n = 2

Consider $f(X_1, X_2)$ represented by the Karnaugh map of Figure 4(a) which satisfies the relevancy condition, with the partial order shown in Figure 4(b) to enforce monotonicity $\{0 \leq \binom{\alpha}{\beta} \leq 1\}$. Since α and β are independent of each other, there are four possibilities for a coherent $f(X_1, X_2)$ as shown in Table 2. All of these are threshold (Rushdi and Alturki, 2015).



(a). Relevancy for n = 2. (b). Monotonicity for n = 2.

Fig. 4. Visual explanation of (a) relevancy, and (b) monotonicity for n = 2.

Table 2. Coherent Functions of n =2.

α	β	$f(X_1, X_2)$	Majority Threshold?
0	0	$X_1\ X_2$	Consensus
0	1	X_2	Dictator
1	0	X_1	Dictator
1	1	$X_1 \vee X_2$	------------

However, the function ($X_1 \vee X_2$) is not a majority-threshold one, it does not allow a threshold that is strictly greater than half the sum of the weights. The other three functions correspond to the two possibilities of minimal winning coalitions with two voters:

(a) Consensus is required (X_1X_2).

(b) The system has a dictator (X_1) or (X_2).

The case n = 3

Consider f (X_1, X_2, X_3) represented by the Karnaugh map of Figure 5(a) which satisfies the relevancy condition, with the partial order shown in Figure 5(b) to enforce monotonicity:

$$0 \leq \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \leq s$$

$$\begin{pmatrix}\alpha\\ \gamma\end{pmatrix} \leq r$$

$$\begin{pmatrix}\beta\\ \gamma\end{pmatrix} \leq t$$

$$\begin{pmatrix}s\\ r\\ t\end{pmatrix} \leq 1$$

Each of α, β, γ can be assigned one of the values 0 and 1, independently of one another. Corresponding possible values for r, s, and t are shown in Table 3, which lists 18 coherent functions f (X_1, X_2, X_3). Out of these, 11 functions are majority threshold, namely:

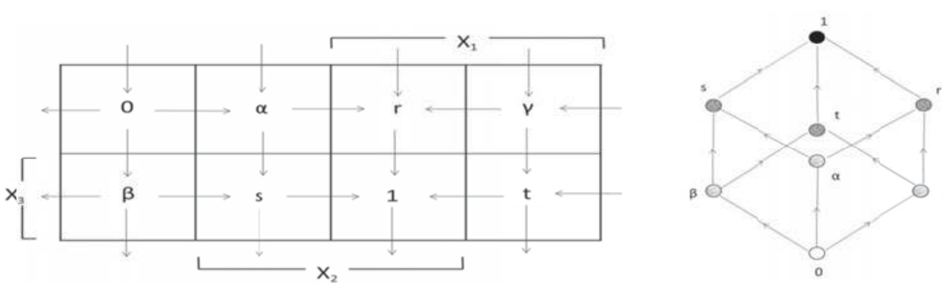
(a) Consensus ($X_1X_2X_3$)

(b) Clique (X_1X_2, X_1X_3, or X_2X_3).

(c) Chair veto (X_3 ($X_1 \vee X_2$), X_2 ($X_1 \vee X_3$), or X_1 ($X_2 \vee X_3$))

(d) Dictator (X_1, X_2, or X_3)

(e) Majority ($X_1X_2 \vee X_1X_3 \vee X_2X_3$)



(a) (b)

Fig. 5. Visual explanation of (a) relevancy, and (b) monotonicity for n = 3.

Table 3. Coherent functions for n = 3.

α β γ	r s t	f (X_1, X_2, X_3)	Majority Threshold?
0 0 0	0 0 0	$X_1X_2X_3$	Consensus
	0 0 1	X_1X_3	Clique
	0 1 0	X_2X_3	Clique
	0 1 1	$X_1X_3 \vee X_2X_3$	Chair 3 veto
	1 0 0	X_1X_2	Clique
	1 0 1	$X_1X_2 \vee X_1X_3$	Chair 1 veto
	1 1 0	$X_1X_2 \vee X_2X_3$	Chair 2 veto
	1 1 1	$X_1X_2 \vee X_1X_3 \vee X_2X_3$	Majority
0 0 1	1 0 1	X_1	Dictator
	1 1 1	$X_1 \vee X_2X_3$	-----------
0 1 0	0 1 1	X_3	Dictator
	1 1 1	$X_3 \vee X_1X_2$	-----------
0 1 1	1 1 1	$X_1 \vee X_2$	-----------
1 0 0	1 1 0	X_2	Dictator
	1 1 1	$X_2 \vee X_1X_3$	-----------
1 0 1	1 1 1	$X_1 \vee X_3$	-----------
1 1 0	1 1 1	$X_2 \vee X_3$	-----------
1 1 1	1 1 1	$X_1 \vee X_2 \vee X_3$	-----------

CONCLUSION AND FUTURE WORK

This paper demonstrated the utility of threshold Boolean functions in the understanding, study and analysis of weighted voting systems. Many important concepts of these systems are given threshold Boolean interpretations, including the concepts of voting decision, winning coalition, losing coalition, minimal winning coalition, least minimal winning coalition, and the Banzhaf index of voting power.

As a sequel to this work, we plan to automate our findings so as to be able to study larger systems whatever their sizes might be. We also plan to study the effect of abstention of some votes on the behavior of the weighted voting system. Further study pertaining to the structure and size of winning coalitions (Butterworth, 1971, 1974; Russell, 1976; Shepsle, 1974a, 1974b; Nurmi, 1997; Axenovich and Roy, 2010; Kirsch and Langner, 2010), is warranted. Detailed comparison is required for the Banzhaf index and measures of importance in reliability (Freixas and Puente, 2002; Freixas and Pons, 2008; Kuo and Zhu, 2012; Zhu and Kuo, 2014). A hot area of potentially fruitful further work is that of mathematics of voting power (Alonso- Meijide et al.

2012; Das & Rezek 2012; Morgan & Várdy 2012; Holler & Nurmi 2013; Jelnov & Taumam 2014; Houy & Zwicker 2014; Freixas & Kaniovski 2014; Michael & Benoit 2015).

The present paper is a theoretical investigation of the topic of weighted voting systems. To further enhance the engineering utility of this work, we need to find mechatronic or electrical devices that can be modeled exactly or partially by the voting scenario in our theoretical examples. In fact, the actuation mechanism of many electro-mechanical systems are triggered by complex voting schemes similar to those outlined in these examples. We hope to report such devices or other practical engineering artifacts in a sequel of this work.

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