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WELL-POSED BOUNDARY CONDITIONS FOR THE NAVIER–STOKES EQUATIONS*

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Abstract. In this article we propose a general procedure that allows us to determine both the number and type of boundary conditions for time dependent partial differential equations. With those, well-posedness can be proven for a general initial-boundary value problem. The procedure is exemplified on the linearized Navier–Stokes equations in two and three space dimensions on a general domain.

Key words. well-posed problems, boundary conditions, Navier–Stokes equations, energy estimates, initial boundary value problems, stability

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1. Introduction.

The problem of well-posed boundary conditions is an essential question in many areas of physics. In fluid dynamics, characteristic boundary conditions for the Euler equations have long been accepted as one way to impose boundary conditions since the specification of the ingoing variable at a boundary implies well-posedness. Often the Euler boundary conditions are used as a guidance when boundary conditions are chosen for the Navier–Stokes equations as well (see [1, 2, 3, 4, 5]). In [6] characteristic boundary conditions for the one-dimensional linearized Navier–Stokes equations were derived.

For the two- and three-dimensional Navier–Stokes equations, the number of boundary conditions implying well-posedness can be obtained using the Laplace transform technique. (See [7] for an introduction of the Laplace transform technique.) Although possible to use, the Laplace transform technique is usually a very complicated procedure for systems of partial differential equations such as the Navier–Stokes equations. However, the exact form of the boundary conditions that lead to a well-posed problem is still an open question and will be the issue addressed in this article.

In this paper we assume that we have unlimited access to accurate boundary data. We do not engage in the elaborate, difficult, and stimulating procedure of deriving artificial (or radiation or absorbing) boundary conditions. Examples of extensive research on these matters are given in [8, 9].

We propose a self-contained procedure to obtain both the number and type of boundary conditions for a general time dependent partial differential equation. The procedure is based on the energy method and has substantial similarities to the derivation of characteristic boundary conditions, since it involves a splitting of the boundary terms into ingoing and outgoing parts by a diagonalization. Compared to the Laplace transform technique, our procedure yields a much simpler analysis.

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As was already mentioned, boundary conditions for the Navier–Stokes equations have been the subject of many investigations, and still there is no theory for the general case. Hence, the linearized and symmetrized Navier–Stokes equations derived in [10] will serve as an example to which our proposed procedure is applied. Since the procedure involves a significant amount of work, we will not treat other equations in this article.

Well-posedness of the continuous problem is a necessary requirement for all numerical methods. Even for well-posed boundary conditions, numerous difficulties arise, and virtually all numerical schemes have their own way of handling boundary conditions. Hence, we will refrain from numerical calculations for a particular discretization technique and focus on the mathematical groundwork.

The contents of this article are divided as follows. In section 2 a general procedure for determining well-posed boundary conditions is presented. Section 3 applies the procedure to the three-dimensional Navier–Stokes equations on a general domain. In section 4 conclusions are drawn.

2. Well-posed boundary conditions. Throughout this paper, the analysis will deal with linear constant coefficient equations. Frequently, the equations of interest are not linear constant coefficient equations but rather variable coefficient or even nonlinear equations (such as the Navier–Stokes equations). We will start with a brief discussion on the relevance of analyzing the constant coefficient case.

Consider a nonlinear initial-boundary value problem on a domain D with boundary ∂D. By linearizing around a solution u and freezing the coefficients, we obtain

$$
\begin{aligned}
\tilde{w}_t &= P(u)\tilde{w} + \delta F(x,t), \quad x \in D,\, t \geq 0,\\
\tilde{w} &= \delta f(x), \quad x \in D,\, t = 0,\\
L\tilde{w} &= \delta g(t), \quad x \in \partial D, t \geq 0,
\end{aligned}
\tag{1}
$$

where P is the (nonlinear) differential operator and L a boundary operator. Here $\delta F, \delta f$, and δg are perturbations of the forcing, initial, and boundary functions. $\tilde{w}$ is the perturbation from the exact solution.

DEFINITION 2.1. *The linear problem* (1) *is well posed if there exists a unique solution bounded by the data* $\delta F, \delta f$, *and* δg.

Remark 1. There are many definitions of well-posedness. Our choice is sometimes referred to as strongly well-posed since it involves all types of data (see, for example, [7]).

Both existence and uniqueness are strongly coupled to the boundedness of the solution. In fact, it suffices to prove that a solution is bounded using a minimal number of boundary conditions; then both existence and uniqueness follow. (See, for example, [11].)

In short, the following principle holds: If (1) is well posed for all values of u, then the original nonlinear problem is well posed (see [12] for more details).

Before considering well-posedness of a problem of the type (1), we will briefly state some additional mathematical theory that is the basis of the forthcoming analysis. First we give a definition from [13].

DEFINITION 2.2. *Let A be a Hermitian matrix. The* inertia *of A is the ordered triple*

$$
i(A) = (i_+(A), i_-(A), i_0(A)),
\tag{2}
$$

where $i_+(A)$ is the number of positive eigenvalues of A, $i_-(A)$ is the number of negative eigenvalues of A, and $i_0(A)$ is the number of zero eigenvalues of A, counting multiplicities.

We will also need the following theorem from [13], and we refer to that textbook for the proof. The theorem is also known as *Sylvester's law of inertia*.

THEOREM 2.3. *Let A, B be Hermitian matrices. There is a nonsingular matrix S such that $A = SBS^*$ if and only if A and B have the same inertia.*

S^* denotes the Hermitian adjoint of S. The following corollary is merely a rephrasing of Theorem 2.3.

COROLLARY 2.4. *Suppose that R is a nonsingular matrix and that A is a real symmetric matrix. Then the number of positive/negative eigenvalues of R^TAR is the same as the number of positive/negative eigenvalues of A.*

Proof. The claim follows immediately from Theorem 2.3 with $B = R^TAR$. ∎

Finally, we state another definition from [13].

DEFINITION 2.5. *If A is a real m-by-n matrix, we set $I(A) = [\mu_{ij}]$, where $\mu_{ij} = 1$ if $a_{ij} \neq 0$ and $\mu_{ij} = 0$ if $a_{ij} = 0$. The matrix $I(A)$ is called the* indicator matrix *of A.*

Now we turn to the main theory of this article. We will give general principles of how to determine boundary conditions such that the constant coefficient problem is well posed. Thus, assuming that linearization and freezing of coefficients have already been carried out, we consider a linear constant coefficient problem with n space dimensions and $\bar{x} = (x_1, \ldots, x_n)$,

$$
\begin{aligned}
\tilde{u}_t + \sum_{i=1}^{n} A_i \tilde{u}_{x_i} &= \sum_{i=1}^{n}\sum_{j=1}^{n} B_{ij}\tilde{u}_{x_i x_j} + F(\bar{x}, t), \quad \bar{x} \in D,\ t \geq 0,\\
\tilde{u}(\bar{x}, 0) &= f(\bar{x}), \quad \bar{x} \in D,\\
L\tilde{u}(\bar{x}, t) &= g(t), \quad \bar{x} \in \partial D,\ t \geq 0.
\end{aligned} \tag{3}
$$

The definition (3) of an initial-boundary value problem covers hyperbolic, parabolic, and incompletely parabolic partial differential equations depending on the rank of the matrices. Let $\|\cdot\|$ denote some norm for functions on D. Our approach of analyzing the well-posedness of (3) comprises the following steps.

(i) Symmetrize (3).

(ii) Apply the energy method. The energy estimate will have the structure

$$
\|\tilde{u}\|_t^2 + c_i \sum_{i=1}^{n} \|\tilde{u}_{x_i}\|^2 + \oint_{\partial D} \tilde{v}^T \mathbf{A} \tilde{v}\, ds \leq 0, \tag{4}
$$

where $c_i \geq 0$, $i = 1, \ldots, n$, are constants and $\tilde{v}$ a vector formed by combinations of $\tilde{u}$ and $\tilde{u}_{x_i}$. Further, $\mathbf{A}$ is reduced to a full rank matrix. The boundedness of $\tilde{u}$ now depends on the boundedness of $\tilde{v}^T\mathbf{A}\tilde{v}$ in boundary data.

(iii) Find a diagonalizing matrix, M, such that $M^T\mathbf{A}M = \Lambda$ is diagonal. ($\mathbf{A}$ is symmetric due to step (i) above.) Then we also have the variable transformation $M^{-1}\tilde{v} = \tilde{w}$.

(iv) Split $\Lambda = \Lambda^+ + \Lambda^-$ such that Λ^+ is positive semidefinite and Λ^- is negative semidefinite. Also, split $\tilde{w}$ into $\tilde{w} = \tilde{w}^+ + \tilde{w}^-$ corresponding to the nonzero entries of $\Lambda^{+,-}$. More precisely, let $\tilde{w}^- = I(\Lambda^-)\tilde{w}$ and $\tilde{w}^+ = \tilde{w} - \tilde{w}^-$.

(v) Supply boundary data to the negative part. That is, specify $\tilde{w}^-$ by g.

Remark 2. In step (iv) the number of boundary conditions is given as the number of negative eigenvalues of $\mathbf{A}$ or Λ. Further, the type of boundary conditions is given by the matrix M, derived in step (iii).

This implies boundedness of $\|\tilde{u}\|_t$ and hence of $\|\tilde{u}\|$. The difficult part of this scheme is step (iii). However, we know that $\mathbf{A}$ is symmetric, and we can prove the following proposition regarding steps (iii)–(v).

PROPOSITION 2.6. *Assume that steps* (i) *and* (ii) *are fulfilled; then the matrix* $\mathbf{A}$ *and the vector* $\tilde{v}$ *can be split such that* $\tilde{v}^T\mathbf{A}\tilde{v} = \tilde{w}^{+T}\Lambda^+\tilde{w}^+ + \tilde{w}^{-T}\Lambda^-\tilde{w}^-$, *where* Λ^+ *is positive semidefinite,* Λ^- *is negative semidefinite, and* $M^{-1}\tilde{v} = \tilde{w} = \tilde{w}^+ + \tilde{w}^-$ *for some matrix* M^{-1}. *Further, by specifying* $\tilde{w}^- = I(\Lambda^-)w$ *at the boundary, we find that* (3) *is well posed.*

Proof. Since $\mathbf{A}$ is symmetric, the eigenvalues are real and there exists a full set of eigenvectors. If Z contains the eigenvectors, we have

$$\tilde{v}^T\mathbf{A}\tilde{v} = \tilde{v}^T ZZ^T\mathbf{A}ZZ^T\tilde{v} = \tilde{w}^T\Lambda_{\mathbf{Z}}\tilde{w} = \tilde{w}^{+T}\Lambda^+_{\mathbf{Z}}\tilde{w}^+ + \tilde{w}^{-T}\Lambda^-_{\mathbf{Z}}\tilde{w}^-, \tag{5}$$

where $\Lambda_{\mathbf{Z}}^{-/+}$ are diagonal negative/positive semidefinite. We define $\tilde{w}^- = I(\Lambda^-)\tilde{w}$ and $\tilde{w}^+ = \tilde{w} - \tilde{w}^-$. This proves the first part of Proposition 2.6.

Another way to prove the first part of Proposition 2.6 is to apply Corollary 2.4, to conclude that any nonsingular matrix R can be used as a transformation, $\mathbf{B} = R^T\mathbf{A}R$, such that $\mathbf{A}$ and $\mathbf{B}$ have the same inertia. By construction, $\mathbf{B}$ is symmetric. Then $\mathbf{B}$ may be diagonalized by its eigenvectors, and we have another diagonalization of $\mathbf{A}$. Denote by X the matrix containing the eigenvectors of $\mathbf{B}$ as columns such that

$$\begin{aligned}\tilde{v}^T\mathbf{A}\tilde{v} &= \tilde{v}^T R^{-1,T}R^T\mathbf{A}RR^{-1}\tilde{v} = \tilde{v}^T R^{-1,T}\mathbf{B}R^{-1}\tilde{v}\\ &= \tilde{v}^T R^{-1,T}X\Lambda_{\mathbf{M}}X^T R^{-1}\tilde{v} = \tilde{w}^T\Lambda^+_{\mathbf{M}}\tilde{w} + \tilde{w}^T\Lambda^-_{\mathbf{M}}\tilde{w}\end{aligned}$$

or

$$\tilde{v}^T M^{-1,T}M^T\mathbf{A}MM^{-1}\tilde{v} = \tilde{w}^T\Lambda_{\mathbf{M}}\tilde{w} = \tilde{w}^{+T}\Lambda^+_{\mathbf{M}}\tilde{w}^+ + \tilde{w}^{-T}\Lambda^-_{\mathbf{M}}\tilde{w}^-, \tag{6}$$

where $\tilde{w} = M^{-1}\tilde{v}$, $M = RX$, and $\Lambda_{\mathbf{M}}^{-/+}$ are diagonal negative/positive semidefinite. Further, $\tilde{w}^- = I(\Lambda^-_{\mathbf{M}})\tilde{w}$ and $\tilde{w}^+ = \tilde{w} - \tilde{w}^-$. We conclude that there are several different ways of diagonalizing $\mathbf{A}$, but in all cases $\Lambda_{\mathbf{Z}}$ and $\Lambda_{\mathbf{M}}$ have the same inertia. The fundamental difference between Z and another diagonalizing matrix, M, is that M is not orthogonal. We may regard Z as a specific M.

Next, we turn to the proof of the second part of the proposition. Specify $\tilde{w}^- = g$ at the boundary. Equation (4) can be rewritten as

$$\|\tilde{u}\|_t^2 + \oint_{\partial D}\tilde{w}^{+T}\Lambda^+_{\mathbf{M}}\tilde{w}^+ ds + c_i\sum_{i=1}^n \|\tilde{u}_{x_i}\|^2 = -\oint_{\partial D} g^T\Lambda^-_{\mathbf{M}}g\, ds. \tag{7}$$

All the terms on the left-hand side of (7) are positive, implying that $\|\tilde{u}\|_t$, and hence $\|\tilde{u}\|$, are bounded. ∎

Remark 3. The assumption that steps (i) and (ii) in Proposition 2.6 can be fulfilled is true for many important partial differential equations. For example, it is true for the Euler, Navier–Stokes, and Maxwell equations.

Remark 4. The procedure that diagonalizes $\mathbf{A}$, with its eigenvectors and bounds the negative part, is what we mean by characteristic boundary conditions.

For Proposition 2.6 to be practically useful, a crucial point is to find a diagonalizing matrix. That is why we gave two examples of diagonalizing matrices. In the first example we used the eigenvalues and eigenvectors directly. For a system of equations, the matrix A can be large (9-by-9 for the Navier–Stokes equations in three dimensions). The eigenvalues of $\mathbf{A}$ are given as the roots of a polynomial of high degree, for which in general there do not exist roots in closed form.

In the second example, we can proceed in a different way. We will seek a diagonalizing matrix to $\mathbf{A}$ that is not orthogonal. By choosing R such that $\mathbf{B}$ has a simpler structure than **A**, we may be able to find the eigenvalues and eigenvectors to **B**. In fact, we will show that this is possible for the three-dimensional Navier–Stokes equations on general domains.

Certainly, not all of the points are novel in the above procedure. For example, in [10] a symmetrization of the linearized Navier–Stokes equations is presented. For the Euler equations, the whole procedure has been carried out when deriving the well-known characteristic boundary conditions. However, the idea of diagonalizing the boundary terms with a nonorthogonal matrix is, to the knowledge of the present authors, new. Furthermore, it is important to formalize the whole procedure since it should be possible to find well-posed boundary conditions to any problem of type (3).

3. The Navier–Stokes equations.

3.1. Step (i): Symmetrize the equations. We will consider the Navier–Stokes equations as an example of how to use the procedure presented above to derive well-posed boundary conditions. We begin by rescaling the three-dimensional Navier–Stokes equations to nondimensional form. Consider the Navier–Stokes equations in primitive variables $\tilde{V} = [\tilde{\rho}, \tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{p}]$ as stated in [10],

$$\begin{aligned} &\tilde{V}_t + \tilde{A}_1^p \tilde{V}_x + \tilde{A}_2^p \tilde{V}_y + \tilde{A}_3^p \tilde{V}_z \\ &= \tilde{B}_{11}^p \tilde{V}_{xx} + \tilde{B}_{22}^p \tilde{V}_{yy} + \tilde{B}_{33}^p \tilde{V}_{zz} + \tilde{B}_{xy}^p \tilde{V}_{xy} + \tilde{B}_{yz}^p \tilde{V}_{yz} + \tilde{B}_{zx}^p \tilde{V}_{zx}, \end{aligned} \tag{8}$$

where the tilde sign emphasizes that the entity depends on the solution. Further, $\tilde{\rho}$ is the density; $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$ are the velocities in the x, y, and z directions, respectively; and $\tilde{p}$ is the pressure. We will also use the ratio between the specific heat capacities, $\gamma = c_p/c_v$, and the speed of sound, c; μ the dynamic viscosity, λ the bulk viscosity, and $\nu = \frac{\mu}{\rho}$ the kinematic viscosity; $Pr = \frac{\nu}{\alpha}$ denoting the Prandtl number, where α is the thermal diffusivity. Let $Re = \frac{\rho_\infty U_\infty L}{\mu_\infty}$ denote the Reynolds number. The infinity subscript denotes free stream conditions, and L is some characteristic length scale.

The equations (8) are nondimensionalized and the coefficients are frozen, which corresponds to the linearization of the Navier–Stokes equations. The tilde signs are dropped on the matrices as they no longer depend on the solution. Using the parabolic symmetrizer S_p derived in [10] and letting $\epsilon = \frac{1}{Re}$ yields

$$\begin{aligned} &\tilde{u}_t + A_1 \tilde{u}_x + A_2 \tilde{u}_y + A_3 \tilde{u}_z \\ &= \epsilon(B_{11}\tilde{u}_{xx} + B_{22}\tilde{u}_{yy} + B_{33}\tilde{u}_{zz} + B_{xy}\tilde{u}_{xy} + B_{yz}\tilde{u}_{uz} + B_{zx}\tilde{u}_{zx}). \end{aligned} \tag{9}$$

The transformed nondimensionalized variables are

$$\begin{aligned} S_p^{-1}\tilde{V} &= \begin{pmatrix} \frac{c}{\sqrt{\gamma}\rho} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{c}{\rho\sqrt{\gamma}\sqrt{\gamma-1}} & 0 & 0 & 0 & \sqrt{\frac{\gamma}{\gamma-1}}\frac{1}{\rho c} \end{pmatrix} \begin{pmatrix} \tilde{\rho} \\ \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ \tilde{p} \end{pmatrix} \\ &= \begin{pmatrix} \frac{c}{\sqrt{\gamma}\rho}\tilde{\rho} \\ \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ -\frac{c}{\sqrt{\gamma}\sqrt{\gamma-1}}\frac{\tilde{\rho}}{\rho} + \sqrt{\frac{\gamma}{\gamma-1}}\frac{1}{\rho c}\tilde{p} \end{pmatrix} = \tilde{u}. \end{aligned} \tag{10}$$

The symmetrized matrices are derived in [10] and are repeated here for convenience. Let $a = \sqrt{\frac{\gamma-1}{\gamma}}c$ and $b = \frac{c}{\sqrt{\gamma}}$. Then

$$A_1 = \begin{pmatrix} u_1 & b & 0 & 0 & 0 \\ b & u_1 & 0 & 0 & a \\ 0 & 0 & u_1 & 0 & 0 \\ 0 & 0 & 0 & u_1 & 0 \\ 0 & a & 0 & 0 & u_1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} u_2 & 0 & b & 0 & 0 \\ 0 & u_2 & 0 & 0 & 0 \\ b & 0 & u_2 & 0 & a \\ 0 & 0 & 0 & u_2 & 0 \\ 0 & 0 & a & 0 & u_2 \end{pmatrix}, \tag{11}$$

$$A_3 = \begin{pmatrix} u_3 & 0 & 0 & b & 0 \\ 0 & u_3 & 0 & 0 & 0 \\ 0 & 0 & u_3 & 0 & 0 \\ b & 0 & 0 & u_3 & a \\ 0 & 0 & 0 & a & u_3 \end{pmatrix}, \qquad B_{xy} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\lambda+\mu}{\rho} & 0 & 0 \\ 0 & \frac{\lambda+\mu}{\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{12}$$

$$B_{yz} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda+\mu}{\rho} & 0 \\ 0 & 0 & \frac{\lambda+\mu}{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B_{zx} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda+\mu}{\rho} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\lambda+\mu}{\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{13}$$

$$B_{11} = \text{diag}\left(0, \frac{\lambda+2\mu}{\rho}, \frac{\mu}{\rho}, \frac{\mu}{\rho}, \frac{\gamma\mu}{Pr\rho}\right), \tag{14}$$

$$B_{22} = \text{diag}\left(0, \frac{\mu}{\rho}, \frac{\lambda+2\mu}{\rho}, \frac{\mu}{\rho}, \frac{\gamma\mu}{Pr\rho}\right), \tag{15}$$

$$B_{33} = \text{diag}\left(0, \frac{\mu}{\rho}, \frac{\mu}{\rho}, \frac{\lambda+2\mu}{\rho}, \frac{\gamma\mu}{Pr\rho}\right). \tag{16}$$

3.2. Step (ii): Apply the energy method. Next, we turn to the analysis of boundary conditions for the Navier–Stokes equations. Consider a general domain D with boundary ∂D in three space dimensions. From (9), the symmetrized and nondimensionalized Navier–Stokes equations are

$$\tilde{u}_t + (A_1\tilde{u} - \epsilon\tilde{F}_v)_x + (A_2\tilde{u} - \epsilon\tilde{G}_v)_y + (A_3\tilde{u} - \epsilon\tilde{H}_v)_z, \tag{17}$$

where

$$\tilde{F}_v = B_{11}\tilde{u}_x + B_{21}\tilde{u}_y + B_{31}\tilde{u}_z, \tag{18}$$

$$\tilde{G}_v = B_{22}\tilde{u}_y + B_{32}\tilde{u}_z + B_{12}\tilde{u}_x, \tag{19}$$

$$\tilde{H}_v = B_{33}\tilde{u}_z + B_{23}\tilde{u}_y + B_{13}\tilde{u}_x, \tag{20}$$

and

$$B_{21} = B_{12} = \frac{B_{xy}}{2}, \quad B_{32} = B_{23} = \frac{B_{yz}}{2}, \quad B_{31} = B_{13} = \frac{B_{zx}}{2}.$$

Applying the energy method (step (ii)),

$$\begin{aligned}(21)\qquad &\int_D \tilde{u}^T\tilde{u}_t dxdydz + \int_D \frac{\partial}{\partial x}\left(\frac{1}{2}\tilde{u}^T A_1\tilde{u} - \epsilon\tilde{u}^T\tilde{F}_v\right)\\ &+\frac{\partial}{\partial y}\left(\frac{1}{2}\tilde{u}^T A_2\tilde{u} - \epsilon\tilde{u}^T\tilde{G}_v\right) + \frac{\partial}{\partial z}\left(\frac{1}{2}\tilde{u}^T A_3\tilde{u} - \epsilon\tilde{u}^T\tilde{H}_v\right) dxdydz\\ &= -\epsilon\int_D (\tilde{u}_x^T\tilde{F}_v + \tilde{u}_y^T\tilde{G}_v + \tilde{u}_z^T\tilde{H}_v)dxdydz.\end{aligned}$$

The right-hand side in (21) is negative definite and denoted by $-DI$.

Remark 5. It is easily verified that the last term in (21) is dissipation,

$$DI = \epsilon\int_D \left(\ \tilde{u}_x^T\ \tilde{u}_y^T\ \tilde{u}_z^T\ \right)\begin{pmatrix} B_{11} & B_{12} & B_{13}\\ B_{21} & B_{22} & B_{23}\\ B_{31} & B_{32} & B_{33}\end{pmatrix}\begin{pmatrix}\tilde{u}_x\\ \tilde{u}_y\\ \tilde{u}_z\end{pmatrix} dxdydz.$$

The matrix is symmetric with positive or zero diagonal entries. With $\lambda \le \mu$, the matrix is diagonally dominant. Thus, it is positive semidefinite.

Denote by $\|\tilde{u}\|^2$ the integral $\int_D \tilde{u}^T\tilde{u}dxdydz$. Using Gauss' theorem, we obtain

$$(22)\qquad \|\tilde{u}\|_t^2 + \oint_{\partial D}\left(\tilde{u}^T(A_1\tilde{u} - 2\epsilon\tilde{F}_v), \tilde{u}^T(A_2\tilde{u} - 2\epsilon\tilde{G}_v), \tilde{u}^T(A_3\tilde{u} - 2\epsilon\tilde{H}_v)\right)\cdot\hat{\mathbf{n}}\,ds = -2DI,$$

where $\hat{\mathbf{n}} = (n_1, n_2, n_3)$ is the outward-pointing unit normal on the surface ∂D and $ds = \sqrt{dx^2 + dy^2 + dz^2}$. Equation (22) can be rewritten as

$$(23)\qquad \|\tilde{u}\|_t^2 + \oint_{\partial D}\begin{pmatrix}\tilde{u}\\ \tilde{F}^V\end{pmatrix}^T\begin{pmatrix} A_1n_1 + A_2n_2 + A_3n_3 & -\epsilon I_5\\ -\epsilon I_5 & 0_5\end{pmatrix}\begin{pmatrix}\tilde{u}\\ \tilde{F}^V\end{pmatrix} ds = -2DI,$$

where I_n denotes the n-by-n identity matrix, and similarly 0_n the n-by-n zero matrix and $\tilde{F}^V = \tilde{F}_v n_1 + \tilde{G}_v n_2 + \tilde{H}_v n_3$.

To prove well-posedness we have to split the matrix in the boundary integral into positive definite and negative definite parts. The negative part of the boundary term in (23) caused by

$$(24)\qquad \mathbf{A_1} = \begin{pmatrix} A_1n_1 + A_2n_2 + A_3n_3 & -\epsilon I_5\\ -\epsilon I_5 & 0_5\end{pmatrix}$$

has to be supplied with boundary conditions, which in turn bounds the growth of $\|\tilde{u}\|_t^2$ in (21).

We note that the first component of $\tilde{F}^V$ is zero, and hence we can reduce the system by omitting that component and denoting the resulting vector by $\tilde{G}^V$. By this procedure $\mathbf{A_1}$ is also reduced from a 10-by-10 matrix to a 9-by-9 matrix by deleting the sixth row and column. With $\mathbf{u} = (u_1, u_2, u_3)$, we have

$$\begin{pmatrix}\tilde{u}\\ \tilde{F}^V\end{pmatrix}^T\begin{pmatrix} A_1n_1 + A_2n_2 + A_3n_3 & -\epsilon I_5\\ -\epsilon I_5 & 0_5\end{pmatrix}\begin{pmatrix}\tilde{u}\\ \tilde{F}^V\end{pmatrix} = \begin{pmatrix}\tilde{u}\\ \tilde{G}^V\end{pmatrix}^T\mathbf{A}\begin{pmatrix}\tilde{u}\\ \tilde{G}^V\end{pmatrix},$$

where

$$\mathbf{A} = \begin{pmatrix} \mathbf{u}\cdot\hat{\mathbf{n}} & bn_1 & bn_2 & bn_3 & 0 & 0 & 0 & 0 & 0 \\ bn_1 & \mathbf{u}\cdot\hat{\mathbf{n}} & 0 & 0 & an_1 & -\epsilon & 0 & 0 & 0 \\ bn_2 & 0 & \mathbf{u}\cdot\hat{\mathbf{n}} & 0 & an_2 & 0 & -\epsilon & 0 & 0 \\ bn_3 & 0 & 0 & \mathbf{u}\cdot\hat{\mathbf{n}} & an_3 & 0 & 0 & -\epsilon & 0 \\ 0 & an_1 & an_2 & an_3 & \mathbf{u}\cdot\hat{\mathbf{n}} & 0 & 0 & 0 & -\epsilon \\ 0 & -\epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & 0_{14} \\ A_{21} & A_{22} & -\epsilon I_4 \\ 0_{41} & -\epsilon I_4 & 0_4 \end{pmatrix}, \tag{25}$$

using the notation 0_{nm} for the n-by-m zero matrix. We will also use the notation $u_n = \mathbf{u}\cdot\hat{\mathbf{n}}$. Since $\hat{\mathbf{n}}$ is the outward-pointing normal, $u_n < 0$ implies inflow. Further, note that A_{11} in (25) is a scalar.

3.3. Step (iii): Find a diagonalizing matrix. Next, we state and prove the following proposition, where $M_n = u_n/c$ is the Mach number.

PROPOSITION 3.1. *If $|M_n| \neq 1, 0$ and $u_n < 0$, there are four positive and five negative eigenvalues of* $\mathbf{A}$. *If $|M_n| \neq 1, 0$ and $u_n > 0$, there are five positive and four negative eigenvalues of* $\mathbf{A}$.

Proposition 3.1 states that an inflow demands five and an outflow four boundary conditions. The number of boundary conditions can also be derived using the Laplace transform technique, which is shown in [14, 15]. However, to prove well-posedness of specific boundary conditions using the Laplace transform technique is algebraically very complex, as shown in [15]. In the proof of Proposition 3.1 we will continue with the procedure outlined in section 2 and find a diagonalizing matrix to A (step (iii)). However, finding the eigenvalues of A corresponds to solving a ninth degree polynomial. Besides the algebraic difficulty of finding roots to ninth degree polynomials, it is probable that the roots in this particular case do not exist in closed form. Instead, we will derive another diagonalizing matrix. That matrix gives the explicit form of the well-posed boundary conditions.

Proof of Proposition 3.1. Rotate $\mathbf{A}$ by

$$R^T\mathbf{A}R = \begin{pmatrix} 1 & 0_{14} & 0_{14} \\ \bar{\alpha}^T & I_4 & 0_4 \\ \bar{\beta}^T & \bar{\gamma}^T & I_4 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & 0_{14} \\ A_{21} & A_{22} & -\epsilon I_4 \\ 0_{41} & -\epsilon I_4 & 0_4 \end{pmatrix} \begin{pmatrix} 1 & \bar{\alpha} & \bar{\beta} \\ 0_{41} & I_4 & \bar{\gamma} \\ 0_{41} & 0_4 & I_4 \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} = \mathbf{E}, \tag{26}$$

where

$$\begin{aligned} E_{11} &= A_{11}, \\ E_{12} &= A_{11}\bar{\alpha} + A_{12}, \\ E_{13} &= A_{11}\bar{\beta} + A_{12}\bar{\gamma}, \\ E_{21} &= \bar{\alpha}^T A_{11} + A_{21}, \\ E_{22} &= \bar{\alpha}^T(A_{11}\bar{\alpha} + A_{12}) + (A_{21}\bar{\alpha} + A_{22}), \end{aligned}$$

$$
\begin{aligned}
E_{23} &= \bar{\alpha}^T(A_{11}\bar{\beta} + A_{12}\bar{\gamma}) + A_{21}\bar{\beta} + A_{22}\bar{\gamma} - \epsilon I_3,\\
E_{31} &= \bar{\beta}^T A_{11} + \bar{\gamma}^T A_{21},\\
E_{32} &= \bar{\beta}^T(A_{11}\bar{\alpha} + A_{12}) + \bar{\gamma}^T(A_{21}\bar{\alpha} + A_{22}) - \epsilon I_3,\\
E_{33} &= \bar{\beta}^T(A_{11}\bar{\beta} + A_{12}\bar{\gamma}) + \bar{\gamma}^T(A_{21}\bar{\beta} + A_{22}\bar{\gamma} - \epsilon I_3) - \epsilon I_3\bar{\gamma}.
\end{aligned}
$$

Using $A_{12}^T = A_{21}$, we cancel the off-diagonal blocks and solve for $\bar{\alpha}, \bar{\beta}$, and $\bar{\gamma}$. We obtain

$$
\bar{\alpha} = -A_{11}^{-1}A_{12}, \quad \bar{\beta} = \epsilon A_{11}^{-1}A_{12}E_{22}^{-1}, \quad \bar{\gamma} = -\epsilon E_{22}^{-1}, \tag{27}
$$

$$
\mathbf{E} = \begin{pmatrix} A_{11} & 0_{14} & 0_{14} \\ 0_{41} & E_{22} & 0_4 \\ 0_{41} & 0_4 & -\epsilon^2 E_{22}^{-1} \end{pmatrix}, \quad E_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12}. \tag{28}
$$

The conditions for this procedure to hold are that $\det(A_{11}) \neq 0$ and $\det(E_{22}) \neq 0$.

We know from Corollary 2.4 that $i(\mathbf{A}) = i(\mathbf{E})$. Thus, we can instead determine the sign of the eigenvalues of $\mathbf{E}$. Note that the upper-left entry of $\mathbf{E}$ is a scalar and hence an eigenvalue. We denote that by

$$
\lambda_1 = A_{11} = u_n. \tag{29}
$$

If $\det(E_{22}) \neq 0$, we know that E_{22} has four real nonzero eigenvalues, since E_{22} is symmetric by construction. The signs of those do not change as E_{22} is inverted such that from the second and third block there are always four negative and four positive eigenvalues of $\mathbf{E}$. Including λ_1, we have for $u_n > 0$ four negative and five positive eigenvalues, and for $u_n < 0$, five negative and four positive eigenvalues of $\mathbf{E}$, as stated in the proposition (assuming that $\det(A_{11}) \neq 0$ and $\det(E_{22}) \neq 0$).

We will now show that $\det(A_{11}) \neq 0$ and $\det(E_{22}) \neq 0$ for $M_n \neq \pm 1, 0$. Since $A_{11} = u_n$, we have $\det(A_{11}) \neq 0$ for $M_n \neq 0$. To evaluate the second condition, we compute the eigenvalues of E_{22} explicitly. From (25) and (28) we have

$$
E_{22} = \begin{pmatrix} -\frac{b^2 n_1^2}{u_n} + u_n & -\frac{b^2 n_1 n_2}{u_n} & -\frac{b^2 n_1 n_3}{u_n} & a n_1 \\ -\frac{b^2 n_1 n_2}{u_n} & -\frac{b^2 n_2^2}{u_n} + u_n & -\frac{b^2 n_2 n_3}{u_n} & a n_2 \\ -\frac{b^2 n_1 n_3}{u_n} & -\frac{b^2 n_2 n_3}{u_n} & -\frac{b^2 n_3^2}{u_n} + u_n & a n_3 \\ a n_1 & a n_2 & a n_3 & u_n \end{pmatrix}, \tag{30}
$$

and the eigenvalues are

$$
\lambda_{2,3} = \frac{-b^2 + 2u_n^2 \pm \sqrt{b^4 + 4a^2 u_n^2}}{2u_n}, \tag{31}
$$

$$
\lambda_4 = \lambda_5 = u_n, \tag{32}
$$

where $n_1^2 + n_2^2 + n_3^2 = 1$ has been used to simplify the expressions. λ_4 and λ_5 obviously shift sign at $u_n = 0$. Also, since $\lambda_4 = \lambda_5 = 0$ with $M_n = u_n = 0$, we have that $\det(E_{22}) = 0$. Thus, to rotate $\mathbf{A}$ by R we once more need $M_n \neq 0$. λ_2 and λ_3 can be expressed as

$$
\lambda_{2,3} = \frac{c}{2\gamma M_n}\left(-1 + 2\gamma M_n^2 \pm \sqrt{1 + 4(\gamma - 1)\gamma M_n^2}\right). \tag{33}
$$

Consider λ_2, and note that $\gamma \geq 1$. Then $\sqrt{1-4\gamma M_n^2+4\gamma^2 M_n^2} \geq 1$ such that the sign of λ_2 is the same as the sign of the denominator, i.e., M_n or u_n. This means that $\lambda_2 \neq 0$ for all $M_n \neq 0$, and $\lambda_2 = 0$ for $M_n = 0$.

At last, λ_3 is considered. λ_3 shifts sign when

$$2\gamma M_n^2 - 1 - \sqrt{1-4\gamma M_n^2+4\gamma^2 M_n^2} = 0.$$

Alternatively, $(2\gamma M_n^2-1)^2 = (1-4\gamma M_n^2+4\gamma^2 M_n^2)$, which has the solutions $M_n = 0, 1, -1$, but $M_n = 0$ is discarded due to the original equality. Thus, $\lambda_3 \neq 0$, and hence $\det(E_{22}) \neq 0$ for $|M_n| \neq 1$. Note that, λ_3 is singular for $M_n = 0$. □

We have now derived the number of positive and negative eigenvalues of $\mathbf{A}$, and hence the number of boundary conditions, and their dependence on M_n. This was done by calculating the eigenvalues of $\mathbf{E}$ explicitly.

To obtain a set of boundary conditions we also need the eigenvectors of $\mathbf{E}$. Given the eigenvectors of $\mathbf{E}$, it is a simple task to derive a diagonalizing matrix to $\mathbf{A}$. The eigenvectors of E_{22} are able to be explicitly derived since the eigenvalues are explicitly given and they are $Y = (y_2, y_3, y_4, y_5)$, where

$$y_2 = \left(n_1, n_2, n_3, -\frac{-b^4 - \sqrt{b^2+4a^2u_n^2}}{2au_n}\right)^T$$
$$= \left(n_1, n_2, n_3, \frac{-\lambda_3+u_n}{a}\right)^T, \tag{34}$$
$$y_3 = \left(n_1, n_2, n_3, -\frac{-b^4 + \sqrt{b^2+4a^2u_n^2}}{2au_n}\right)^T$$
$$= \left(n_1, n_2, n_3, \frac{-\lambda_2+u_n}{a}\right)^T, \tag{35}$$
$$y_4 = (-n_2, n_1, 0, 0)^T, \tag{36}$$
$$y_5 = (-n_3, 0, n_1, 0). \tag{37}$$

Remark 6. We omit the normalization of the eigenvectors to keep the expressions (34)–(37) simple.

Now, we can derive a specific diagonalizing matrix M and conclude step (iii). For convenience, we restate (6),

$$\tilde{v}^T M^{-1,T} M^T \mathbf{A} M M^{-1} \tilde{v} = \tilde{w}^T \Lambda_{\mathbf{M}} \tilde{w},$$

where $M = RX$ and $\tilde{v} = (\tilde{u}^T\, (\tilde{G}^V)^T)^T$. R is given in (26), (27), and (28). Further,

$$X = \begin{pmatrix} 1 & 0_{14} & 0_{14} \\ 0_{41} & Y & 0_4 \\ 0_{41} & 0_4 & Y \end{pmatrix}, \qquad \Lambda_{\mathbf{M}} = \begin{pmatrix} u_n & 0_{14} & 0_{14} \\ 0_{41} & \Lambda & 0_4 \\ 0_{41} & 0_4 & -\epsilon^2\Lambda^{-1} \end{pmatrix},$$

where $\Lambda = \mathrm{diag}(\lambda_2, \lambda_3, \lambda_4, \lambda_5)$. Inverting R and M yields

$$R^{-1} = \begin{pmatrix} 1 & -\bar{\alpha} & 0_{14} \\ 0_{41} & I_4 & -\bar{\gamma} \\ 0_{41} & 0_4 & I_4 \end{pmatrix}, \qquad M^{-1} = X^T R^{-1} = \begin{pmatrix} 1 & -\bar{\alpha} & 0_{14} \\ 0_{41} & Y^T & -Y^T\bar{\gamma} \\ 0_{41} & 0_4 & Y^T \end{pmatrix}. \tag{38}$$

To simplify the computation of M^{-1} we use (27) and obtain

$$-Y^T\gamma = \epsilon Y^T E_{22}^{-1} = \epsilon Y^T Y \Lambda^{-1} Y^T = \epsilon \Lambda^{-1} Y^T = \epsilon \begin{pmatrix} \lambda_2^{-1} y_2^T \\ \lambda_3^{-1} y_3^T \\ \lambda_4^{-1} y_4^T \\ \lambda_5^{-1} y_5^T \end{pmatrix}, \tag{39}$$

yielding

$$M^{-1} = \begin{pmatrix} 1 & -\bar{\alpha} & 0_{14} \\ 0_{41} & Y^T & \epsilon\Lambda^{-1}Y^T \\ 0_{41} & 0_4 & Y^T \end{pmatrix}, \quad \text{where } \bar{\alpha} = \left(-\frac{b}{u_n}\hat{\mathbf{n}}, 0\right). \tag{40}$$

We proceed by computing the variables, $\tilde{w} = X^T R^{-1}\tilde{v} = M^{-1}\tilde{v}$, to which boundary conditions should be applied. Let $\tilde{G}_i^V$ be the ith component of $\tilde{G}^V$. Define $\tilde{v}_{i\ldots j} = (\tilde{v}_i, \ldots, \tilde{v}_j)^T$ and $\tilde{u}_n = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \cdot \hat{\mathbf{n}}$. For convenience, we restate $\tilde{v}$,

$$\tilde{v} = \left(\frac{b}{\rho}\tilde{\rho}, \tilde{u}_1, \tilde{u}_2, \tilde{u}_3, -\frac{b}{\sqrt{\gamma-1}}\frac{\tilde{\rho}}{\rho} + \frac{1}{\rho a}\tilde{p}, \tilde{G}_1^V, \tilde{G}_2^V, \tilde{G}_3^V, \tilde{G}_4^V\right)^T. \tag{41}$$

Then,

$$\tilde{w} = M^{-1}\tilde{v} = \begin{pmatrix} \tilde{v}_1 - \bar{\alpha}\cdot\tilde{v}_{2\ldots5} \\ y_2^T(\tilde{v}_{2\ldots5} - \epsilon\lambda_2^{-1}\tilde{G}^V) \\ y_3^T(\tilde{v}_{2\ldots5} - \epsilon\lambda_3^{-1}\tilde{G}^V) \\ y_4^T(\tilde{v}_{2\ldots5} - \epsilon\lambda_4^{-1}\tilde{G}^V) \\ y_5^T(\tilde{v}_{2\ldots5} - \epsilon\lambda_5^{-1}\tilde{G}^V) \\ y_2^T\tilde{G}^V \\ y_3^T\tilde{G}^V \\ y_4^T\tilde{G}^V \\ y_5^T\tilde{G}^V \end{pmatrix}, \tag{42}$$

by using (34)–(37).

For completeness we also give the reverse transformation. It is $\tilde{v} = RX\tilde{w} = M\tilde{w}$,

$$M = \begin{pmatrix} 1 & \bar{\alpha}Y & \bar{\beta}Y \\ 0_{31} & Y & \bar{\gamma}Y \\ 0_{31} & 0_3 & Y \end{pmatrix} = \begin{pmatrix} 1 & \bar{\alpha}Y & \bar{\alpha}\Lambda^{-1}Y^T \\ 0_{31} & Y & -\epsilon Y\Lambda \\ 0_{31} & 0_3 & Y \end{pmatrix}. \tag{43}$$

The corresponding diagonalizing matrices in the two-dimensional case are given in Appendix A.

Remark 7. Note that we have found one of possibly several diagonalizing matrices. M is not orthogonal, which means that $\Lambda_{\mathbf{M}}$ does not hold the eigenvalues of $\mathbf{A}$.

Remark 8. Note that the only condition involved with finding a diagonalizing matrix M is that $\mathbf{A}$ be nonsingular. Then we can choose to rotate $\mathbf{A}$ to block diagonal form with blocks of arbitrary size. If the blocks are small enough, we can derive their eigenvalues analytically.

3.4. Step (iv) and (v): Split $\Lambda_{\mathbf{M}}$ and $\tilde{w}$. In order to know which components of $\tilde{w}$ to bound with boundary conditions we need to investigate the sign of the diagonal entries of $\Lambda_{\mathbf{M}}$, i.e., the eigenvalues of $\mathbf{E}$ (step (iv)).

Table 1
The sign of the eigenvalues for different Mach numbers.

Eigenvalue	$M_n < -1$	$-1 < M_n < 0$	$0 < M_n < 1$	$M_n > 1$
λ_1	$-$	$-$	$+$	$+$
λ_2	$-$	$-$	$+$	$+$
λ_3	$-$	$+$	$-$	$+$
λ_4	$-$	$-$	$+$	$+$
λ_5	$-$	$-$	$+$	$+$
λ_6	$+$	$+$	$-$	$-$
λ_7	$+$	$-$	$+$	$-$
λ_8	$+$	$+$	$-$	$-$
λ_9	$+$	$+$	$-$	$-$

Table 2
The number of boundary conditions to be specified at different flow cases for the three-dimensional Navier–Stokes equations.

Supersonic inflow	5
Subsonic inflow	5
Subsonic outflow	4
Supersonic outflow	4

Table 3
The number of boundary conditions to be specified at different flow cases for the three-dimensional Euler equations.

Supersonic inflow	5
Subsonic inflow	4
Subsonic outflow	1
Supersonic outflow	0

In the proof of Proposition 3.1, λ_3 given by (33) was analyzed. It was shown that λ_3 changes sign at $M_n = 0$ and $|M_n| = 1$. The eigenvalues $\lambda_1, \lambda_2, \lambda_4$, and λ_5 only change signs at $M_n = 0$. Thus, the different cases are inflow or outflow and sub- or supersonic flow. A consequence is that sub- or supersonic flow affects which boundary conditions to choose but not the number of them. In fact, only the boundary condition corresponding to λ_3 (and hence $-\epsilon^2\lambda_3^{-1} \equiv \lambda_7$) changes sign at $|M_n| = 1$. With $\Lambda = \mathrm{diag}(\lambda_2, \lambda_3, \lambda_4, \lambda_5)$, the diagonal form of $\mathbf{E}$ is $\Lambda_{\mathbf{M}} = \mathrm{diag}(\lambda_1, \Lambda, -\epsilon^2\Lambda^{-1})$. In Table 1 the signs of the different eigenvalues are summarized, where $\lambda_6, \dots, \lambda_9$ denotes the diagonal entries of $-\epsilon^2\Lambda^{-1}$. Those with negative signs have to be supplied with boundary conditions. As mentioned above, since $\hat{\mathbf{n}}$ is the outward-pointing normal, negative values of M_n indicate inflow and positive values mean outflow.

In Table 2 the numbers of boundary conditions deduced from Table 1 for different flow cases are shown. They are in full agreement with the results from the Laplace transform technique derived in [14] and also in [15]. Note that in the Euler limit, i.e., $\epsilon \to 0$, the last four eigenvalues will become zero, and there are five nontrivial eigenvalues. In Table 3 the numbers of boundary conditions are displayed for the Euler case, $\epsilon \to 0$. The result agrees with the well-known theory for Euler equations.

At last, we can split $\tilde{w}$ given by (42) into $\tilde{w}^+$ and w^- corresponding to the positive and negative eigenvalues and perform step (v), such that well-posedness follows.

Remark 9. Though there are no numerical computations in this article, we would like to comment on some computational aspects. We assume that we know the exact boundary data ahead of time. This implies knowledge of the type of boundary (inflow/outflow, subsonic/supersonic) that we have at each point on the boundary as well as when one boundary type changes to another.

However, in computations, the numerical result might indicate that the assumed data are erroneous. In such a case, this procedure as well as other boundary condition procedures require an adjustment of the given data or location of the boundary for better accuracy.

3.5. Special case: $\boldsymbol{u_n = 0}$. The above derivation gives a set of boundary conditions that leads to a well-posed mathematical problem. However, it is assumed that $u_n \neq 0$, which excludes two cases: tangential flow and the important solid wall condition. We will treat the case $u_n = 0$ separately and redo the steps (iii)–(v). Throughout this paper, we have considered the Navier–Stokes equations linearized around the solution at the boundary, in this case $u_n = 0$. We obtain

$$\mathbf{A} = \begin{pmatrix} 0 & bn_1 & bn_2 & bn_3 & 0 & 0 & 0 & 0 & 0 \\ bn_1 & 0 & 0 & 0 & an_1 & -\epsilon & 0 & 0 & 0 \\ bn_2 & 0 & 0 & 0 & an_2 & 0 & -\epsilon & 0 & 0 \\ bn_3 & 0 & 0 & 0 & an_2 & 0 & 0 & -\epsilon & 0 \\ 0 & an_1 & an_2 & an_3 & 0 & 0 & 0 & 0 & -\epsilon \\ 0 & -\epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{44}$$

to which the previous rotation does not apply, since $\mathbf{A}$ is now singular. This leaves us with no other choice but to seek the eigenvalues and eigenvectors of this matrix. It turns out that it is now a simpler task than with $u_n \neq 0$. The result is presented below, and the details of the derivation are found in Appendix B.

Define $\mathbf{m_1}$ and $\mathbf{m_2}$ such that $\hat{\mathbf{n}}^T\mathbf{m_1} = 0$, $\hat{\mathbf{n}}^T\mathbf{m_2} = \mathbf{m_1}^T\mathbf{m_2} = 0$, and

$$\mu_{1,2} = -\frac{c^2}{2} \pm \sqrt{\frac{c^4}{4} + a^2\epsilon^2}.$$

Then,

$$\begin{aligned} \lambda_1 &= -\epsilon, & e_1 &= (0, \mathbf{m_1}^T, 0, \mathbf{m_1}^T, 0)^T, \\ \lambda_2 &= -\epsilon, & e_2 &= (0, \mathbf{m_2}^T, 0, \mathbf{m_2}^T, 0)^T, \\ \lambda_3 &= \epsilon, & e_3 &= (0, \mathbf{m_1}^T, 0, -\mathbf{m_1}^T, 0)^T, \\ \lambda_4 &= \epsilon, & e_4 &= (0, \mathbf{m_2}^T, 0, -\mathbf{m_2}^T, 0)^T, \\ \lambda_5 &= 0, & e_5 &= \left(1, 0, 0, 0, 0, \frac{b}{\epsilon}\hat{\mathbf{n}}, 0^T\right), \\ \lambda_6 &= \sqrt{\epsilon^2 - \mu_1}, & e_6 &= \left(b, \lambda_6\hat{\mathbf{n}}^T, -\frac{a\lambda_6^2}{\mu_1^2}, -\epsilon\hat{\mathbf{n}}^T, \frac{\epsilon a\lambda_6}{\mu_1^2}\right)^T, \\ \lambda_7 &= -\sqrt{\epsilon^2 - \mu_1}, & e_7 &= \left(b, \lambda_7\hat{\mathbf{n}}^T, -\frac{a\lambda_7^2}{\mu_1^2}, -\epsilon\hat{\mathbf{n}}^T, \frac{\epsilon a\lambda_7}{\mu_1^2}\right)^T, \\ \lambda_8 &= \sqrt{\epsilon^2 - \mu_2}, & e_8 &= \left(b, \lambda_8\hat{\mathbf{n}}^T, -\frac{a\lambda_8^2}{\mu_2^2}, -\epsilon\hat{\mathbf{n}}^T, \frac{\epsilon a\lambda_8}{\mu_2^2}\right)^T, \\ \lambda_9 &= -\sqrt{\epsilon^2 - \mu_2}, & e_9 &= \left(b, \lambda_9\hat{\mathbf{n}}^T, -\frac{a\lambda_9^2}{\mu_2^2}, -\epsilon\hat{\mathbf{n}}^T, \frac{\epsilon a\lambda_9}{\mu_2^2}\right)^T. \end{aligned} \tag{45}$$

Remark 10. With some algebra one can show that $\epsilon^2 \geq \mu_{1,2}$ such that the eigenvalues $\lambda_6, \dots, \lambda_9$ are real. In fact, since $\mathbf{A}$ is symmetric and the vectors $e_1, \dots, e_9$ are orthogonal and diagonalize $\mathbf{A}$, $\lambda_1, \dots, \lambda_9$ have to be real.

Above, step (iii) is performed and we turn to step (iv). We have

$$\Lambda^- = \mathrm{diag}(\lambda_1, \lambda_2, 0, 0, 0, 0, \lambda_7, 0, \lambda_9),$$
$$\Lambda^+ = \mathrm{diag}(0, 0, \lambda_3, \lambda_4, 0, \lambda_6, 0, \lambda_8, 0).$$

Remark 11. Note that we have four negative eigenvalues. This means that a boundary with $u_n = 0$ is classified as an outflow boundary.

Further, $\tilde{w} = X^T v$, where the column vectors of X are the eigenvectors. With $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)^T$, $\tilde{G}^V_{i\dots j} = (\tilde{G}^V_i, \dots, \tilde{G}^V_j)^T$, and the ith component of $\tilde{v}$ denoted by $\tilde{v}_i$, we obtain

$$\tilde{w} = \begin{pmatrix} \mathbf{m_1}^T(\tilde{\mathbf{u}} + \tilde{G}^V_{1\dots3}) \\ \mathbf{m_2}^T(\tilde{\mathbf{u}} + \tilde{G}^V_{1\dots3}) \\ \mathbf{m_1}^T(\tilde{\mathbf{u}} - \tilde{G}^V_{1\dots3}) \\ \mathbf{m_2}^T(\tilde{\mathbf{u}} - \tilde{G}^V_{1\dots3}) \\ v_1 + \frac{b}{\epsilon}\hat{\mathbf{n}}^T(\tilde{G}^V)_{1\dots3} \\ bv_1 + \hat{\mathbf{n}}^T(\lambda_6\tilde{\mathbf{u}} - \epsilon\tilde{G}^V_{1\dots3}) - \frac{a\lambda_6}{\mu_1^2}(\lambda_6 v_4 - \epsilon\tilde{G}^V_4) \\ bv_1 + \hat{\mathbf{n}}^T(\lambda_7\tilde{\mathbf{u}} - \epsilon\tilde{G}^V_{1\dots3}) - \frac{a\lambda_7}{\mu_1^2}(\lambda_7 v_4 - \epsilon\tilde{G}^V_4) \\ bv_1 + \hat{\mathbf{n}}^T(\lambda_8\tilde{\mathbf{u}} - \epsilon\tilde{G}^V_{1\dots3}) - \frac{a\lambda_8}{\mu_2^2}(\lambda_8 v_4 - \epsilon\tilde{G}^V_4) \\ bv_1 + \hat{\mathbf{n}}^T(\lambda_9\tilde{\mathbf{u}} - \epsilon\tilde{G}^V_{1\dots3}) - \frac{a\lambda_9}{\mu_2^2}(\lambda_9 v_4 - \epsilon\tilde{G}^V_4) \end{pmatrix}. \tag{46}$$

Finally, we can split $\tilde{w}$ into $\tilde{w}^+$ and $\tilde{w}^-$ as before and perform step (v), i.e., supply $\tilde{w}^-$ with boundary conditions to obtain a well-posed system.

Remark 12. There are two more cases where $u_n \neq 0$. Those are tangential flows with $|M_n| = 1$. To find the eigenvalues of $\mathbf{A}$ directly for $M_n = 1, -1$ is equally difficult as the general case, and we did not find roots in closed form.

3.6. Curvilinear coordinates. Until now, we have analyzed well-posed boundary conditions for the Navier–Stokes equations in a Cartesian coordinate system and a general domain. Considering numerical computations, that derivation suffices when using unstructured methods such as finite volume schemes. However, for structured methods, such as finite difference schemes, the Navier–Stokes equations are usually expressed in a curvilinear coordinate system. We have included a brief analysis in Appendix C showing that the Cartesian results are directly applicable in the curvilinear case through metric transformations.

4. Conclusions. We have proposed a step-by-step procedure to analyze a general time dependent partial differential equation in terms of well-posedness including boundary conditions. The procedure applied to the Euler equations results in the well-known characteristic boundary conditions. In this article we have applied the procedure to the three-dimensional Navier–Stokes equations on a general domain and obtained a novel set of well-posed boundary conditions.

Appendix A. The two-dimensional matrices. With very few comments and leaving out most details, we show the differences of the derivation in section 3 for the two-dimensional case.

With

$$B_{21} = B_{12} = \frac{B_{xy}}{2},$$

the symmetrized equations are

$$\tilde{u}_t + A_1\tilde{u}_x + A_2\tilde{u}_y = \epsilon(B_{11}\tilde{u}_{xx} + B_{22}\tilde{u}_{yy} + B_{12}\tilde{u}_{xy} + B_{21}\tilde{u}_{yx}).$$

The matrices are obtained by deleting the row and column referring to the u_3 component (see [10]). We introduce

$$\tilde{F}_v = B_{11}\tilde{u}_x + B_{21}\tilde{u}_y, \quad \tilde{G}_v = B_{22}\tilde{u}_y + B_{12}\tilde{u}_x,$$

such that

$$\frac{1}{2}\|\tilde{u}\|_t^2 + \oint_{\partial D} \frac{1}{2} \begin{pmatrix} \tilde{u} \\ \tilde{F}^V \end{pmatrix}^T \begin{pmatrix} A_1n_1 + A_2n_2 & -\epsilon I_4 \\ -\epsilon I_4 & 0_4 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{F}^V \end{pmatrix} = DI,$$

where $\hat{\mathbf{n}} = [n_1, n_2]$, $ds = \sqrt{dx^2 + dy^2}$, and $\tilde{F}^V = \tilde{F}_v n_1 + \tilde{G}_v n_2$.

By deleting the first component of $\tilde{F}^V$ yielding $\tilde{G}^V$, the matrix is reduced from an 8-by-8 matrix to a 7-by-7 matrix. With $\mathbf{u} = (u_1, u_2)$, we obtain

$$\begin{pmatrix} \tilde{u} \\ \tilde{F}^V \end{pmatrix}^T \begin{pmatrix} A_1n_1 + A_2n_2 & -\epsilon I_4 \\ -\epsilon I_4 & 0_4 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{F}^V \end{pmatrix} = \begin{pmatrix} \tilde{u} \\ \tilde{G}^V \end{pmatrix}^T \mathbf{A} \begin{pmatrix} \tilde{u} \\ \tilde{G}^V \end{pmatrix},$$

where

$$\mathbf{A} = \begin{pmatrix} \mathbf{u}\cdot\hat{\mathbf{n}} & bn_1 & bn_2 & 0 & 0 & 0 & 0 \\ bn_1 & \mathbf{u}\cdot\hat{\mathbf{n}} & 0 & an_1 & -\epsilon & 0 & 0 \\ bn_2 & 0 & \mathbf{u}\cdot\hat{\mathbf{n}} & an_2 & 0 & -\epsilon & 0 \\ 0 & an_1 & an_2 & \mathbf{u}\cdot\hat{\mathbf{n}} & 0 & 0 & -\epsilon \\ 0 & -\epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon & 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} A_{11} & A_{12} & 0_{14} \\ A_{21} & A_{22} & -\epsilon I_4 \\ 0_{41} & -\epsilon I_4 & 0_4 \end{pmatrix}.$$

The rotation of $\mathbf{A}$ is precisely similar,

$$\mathbf{R}^T\mathbf{A}\mathbf{R} = \begin{pmatrix} 1 & 0_{13} & 0_{13} \\ \bar{\alpha}^T & I_3 & 0_3 \\ \bar{\beta}^T & \bar{\gamma}^T & I_3 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & 0_{13} \\ A_{21} & A_{22} & -\epsilon I_3 \\ 0_{31} & -\epsilon I_3 & 0_3 \end{pmatrix} \begin{pmatrix} 1 & \bar{\alpha} & \bar{\beta} \\ 0_{31} & I_3 & \bar{\gamma} \\ 0_{31} & 0_3 & I_3 \end{pmatrix}$$
$$= \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} = \mathbf{E}.$$

The same solution is obtained,

$$\bar{\alpha} = -A_{11}^{-1}A_{12}, \quad \bar{\beta} = A_{11}^{-1}A_{12}E_{22}^{-1}, \quad \bar{\gamma} = -\epsilon E_{22}^{-1},$$

$$\mathbf{E} = \begin{pmatrix} A_{11} & 0_{14} & 0_{14} \\ 0_{41} & E_{22} & 0_4 \\ 0_{41} & 0_4 & -\epsilon^2 E_{22}^{-1} \end{pmatrix}, \qquad E_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12}.$$

The first eigenvalue of $\mathbf{E}$ is $\lambda_1 = A_{11} = u_n$, and the others are given by the eigenvalues of E_{22},

$$E_{22} = \begin{pmatrix} -\frac{b^2 n_1^2}{u_n} + u_n & -\frac{b^2 n_1 n_2}{u_n} & an_1 \\ -\frac{b^2 n_1 n_2}{u_n} & -\frac{b^2 n_2^2}{u_n} + u_n & an_2 \\ an_1 & an_2 & u_n \end{pmatrix},$$

$$\lambda_{2,3} = \frac{-b^2 + 2u_n^2 \pm \sqrt{b^4 + 4a^2u_n^2}}{2u_n}, \quad \lambda_4 = u_n,$$

where $n_1^2 + n_2^2 = 1$ and $u_n = \mathbf{u} \cdot \hat{\mathbf{n}}$. These can be simplified similarly as for the three-dimensional case.

The eigenvectors $Y = (y_2, y_3, y_4)$ are

$$y_2 = \begin{pmatrix} n_1 \\ n_2 \\ \frac{-\lambda_3 + u_n}{a} \end{pmatrix}, \quad y_3 = \begin{pmatrix} n_1 \\ n_2 \\ \frac{-\lambda_2 + u_n}{a} \end{pmatrix}, \quad y_4 = \begin{pmatrix} -n_2, \\ n_1 \\ 0 \end{pmatrix}. \tag{47}$$

Introduce the block matrix, $X = \text{diag}(1, Y, Y)$, such that $X^T \mathbf{E} X = \Lambda$, where $\Lambda = \text{diag}(u_n, \Lambda, -\epsilon^2 \Lambda)$. Let $\tilde{v} = (\tilde{u}^T, (\tilde{G}^V)^T)^T$; then $\tilde{v}^T \mathbf{A} \tilde{v} = \tilde{w}^T \Lambda \tilde{w}$, where $\tilde{w} = X^T R^{-1} \tilde{v} = M^{-1}\tilde{v}$ and $\Lambda = M^T \mathbf{A} M$. The matrices are

$$R^{-1} = \begin{pmatrix} 1 & -\bar{\alpha} & 0_{13} \\ 0_{31} & I_3 & -\bar{\gamma} \\ 0_{31} & 0_3 & I_3 \end{pmatrix}, \qquad M^{-1} = \begin{pmatrix} 1 & -\bar{\alpha} & 0_{14} \\ 0_{41} & Y^T & \epsilon \Lambda^{-1} Y^T \\ 0_{41} & 0_4 & Y^T \end{pmatrix},$$

where

$$\Lambda^{-} Y^T = \begin{pmatrix} \lambda_2^{-1} y_2 \\ \lambda_3^{-1} y_3 \\ \lambda_4^{-1} y_4 \end{pmatrix}, \quad \bar{\alpha} = \left(-\frac{b}{u_n} \hat{\mathbf{n}}, 0 \right), \quad M = \begin{pmatrix} 1 & \bar{\alpha} Y & \bar{\alpha} \Lambda^{-1} Y^T \\ 0_{31} & Y & -\epsilon Y \Lambda \\ 0_{31} & 0_3 & Y \end{pmatrix}.$$

In two dimensions, $\tilde{v}$ is

$$\tilde{v} = \left(\frac{b}{\rho} \frac{\tilde{\rho}}{\rho}, \tilde{u}_1, \tilde{u}_2, -\frac{b}{\sqrt{\gamma - 1}} \tilde{\rho} + \frac{1}{\rho a} \tilde{p}, \tilde{G}_1^V, \tilde{G}_2^V, \tilde{G}_3^V \right).$$

Then,

$$\tilde{w} = M^{-1} \tilde{v} = \begin{pmatrix} \tilde{v}_1 - \bar{\alpha} \cdot \tilde{v}_{2\ldots4} \\ y_2^T (\tilde{v}_{2\ldots4} - \epsilon \lambda_2^{-1} \tilde{G}^V) \\ y_3^T (\tilde{v}_{2\ldots4} - \epsilon \lambda_3^{-1} \tilde{G}^V) \\ y_4^T (\tilde{v}_{2\ldots4} - \epsilon \lambda_4^{-1} \tilde{G}^V) \\ y_2^T \tilde{G}^V \\ y_3^T \tilde{G}^V \\ y_4^T \tilde{G}^V \end{pmatrix}. \tag{48}$$

Appendix B. Diagonalization with $\boldsymbol{u_n = 0}$. Consider

$$\mathbf{A}e = \lambda e, \tag{49}$$

where $\mathbf{A}$ is given by (44), repeated here for convenience,

$$\mathbf{A} = \begin{pmatrix} 0 & bn_1 & bn_2 & bn_3 & 0 & 0 & 0 & 0 & 0 \\ bn_1 & 0 & 0 & 0 & an_1 & -\epsilon & 0 & 0 & 0 \\ bn_2 & 0 & 0 & 0 & an_2 & 0 & -\epsilon & 0 & 0 \\ bn_3 & 0 & 0 & 0 & an_2 & 0 & 0 & -\epsilon & 0 \\ 0 & an_1 & an_2 & an_3 & 0 & 0 & 0 & 0 & -\epsilon \\ 0 & -\epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\epsilon & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{50}$$

The structure of $\mathbf{A}$ suggests the following ansatz:

$$e_1 = (0, m_1, m_2, m_3, 0, m_1, m_2, m_3, 0)^T, \tag{51}$$

$$e_2 = (0, m_1, m_2, m_3, 0, -m_1, -m_2, -m_3, 0)^T, \tag{52}$$

$$e_3 = (m_4, m_5n_1, m_5n_2, m_5n_3, m_6, m_7n_1, m_7n_2, m_7n_3, m_8). \tag{53}$$

We will use the notation $\mathbf{m} = (m_1, m_2, m_3)^T$. With (51), equation (49) becomes

$$\begin{pmatrix} b\mathbf{\hat{n}^T m} \\ -\epsilon m_1 \\ -\epsilon m_2 \\ -\epsilon m_3 \\ a\mathbf{\hat{n}^T m} \\ -\epsilon m_1 \\ -\epsilon m_2 \\ -\epsilon m_3 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ m_1 \\ m_2 \\ m_3 \\ 0 \\ m_1 \\ m_2 \\ m_3 \\ 0 \end{pmatrix}. \tag{54}$$

With $\lambda = \lambda_1$ and $\mathbf{m} = \mathbf{m}_1$, the following choice satisfies the above equation, $\mathbf{\hat{n}}^T\mathbf{m_1} = 0$, and $\lambda_1 = -\epsilon$. Further, we may also choose a second solution $\mathbf{m} = \mathbf{m_2}$ and $\lambda_2 = -\epsilon$ such that $\mathbf{\hat{n}}^T\mathbf{m_2} = 0$ and $\mathbf{m_2}^T\mathbf{m_1} = 0$. Similarly, ansatz (52) yields

$$\lambda_3 = \epsilon, \quad \mathbf{\hat{n}}^T\mathbf{m_3} = 0, \tag{55}$$

$$\lambda_4 = \epsilon, \quad \mathbf{\hat{n}}^T\mathbf{m_4} = 0, \quad \mathbf{m_3}^T\mathbf{m_4} = 0. \tag{56}$$

In fact, we can let $\mathbf{m}_1 = \mathbf{m}_3$ and $\mathbf{m_2} = \mathbf{m_4}$. It is obvious that the vectors (51) and (52) will be orthogonal, and, by definition, they are orthogonal to (53). So far, four eigenvalues and eigenvectors out of nine are derived when we turn to the last ansatz (53). In this case (49) becomes

$$\begin{pmatrix} m_5 b \\ (bm_4 + am_6 - \epsilon m_7)n_1 \\ (bm_4 + am_6 - \epsilon m_7)n_2 \\ (bm_4 + am_6 - \epsilon m_7)n_3 \\ am_5 - \epsilon m_8 \\ -\epsilon m_5 n_1 \\ -\epsilon m_5 n_2 \\ -\epsilon m_5 n_3 \\ -\epsilon m_6 \end{pmatrix} = \lambda \begin{pmatrix} m_4 \\ m_5n_1 \\ m_5n_2 \\ m_5n_3 \\ m_6 \\ m_7n_1 \\ m_7n_2 \\ m_7n_3 \\ m_8 \end{pmatrix}, \tag{57}$$

where $n_1^2+n_2^2+n_3^3$ has been used. Note that the above system of equations reduces to only five equations by the choice of the eigenvector. Further, we have five unknowns, including λ. (One of the unknowns of the eigenvector drops out since it should only enter as a scaling.) We have

$$m_5 b = \lambda m_4, \tag{58}$$

$$b m_4 + a m_6 - \epsilon m_7 = \lambda m_5, \tag{59}$$

$$a m_5 - \epsilon m_8 = \lambda m_6, \tag{60}$$

$$-\epsilon m_5 = \lambda m_7, \tag{61}$$

$$-\epsilon m_6 = \lambda m_8. \tag{62}$$

In this case it turns out that the ansatz was satisfactory since five solutions to the system (58)–(62) exist.

The case we examine is the marginal case with $u_n = 0$, which leads us to expect one eigenvalue to be zero. Thusly, with $\lambda_5 = 0$ the following eigenvector is obtained:

$$e_5 = \left(1, 0, 0, 0, 0, \frac{b}{\epsilon} n_1, \frac{b}{\epsilon} n_2, \frac{b}{\epsilon} n_3, 0\right)^T. \tag{63}$$

Next, we solve full system (58)–(62) without assumptions on the solution. With $\mu = \epsilon^2 - \lambda^2$, a second degree equation in μ is obtained,

$$\mu^2 + (b + a^2)\mu - a^2\epsilon^2 = 0, \tag{64}$$

with the solutions

$$\mu_{1,2} = -\frac{b+a^2}{2} \pm \sqrt{\frac{(b+a^2)^2}{4} + a^2\epsilon^2} = -\frac{c^2}{2} \pm \sqrt{\frac{c^4}{4} + a^2\epsilon^2} \tag{65}$$

such that $\lambda_{6,7} = \pm\sqrt{\epsilon^2 - \mu_1}$ and $\lambda_{8,9} = \pm\sqrt{\epsilon^2 - \mu_2}$. For any of these λ's the eigenvector is given by

$$e = \begin{pmatrix} b \\ \lambda n_1 \\ \lambda n_2 \\ \lambda n_3 \\ -\frac{a\lambda^2}{\epsilon^2 - \lambda^2} \\ -\epsilon n_1 \\ -\epsilon n_2 \\ -\epsilon n_3 \\ \frac{\epsilon a \lambda}{\epsilon^2 - \lambda^2} \end{pmatrix}. \tag{66}$$

Next, we have to show that the different eigenvectors obtained from (66) are orthogonal to each other. We distinguish between two cases: 1. any of the eigenvalues derived from μ_1, denoted by ξ_1, and another eigenvalue ξ_2 derived from μ_2; 2. both eigenvalues $\xi_{1,2}$ derived from the same μ.

The scalar product is

$$e(\xi_1)^T \cdot e(\xi_2) = b + \xi_1\xi_2 + \frac{a^2\xi_1^2\xi_2^2}{(\epsilon^2 - \xi_1^2)(\epsilon^2 - \xi_2^2)} + \epsilon^2 + \frac{\epsilon^2 a^2 \xi_1 \xi_2}{(\epsilon^2 - \xi_1^2)(\epsilon^2 - \xi_2^2)}. \tag{67}$$

Case 1. For a general quadratic equation $x^2+px+q=0$ the roots fulfill $x_1x_2=q$ and $x_1+x_2=-p$. When applied to (64) this implies

$$\mu_1\mu_2 = (\epsilon^2-\xi_1^2)(\epsilon^2-\xi_2^2) = -a^2\epsilon^2, \tag{68}$$

$$\mu_1+\mu_2 = -(b+a^2). \tag{69}$$

Thus, (67) is

$$\begin{aligned} b+\xi_1\xi_2 - \frac{\xi_1^2\xi_2^2}{\epsilon^2} + \epsilon^2 - \xi_1\xi_2 \\ = b+\epsilon^2 + \frac{(\epsilon^2-\mu_1)(\epsilon^2-\mu_2)}{\epsilon^2} \\ = b+\epsilon^2-(\epsilon^2-(\mu_1+\mu_2)-a^2) \\ = b-(b+a^2)+a^2 = 0. \end{aligned} \tag{70}$$

Case 2. In this case the following relations hold:

$$\lambda^2 = \xi_1^2 = \xi_2^2, \tag{71}$$

$$\lambda = \xi_1 = -\xi_2, \tag{72}$$

$$\begin{aligned} \lambda^2 = -\xi_1\xi_2 = (\mu-\epsilon^2), \\ (\epsilon^2-\xi_{1,2}^2) = \mu. \end{aligned} \tag{73}$$

Then (67) becomes, after multiplying by $(\epsilon^2-\lambda^2)^2$,

$$\begin{aligned} (\epsilon^2-\lambda^2)^2(b-\lambda^2+\epsilon^2)+a^2\lambda^4-\epsilon^2a^2\lambda^2 \\ = (b-\lambda^2+\epsilon^2)(\epsilon^2-\lambda^2)^2+a^2\lambda^2(\lambda^2-\epsilon^2) \\ = (\lambda^2-\epsilon^2)((b+(\epsilon-\lambda^2))(\epsilon^2-\lambda^2)+a^2\lambda^2) \\ = -\mu((b+\mu)\mu+a^2(\mu-\epsilon^2)) \\ = -\mu(\mu^2+(b+a^2)\mu-a^2\epsilon^2) = 0, \end{aligned}$$

where the last equality is due to (64).

One should also normalize these vectors to formally obtain the eigenvectors of the matrix **A**. With this done, we conclude that in the case of neither inflow nor outflow, the above derivation gives the eigenvalues and eigenvectors of the linearized Navier–Stokes equations in three dimensions.

Appendix C. Curvilinear coordinates.

C.1. Metric relations. Let x, y, z denote the usual Cartesian coordinates. Consider the following coordinate transformation:

$$\xi = \xi(x,y,z), \quad \eta = \eta(x,y,z), \quad \zeta = \zeta(x,y,z).$$

The Jacobian is defined as

$$\mathbf{J} = \begin{pmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{pmatrix}. \tag{74}$$

Let $\bar{x} = (x,y,z) = (x_1,x_2,x_3)$ and $\bar{\xi} = (\xi,\eta,\zeta) = (\xi_1,\xi_2,\xi_3)$. Then we can formally express the Jacobian as $\mathcal{D}_{\bar{\xi}}\bar{x} = \mathbf{J}$. The following relation holds:

$$I = \mathcal{D}_{\bar{x}}\bar{x}(\bar{\xi}) = \mathcal{D}_{\bar{\xi}}\bar{x}\mathcal{D}_{\bar{x}}\bar{\xi}. \tag{75}$$

Hence,

$$\mathbf{J}^{-1} = \mathcal{D}_{\bar{x}}\xi = \begin{pmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{pmatrix}. \tag{76}$$

However, $\mathbf{J}^{-1}$ can also be obtained directly by inverting (74),

$$\mathbf{J}^{-1} = \mathcal{D}_{\bar{x}}\xi = \frac{1}{J}\begin{pmatrix} y_\eta z_\zeta - y_\zeta z_\eta & -(x_\eta z_\zeta - x_\zeta z_\eta) & x_\eta y_\zeta - x_\zeta y_\eta \\ -(y_\xi z_\zeta - y_\zeta z_\xi) & x_\xi z_\zeta - x_\zeta z_\xi & -(x_\xi y_\zeta - x_\zeta y_\xi) \\ y_\xi z_\eta - y_\eta z_\xi & -(x_\xi z_\eta - x_\eta z_\xi) & x_\xi y_\eta - x_\eta y_\xi \end{pmatrix}, \tag{77}$$

where J denotes the determinant of the Jacobian. Then (76) and (77) give relations between the different metric coefficients. For example, we note that

$$\begin{aligned} (J\xi_x)_\xi + (J\eta_x)_\eta + (J\zeta_x)_\zeta &= (y_\eta z_\zeta - y_\zeta z_\eta)_\xi - (y_\xi z_\zeta - y_\zeta z_\xi)_\eta + (y_\xi z_\eta - y_\eta z_\xi)_\zeta = 0, \\ (J\xi_y)_\xi + (J\eta_y)_\eta + (J\zeta_y)_\zeta &= -(x_\eta z_\zeta - x_\zeta z_\eta)_\xi + (x_\xi z_\zeta - x_\zeta z_\xi)_\eta - (x_\xi z_\eta - x_\eta z_\xi)_\zeta = 0, \\ (J\xi_z)_\xi + (J\eta_z)_\eta + (J\zeta_z)_\zeta &= (x_\eta y_\zeta - x_\zeta y_\eta)_\xi - (x_\xi y_\zeta - x_\zeta y_\xi)_\eta + (x_\xi y_\eta - x_\eta y_\xi)_\zeta = 0, \end{aligned} \tag{78}$$

which will be used below.

C.2. Curvilinear Navier–Stokes equations. Consider the linearized and symmetrized Navier–Stokes equations (9), restated here for convenience,

$$\begin{aligned} \tilde{u}_t + \quad & (A_1\tilde{u} - \epsilon(B_{11}\tilde{u}_x + B_{12}\tilde{u}_y + B_{13}\tilde{u}_z))_x \\ & + (A_2\tilde{u} - \epsilon(B_{22}\tilde{u}_y + B_{23}\tilde{u}_z + B_{12}\tilde{u}_x))_y \\ & + (A_3\tilde{u} - \epsilon(B_{33}\tilde{u}_z + B_{32}\tilde{u}_y + B_{13}\tilde{u}_x))_z = 0 \end{aligned}$$

or

$$\begin{aligned} \tilde{u}_t + (F^I - \epsilon\tilde{F}_v)_x + (G^I - \epsilon\tilde{G}_v)_y + (H^I - \epsilon\tilde{H}_v)_z \\ = \tilde{u}_t + F_x + G_y + H_z = 0. \end{aligned} \tag{79}$$

Multiply (79) by J and make the change of coordinates,

$$\begin{aligned} 0 &= (J\tilde{u})_t + JF_x + JG_y + JH_z \\ &= (J\tilde{u})_t + J\xi_x F_\xi + J\eta_x F_\eta + J\zeta_x F_\zeta \\ &\quad + J\xi_y G_\xi + J\eta_y G_\eta + J\zeta_y G_\zeta \\ &\quad + J\xi_z H_\xi + J\eta_z H_\eta + J\zeta_z H_\zeta. \end{aligned} \tag{80}$$

Reformulating (80) yields

$$\begin{aligned} (J\tilde{u})_t &+ (J\xi_x F + J\xi_y G + J\xi_z H)_\xi - R_1 \\ &+ (J\eta_x F + J\eta_y G + J\eta_z H)_\eta - R_2 \\ &+ (J\zeta_x F + J\zeta_y G + J\zeta_z H)_\zeta - R_3, \end{aligned}$$

where

$$
\begin{aligned}
R_1 &= (J\xi_x)_\xi F + (J\xi_y)_\xi G + (J\xi_z)_\xi H,\\
R_2 &= (J\eta_x)_\eta F + (J\eta_y)_\eta G + (J\eta_z)_\eta H,\\
R_3 &= (J\zeta_x)_\zeta F + (J\zeta_y)_\zeta G + (J\zeta_z)_\zeta H.
\end{aligned}
$$

By using the metric relations in (78), we obtain

$$
\begin{aligned}
R_1 + R_2 + R_3 = F(&(J\xi_x)_\xi + (J\eta_x)_\eta + (J\zeta_x)_\zeta)\\
+ G(&(J\xi_y)_\xi + (J\eta_y)_\eta + (J\zeta_y)_\zeta)\\
+ H(&(J\xi_z)_\xi + (J\eta_z)_\eta + (J\zeta_z)_\zeta) = 0.
\end{aligned}
$$

Define

$$
\begin{aligned}
\hat{F} &= (J\xi_x F + J\xi_y G + J\xi_z H),\\
\hat{G} &= (J\eta_x F + J\eta_y G + J\eta_z H),\\
\hat{H} &= (J\zeta_x F + J\zeta_y G + J\zeta_z H)
\end{aligned}
$$

such that

$$
0 = (J\tilde{u})_t + JF_x + JG_y + JH_z = (J\tilde{u})_t + \hat{F}_\xi + \hat{G}_\eta + \hat{H}_\zeta. \tag{81}
$$

Next, we express the new fluxes in curvilinear coordinates. We obtain

$$
\begin{aligned}
\hat{F}^I &= (J\xi_x F^I + J\xi_y G^I + J\xi_z H^I) = J(\xi_x A_1 + \xi_y A_2 + \xi_z A_3)u,\\
\hat{G}^I &= (J\eta_x F^I + J\eta_y G^I + J\eta_z H^I) = J(\eta_x A_1 + \eta_y A_2 + \eta_z A_3)u,\\
\hat{H}^I &= (J\zeta_x F^I + J\zeta_y G^I + J\zeta_z H^I) = J(\zeta_x A_1 + \zeta_y A_2 + \zeta_z A_3)u,
\end{aligned} \tag{82}
$$

and

$$
\begin{aligned}
\hat{F}_v &= (J\xi_x \tilde{F}_v + J\xi_y \tilde{G}_v + J\xi_z \tilde{H}_v),\\
\hat{G}_v &= (J\eta_x \tilde{F}_v + J\eta_y \tilde{G}_v + J\eta_z \tilde{H}_v),\\
\hat{H}_v &= (J\zeta_x \tilde{F}_v + J\zeta_y \tilde{G}_v + J\zeta_z \tilde{H}_v),
\end{aligned} \tag{83}
$$

where

$$
\begin{aligned}
\tilde{F}_v &= \tilde{B}_{11}\tilde{u}_\xi + \tilde{B}_{12}\tilde{u}_\eta + \tilde{B}_{13}\tilde{u}_\zeta,\\
\tilde{G}_v &= \tilde{B}_{22}\tilde{u}_\eta + \tilde{B}_{23}\tilde{u}_\zeta + \tilde{B}_{12}\tilde{u}_\xi,\\
\tilde{H}_v &= \tilde{B}_{33}\tilde{u}_\zeta + \tilde{B}_{32}\tilde{u}_\eta + \tilde{B}_{13}\tilde{u}_\xi,
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{B}_{11} &= B_{11}\xi_x + B_{12}\xi_y + B_{13}\xi_z, & \tilde{B}_{12} &= B_{11}\eta_x + B_{12}\eta_y + B_{13}\eta_z,\\
\tilde{B}_{13} &= B_{11}\zeta_x + B_{12}\zeta_y + B_{13}\zeta_z, & \tilde{B}_{22} &= B_{22}\xi_y + B_{23}\xi_z + B_{12}\xi_x,\\
\tilde{B}_{23} &= B_{22}\eta_y + B_{23}\eta_z + B_{12}\eta_x, & \tilde{B}_{21} &= B_{22}\zeta_y + B_{23}\zeta_z + B_{12}\zeta_x,\\
\tilde{B}_{33} &= B_{33}\xi_z + B_{32}\xi_y + B_{13}\xi_x, & \tilde{B}_{32} &= B_{33}\eta_z + B_{32}\eta_y + B_{13}\eta_x,\\
\tilde{B}_{31} &= B_{33}\zeta_z + B_{32}\zeta_y + B_{13}\zeta_x.
\end{aligned}
$$

C.3. Energy estimate. Next, we turn to the well-posedness of (81). We apply the energy method and derive the boundary terms. Our aim is to relate the boundary terms in curvilinear coordinates to those derived in $\bar{x}$-space.

First we note that

$$dxdydz = Jd\xi d\eta d\zeta. \tag{84}$$

Further, we use the notation $D_{\bar{\xi}}$ in $\bar{\xi}$-space for the image of the domain $D_{\bar{x}}$ in $\bar{x}$-space.

Apply the energy method to (81) to obtain

$$\begin{aligned} 0 = &\int_{D_{\bar{\xi}}} \tilde{u}^T \tilde{u}_t J d\xi d\eta d\zeta + \int_{D_{\bar{\xi}}} \tilde{u}^T (\hat{F}^I_\xi + \hat{G}^I_\eta + \hat{H}^I_\zeta) d\xi d\eta d\zeta \\ &- \epsilon \int_{D_{\bar{\xi}}} \tilde{u}^T ((\hat{F}_v)_\xi + (\hat{G}_v)_\eta + (\hat{H}_v)_\zeta) d\xi d\eta d\zeta = \int_{D_{\bar{x}}} \tilde{u}^T \tilde{u}_t dxdydz + I_1 - \epsilon I_2, \end{aligned} \tag{85}$$

$$\begin{aligned} I_2 &= \int_{D_{\bar{\xi}}} (\tilde{u}^T \hat{F}_v)_\xi + (\tilde{u}^T \hat{G}_v)_\eta + (\tilde{u}^T \hat{H}_v)_\zeta d\xi d\eta d\zeta \\ &\quad - \int_{D_{\bar{\xi}}} \tilde{u}^T_\xi (\hat{F}_v)_\xi + \tilde{u}^T_\eta (\hat{G}_v)_\eta + \tilde{u}^T_\zeta (\hat{H}_v)_\zeta d\xi d\eta d\zeta \\ &= \int_{D_{\bar{\xi}}} (\tilde{u}^T \hat{F}_v)_\xi + (\tilde{u}^T \hat{G}_v)_\eta + (\tilde{u}^T \hat{H}_v)_\zeta d\xi d\eta d\zeta - DI \\ &= \oint_{\Gamma_{\bar{\xi}}} (\tilde{u}^T \hat{F}_v, \tilde{u}^T \hat{G}_v, \tilde{u}^T \hat{H}_v) \cdot \mathbf{n}_{\bar{\xi}} ds_{\bar{\xi}} - DI \\ &= \oint_{\Gamma_{\bar{\xi}}} \tilde{u}^T \hat{F}^V ds_{\bar{\xi}} - DI, \end{aligned} \tag{86}$$

where $\mathbf{n}_{\bar{\xi}} = (n_\xi, n_\eta, n_\zeta)$ and $ds_{\bar{\xi}}$ denote the outward-pointing normal and surface element in $\bar{\xi}$-space, respectively. Further, $\hat{F}^V = \hat{F}_v n_\xi + \hat{G}_v n_\eta + \hat{H}_v n_\zeta$. DI denotes a dissipative term and is equal to DI defined in subsection 3.2.

$$\begin{aligned} I_1 &= \int_{D_{\bar{\xi}}} \tilde{u}^T (\hat{F}^I_\xi + \hat{G}^I_\eta + \hat{H}^I_\zeta) d\xi d\eta d\zeta \\ &= \int_{D_{\bar{\xi}}} \tilde{u}^T (J\xi_x A_1 \tilde{u} + J\xi_y A_2 \tilde{u} + J\xi_z A_3 \tilde{u})_\xi \\ &\quad + \tilde{u}^T (J\eta_x A_1 \tilde{u} + J\eta_y A_2 \tilde{u} + J\eta_z A_3 \tilde{u})_\eta \\ &\quad + \tilde{u}^T (J\zeta_x A_1 \tilde{u} + J\zeta_y A_2 \tilde{u} + J\zeta_z A_3 \tilde{u})_\zeta d\xi d\eta d\zeta. \end{aligned}$$

Next, we use relations of the type

$$\begin{aligned} \tilde{u}^T (J\xi_x A_1 \tilde{u})_\xi &= (J\xi_x)_\xi \tilde{u}^T A_1 \tilde{u} + (J\xi_x) \left(\frac{1}{2} \tilde{u}^T A_1 \tilde{u} \right)_\xi \\ &= (J\xi_x)_\xi \tilde{u}^T A_1 \tilde{u} + \left(J\xi_x \frac{1}{2} \tilde{u}^T A_1 \tilde{u} \right)_\xi - (J\xi_x)_\xi \left(\frac{1}{2} \tilde{u}^T A_1 \tilde{u} \right) \\ &= \left(J\xi_x \frac{1}{2} \tilde{u}^T A_1 \tilde{u} \right)_\xi + (J\xi_x)_\xi \left(\frac{1}{2} \tilde{u}^T A_1 \tilde{u} \right) \end{aligned}$$

to obtain

$$
\begin{aligned}
I_1 = \int_{D_{\bar\xi}} & \left(J\xi_x \frac{1}{2}\tilde u^T A_1 \tilde u\right)_\xi + \left(J\xi_y \frac{1}{2}\tilde u^T A_2 \tilde u\right)_\xi + \left(J\xi_z \frac{1}{2}\tilde u^T A_3 \tilde u\right)_\xi \\
& + \left(J\eta_x \frac{1}{2}\tilde u^T A_1 \tilde u\right)_\eta + \left(J\eta_y \frac{1}{2}\tilde u^T A_2 \tilde u\right)_\eta + \left(J\eta_z \frac{1}{2}\tilde u^T A_3 \tilde u\right)_\eta \\
& + \left(J\zeta_x \frac{1}{2}\tilde u^T A_1 \tilde u\right)_\zeta + \left(J\zeta_y \frac{1}{2}\tilde u^T A_2 \tilde u\right)_\zeta + \left(J\zeta_z \frac{1}{2}\tilde u^T A_3 \tilde u\right)_\zeta \\
& + \frac{1}{2}\tilde u^T A_1 \tilde u (J\xi_x)_\xi + \frac{1}{2}\tilde u^T A_1 \tilde u (J\eta_x)_\eta + \frac{1}{2}\tilde u^T A_1 \tilde u (J\zeta_x)_\zeta \\
& + \frac{1}{2}\tilde u^T A_2 \tilde u (J\xi_y)_\xi + \frac{1}{2}\tilde u^T A_2 \tilde u (J\eta_y)_\eta + \frac{1}{2}\tilde u^T A_2 \tilde u (J\zeta_y)_\zeta \\
& + \frac{1}{2}\tilde u^T A_3 \tilde u (J\xi_z)_\xi + \frac{1}{2}\tilde u^T A_3 \tilde u (J\eta_z)_\eta + \frac{1}{2}\tilde u^T A_3 \tilde u (J\zeta_z)_\zeta \, d\xi d\eta d\zeta.
\end{aligned} \tag{87}
$$

Hence, by using (78), the last three rows of (87) are identically zero:

$$
I_1 = \oint_{\Gamma_{\bar\xi}} \frac{1}{2}(\tilde u^T(\hat A_1)\tilde u, \tilde u^T(\hat A_2)\tilde u, \tilde u^T(\hat A_3)\tilde u) \cdot \mathbf{n}_{\bar\xi} ds_{\bar\xi}, \tag{88}
$$

where

$$
\begin{aligned}
\hat A_1 &= (A_1 J\xi_x + A_2 J\xi_y + A_3 J\xi_z), \\
\hat A_2 &= (A_1 J\eta_x + A_2 J\eta_y + A_3 J\eta_z), \\
\hat A_3 &= (A_1 J\zeta_x + A_2 J\zeta_y + A_3 J\zeta_z).
\end{aligned}
$$

By inserting (86) and (88) into (85), we obtain

$$
\begin{aligned}
& 2\int_{D_{\bar x}} \tilde u^T \tilde u_t dx dy dz \\
& + \oint_{\Gamma_{\bar\xi}} (\tilde u^T(\hat A_1)\tilde u, \tilde u^T(\hat A_2)\tilde u, \tilde u^T(\hat A_3)\tilde u) \cdot \mathbf{n}_{\bar\xi} ds_{\bar\xi} - \epsilon\left(\oint_{\Gamma_{\bar\xi}} 2\tilde u^T \hat F^V ds_{\bar\xi} - DI\right) \\
& = \|\tilde u\|_t^2 + \oint_{\Gamma_{\bar\xi}} \begin{pmatrix} \tilde u \\ \hat F^V \end{pmatrix} \begin{pmatrix} (\hat A_1, \hat A_2, \hat A_3)\cdot \mathbf{n}_{\bar\xi} & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde u \\ \hat F^V \end{pmatrix} ds_{\bar\xi} - DI \\
& = \|\tilde u\|_t^2 + \oint_{\Gamma_{\bar\xi}} \begin{pmatrix} \tilde u \\ \hat F^V \end{pmatrix} \hat{\mathbf{A}} \begin{pmatrix} \tilde u \\ \hat F^V \end{pmatrix} ds_{\bar\xi} - DI = 0.
\end{aligned} \tag{89}
$$

The form (89) is completely similar to the one in the $\bar x$-system. As mentioned earlier, the domain in $\bar\xi$-space is a cube. Hence, $\mathbf{n}_{\bar\xi}$ is particularly simple. It is a unit vector in the coordinate directions, $\pm e_\xi, \pm e_\eta, \pm e_\zeta$, on the boundary of the computational domain, $0 \le \xi \le 1$, $0 \le \eta \le 1$, $0 \le \zeta \le 1$. The full formulation for the cube is

$$
\begin{aligned}
\|\tilde{u}\|_t^2 - & \int_{\xi=0} \begin{pmatrix} \tilde{u} \\ \hat{\mathbf{F}}_{\mathbf{v}} \end{pmatrix} \begin{pmatrix} \hat{A}_1 & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} ds_{\bar{\xi}} \\
& + \int_{\xi=1} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} \begin{pmatrix} \hat{A}_1 & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} ds_{\bar{\xi}} \\
& - \int_{\eta=0} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} \begin{pmatrix} \hat{A}_2 & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} ds_{\bar{\xi}} \\
& + \int_{\eta=1} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} \begin{pmatrix} \hat{A}_2 & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} ds_{\bar{\xi}} \\
& - \int_{\zeta=0} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} \begin{pmatrix} \hat{A}_3 & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} ds_{\bar{\xi}} \\
& + \int_{\zeta=1} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} \begin{pmatrix} \hat{A}_3 & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} ds_{\bar{\xi}} = DI.
\end{aligned} \tag{90}
$$

Note that $ds_{\bar{\xi}}$ is different in the different coordinate directions. As a last step we will express one of the integrals in (90) in $\bar{x}$- space. Consider, for example,

$$
\begin{aligned}
& - \int_{\xi=0} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} \begin{pmatrix} \hat{A}_1 & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} ds_{\bar{\xi}} \\
&= \int_{\xi=0} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} \begin{pmatrix} -A_1 J\xi_x - A_2 J\xi_y - A_3 J\xi_z & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \hat{F}^V \end{pmatrix} ds_{\bar{\xi}} \\
&= \int_{\xi=0} \begin{pmatrix} \tilde{u} \\ \frac{\hat{F}^V}{JT_1} \end{pmatrix} \begin{pmatrix} -A_1 \frac{\xi_x}{T_1} - A_2 \frac{\xi_y}{T_1} - A_3 \frac{\xi_z}{T_1} & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \frac{\hat{F}^V}{JT_1} \end{pmatrix} JT_1 ds_{\bar{\xi}} \\
&= \int_{\xi=0} \begin{pmatrix} \tilde{u} \\ \frac{\hat{F}^V}{JT_1} \end{pmatrix} \begin{pmatrix} A_1 n_1 + A_2 n_2 + A_3 n_3 & -\epsilon I \\ -\epsilon I & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \frac{\hat{F}^V}{JT_1} \end{pmatrix} JT_1 ds_{\bar{\xi}},
\end{aligned} \tag{91}
$$

where $T_1 = \sqrt{(\xi_x)^2 + (\xi_y)^2 + (\xi_z)^2}$ and $n_1^2 + n_2^2 + n_3^2 = 1$. In fact, (n_1, n_2, n_3) is equal to the normal in the $\bar{x}$- system. This is easily seen by the following. Denote by $\mathbf{r} = (x, y, z)$ a position vector in space. The unnormalized normal vector at $\xi = 0$ is

$$
\begin{aligned}
& \frac{\partial \mathbf{r}}{\partial \eta} \times \frac{\partial \mathbf{r}}{\partial \zeta} = (x_\eta, y_\eta, z_\eta) \times (x_\zeta, y_\zeta, z_\zeta) \\
&= (y_\eta z_\zeta - z_\eta y_\zeta, -(x_\eta z_\zeta - z_\eta x_\zeta), x_\eta y_\zeta - y_\eta x_\zeta) = JT_1(n_1, n_2, n_3),
\end{aligned} \tag{92}
$$

where (76) and (77) have been used. Hence the matrices appearing in (91) and (23) are equal. Next, we will show that the vectors in (91) and (23) are also equal. We have

$$
\begin{aligned}
& \frac{\hat{F}^V}{JT_1} = \frac{\hat{F}_v \cdot 1 + \hat{G}_v \cdot 0 + \hat{H}_v \cdot 0}{JT_1} = \frac{\hat{F}_v}{JT_1} \\
&= \frac{(\xi_x \tilde{F}_v + \xi_y \tilde{G}_v + \xi_z \tilde{H}_v)}{T_1} = \tilde{F}_v n_1 + \tilde{G}_v n_2 + \tilde{H}_v n_3 = \tilde{F}^V.
\end{aligned}
$$

At last, we find

$$
ds_{\bar{x}} = \left| \frac{\partial \mathbf{r}}{\partial \eta} \times \frac{\partial \mathbf{r}}{\partial \zeta} \right| ds_{\bar{\xi}} = JT_1 ds_{\bar{\xi}}, \tag{93}
$$

implying that (91) and (23) are equal.

The other boundaries can be treated similarly. To summarize, we have shown that the relations in $\bar{x}$-space are completely equivalent to those in $\bar{\xi}$-space.

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