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Weihrauch degrees of finding equilibria in sequential games (Extended Abstract*)

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Abstract. We consider the degrees of non-computability (Weihrauch degrees) of finding winning strategies (or more generally, Nash equilibria) in infinite sequential games with certain winning sets (or more generally, outcome sets). In particular, we show that as the complexity of the winning sets increases in the difference hierarchy, the complexity of constructing winning strategies increases in the effective Borel hierarchy.

1 Overview

We consider questions of (non)computability related to infinite sequential games played by any countable number of players. The best-known example of such games are Gale-Stewart games [10], which are two-player win/lose games. The existence of winning strategies in (special cases of) Gale-Stewart games is often employed to show that truth-values in certain logics are well-determined. The degrees of noncomputability of variations of (Borel) determinacy [17] can be studied using our techniques, and several are fully classified.

This work falls within the research programme to study the computational content of mathematical theorems in the Weihrauch lattice, which was outlined by Brattka and Gherardi in [3]. In particular, it continues the investigation of the Weihrauch degrees of operations mapping games to their equilibria started in [21]. There, finding pure and mixed Nash equilibria in two-player games with finitely many actions in strategic form were classified.

One motivation for this line of inquiry is the general stance that solution concepts in game theory can only be convincing if the players are capable of (at least jointly) computing them, taken e.g. in [22]. Even if we allow for some degree of hypercomputation, or are, e.g., willing to tacitly replace actually attaining a solution concept by some process (slowly) converging to it, we still have to reject solution concepts with too high a Weihrauch degree.

The results for determinacy of specific pointclasses that we provide are a refinement of results obtained in reverse mathematics by Nemoto, MedSalem and Tanaka [20]; the first is also a uniformization of a result by Cenzer and Remmel [7]. For some represented pointclass Γ, let $\mathrm{Det}_\Gamma : \Gamma \rightrightarrows \{0,1\}^{\mathbb{N}}$ be

* A full version is available as [16].

the map taking a Γ-subset A of Cantor space to a (suitably encoded) Nash equilibrium in the sequential two-player game with alternating moves where the first player wins if the induced play is in A, and the second player wins otherwise. Let $\mathcal{A}$ be the closed subsets of Cantor space, and $\mathfrak{D} := \{U \setminus U' \mid U, U' \in \mathcal{A}\}$. Some of our results are:

Theorem. *$Det_{\mathcal{A}} \equiv_W C_{\{0,1\}^{\mathbb{N}}}$ and $Det_{\mathfrak{D}} \equiv_W C_{\{0,1\}^{\mathbb{N}}} \star \lim$.*

We have two remarks. One, by combining the preceding theorem with the main result of [6], we find that $\mathrm{Det}_{\mathfrak{D}}$ is equivalent to the Bolzano-Weierstrass-Theorem. This may be a bit unexpected in particular seeing that $\mathrm{C}_{\{0,1\}^{\mathbb{N}}} \star \lim$ is not (yet) known to contain a plethora of mathematical theorems (unlike, e.g., $\mathrm{C}_{\{0,1\}^{\mathbb{N}}}$). Two, we already need to use a limit operator in order to move up one level of the difference hierarchy – rather than being able to move up one level in the Borel hierarchy as one may have expected naively. Thus, this observation may complement Harvey Friedman's famous result [9] that proving Borel determinacy requires repeated use of the axiom of replacement.

Another group of results is based on inspecting the various results extending Borel determinacy to more general classes of games (and solution concepts) in [13–15]. If we instantiate these generic results with specific determinacy version as above, we can prove for some of them that they are actually optimal w.r.t. Weihrauch reducibility. We shall state two such classifications explicitly.

Consider two-player sequential games with finitely many *outcomes* and antagonistic (inverse of each other) linear preferences over the outcomes. For any upper set of outcomes w.r.t. some player's preference let the corresponding set of plays be open or closed. Let $\mathrm{NE}^{ap}_{\mathcal{O}\cup\mathcal{A}}$ be the operation taking such a game (suitably encoded) and producing a Nash equilibrium. Then:

Theorem. *$NE^{ap}_{\mathcal{O}\cup\mathcal{A}} \equiv_W C_{\{0,1\}^{\mathbb{N}}} \times LPO^*$*

Next, we restrict the aforementioned class of games to antagonistic games, that is, games where the preferences of one player are the inverse of the preferences of the other player. Those games will have subgame-perfect equilibria, and we let $\mathrm{SPE}_{\mathcal{O}\cup\mathcal{A}}$ be the operation mapping such games to a subgame-perfect equilibrium.

Theorem. *$SPE_{\mathcal{O}\cup\mathcal{A}} \equiv_W \lim$*

2 Fundamentals

We proceed to give brief, informal introductions to represented spaces, Weihrauch reducibility and infinite sequential games. For a formal treatment and further references, we refer to the extended version of the present paper [16].

2.1 Informal background on represented spaces and Weihrauch reducibility

We use representations to induce computability notions on the spaces of interest to us, in particular on pointclasses (i.e. sets of subsets of Cantor space) and derived from that, infinite sequential games. A represented space is a set X together with a partial surjection $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$. A function between represented spaces is computable, iff there is a matching computable function on Baire space.

An open subset U of Cantor space is represented by some list of finite words $(w_i)_{i\in\mathbb{N}}$ such that $U = \bigcup_{i\in\mathbb{N}} w_i\{0,1\}^{\mathbb{N}}$. For other derived pointclasses their representation follows directly from how they are defined; i.e. closed sets are given via their complement as an open set, a Σ^0_2-set is given by a sequence of closed sets whose union it is, etc. (this idea is explored in more detail in [2]).

Weihrauch reducibility is a relation between multivalued functions on represented spaces, where $f \leq_{\mathrm{W}} g$ means that f can be computed with the help of a single oracle call to g. We make use of some operations on Weihrauch degrees: With $f \times g$ we denote the ability to make a call to f and an independent call to g. Having f^* means being able to make any given finite number of independent calls to f, whereas $\widehat{f}$ allows countably many parallel calls. Given $f \star g$, one can first use g, and then use the answer to choose the query for f. By $f^{(n)}$ we denote the n-th fold iteration of $\star$ on f. Finally, $f^{[n]}$ means that one does not have to provide the input to f explicitly, but merely a sequence converging to a sequence... (n-times) converging to the input to f (so $f^{[0]} = f$, $f^{[1]} = f'$ with f' as in [6]).

There is a zoo of Weihrauch degrees commonly appearing in the classification of theorems. Relevant for us are $\mathrm{C}_{\{0,1\}^{\mathbb{N}}}$, which takes a non-empty closed subset of Cantor space (i.e. an infinite binary tree) and produces a point in it (i.e. an infinite path through the tree), LPO, which decides whether a sequence is constant 0 or not, and lim, which computes the limit of a sequence in Cantor space.

2.2 Informal background on infinite sequential games

We use the formal definitions of sequential games and related concepts from [15] and [13]. Informally, given a fixed (wlog) set C, we let the players sequentially choose elements in C until an infinite sequence in C^ω is generated. Whose turn it is depends on the finite history of choices. The outcome (from a set O) of the game depends on the generated sequence in C^ω, and each player may compare outcomes via a binary relation over O, called preference. A strategy of a player is an object that fully specifies what the choice of the player would be for each possible finite history that requires this player to play. A combination of one strategy per player is called a strategy profile and it induces one unique infinite sequence in C^ω, and thus one unique outcome. So, preferences may be lifted from outcomes to strategy profiles. A Nash equilibrium is a profile such that no player can unilaterally change strategies and induce a (new) outcome that he or she prefers over the old one. We also consider a refinement of the concept of a Nash equilibrium, namely subgame-perfect Nash equilibria. Intuitively, a strategy profile is subgame-perfect, if it still forms an equilibrium if the game were started at an arbitrary history.

As a important special case we consider win/lose games. These are games with two players a, b and two outcomes w_a, w_b, where a prefers w_a to w_b and b

prefers w_b to w_a. We say that a wins the game, if outcome w_a is reached, and call the set of all plays that induce outcome w_a as the winning set for a (likewise for b and w_b).

2.3 Defining the problems of interest

Let Γ be a represented pointclass over $\{0,1\}^{\mathbb{N}}$. In a straightforward fashion, we can obtain a representation of the infinite sequential games with countably many agents, countably many outcomes, sets of choices $C = \{0,1\}$ and Γ-measurable valuation function $v : \{0,1\}^{\mathbb{N}} \to O$. The representation encodes the number of agents and outcomes available, for each upper set of outcome the Γ-set of plays resulting in it, the map d as a look-up table, and the relations $\prec_a$ as look-up tables. We always assume that the inverse of any preference relation is well-founded (this guarantees that equilibria exist). Using a canonic isomorphism $\{0,1\}^* \cong \mathbb{N}$, we will pretend that the space of strategy profiles in such a game is $\{0,1\}^{\mathbb{N}}$.

We now consider the following multivalued functions:

1. Det_Γ takes a two-player win/lose game as input, where the first player has a winning set in Γ. Valid outputs are the Nash equilibria, i.e. the pairs of strategies where one strategy is a winning strategy.
2. Win_Γ has the same inputs as Det_Γ, and decides which player (if any) has a winning strategy.
3. FindWS_Γ is the restriction of Det_Γ to games where the first player has a winning strategy.
4. NE_Γ takes as input a game with countably many players, finitely many outcomes, and linear preferences, where each upper set of outcomes (w.r.t. each player preference) comes from a Γ-set. The valid outputs are the Nash equilibria.
5. NE_Γ^{ap} is the restriction of NE_Γ to the two-player games with antagonistic preferences (i.e. $\prec_a = \prec_b^{-1}$).
6. SPE_Γ takes as input a two-player game with finitely many outcomes and antagonistic preferences, where each upper set of outcomes comes from a Γ-set. Valid outputs are the subgame perfect equilibria.

We abbreviate $\overline{\Gamma} := \{U^C \mid U \in \Gamma\}$. Some trivial reducibilities between these problems are: $\mathrm{Win}_\Gamma \equiv_{\mathrm{W}} \mathrm{Win}_{\overline{\Gamma}}$, $\mathrm{Det}_\Gamma \equiv_{\mathrm{W}} \mathrm{Det}_{\overline{\Gamma}}$, $\mathrm{FindWS}_\Gamma \leq_{\mathrm{W}} \mathrm{Det}_\Gamma \leq_{\mathrm{W}} \mathrm{NE}^{ap}_{\Gamma\cup\overline{\Gamma}} \leq_{\mathrm{W}} \mathrm{SPE}_{\Gamma\cup\overline{\Gamma}}$ and $\mathrm{NE}^{ap}_\Gamma \leq_{\mathrm{W}} \mathrm{NE}_\Gamma$.

Throughout the paper we assume that Γ is determined (which implies that all operations are well-defined in the first place), closed under rescaling and finite intersection with clopens, and that $\emptyset, \{0,1\}^{\mathbb{N}} \in \Gamma$. All such closure properties (including those appearing as conditions in the results) are assumed to hold in a uniformly computable way, e.g. given a name for a set in Γ and a clopen, we can compute a name for the intersection of the set with the clopen. With rescaling we refer to the operation $(w, A) \mapsto \{wp \mid p \in A\} : \{0,1\}^* \times \Gamma \to \Gamma$ and its inverse.

2.4 The difference hierarchy

The pointclasses we shall study in particular are the levels of the Hausdorff difference hierarchy. Intuitively, these are the sets that can be obtained as boolean combinations of open sets; and their level denotes the least complexity of a suitable term. Roughly following [12, Section 22.E], we shall recall the definition of the difference hierarchy. We define a function par from the countable ordinals to $\{0,1\}$ by $\operatorname{par}(\alpha) = 0$, if there is a limit ordinal β and a number $n \in \mathbb{N}$ such that $\alpha = \beta + 2n$; and $\operatorname{par}(\alpha) = 1$ otherwise. For a fixed ordinal α, we let $\mathfrak{D}_\alpha$ be the collection of sets D definable in terms of a family $(U_\lambda)_{\lambda<\alpha}$ of open sets via:

$$x \in D \Leftrightarrow \operatorname{par}\left(\inf\{\beta \mid x \in U_\beta\}\right) \neq \operatorname{par}(\alpha)$$

In the preceding formula, we understand that $\inf \emptyset = \alpha$. In particular, $\mathfrak{D}_0 = \{\emptyset\}$ and $\mathfrak{D}_1 = \mathcal{O}$.

For our constructions, a different characterization is more useful, though: For some pointclass Γ, let $\mathfrak{D}(\Gamma) := \{\bigcup_{i\in I} v_i U_i \mid \forall i,j \in I v_i \in \{0,1\}^* \wedge U_i \in \Gamma \wedge v_i \not\prec v_j\}$.

Lemma 1. *$\mathfrak{D}_{\alpha+1} = \mathfrak{D}(\overline{\mathfrak{D}_\alpha})$ and, more generally, $\mathfrak{D}_\alpha = \mathfrak{D}\left(\overline{\bigcup_{\lambda<\alpha} \mathfrak{D}_\lambda}\right)$.*

Observation 1. *If A_n is in $\mathfrak{D}_\alpha$ for all $n \in \mathbb{N}$, so is $A := \cup_{n\in\mathbb{N}} 0^n 1 A_n$.*

Corollary 1. *If B_n is in $\overline{\mathfrak{D}_\alpha}$ for all $n \in \mathbb{N}$, so is $B := \{0^\mathbb{N}\} \cup \bigcup_{n\in\mathbb{N}} 0^n 1 B_n$.*

A fundamental result on the difference hierarchy is the Hausdorff-Kuratowski theorem stating that $\bigcup_{\alpha<\omega_1} \mathfrak{D}_\alpha = \Delta^0_2$ (where ω_1 is the smallest uncountable ordinal), see e.g. [12, Theorem 22.27].

3 The computational content of some determinacy principles

We begin by classifying the simplest non-computable games, namely games where the first player wants to reach some closed set. This classification essentially is a uniform version of a result by Cenzer and Remmel [7].

Theorem 2. *$\mathit{FindWS}_\mathcal{A} \equiv_W \mathit{Det}_\mathcal{A} \equiv_W C_{\{0,1\}^\mathbb{N}}$.*

Proof. $\mathbf{C}_{\{0,1\}^\mathbb{N}} \leq_\mathbf{W} \mathbf{FindWS}_\mathcal{A}$ Given a closed subset $A \in \mathcal{A}(\{0,1\}^\mathbb{N})$, we can easily obtain the game where only player 1 moves, and player 1 wins iff the induced play falls in A. If A is non-empty, then player 1 has a winning strategy: Play any infinite sequence in A.

$\mathbf{FindWS}_\mathcal{A} \leq_\mathbf{W} \mathbf{Det}_\mathcal{A}$ Trivial.

$\mathbf{Det}_\mathcal{A} \leq_\mathbf{W} \mathbf{C}_{\{0,1\}^\mathbb{N}}$ Given the open winning set of player 2, we can modify the game tree by ending the game once we know for sure that player 2 will win. Now the set of strategy profiles where either player 1 wins and player 2 cannot win, or player 2 wins and player 1 cannot prolong the game, is a closed set effectively obtainable from the game. Moreover, it is non-empty, and any such strategy profile is a Nash equilibrium.

□

Proposition 1. *$Win_{\mathcal{A}} \equiv_W LPO$.*

Proof. This follows by combining the constructions from the preceding theorem with the fact that IsEmpty : $\mathcal{A}(\{0,1\}^{\mathbb{N}}) \to \{0,1\}$ is equivalent to LPO. □

We can use the results for $\mathcal{A}$ as the base case for classifying the strength of determinacy for the difference hierarchy.

Lemma 2 ([3]). *$Det_{\mathfrak{D}(\Gamma)} \leq_W C_{\{0,1\}^{\mathbb{N}}} \star (\widehat{Det_\Gamma \times Win_\Gamma})$ and $Win_{\mathfrak{D}(\Gamma)} \leq_W LPO \star \widehat{Win_\Gamma}$.*

We will relate deciding the winner and finding a winning strategy for games induced by sets from some level of the difference hierarchy to the *lessor limited principle of omniscience* and the *law of excluded middle* for Σ^0_n-formulae of the corresponding level. These principles were studied in [1, 6, 11] (among others). Let $\left(\Sigma^0_n - \text{LLPO}\right) :\subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \rightrightarrows \{0,1\}$ be defined via $i \in \left(\Sigma^0_n - \text{LLPO}\right)(p_0, p_1)$ iff $\forall k_1 \exists k_2 \ldots \natural k_n\ p_i(\langle k_1, \ldots, k_n\rangle) = 1$ (where $\natural = \forall$ if n is odd and $\natural = \exists$ otherwise). Let $\left(\Sigma^0_n - \text{LEM}\right) : \{0,1\}^{\mathbb{N}} \to \{0,1\}$ be defined via $\left(\Sigma^0_n - \text{LEM}\right)(p) = 1$ iff $\forall k_1 \exists k_2 \ldots \natural k_n\ p(\langle k_1, \ldots, k_n\rangle) = 1$ and $\left(\Sigma^0_n - \text{LEM}\right)(p) = 0$ otherwise. Then:

Proposition 2. *$\left(\Sigma^0_{n+1} - LLPO\right) \equiv_W LLPO^{[n]}$ and $\left(\Sigma^0_{n+1} - LEM\right) \equiv_W LPO^{[n]}$.*

Lemma 3. *$(\widehat{\Sigma^0_n - LLPO}) \leq_W Det_{\mathfrak{D}_n}$ and $\left(\Sigma^0_n - LEM\right) \leq_W Win_{\mathfrak{D}_n}$.*

Partial proof. The game for the first claim works as follows: The second player may pick some $k_1 \in \mathbb{N}$, or refuse to play. If the second player picks a number, then the first player may pick $k_2 \in \mathbb{N}$ or refuse to play. This alternating choice continues until k_{n-1} has been chosen, or a player refuses to pick. A player refusing to pick a number loses. If all numbers are picked, the winner depends on the input p to $\Sigma^0_n -$LEM as follows: If n is even and $\exists k_n\ p(\langle k_1, \ldots, k_n\rangle) = 1$, then player 1 wins. If n is odd, and $\exists k_n\ p(\langle k_1, \ldots, k_n\rangle) = 0$, then player 2 wins. Note that this always describes an open component U_{picked} of the winning set of the respective player.

Furthermore, note that the set of plays U_j where a value for k_j was chosen is always an open set. Now the condition that the second player refused to pick first is $U_1^C \cup (U_2 \cap U_3) \cup (U_4 \cap U_5) \cup \ldots$. This makes for a winning set in $\mathfrak{D}_n$, as required. If player 1 has a winning strategy in the game, the answer to $\left(\Sigma^0_n - \text{LEM}\right)(p)$ is 1, if player 2 wins, it is 0. □

Combining the results above yields:

Theorem 3. *$Det_{\mathfrak{D}_{n+1}} \equiv_W C^{[n]}_{\{0,1\}^{\mathbb{N}}}$ and $Win_{\mathfrak{D}_{n+1}} \equiv_W LPO^{[n]}$.*

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[3] This is a generalization of the proof idea for [20, Theorem 3.7] by Nemoto, MedSalem and Tanaka. [20, Theorem 3.7] states that ACA_0 proves determinacy for $\mathfrak{D}(\Sigma^0_1)$.

Knowing the Weihrauch degree of a mapping entails some information about the Turing degrees of outputs relative to the Turing degrees of inputs, this was explored in e.g. [4–6, 22]. Thus, we can obtain the following corollaries:

Corollary 2. *Any computable game with a winning condition in* $\mathfrak{D}_{n+1}$ *has a winning strategy* s *such that* s' *is computable relative to* $\emptyset^{(n+1)}$*, and there is a computable game of this type such that any winning strategy computes the* n*-th Turing jump of a completion of Peano arithmetic.*

Corollary 3. *Let* $(G_i)_{i\in\mathbb{N}}$ *be an effective enumeration of computable games with winning conditions in* $\mathfrak{D}_{n+1}$*, and define* $w \in \{0,1\}^{\mathbb{N}}$ *via* $w(i) = 1$ *iff the first player has a winning strategy in* G_i*. Then* $w \leq_T \emptyset^{(n+1)}$*, and there is an enumeration* $(G_i)_{i\in\mathbb{N}}$ *such that* $w \equiv_T \emptyset^{(n+1)}$*.*

Corollary 4. *There is a* Σ^0_{n+1}*-measurable function mapping games with winning conditions in* $\mathfrak{D}_n$ *to winning strategies, but no* Σ^0_n*-measurable such function.*

4 The complexity of equilibrium transfer

In [13–15], various results were provided that transfer Borel determinacy (or, somewhat more general, determinacy for some pointclass), to prove the existence of Nash equilibria (and sometimes even subgame-perfect equilibria) in multi-player multi-outcome infinite sequential games. In this section, we shall inspect those constructions and extract Weihrauch reductions from them.

In [14], the first author gave a very general construction that allows to extend determinacy of win/lose games to the existence of Nash equilibria for two-player games of the same type. For brevity, we only consider the strength of the toy example from [14] here:

Theorem 4 (Equilibrium transfer). $NE^{ap}_\Gamma \leq_W Det^*_\Gamma \times Win^*_\Gamma$.

Proof. For any upper set of outcomes (for either players preferences), we construct the win/lose derived game where that player wins, iff he enforces the set, and loses otherwise. There are finitely many such games, so we can use Win^*_Γ to decide which are won and which are lost. As shown in [14], there will be a combination of upper sets of outcomes for each player, such that if both players enforce their upper set, this forms a Nash equilibrium. We use Det^*_Γ to compute Nash equilibria for all derived games in parallel, and then simply select the suitable strategies. □

Techniques suitable for multiplayer sequential games were then introduced in [13], again by the first author. Again for brevity, we only consider the version with finitely many outcomes:

Theorem 5 (Constructing Nash equilibria). $NE_\Gamma \leq_W \widehat{Win_\Gamma} \times \widehat{Det_\Gamma}$.

A further improvement on the techniques in [13] were provided by the authors in [15]. These techniques in particular suffice to prove the existence of subgame-perfect equilibria in antagonistic games (this implies two players and finitely many outcomes).

Theorem 6. *$SPE_\Gamma \leq_W \widehat{Win_\Gamma} \times \widehat{Det_\Gamma}$.*

5 Deciding the winner and finding Nash equilibria

The results in Section 3 show that for many concrete examples of Γ, the problem Det_Γ is inherently multivalued, i.e. not equivalent to any functions between admissible spaces. On the other hand, the upper bounds provided in Section 4 all include Win_Γ, which is of course single-valued. In the current section, we will explore some converse reductions, from deciding the winner to finding Nash equilibria. This generally requires some (rather tame) requirements on the pointclasses involved.

Lemma 4. *Let Γ be obtained by Γ_1 by first closing under finite union, rescaling and union with clopens; and then adding complements. Then:*

$$Win^*_{\Gamma_1} \leq_W NE^{ap}_\Gamma$$

Proof sketch. Given n win/lose games, the first player starts by announcing which of these games she believes she can win. Then the second player can choose one of the listed games to play. If the first player did not claim any winnable games, the game ends and the outcome is 0. If the first player claimed to be able to win k out of n games, then the outcomes of the games subsequently chosen by the second player are scaled up to $k, -k$. Thus, the first player has every reason to list precisely those games she can actually win: If she would not list a game she could win, she trades payoff $k-1$ for payoff k. If she lists a game she cannot win, the second player will subsequently chose and win it, and then the first player is punished by $-k$. □

Lemma 5. *Let Γ be closed under taking unions with Γ_1 and $\overline{\Gamma_1}$. Then:*

$$NE^{ap}_\Gamma \times FindWS_{\Gamma_1} \equiv_W NE^{ap}_\Gamma$$

Lemma 6. *Let Γ be obtained by Γ_1 by first closing under finite union, rescaling and union with clopens; and then adding complements. Then:*

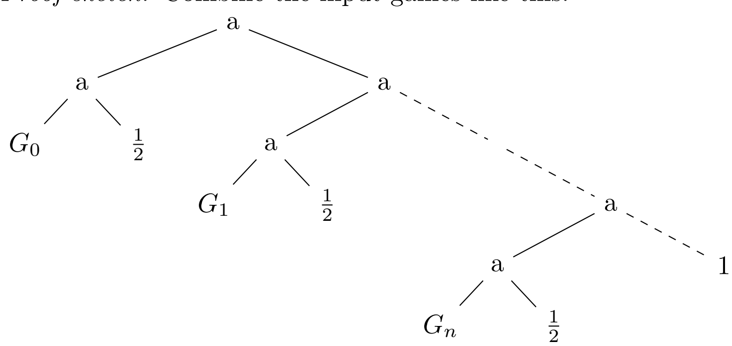
$$FindWS^*_{\Gamma_1} \times Win^*_{\Gamma_1} \leq_W NE^{ap}_\Gamma$$

If we have access to subgame perfect equilibria (and are in a context where they are guaranteed to exist), then we can even decide the winner of countably many games in parallel:

Lemma 7. *Let Γ_1 contain the closed sets and be closed under finite unions and the operation $(A_n)_{n\in\mathbb{N}} \mapsto \left(\{0^{\mathbb{N}}\} \cup \bigcup_{n\in\mathbb{N}} 0^n 1 A_n\right)$. Let Γ be obtained from Γ_1 by closing under complements. Then:*

$$\widehat{Win_{\Gamma_1}} \leq_W SPE_\Gamma$$

Proof sketch. Combine the input games like this:



□

6 General games with concrete pointclasses

The general constructions put together with the classifications for specific pointclasses allow us to obtain some concrete Weihrauch degrees. First, we shall see that moving from a win/lose game with closed and open outcomes to a two-player game with several outcomes just complicates the operation of finding Nash equilibria by finitely many uses of LPO in parallel:

Theorem 7. $NE^{ap}_{\mathcal{O}\cup\mathcal{A}} \equiv_W C_{\{0,1\}^{\mathbb{N}}} \times LPO^*$.

Proof. For the reduction $\mathrm{NE}^{ap}_{\mathcal{O}\cup\mathcal{A}} \leq_\mathrm{W} \mathrm{C}_{\{0,1\}^{\mathbb{N}}} \times \mathrm{LPO}^*$, instantiate Theorem 4 with the results from Theorem 2 and Proposition 1.

For the other direction, note that $\mathrm{FindWS}_{\mathcal{A}} \equiv_\mathrm{W} \mathrm{FindWS}_{\mathcal{O}\cup\mathcal{A}} \equiv_\mathrm{W} \mathrm{C}_{\{0,1\}^{\mathbb{N}}}$ as in Theorem 2; and that $\Gamma_1 := \mathcal{A}$ and $\Gamma := \mathcal{O} \cup \mathcal{A}$ satisfy the requirements of Lemma 6, which then provides the desired result. □

The result can actually be strengthened into the following (by noting that the second game constructed in Lemma 3 is always won by the first player):

Theorem 8. $NE^{ap}_{\mathfrak{D}_{n+1}\cup\overline{\mathfrak{D}_{n+1}}} \equiv_W C^{[n]}_{\{0,1\}^{\mathbb{N}}} \times \left(LPO^{[n]}\right)^*$.

If one wishes to have subgame-perfect equilibria instead of mere Nash equilibria, then countably many uses of LPO become necessary, and the problem becomes equivalent to lim. Note that as long as there are at least three distinct outcomes, the number of outcomes has no further impact on the Weihrauch degree (due to the nature of the construction used to prove Lemma 7)– unlike the situation in Theorem 7, where the number of outcomes is related to the number of times that LPO is used.

Theorem 9. $SPE_{\mathfrak{D}_n \cup \overline{\mathfrak{D}_n}} \equiv_W \lim^{(n)}$.

Proof. For $\mathrm{SPE}_{\mathfrak{D}_n \cup \overline{\mathfrak{D}_n}} \leq_{\mathrm{W}} \lim^{(n)}$, instantiate Theorem 6 with the results from Theorem 3, and note that $\widehat{\mathrm{LPO}^{(n)}} \equiv_{\mathrm{W}} \lim^{(n)}$ and $\mathrm{C}^{[n]}_{\{0,1\}^{\mathbb{N}}} \leq_{\mathrm{W}} \lim^{(n+1)}$.

For the other direction, we use Lemma 7 (applicable by Corollary 1) together with Proposition 1. □

Regarding Theorem 5, we do not (yet?) have matching lower bounds for any particular pointclass. The gap is exemplified by the following:

Corollary 5. $C_{\{0,1\}^{\mathbb{N}}} \times LPO^* \leq_W NE_{\mathcal{O} \cup \mathcal{A}} \leq_W \lim$.

7 Conclusions and Outlook

With Theorem 3, we have shown that the computational strength of determinacy provides a tight connection between the difference hierarchy and the Borel hierarchy (in form of Corollary 4). Note that winning sets from the difference hierarchy correspond to Boolean combinations of reachability and safety conditions. Corollary 3 then provides an upper bound and a worst case for corresponding decidability questions for logic. Theorem 3 also shows that the computational powers of the players required to find a winning strategy vastly exceeds the computational power required to determine the outcome, thus casting doubt on the adequateness of winning strategies (or Nash equilibria) as adequate solution concepts for infinite sequential games.

The results in Section 4 contrasted with those in Section 6 essentially show that the proofs in [13–15] are not too wasteful from a constructive perspective – i.e. the constructions employed are not far less constructive than the theorems proven with them.

There are several immediate avenues for extending the work presented here: The restriction to finite action sets (i.e. finitely branching trees) can mostly be lifted without a significant impact on the proof techniques. Note though that the concrete Weihrauch degrees would change drastically, as in Theorem 2 we would need to replace $\mathrm{C}_{\{0,1\}^{\mathbb{N}}}$ by $\mathrm{C}_{\mathbb{N}^{\mathbb{N}}}$, with the latter residing in a less explored part of the Weihrauch lattice. The results in [15] are more general than covered here, too (with the same proof complexity).

The study of the strength of determinacy for particular pointclasses in reverse mathematics presumably offers further proofs adaptable into the framework of Weihrauch reducibility, e.g. [8, 18, 19].

Further afield, understanding the Weihrauch degrees of determinacy principles may be a contribution to the development of descriptive set theory in computational/category-theoretical terms as suggested in [23].

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