WELFARE DOMINANCE: AN
APPLICATION TO COMMODITY TAXATION

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Welfare Dominance: An Application to Commodity Taxation

ABSTRACT

In this paper, we suggest a method which enables the user to identify

commodities that all individuals who can agree on certain weak assumptions

with regard to the social welfare function will agree upon as worth

subsidizing or taxing in the absence of efficiency considerations. The method

is based on an extension of the stochastic dominance criteria and is

illustrated using data from Israel.

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I. Introduction

A principal weakness of the theory of optimal taxation with heterogeneous taxpayers is the dependence of the optimum tax rates on the exact properties of the social welfare function. While other components of the problem, such as the excess burden of the tax system, can presumably be recovered from empirical observations on the behavior of consumers, it is clear that estimating the social welfare function is not an easy task. Although there have been several attempts to recover the social welfare function using the revealed preferences of governments, (usually by assuming that governments are acting optimally according to a "just" principal of taxation, such as equal sacrifice), all of these methods require very strong assumptions which are unlikely to command wide support. [1]

This problem is especially disturbing for developing countries which rely heavily on commodity taxes as a major policy instrument for raising revenue and changing the income distribution. In most of these countries data is not available even for estimating the excess burden of the tax system. Therefore, it seems that the theory of optimal commodity taxation is not very helpful as an input into policy formulation in this context. This problem is even more complicated from the perspective of an economic adviser whose role is to advise the government of a specific country. If, in an ideal case, the government can supply him with all the necessary data, then the "cost" or the

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1. See Musgrave (1959), Mera (1969), Weisbrod (1968), Piggott (1982), Yaari (1986).

inefficiency caused by the tax system can be estimated. However, in order to make recommendations about optimal tax design, the adviser must be aware of the preferences of the government (or the society) involved. Without this information the advice rendered represents only the adviser's preferences, which need not be the same as the preferences of the government seeking advice. Hence, it will be convenient to see whether it is possible to overcome this difficulty by statistical analysis.

The aim of this paper is to suggest a method which enables the user to identify commodities that all individuals who can agree on certain weak assumptions with regard to social policy will agree upon as worth subsidizing or taxing in the absence of efficiency considerations. If such commodities can be identified, then the task of advising the government on optimal directions of tax reform will be rendered nearly value-free. In other words, all individuals would agree that the taxation of one commodity should be reduced in favor of heavier taxation on another, if the marginal excess burden is equal for the two commodities.

The specific question this paper addresses is the following: assume that the government wants to make an equal yield change in its commodity tax system by subsidizing one commodity and taxing another commodity by an additional dollar -- is it possible to identify two such commodities such that social welfare increases for all concave Paretian social welfare functions? If such situations can be identified, the preferred direction of tax reform can be located without detailed knowledge of preferences regarding inter- personal transfers. Alternatively, if such commodities cannot be found, it

will be clear that it is impossible to make recommendations in the absence of further information about the governing social welfare function.

The methodology which enables us to answer this question was originally developed in the finance literature, where it is referred to as the Second-Degree Stochastic Dominance Criteria. The main idea is to assume that the criteria which is used to rank prospects (portfolios) is expected utility, and that the investigator assumes only that the marginal utility of income is non-negative and non-increasing. Based on these assumptions, rules for ranking prospects have been developed (e. g., Hadar and Russell (1969), Hanoch and Levy (1969)).

Our intention is to use the methodology of stochastic dominance for ranking taxes on different commodities. As has been demonstrated by Atkinson (1970), there is a formal similarity between the ranking of income distributions and the ranking of prospects. Hence, the use of stochastic dominance rules in welfare economics is a natural development following from Atkinson's observation. However, since taxation and in particular commodity taxation affect social welfare in a slightly different way than the effect of a prospect on the utility function, several changes must be made to the methodology; we refer to these adapted rules as welfare dominance. [2] The major changes are the following:

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2. This term was coined by Shorrocks (1983).

(a)   In the finance literature, the main interest is to rank portfolios, the analogy to which in our study is the ranking of income distributions. Our goal, though, is the ranking of (taxes on) commodities, expenditure on which is a component of total income. Therefore, we have to use conditional stochastic dominance rules, which can be translated into the finance liter- ature as asking whether asset A dominates asset B given that the investor has also to hold portfolio C. In the case of welfare dominance, the same formal question can be interpreted as whether subsidizing expenditure on commodity A instead of expenditure on commodity B improves welfare, given that the income distribution is C.

(b)   We are interested in dominance at the margin. The analogy to finance is whether a small increase in the share of asset A at the expense of asset B (given that the rest of the portfolio held is C) increases expected utility. In the case of taxation, the same formal question can be interpreted as whether a small decrease in tax on A financed by a small increase in tax on B, (given that the income distribution is C), increases expected welfare.

As we show in the next section, this question can be answered by comparing concentration curves. The concentration curve is a diagram which is similar to the Lorenz curve. On the horizontal axis the households are ordered according to their income, while the vertical axis describes the cumulative percentage of the total expenditure on a specific commodity that is spent by the families whose incomes are less than or equal to the specified income level. The concentration curve, like the Lorenz curve, passes through the origin. But, unlike the Lorenz curve, it need not always be increasing,

and its curvature depends on the income elasticity of the commodity. In particular, if the curve is convex (concave) to the origin then the income elasticity is negative (positive). [3]

If the concentration curve of one commodity is above the concentration curve of another commodity, then the first commodity dominates the second. However, if the concentration curves intersect, then it is impossible to show dominance. In other words, only if concentration curves do not intersect will all social welfare functions show that the tax change increases welfare. [4] We refer to these rules as Marginal Conditional Stochastic Dominance rules (hereafter MCSD rules) and in the rest of the paper, dominates stands for marginal conditional stochastic dominates).

The structure of the paper is the following: the next section provides an intuitive proof for MCSD rules. In the third section, additional insight is gained by relating these rules to a methodology based on the decomposition of the Gini coefficient. Section IV uses data from Israel in order to illustrate the methodology. The paper concludes with suggestions for further research.

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3. For a detailed analysis of the curvature of concentration curves, see Kakwani (1981) and Yitzhaki and Olkin (1987).

4. It is worth noting that concentration curves have been used to describe the progressivity of taxes by many investigators. See, among others, Suits (1977a, 1977b), Clotfelter (1977), Kakwani (1977,1981,1984), Kiefer (1984), Rock (1983), Formby, Smith and Sykes (1986). However, the use of concentration curves to identify tax changes which are welfare dominating is new.

II. Intuitive Derivation of the Methodology. [5]

Assume that tax policies are evaluated according to an additively separable social welfare function, which is the sum of identical individual utility functions. All that is known about the social welfare function is that the marginal utility of income is positive and declining. Formally,

$$W = \Sigma_i V(I_i, P_1, \ldots P_n)$$

where I is income and P are prices and all that is known about V is

that $\frac{\partial V}{\partial I} > 0$ and $\frac{\partial^2 V}{\partial I^2} < 0$.

Now suppose that the government is considering a small increase in the tax on commodity i and a small decrease in the tax on commodity j so that total revenue does not change. Let x_i^h denote the consumption of commodity i by the h_{th} individual, where the individuals are arranged in a non-decreasing order of income. Let X_i denotes total consumption of commodity i. Since the change in revenue raised is zero, there is a link between the change in prices of commodity i and j. Assuming no change in the quantities consumed or in the producer prices, the relationship is:

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5. In Yitzhaki and Olkin (1987), Marginal Conditional Stochastic Dominance rules are developed in the context of portfolio analysis. Since the proofs are identical to those required in this paper, we refer the interested reader to that paper. In this section an intuitive proof is given.

$$dP_i \ X_i \ + \ dP_j \ X_j \ = \ 0$$

hence

$$(1) \qquad dP_i \ = \ - \ \{X_j / \ X_i\} \ dP_j$$

Consider the welfare of the poorest person in the society. Denote his consumption of commodity i and commodity j by x_i^1 and x_j^1. A necessary condition that all social welfare functions show that the suggested reform is welfare increasing is that the reform does not worsen the utility of the poorest individual. (Otherwise a Rawlsian decision maker will judge that the suggested reform decreases welfare). The effect of the reform on the welfare of the poorest in the society depends on the compensation needed to allow him to have the same utility as before. That is, it depends on the sign of

$$(2) \qquad x_i^1 \ dP_i \ + \ x_j^1 \ dP_j$$

and if (2) is negative, then the welfare of the poorest person has been increased as a result of the reform.

Substituting for dP_i from equation (1), we get:

$$(3) \qquad [x_j^1 \ / \ X_j \ - \ x_i^1 \ / \ X_i] \ X_j \ dP_j$$

Since X_j is positive, and dP_j is negative by assumption, the sign of (3) depends on whether the term in the square brackets is positive. But this expression is the difference between the poorest's individual's share of consumption of commodity j to his share of commodity i. Hence, a necessary

condition that all social welfare functions show an increase in social welfare
is that the share of the expenditure of the poorest individual on commodity i
is lower than his share in the expenditure on commodity j. Having established
the condition for a socially acceptable reform with regard to the poorest
individual, we next check the condition applying to the second individual.
The marginal utility of income of the second individual is lower than the
marginal utility of income of the first individual, hence all social welfare
functions will show an increase in social welfare if the gain in income for
the first individual is higher than the loss for the second individual. By
repeating the same considerations which led to (3), the condition with regard
to the second individual is:

$$(4) \qquad \{[x_j^1 + x_j^2]/X_j - [x_i^1 + x_i^2]/X_i\} \ X_j \ dP_j < 0$$

and the condition for the k_{th} individual is

$$(5) \qquad \{\Sigma_{h=1}^{k} \ x_j^h/X_j - \Sigma_{h=1}^{k} \ x_i^h/X_i\} \ X_j \ dP_j < 0$$

Condition (5) is easy to interpret. In brackets is the difference
between the relative concentration curve of commodity i and the relative
concentration curve of commodity j. The condition implies that for all social
welfare functions to show that this tax change increases social welfare it is
necessary that the concentration curve of commodity i with respect to income
is at least as high as the concentration curve of commodity j at each point in
the income distribution. Condition (5) is also a sufficient one because if
the condition is violated it is possible to construct a concave Paretian
welfare function which indicates that the tax reform decreases social welfare.

Note that condition (5) does not constitute a complete ordering of (commodity) taxes. That is, if two concentration curves intersect, then it will be impossible to find a small change in the taxation of the commodities involved that all social welfare functions judge to be an increase in welfare. If we insist on having a complete ordering, then we have to further investigate the concentration curves to see whether restricting the class of social welfare function enables us to classify policies. This issue is discussed in the next section.

Although the ordering is not complete, it is clearly transitive: if commodity A dominates commodity B and commodity B dominates C, then A dominates C. $\underline{6/}$ This property enables us to establish an ordering among the subgroup of commodities in which dominance is found.

Figure 1 presents some typical cases. On the horizontal axis of each graph the cumulative distribution of income is plotted while the vertical axis represents the difference between the concentration curve of commodity A and the concentration curve of commodity B. In other words, on the vertical axis we present the difference between the share of total expenditure on A and the share of total expenditure on B for the poorest F families. All of the "difference in concentration curves" (DCC_{ij} curve hereafter, where the indices indicate the commodities) include the origin and end up at [0,1]. If the curve is above the horizontal axes then A dominates B, while if it is entirely below the horizontal axes then B dominates A. If the curve crosses the

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6. To see this, notice that if the concentration curve of A is everywhere above the concentration curve of B, and the concentration curve of B is everywhere above the concentration curve of C, then it is clear that the concentration curve of A is everywhere above the concentration curve of C.

horizontal axes then neither B dominates A nor A dominates B. In the next section we show that the area between the difference in concentration curves and the horizontal axis is equal to the difference in the income elasticities of the two commodities.

Until now, we have been interested in an increase of the subsidy to one commodity that is financed by an increase of the tax on another commodity. The same methodology can be used to ask whether an increase in a subsidy that is financed by a proportional income tax increases welfare. In this case we treat total income as a commodity and look at the difference between the concentration curve for commodity i and the Lorenz curve. If the DCC_{i0}, where the index 0 indicates the Lorenz curve, is always positive, then commodity i dominates the proportional tax.

III. Restricting the Social Welfare Functions.

As noted above, the MCSD rules do not form a complete ordering over the taxable commodities. Moreover, we suspect that there are many commodities that will be impossible to order by this method. In these cases one may wish

to investigate further restrictions on the set of possible welfare functions, such as third degree stochastic dominance rules. [7]/

The main problem with such rules is that their use does not ensure a complete ordering in all cases. Therefore, their use may increase the set of commodities over which an ordering is defined, but it does not eliminate the problem of incomplete ordering. An alternative way that ensures a complete ordering is to restrict ourselves to necessary conditions for stochastic dominance. This is the case where the analysis of tax reforms is carried out with the use of the Gini coefficient (Yitzhaki (1987)). This procedure provides a complete ordering, however, it will be impossible to show that the suggested ordering may be rejected by all social welfare functions. Another approach which ensures a complete ordering is to use a specific welfare function, but then we may abandon the connection to MCSD rules.

The area below the forty-five degree line minus the area below the concentration curve is defined as one-half of the concentration ratio. As shown in Yitzhaki and Olkin (1987), this area is also equal to:

(6) $C_i = cov(X_i, F(Y)) / m_i$

where C_i is one half of the concentration ratio, m_i is the mean expenditure on commodity i, and $F(Y)$ is the cumulative distribution of income. In other words, the concentration ratio is equal to twice the covariance between the expenditure on commodity i and the cumulative distribution of income divided

7. For a definition of third degree stochastic dominance, see Whitmore (1970).

by the mean expenditure on commodity i. Hence the area between the

concentration curve of commodity i and the concentration curve of commodity j

(i. e. between the DCC_{ij} curve and the horizontal axis) is:

$$(7) \qquad C_i - C_j = cov(X_i, F(Y))/m_i - cov(X_j, F(Y)) / m_j$$

By dividing and multiplying equation (7) by $cov(Y, F(Y))$ and m_Y, respectively,

we can rewrite (7) as:

$$(8) \qquad \int_0^1 DCC_{ij}(F) \, dF = \{b_i/S_i - b_j/S_j\} G_Y$$

where G_Y is the Gini coefficient of income, S_i is the share of the

expenditure on X_i and

$$(9) \qquad b_i = cov(X_i, F(Y))/cov(Y, F(Y))$$

is Siever's (1983) non-parametric estimator of the slope of the regression

line of X_i on Y. In our context b_i is a weighted mean of the marginal

propensity to spend on commodity i. As argued in Yitzhaki (1987), b_i/S_i can

be interpreted as the weighted average income elasticity of commodity i.

Hence equation (9) tells us that the sign of the area below the DCC_{ij} curve is

determined by the difference between the weighted average income elasticities

of the commodities. Because for commodity i to dominate commodity j DCC_{ij}

must be positive, it is clear that a necessary condition (but, of course, not

sufficient) for welfare dominance is that the income elasticity of commodity i is lower than the income elasticity of commodity j.

One can repeat the same argument using the extended Gini. The extended Gini is a weighted integral of the area between the forty-five degree line and the Lorenz curve. The formula for the extended Gini is:

$$(10) \qquad G_v(Y) = - v \ cov(Y, [1-F(Y)]^{v-1}) \ / \ m_Y \qquad v>1$$

where v is a parameter chosen by the investigator. The extended Gini is similar to the Gini coefficient except that it uses a different weighting scheme. The Gini is a special case of the extended Gini where v is 2. The higher is v, the more the bottom of the income distribution is stressed. [8]

The above analysis using the Gini can be carried out using the extended Gini. In the appendix we show that an increase in the subsidy to commodity i that is financed by an increase in the tax on commodity j decreases the extended Gini inequality index, if

$$(11) \qquad \int_0^1 \{\phi_i(F) - \phi_j(F)\} \ (1-F)^{v-2} \ dF > 0,$$

where $\phi(F)$ is the concentration curve. Since $\phi_i(F) - \phi_j(F)$ is the DCC_{ij} curve, then the analysis of tax reform with the extended Gini coefficient provides additional necessary conditions for welfare dominance. If commodity i dominates commodity j, then a shift from taxing commodity i to commodity j

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8. See Yitzhaki (1983) for a discussion of the properties of the extended Gini.

must decrease the extended Gini inequality index for all v, including the standard Gini case where v=2. These necessary conditions are useful in empirical investigation of welfare dominance because they are fairly easy to calculate and can be used to identify the pairs of commodities for which welfare dominance is possible.

IV. An Illustration with Israeli Data.

This section illustrates the methodology of MCSD rules using cross-section data on consumption of subsidized commodities in Israel. The data set is the Survey of Family Expenditure (1979/80) conducted by the Central Bureau of Statistics. This Survey consists of a random stratified sample of 2271 urban households. Since we are interested in level of economic well-being, the concept of income per standard adult is used. [9] The households are ordered according to total net income per standard adult, where total net income is defined as monetary income plus imputed income from ownership of housing and vehicles minus income and social security taxes. [10]

Before presenting the results, two technical points are worth

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9. The concept of standard adult is an equivalence scale intended to take into account the effect of the size of the household on consumption needs. The scale that is used is the following: Single 1.25 standard adults, a couple without children 2.0, a couple with one child 2.65, with two children 3.2, with three children 3.75, with four children 4.2, and .4 for each additional child. It is used in many official publications of the National Insurance Institute and the Central Bureau of Statistics of Israel.

10. The sample is a weighted sample, which means that each household in the sample represents a different number of households in the population. The equations given in this paper have therefore been adjusted in a straight-forward way to account for this, See Lerman and Yitzhaki (1986)

making. In general, if there are n commodities, one would have to plot n • (n-1)/2 pairs of concentration curves to investigate the existence of welfare dominance. Since cross-section samples usually contain thousands of observations, this can clearly become a cumbersome procedure when more than a few commodities are being studied. As suggested above, comparisons of the magnitude of the weighted income elasticities, according to the Gini or extended Gini coefficient, yield necessary conditions for dominance, thus reducing the number of comparisons of curves needed. The second issue arises because of the use of a sample instead of the whole population. As shown by Goldie (1967), sample-based Lorenz curves do converge to the population Lorenz curve, but it is clear that our results may be affected by the sample variability. One way to reduce the sample variability is to average observations. Hence in what follows we plot concentration curves based on the whole sample and also report the results when concentration curves are based on averaging consecutive pairs of observations.

Table 1 presents weighted average income elasticities, estimated by using several variants of the extended Gini coefficient. As can be seen, there are two inferior commodities, bread and cooking oil, while the other commodities are normal.

As mentioned above, the higher is v the more weight that is given to the lower portion of the income distribution. Hence, we can conclude from the table that the income elasticity of public transportation declines as income

Table 1: Income Elasticities of Several Subsidized Commodities
Israel, 1979/80

Gini parameter	v = 1.5	v = 2	v= 4
Cooking Oil	-.13	-.14	-.11
Bread	-.06	-.07	-.09
Sugar	.07	.06	.08
Public Transportation	.12	.15	.25
Water for Household Consumption	.31	.30	.33

increases while bread tend to be less inferior, the higher the income. Since the income elasticity of cooking oil is lower than the income elasticity of bread for all v, the necessary conditions for cooking oil to dominate bread are met, and it is reasonable to check whether cooking oil dominates bread. By the same reasoning we conclude from Table 1 that, if there is dominance, then the ordering must be: cooking oil, bread, sugar, public transportation and finally water for household consumption. It is impossible that a good will be dominated by another good lower in the table.

Figure 2 presents the curve of the difference between the concentration curves (DCC curve) of cooking oil and bread. As expected from Table 1, the area above the X-axis is larger than the area below it. (The difference between the areas is equal to the income elasticity of cooking oil minus bread calculated by the Gini index). Hence, we can see that the cooking oil is more inferior than bread when the weighting scheme of the Gini is used. However, the curve intersects the X-axis. This implies that for the lower two deciles of the income distribution, bread has a lower income elasticity than cooking oil, while for the other deciles bread has a higher income elasticity. If the social welfare function is concave only for the lower two deciles then taxation policy should shift to subsidize bread at the expense cooking oil. The conclusion is that cooking oil does not dominate bread.

Figure 3 presents the DCC curve between bread and water for household consumption. As can be seen, the curve is always above the X-axis, and hence bread dominates water. [11]

Figure 4 presents the DCC of bread and public transportation. As in Figure 3, except for one observation, bread dominates public transportation.

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11. A close examination of the individual observations reveals that the curve intersects the X-axis for the very last observation, so that the share of the richest household in the sample, in total expenditure of bread, is higher than his share in the overall expenditure in water. If this sample exactly portrayed the population, a social welfare function that is linear over the whole range of the distribution and concave between the second richest family and the richest one would show that subsidizing bread at the expense of water would be welfare decreasing. However, we suspect that this is a result of the sampling error. If we take the average of two consecutive observations, then this proviso disappears.

If we plot the DCC averaging every two consecutive observations, then bread dominates public transportation with no exceptions.

This phenomenon illustrates just how strong the condition of MCSD is. Even if the entire population is studied, the consumption patterns of the richest and poorest members of society are critical in determining the existence of welfare dominance.

Figure 5 presents the DCC of bread minus the Lorenz curve. This figure is intended to show whether an increase in the subsidy on bread that is financed by an increase in proportional income tax increases welfare for all additive concave Paretian social welfare functions. Because the DCC curve is above the X-axis over the whole income range, the answer to this question is yes.

V. Conclusions.

Since the exact properties of social welfare functions are not known, and it is doubtful whether they can be recovered by future research, it would be valuable to make judgments about potential tax reforms that depend only on uncontroversial characteristics of the social welfare function. The methodology provided in this paper is a first step in this direction. It states the conditions (called marginal conditional stochastic dominance rules) required for all individuals with Paretian concave social welfare functions to agree that an increase in the subsidy on one commodity, which is financed by an increase of the tax on another commodity (or a proportional income tax),

increases social welfare. If this reform does not increase excess burden, then all individuals will agree on the preferred direction of tax reform. An inspection of Israeli data suggests that these conditions are quite commonly observed in practice, making this a practically relevant point.

One direction for future research is to apply this methodology to a weighted combination of commodities. In this case a set of changes in taxes and subsidies can be compared to another set in order to see whether all individuals can agree on the welfare implications of particular tax reforms. Another interesting challenge is to introduce efficiency considerations into the analysis. In the present setting of MCSD rules, the permissable set of welfare functions includes welfare functions which are almost linear (constant marginal utility). The permissability of such welfare functions means that it will be impossible to find unanimous preference for any redistribution that increases efficiency costs. Therefore only by limiting the set of the admissible welfare functions will it be possible to find MCSD rules for costly redistribution. One possible way of doing that is to restrict the set such that only welfare functions with some minimum concavity are included.

Figure 1: Alternative Dominance Situations
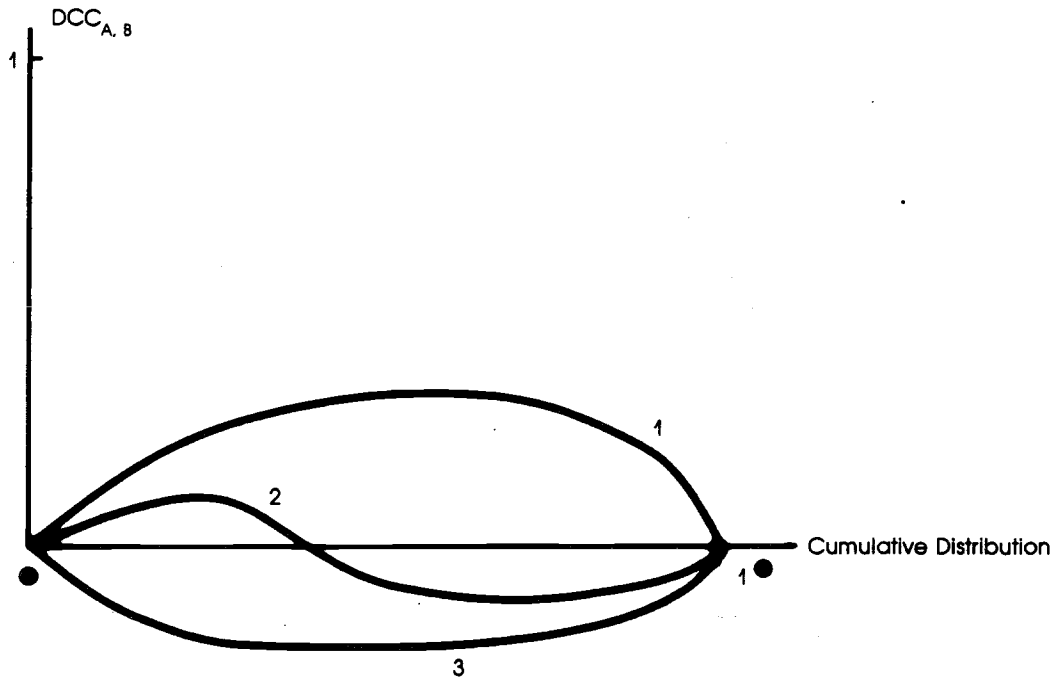
1. A dominates B
2. No dominance
3. B dominates A

Figure 2

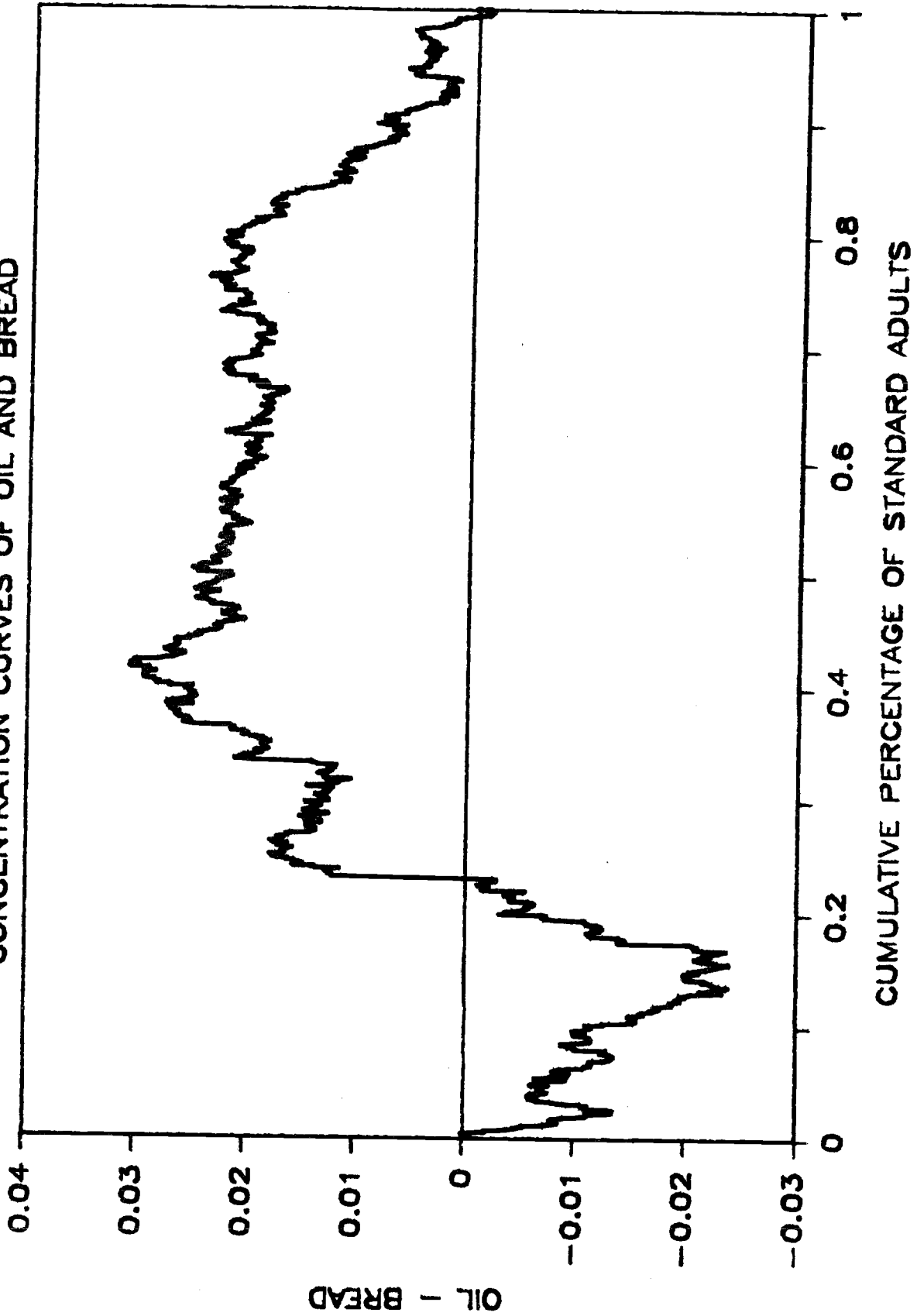THE DIFFERENCE BETWEEN
CONCENTRATION CURVES OF OIL AND BREAD

Figure 3

THE DIFFERENCE BETWEEN
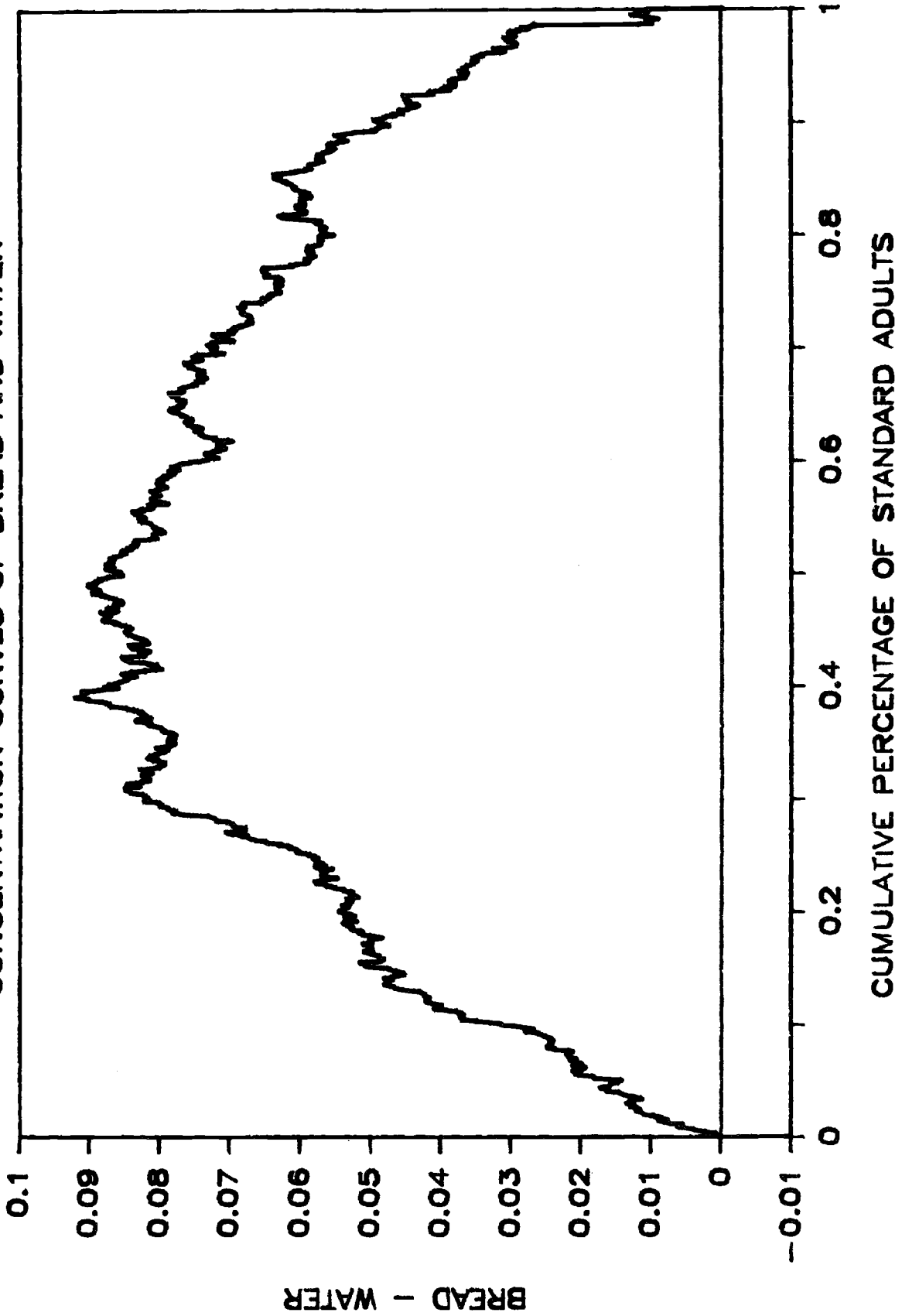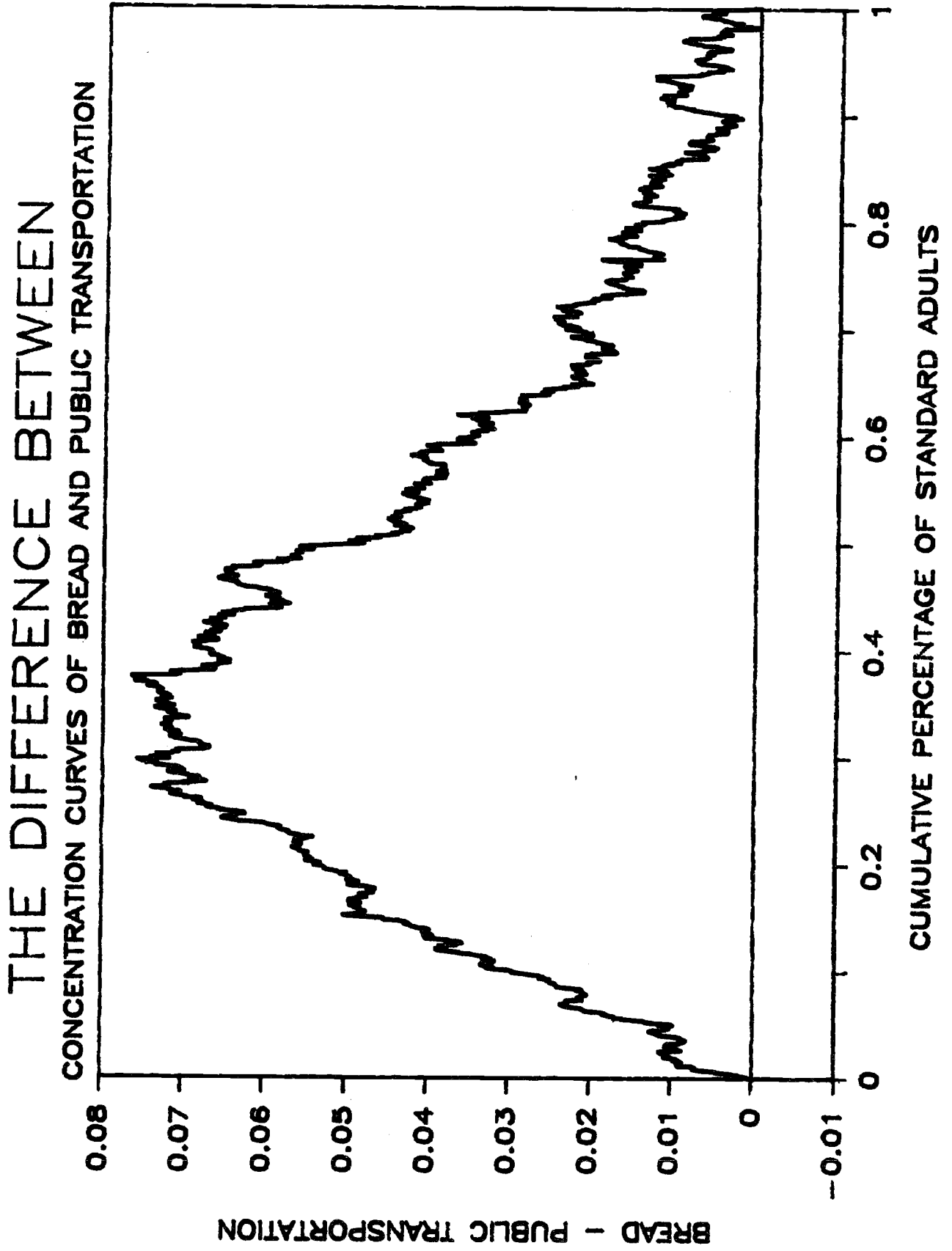## CONCENTRATION CURVES OF BREAD AND WATER

Figure 4

THE DIFFERENCE BETWEEN
CONCENTRATION CURVES OF BREAD AND PUBLIC TRANSPORTATION

Figure 5

THE DIFFERENCE BETWEEN
CONCENTRATION CURVES OF BREAD AND LORENZ

CUMULATIVE PERCENTAGE OF STANDARD ADULTS

BREAD — LORENZ
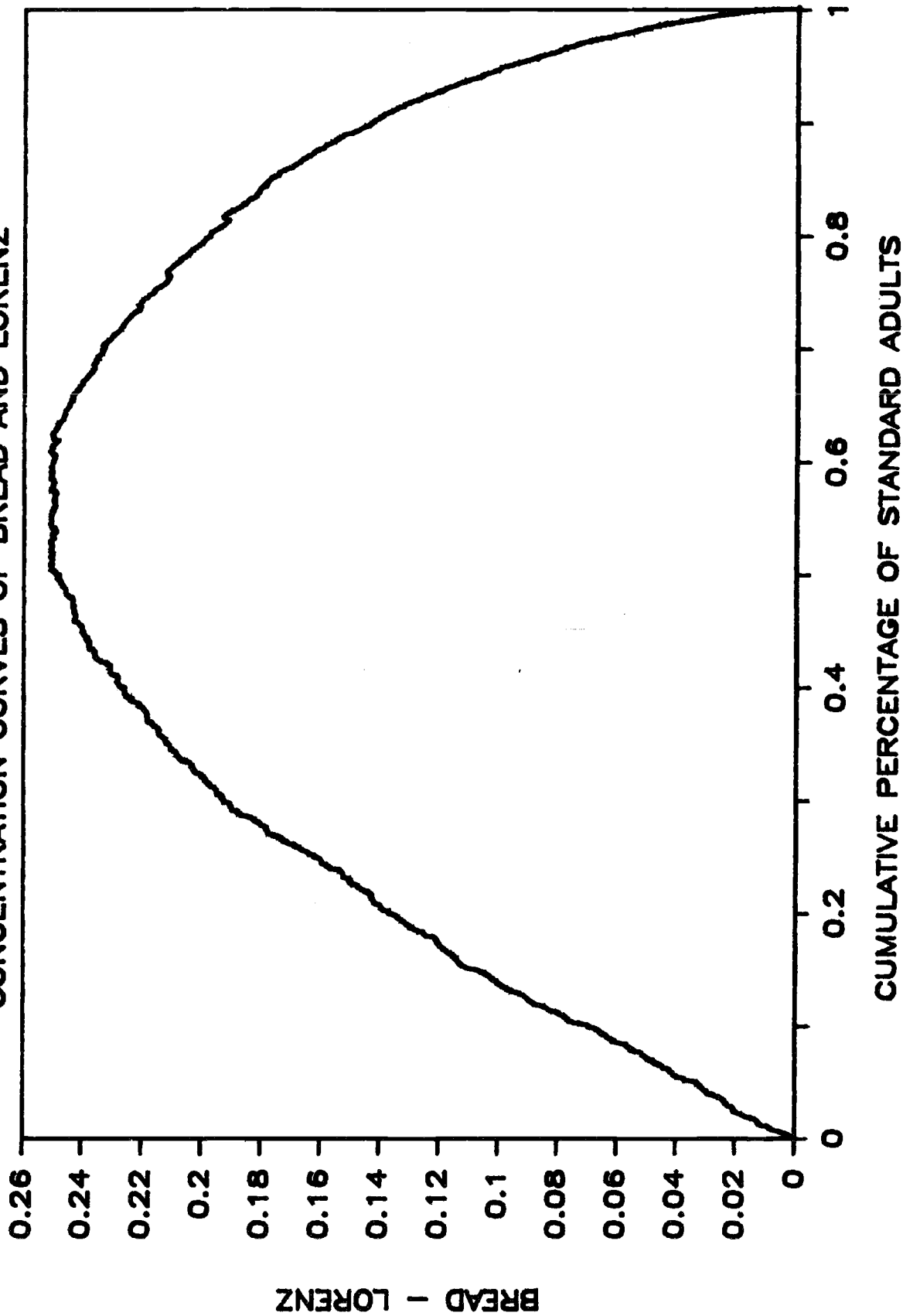
Appendix

In this appendix we prove the following results:

a. The derivative of the extended Gini of overall income with respect to constant revenue tax changes is equal to the difference in the (weighted) concentration ratio of the commodities involved in the tax changes.

b. The (weighted) concentration ratio is equal to the (weighted) area below the concentration curve.

Properties (a) and (b) together with equation (2) ensure that a necessary condition for a commodity to dominate another commodity is that the extended Gini of inequality declines as a result of the tax changes.

Proof of property (a)

For simplicity of presentation, we assume continuous distributions, and two commodities

(A.1)   Let $y = p_1 x_1 + p_2 x_2$

where $p_i = 1$ is the price of commodity i, y is income and x represents commodities.

The extended Gini of inequality is

(A.2) $G_y(v) = - v \text{ cov } (y, [1-F(y)]^{v-1}) / m_y$

where m_y is mean income.

The derivative of $G_y(v)$ with respect to p_1, evaluated at $p_1 = 1$ can be interpreted as the effect of a change in the tax on commodity 1 on the extended Gini. Note that p_1 affects both the nominator and denominator of (A.2). Let us start with the derivative of the nominator.

Let $A(y) = 1-F(y)$.

Note that

$$E_y \{A^{v-1}(y)\} = {}_{.\infty}\!\!\int^{\infty} [1-F(y)]^{v-1} f(y) \, dy$$

and by transformations of variables with $F = F(y)$

we get

(A.3) $Ey (A^{v-1}(y)) = \int_0^1 (1-F)^{v-1} \, dF = \frac{1}{v}$

Using (A.3), the nominator of (A.2) can be written as

$$\text{cov}(y, A^{v-1}(y)) = \int_{.\infty}^{\infty} y \, (v \, A^{v-1}(y) - 1) \, f(y) \, dy$$

and again using transformation of variables where $F = F(y)$

(A.4) $\quad$ cov $(y, A^{v-1}(y)) = \int y(F) [v(1-F)^{v-1} - 1] dF$

where $\quad y(F) = \inf \{y : F(y) \geq F\}$

is the inverse of the cumulative distribution function.

Differentiating (A.4), while using (A.1) yields

(A.5) $\quad \dfrac{\partial \ \text{cov}(y, A^{v-1}(y))}{\partial p_1} = {}_0\!\int^1 x_1(F) [v(1-F)^{v-1} - 1] dF$

and by reversing the procedure that led from (A.2) to (A.4) we get

(A.6) $\quad \dfrac{\partial \ \text{cov}(y, A^{v-1}(y))}{\partial p_1} = \text{cov} (x_1, [1-F(y)]^{v-1})$

Now that the derivative of the numerator is known, we can derive the derivative of the extended Gini with respect to a change in p_1.

(A.7) $\quad \dfrac{\partial G_y(v)}{\partial p_1} = S_1 \ C(x_1, y, v) - S_1 \ G_y(v)$

where $S_1 = m_1/m_y$ is the share of x_1 and

$$C(x_1,y,v) = -v \; cov(x_1,[1-F(y)]^{v-1})/m_1$$

is the (weighted) concentration index.

The effect of an equal revenue tax change on the extended Gini can be evaluated by using (A.7). Let $\dfrac{dG}{dR}$ denote the derivative of the extended Gini with respect to the tax reform. That is

(A.8) $\qquad \dfrac{dG}{dR} = \dfrac{\partial G_y(v)}{\partial P_1} \; dp_1 + \dfrac{\partial G_y(v)}{\partial P_2} \; dp_2$

and by using the restriction $m_1 \; dp_1 + m_2 \; dp_2 = 0$

and (A.7) we get

(A.9) $\qquad \dfrac{dG}{dR} = \{C(x_1,y,v) - C(x_2,y,v)\} \; s_1 \; dp_1$

which means that the effect of the tax changes on the extended Gini depends on the difference between the concentration ratios of the commodities. This completes the proof of (a).

Proof of Property (b).

Let x represent an expenditure on a commodity, then

(A.10) $\qquad c(x,y,v) = -v \; cov(x,[1-F(y)]^{v-1}) \, / \, m_x$

writing the covariance explicitly and eliminating zeros yield

(A.11) $c(x,y,v) = -\left(v/m_x\right) E_y E_x \left\{(x - m_x)[1-F(y)]^{v-1} f(x,y) \, dx \, dy\right.$

Let $g(y) = E_{x_1}(x_1|y))$, then

(A.12) $c(x_1,y,v) = \left(\frac{-v}{m_x}\right) {}_0\!\int^\infty \left(g(y)-m_x\right) f(y) \, [1-F(y)]^{v-1} \, dy$

and by integration by parts where

$v'(y) = \left(g(y) - m_x\right) f(y)$ $v(y) = {}_0\!\int^y \left(g(t) - m_x\right) f(t) \, dt$

$V(y) = \left(1-F(y)\right)^{v-1} dy$ $V'(y) = -(v-1)\left(1-F(y)\right)^{v-1} f(y) \, dy$

and rearranging terms, we get

(A.13) $c(x,y,v) = \dfrac{v(v-1)}{m_x} {}_0\!\int^\infty {}_0\!\int^y \left(g(t) - m_x\right) f(t) \, dt \, [1-F(y)]^{v-2} f(y) \, dy$

By transformation of variables, where $F = F(y)$

and $\phi_x(F) = \dfrac{1}{m_x} {}_0\!\int^{y(F)} g(t) f(t) \, dt$ is the concentration curve

we get

$$c(x,y,v) = v(v-1) \int_0^1 [\phi_x(F) - F] [1-F]^{v-2} dp$$

which means that the concentration ratio is equal to the area between the concentration curve and the 45° line.

Note that the difference between the concentration ratios is hence the (weighted) difference between the concentration curves.

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