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ORIGINAL ARTICLE

Well-posedness and regularity for an Euler–Bernoulli plate with variable coefficients and boundary control and observation

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Abstract The open loop system of an Euler–Bernoulli plate with variable coefficients and partial boundary Neumann control and collocated observation is considered. Using the geometric multiplier method on Riemannian manifolds, we show that the system is well-posed in the sense of D. Salamon and regular in the sense of G. Weiss. Moreover, we determine that the feedthrough operator of this system is zero. The result implies in particular that the exact controllability of the open-loop system is equivalent to the exponential stability of the closed-loop system under proportional output feedback.

Keywords Euler-Bernoulli plate · Well-posed and regular system · Boundary control and observation

1 Introduction

Well-posed and regular linear systems are a quite general class of linear infinite-dimensional systems, which cover many control systems described by partial differential equations with actuators and sensors supported at isolated points, sub-domain,

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or on a part of the boundary of the spatial region. This class of infinite-dimensional systems, although the input and output operators are allowed to be unbounded, possess many properties that make them similar in many ways to finite-dimensional systems.

The abstract theory for well-posed and regular linear systems has already been quite fruitful. However, the well-posedness and regularity are verified only for a few control systems described by multi-dimensional partial differential equations and in particular, to our best knowledge, there are no examples with variable coefficients analyzed in the literature. Concerning systems with constant coefficients, the well-posedness and regularity of a multi-dimensional heat equation with both Dirichlet and Neumann type boundary controls were established in [3]. For a wave equation with boundary Dirichlet input and collocated output, the well-posedness was proved in [1] and the regularity was proved in [9]. The well-posedness and regularity for multi-dimensional Schrödinger and Euler–Bernoulli equations were established in [7,8,12]. Although Remark 4.1 of [12] mentioned some references for PDEs with variable coefficients, these earlier results mainly concern with observability/controllability and stability, not well-posedness and regularity.

The objective of this paper is to generalize the results for the Euler–Bernoulli plate [8,12] to the variable coefficients case, which occurs often for the plate in practice when the material consisting of the plate is not uniform. The system is described by the following Euler–Bernoulli plate with partial boundary Neumann control and collocated observation:

$$\begin{cases} w_{tt}(x,t) + \mathcal{A}^2 w(x,t) = 0, & x = (x_1, x_2, \dots, x_n) \in \Omega, \quad t > 0, \\ w(x,t) = 0, & x \in \partial\Omega, \quad t \geq 0, \\ \dfrac{\partial w(x,t)}{\partial \nu_{\mathcal{A}}} = 0, & x \in \Gamma_1, \quad t \geq 0, \\ \dfrac{\partial w(x,t)}{\partial \nu_{\mathcal{A}}} = u(x,t), & x \in \Gamma_0, \quad t \geq 0, \\ y(x,t) = -\mathcal{A}(\mathscr{A}^{-1} w_t(x,t)), & x \in \Gamma_0, \quad t \geq 0, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 2)$ is an open bounded region with smooth boundary $\partial\Omega =: \Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$. Γ_0, Γ_1 are disjoint parts of the boundary relatively open in $\partial\Omega$, $\text{int}(\Gamma_0) \neq \emptyset$.

$$\mathcal{A}w(x,t) := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial w(x,t)}{\partial x_j}\right), \quad D(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega),$$
$$\mathscr{A}\psi := \mathcal{A}^2\psi, \; D(\mathscr{A}) = H^4(\Omega) \cap H_0^2(\Omega),$$

and for some constant $a > 0$,

$$a_{ij}(x) = a_{ji}(x) \in C^4(\mathbb{R}^n), \quad \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\overline{\xi_j} \geq a\sum_{i=1}^{n} |\xi_i|^2, \qquad \forall\, x \in \Omega,$$
$$\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{C}^n, \tag{2}$$

$$\nu_{\mathcal{A}} := \left(\sum_{k=1}^{n} \nu_k a_{k1}(x), \sum_{k=1}^{n} \nu_k a_{k2}(x), \ldots, \sum_{k=1}^{n} \nu_k a_{kn}(x) \right),$$
$$\frac{\partial}{\partial \nu_{\mathcal{A}}} := \sum_{i,j=1}^{n} \nu_i a_{ij}(x) \frac{\partial}{\partial x_j}, \tag{3}$$

where $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ is the unit normal of $\partial\Omega$ pointing towards the exterior of Ω. u is the input function (or control) and y is the output function (or observation).

Let $\mathcal{H} = L^2(\Omega) \times H^{-2}(\Omega)$ and $U = L^2(\Gamma_0)$. The following theorem is the generalization of Theorem 4.15 of [12], where the coefficients of the system (1) are considered constant.

Theorem 1.1 *Let $T > 0$, $(w_0, w_1) \in \mathcal{H}$ and $u \in L^2(0, T; U)$. Then there exists a unique solution $(w, w_t) \in C([0, T]; \mathcal{H})$ to the system (1), which satisfies $w(\cdot, 0) = w_0$ and $w_t(\cdot, 0) = w_1$. Moreover, there exists a constant $C_T > 0$, independent of (w_0, w_1, u), such that*

$$\|(w(\cdot, T), w_t(\cdot, T))\|_{\mathcal{H}}^2 + \|y\|_{L^2(0,T;U)}^2 \le C_T \left[\|(w_0, w_1)\|_{\mathcal{H}}^2 + \|u\|_{L^2(0,T;U)}^2 \right].$$

Theorem 1.1 implies that the open-loop system (1) is well-posed in the sense of D. Salamon with the state space $\mathcal{H}$, input and output space U [15]. From this result and Theorem 2.2 of [2] (see also Theorem 3 of [6]), we have immediately the following corollary.

Corollary 1.1 *The system (1) is exactly controllable in some time interval $[0, T]$ if and only if its closed-loop system under the proportional output feedback $u = -ky$, $k > 0$ is exponentially stable.*

For the conditions of the exact controllability for the system (1), we refer reader to Theorem 1.3 of [21]. The above equivalent result is new for the exponential stability of the system (1). The following theorem is the generalization of Theorem 1.2 of [8], where the coefficients of the system (1) are considered constant.

Theorem 1.2 *The system (1) is regular. More precisely, if $w(\cdot, 0) = w_t(\cdot, 0) = 0$ and u is a step input: $u(\cdot, t) \equiv u(\cdot) \in U$, then the corresponding output y satisfies*

$$\lim_{\sigma \to 0} \int_{\Gamma_0} \left| \frac{1}{\sigma} \int_0^{\sigma} y(x, t) \mathrm{d}t \right|^2 \mathrm{d}x = 0. \tag{4}$$

Theorems 1.1 and 1.2 ensure that the system (1) is a well-posed regular linear system with feedthrough operator zero. This makes the system (1) similar to a linear finite-dimensional system in many ways. The property claimed by Corollary 1.1 is one of them.

It should be pointed out that although Theorems 1.1 and 1.2 are generalizations of [8,12] where the coefficients are constant, such a generalization is not direct. In order to prove the results, some computations on Riemannian manifolds are required. As it was used in [20,21], the classical multiplier for a domain in Euclidean space is inadequate to deal with variable coefficients. Standard microlocal analysis may be used as an alternative to prove Theorems 1.1 and 1.2, but the geometric multiplier method is more natural to these proofs since it is parallel to the classical multiplier method in Euclidean space for the system with constant coefficients.

The remaining part of the paper are organized as follows. In Sect. 2, we cast the system (1) into an abstract setting studied in [6] and give some basic background on Riemannian geometry. The proofs of Theorems 1.1 and 1.2 are given in Sects. 3 and 4, respectively.

2 Abstract formulation and preliminaries

Let $H = H^{-2}(\Omega)$ be the dual space of the Sobolev space $H_0^2(\Omega)$ with usual inner product. Let A be the positive self-adjoint operator in H induced by the bilinear form $a(\cdot,\cdot)$ defined by

$$\langle Af, g\rangle_{H^{-2}(\Omega)\times H_0^2(\Omega)} = a(f,g) = \int_\Omega \mathcal{A}f(x)\cdot\overline{\mathcal{A}g(x)}dx, \quad \forall f, g \in H_0^2(\Omega).$$

By means of the Lax-Milgram theorem, A is a canonical isomorphism from $D(A) = H_0^2(\Omega)$ onto H. It is easy to show that $Af = \mathscr{A}f$ whenever $f \in H^4(\Omega)\cap H_0^2(\Omega)$ and that $A^{-1}g = \mathscr{A}^{-1}g$ for any $g \in L^2(\Omega)$. Hence A is an extension of $\mathscr{A}$ to the space $H_0^2(\Omega)$.

Same as [8], it can be shown that $D(A^{1/2}) = L^2(\Omega)$ and $A^{1/2}$ is an isomorphism from $L^2(\Omega)$ onto H. Define the map $\Upsilon \in \mathcal{L}(L^2(\Gamma_0), H^{3/2}(\Omega))$([14], p. 189) so that $\Upsilon u = v$ if and only if

$$\begin{cases} \mathcal{A}^2 v(x) = 0, \quad x \in \Omega, \\ v(x)|_{\partial\Omega} = \left.\dfrac{\partial v(x)}{\partial \nu_{\mathcal{A}}}\right|_{\Gamma_1} = 0, \quad \left.\dfrac{\partial v(x)}{\partial \nu_{\mathcal{A}}}\right|_{\Gamma_0} = u(x). \end{cases} \tag{5}$$

By virtue of the above map, one can write (1) as

$$\ddot{w} + A(w - \Upsilon u) = 0. \tag{6}$$

Since $D(A)$ is dense in H, so is $D(A^{1/2})$. We identify H with its dual H'. Then the following relations hold:

$$D(A^{1/2}) \hookrightarrow H = H' \hookrightarrow (D(A^{1/2}))'.$$

An extension $\tilde{A} \in \mathcal{L}(D(A)^{1/2}, (D(A^{1/2}))')$ of A is defined by

$$\langle \tilde{A}f, g\rangle_{(D(A^{1/2}))'\times D(A^{1/2})} = \langle A^{1/2}f, A^{1/2}g\rangle_H, \quad \forall\, f, g \in D(A^{1/2}). \tag{7}$$

So (6) can be further written in $(D(A^{1/2}))'$ as

$$\ddot{w} + \tilde{A}w + Bu = 0,$$

where $B \in \mathcal{L}(U, (D(A^{1/2}))')$ is given by

$$Bu = -\tilde{A}\Upsilon u, \quad \forall\, u \in U. \tag{8}$$

Define $B^* \in \mathcal{L}(D(A^{1/2}), U)$ by

$$\langle B^* f, u\rangle_U = \langle f, Bu\rangle_{D(A^{1/2})\times(D(A^{1/2}))'}, \quad \forall f \in D(A^{1/2}), u \in U.$$

Then for any $f \in D(A^{1/2})$ and $u \in C_0^\infty(\Gamma_0)$, we have

$$\begin{aligned}\langle f, Bu\rangle_{D(A^{1/2})\times(D(A^{1/2}))'} &= \langle f, \tilde{A}\tilde{A}^{-1}Bu\rangle_{D(A^{1/2})\times(D(A^{1/2}))'}\\ &= \langle A^{1/2}f, A^{1/2}\tilde{A}^{-1}Bu\rangle_H\\ &= -\langle A^{1/2}f, A^{1/2}\Upsilon u\rangle_H = -\langle f, \Upsilon u\rangle_{L^2(\Omega)}\\ &= -\langle \mathscr{A}\mathscr{A}^{-1}f, \Upsilon u\rangle_{L^2(\Omega)} = \left\langle \mathcal{A}(\mathscr{A}^{-1}f), u\right\rangle_U.\end{aligned}$$

In the last step, we have used the fact that $\Upsilon^*\mathscr{A} = -\mathcal{A}\cdot|_{\Gamma_0}$ on $H^4(\Omega)\cap H_0^2(\Omega)$. Indeed, by Lemma 2.2 of the next section, for any $\psi \in H^4(\Omega)\cap H_0^2(\Omega),\ u \in L^2(\Gamma_0)$, we have

$$\begin{aligned}\left\langle \Upsilon^*\mathscr{A}\psi, u\right\rangle_{L^2(\Gamma_0)} &= \langle \mathscr{A}\psi, \Upsilon u\rangle_{L^2(\Omega)} = \left\langle \mathcal{A}^2\psi, \Upsilon u\right\rangle_{L^2(\Omega)}\\ &= \int_\Omega \mathcal{A}(\mathcal{A}\psi)\overline{\Upsilon u}\mathrm{d}x\\ &= \int_{\partial\Omega} \overline{\Upsilon u}\frac{\partial(\mathcal{A}\psi)}{\partial\nu_{\mathcal{A}}}\mathrm{d}\Gamma - \int_{\partial\Omega}\mathcal{A}\psi\cdot\frac{\partial(\overline{\Upsilon u})}{\partial\nu_{\mathcal{A}}}\mathrm{d}\Gamma\\ &\quad -\int_\Omega \mathcal{A}\psi\cdot\mathcal{A}(\overline{\Upsilon u})\mathrm{d}x = -\int_{\Gamma_0}\mathcal{A}\psi\cdot\overline{u}\mathrm{d}\Gamma = \langle -\mathcal{A}\psi, u\rangle_{L^2(\Gamma_0)}.\end{aligned}$$

Hence $\Upsilon^*\mathscr{A} = -\mathcal{A}\cdot|_{\Gamma_0}$ on $H^4(\Omega)\cap H_0^2(\Omega)$. Since $C_0^\infty(\Gamma_0)$ is dense in $L^2(\Gamma_0)$, we finally obtain that

$$B^*f = \mathcal{A}(\mathscr{A}^{-1}f)\Big|_{\Gamma_0}, \quad \forall f \in D(A^{1/2}) = L^2(\Omega). \tag{9}$$

Now, we have formulated the open-loop system (1) into an abstract form of a second-order system in $\mathcal{H}$:

$$\begin{cases}\ddot{w}(\cdot,t) + \tilde{A}w(\cdot,t) + Bu(\cdot,t) = 0,\\ y(\cdot,t) = -B^*\dot{w}(\cdot,t),\end{cases} \tag{10}$$

where B and B^* are defined by (8) and (9), respectively. The abstract system (10) has been studied in detail in [2,6].

To end this section, we list some basic facts in Riemannian geometry that we need in the following sections. Notice the hypothesis (2) and let $A(x)$ and $G(x)$ be, repectively, the coefficient matrix and its inverse:

$$A(x) := (a_{ij}(x)), \quad G(x) := (g_{ij}(x)) = A(x)^{-1}, \quad \mathcal{G}(x) := \det(g_{ij}(x)).$$

Let $\mathbb{R}^n$ be the usual Euclidean space. For each $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, define the inner product and norm over the tangent space $\mathbb{R}^n_x$ of $\mathbb{R}^n$ by

$$g(X, Y) := \langle X, Y\rangle_g = \sum_{i,j=1}^{n} g_{ij}\alpha_i\beta_j,$$

$$|X|_g := \langle X, X\rangle_g^{1/2}, \quad \forall X = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial x_i}, \; Y = \sum_{i=1}^{n} \beta_i \frac{\partial}{\partial x_i} \in \mathbb{R}^n_x.$$

Then $(\mathbb{R}^n, g)$ is a Riemannian manifold with Riemannian metric g [20,21]. Denote by D the Levi-Civita connection with respect to g. Let N be a vector field on $(\mathbb{R}^n, g)$. Then for each $x \in \mathbb{R}^n$, the covariant differential DN of N determines a bilinear form on $\mathbb{R}^n_x \times \mathbb{R}^n_x$:

$$DN(X, Y) = \langle D_X N, Y\rangle_g, \quad \forall X, Y \in \mathbb{R}^n_x,$$

where $D_X N$ stands for the covariant derivative of the vector field N with respect to X.

For any $\varphi \in C^2(\mathbb{R}^n)$ and $N = \sum_{i=1}^{n} \gamma_i(x)\frac{\partial}{\partial x_i}$, denote

$$\begin{aligned}
\operatorname{div}_0(N) &:= \sum_{i=1}^{n} \frac{\partial \gamma_i(x)}{\partial x_i},\\
D\varphi &= \nabla_g\varphi := \sum_{i,j=1}^{n} \frac{\partial\varphi}{\partial x_i} a_{ij}(x)\frac{\partial}{\partial x_j},\\
\operatorname{div}_g(N) &:= \sum_{i=1}^{n} \frac{1}{\sqrt{\mathcal{G}(x)}}\frac{\partial}{\partial x_i}\left(\sqrt{\mathcal{G}(x)}\gamma_i(x)\right),\\
\Delta_g\varphi &:= \sum_{i,j=1}^{n} \frac{1}{\sqrt{\mathcal{G}(x)}}\frac{\partial}{\partial x_i}\left(\sqrt{\mathcal{G}(x)}a_{ij}(x)\frac{\partial\varphi}{\partial x_j}\right) = \mathcal{A}\varphi - (Df)\varphi,\\
f(x) &= \frac{1}{2}\log\det(a_{ij}(x)),
\end{aligned}$$

where Δ_g is the Beltrami-Laplace operator.

Let μ be the unit outward-pointing normal to $\partial\Omega$ in terms of the Riemannian metric g. The following Lemma 2.1 ([16], p. 128, 138) and Lemma 2.2 provide some useful identities.

Lemma 2.1 *Let $\varphi, \psi \in C^2(\overline{\Omega})$ and N be a vector field on $(\mathbb{R}^n, g)$. Then we have*

(i) *divergence formula and theorem*:

$$\mathrm{div}_0(\varphi N) = \varphi \mathrm{div}_0(N) + N(\varphi), \quad \mathrm{div}_g(\varphi N) = \varphi \mathrm{div}_g(N) + N(\varphi),$$
$$\int_\Omega \mathrm{div}_0(N)\mathrm{d}x = \int_{\partial\Omega} N \cdot \nu \mathrm{d}\Gamma, \quad \int_\Omega \mathrm{div}_g(N)\mathrm{d}x = \int_{\partial\Omega} \langle N, \mu\rangle_g \mathrm{d}\Gamma;$$

(ii) *Green's identities*:

$$\int_\Omega \Delta_g\varphi \cdot \psi \mathrm{d}x = \int_{\partial\Omega} \psi \frac{\partial\varphi}{\partial\mu}\mathrm{d}\Gamma - \int_\Omega \langle \nabla_g\varphi, \nabla_g\psi\rangle_g \mathrm{d}x,$$
$$\int_\Omega \Delta_g\varphi \cdot \psi \mathrm{d}x - \int_\Omega \varphi\Delta_g\psi \mathrm{d}x = \int_{\partial\Omega} \psi \frac{\partial\varphi}{\partial\mu}\mathrm{d}\Gamma - \int_{\partial\Omega} \varphi \frac{\partial\psi}{\partial\mu}\mathrm{d}\Gamma.$$

Lemma 2.2 *Let $\varphi, \psi \in C^2(\overline{\Omega})$, then*

$$\int_\Omega \mathcal{A}\varphi \cdot \psi \mathrm{d}x = \int_{\partial\Omega} \psi \frac{\partial\varphi}{\partial\nu_{\mathcal{A}}}\mathrm{d}\Gamma - \int_\Omega \langle \nabla_g\varphi, \nabla_g\psi\rangle_g \mathrm{d}x,$$
$$\int_\Omega \mathcal{A}\varphi \cdot \psi \mathrm{d}x - \int_\Omega \varphi\mathcal{A}\psi \mathrm{d}x = \int_{\partial\Omega} \psi \frac{\partial\varphi}{\partial\nu_{\mathcal{A}}}\mathrm{d}\Gamma - \int_{\partial\Omega} \varphi \frac{\partial\psi}{\partial\nu_{\mathcal{A}}}\mathrm{d}\Gamma,$$
$$\mathcal{A}(\varphi\psi) = \mathcal{A}\varphi \cdot \psi + 2\langle \nabla_g\varphi, \nabla_g\psi\rangle_g + \varphi\mathcal{A}\psi.$$

Proof

$$\begin{aligned}\int_\Omega \mathcal{A}\varphi \cdot \psi \mathrm{d}x &= \int_\Omega \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial\varphi}{\partial x_j}\right) \cdot \psi \mathrm{d}x \\ &= \int_{\partial\Omega} \sum_{i,j=1}^n a_{ij}(x)\frac{\partial\varphi}{\partial x_j} \cdot \nu_i \cdot \psi \mathrm{d}\Gamma - \int_\Omega \sum_{i,j=1}^n a_{ij}(x)\frac{\partial\varphi}{\partial x_j} \cdot \frac{\partial\psi}{\partial x_i}\mathrm{d}x \\ &= \int_{\partial\Omega} \psi \frac{\partial\varphi}{\partial\nu_{\mathcal{A}}}\mathrm{d}\Gamma - \int_\Omega \langle \nabla_g\varphi, \nabla_g\psi\rangle_g \mathrm{d}x,\end{aligned}$$

where we have used the identity (formula (2.5) of [20]):

$$\langle \nabla_g\varphi, \nabla_g\psi\rangle_g = \left(\frac{\partial\varphi}{\partial x_1}, \ldots, \frac{\partial\varphi}{\partial x_n}\right) \cdot A(x) \cdot \left(\frac{\partial\psi}{\partial x_1}, \ldots, \frac{\partial\psi}{\partial x_n}\right)^\tau.$$

The second identity follows from the first one directly.

$$\begin{aligned}\mathcal{A}(\varphi\psi) &= \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial(\varphi\psi)}{\partial x_j}\right)\\ &= \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial\varphi}{\partial x_j}\cdot\psi + a_{ij}(x)\cdot\varphi\frac{\partial\psi}{\partial x_j}\right)\\ &= \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial\varphi}{\partial x_j}\right)\cdot\psi + \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial\varphi}{\partial x_j}\cdot\frac{\partial\psi}{\partial x_i}\\ &\quad + \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial\psi}{\partial x_j} + \sum_{i,j=1}^{n}\varphi\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial\psi}{\partial x_j}\right)\\ &= \mathcal{A}\varphi\cdot\psi + 2\langle\nabla_g\varphi,\nabla_g\psi\rangle_g + \varphi\mathcal{A}\psi.\end{aligned}$$

□

Denote by $T^2(\mathbb{R}^n_x)$ the set of all covariant tensors of order 2 on $\mathbb{R}^n_x$. Then $T^2(\mathbb{R}^n_x)$ in an inner product space of dimension n^2 with inner product of the following:

$$\langle F, G\rangle_{T^2(\mathbb{R}^n_x)} = \sum_{i,j=1}^{n} F(e_i, e_j)G(e_i, e_j), \quad \forall\, F, G \in T^2(\mathbb{R}^n_x),$$

where $\{e_1, e_2, \ldots, e_n\}$ is an arbitrarily chosen orthonormal basis of $(\mathbb{R}^n_x, g)$.

Let $\mathfrak{X}(\mathbb{R}^n)$ be the set of all vector fields on $\mathbb{R}^n$. Denote by $\triangle : \mathfrak{X}(\mathbb{R}^n) \longrightarrow \mathfrak{X}(\mathbb{R}^n)$ the Hodge-Laplace operator. Then it has ([21], formulae (2.2.7), (2.2.14)):

$$\Delta_g(N(\varphi)) = (\triangle N)(\varphi) + 2\langle DN, D^2\varphi\rangle_{T^2(\mathbb{R}^n_x)} + N(\Delta_g\varphi) + \mathrm{Ric}(N, D\varphi), \tag{11}$$

$$N(\Delta_g\varphi) = N(\mathcal{A}\varphi) - D^2 f(N, D\varphi) - D^2\varphi(N, Df), \quad \forall\varphi \in C^2(\mathbb{R}^n), \tag{12}$$

where $\mathrm{Ric}(\cdot,\cdot)$ is the Ricci curvature tensor of the Riemannian metric g, $D^2 f$ and $D^2\varphi$ are the Hessian of f and φ in the Riemannian metric g, respectively. The following lemma is straightforward. Actually, these inequalities have been used frequently in literature (see for instance inequality (2.3.6) of [21]).

Lemma 2.3 *Let $\varphi \in C^2(\overline{\Omega})$. Then there is a constant C depending on g, N and Ω only such that*

(i) $\sup_{x\in\overline{\Omega}} \left|\mathcal{A}(\mathrm{div}_g(N))\right| \le C, \quad \sup_{x\in\overline{\Omega}} \left|Df(\mathrm{div}_g(N))\right| \le C,$

$\sup_{x\in\overline{\Omega}} |\mathrm{div}_0(Df)| \le C, \quad \sup_{x\in\overline{\Omega}} |\mathrm{div}_0(N)| \le C,$

$\sup_{x\in\overline{\Omega}} \left|\mathrm{div}_g(N)\right| \le C, \quad \sup_{x\in\partial\Omega} \left|\frac{1}{|\nu_{\mathcal{A}}|_g}\right| \le C,$

$\sup_{x\in\partial\Omega} |Df\cdot\nu| \le C.$

(ii) $\left|\langle D\varphi, D(\mathrm{div}_g(N))\rangle_g\right| \le |D\varphi|_g \left|D(\mathrm{div}_g(N))\right|_g \le C\,|D\varphi|_g\,,$

$$|(\triangle N)\varphi|_g \le C\,|\triangle N|_g\,|D\varphi|_g \le C\,|D\varphi|_g\,,$$

$$\left|\langle DN, D^2\varphi\rangle_{T^2(\mathbb{R}^n_x)}\right| \le C\,|DN|_g \left|D^2\varphi\right|_g \le C\left|D^2\varphi\right|_g,$$

$$\left|D^2 f(N, D\varphi)\right| \le \left|D^2 f\right|_g |N|_g\,|D\varphi|_g \le C\,|D\varphi|_g\,,$$

$$\left|D^2\varphi(N, Df)\right| \le \left|D^2\varphi\right|_g |N|_g\,|Df|_g \le C\left|D^2\varphi\right|_g,$$

$$|\mathrm{Ric}(N, D\varphi)| \le |\mathrm{Ric}|_g\,|N|_g\,|D\varphi|_g \le C\,|D\varphi|_g\,,\quad |Df(\varphi)| \le C\,|D\varphi|_g\,,$$

$$|N(\varphi)| \le C\,|D\varphi|_g\,,\quad |Df(N(\varphi))| \le C\left|D^2\varphi\right|_g,\quad |\mathcal{A}\varphi| \le C\left|D^2\varphi\right|_g.$$

(iii) $\displaystyle\int_\Omega |\varphi|^2\,dx \le C\|\varphi\|^2_{H^2(\Omega)},\quad \int_\Omega |D\varphi|^2_g\,\mathrm{d}x \le C\|\varphi\|^2_{H^2(\Omega)},$

$$\int_\Omega \left|D^2\varphi\right|^2_g \mathrm{d}x \le C\|\varphi\|^2_{H^2(\Omega)}.$$

3 The proof of Theorem 1.1

In this section, we use C_T to denote some positive constant that is independent of (y, u) although it may change values from different contexts. We rewrite (1) with zero initial data as follows:

$$\begin{cases} w_{tt}(x,t) + \mathcal{A}^2 w(x,t) = 0, & x \in \Omega,\ t > 0, \\ w(x,0) = w_t(x,0) = 0, & x \in \Omega, \\ w(x,t) = 0, & x \in \partial\Omega,\ t \ge 0, \\ \dfrac{\partial w(x,t)}{\partial \nu_{\mathcal{A}}} = 0, & x \in \Gamma_1,\ t \ge 0, \\ \dfrac{\partial w(x,t)}{\partial \nu_{\mathcal{A}}} = u(x,t), & x \in \Gamma_0,\ t \ge 0, \\ y(x,t) = -\mathcal{A}(\mathscr{A}^{-1} w_t(x,t)), & x \in \Gamma_0,\ t \ge 0, \end{cases} \tag{13}$$

By Propositions 3.2 and 3.3 of [2] (see also [18]), Theorem 1.1 is equivalent to saying that the solution to (13) satisfies

$$\|y\|^2_{L^2(0,T;U)} \le C_T\|u\|^2_{L^2(0,T;U)},\quad \forall\, u \in L^2(0,T;U).$$

We consider the system (13) in the smoother space $H_0^2(\Omega) \times L^2(\Omega)$ by the transformation

$$z = A^{-1} w_t.$$

Then z satisfies

$$\begin{cases} z_{tt}(x,t) + \mathcal{A}^2 z(x,t) = \Upsilon u_t(x,t), & x \in \Omega,\ t > 0, \\ z(x,0) = z_0(x),\ z_t(x,0) = z_1(x), & x \in \Omega,\ t \geq 0, \\ z(x,t) = \dfrac{\partial z(x,t)}{\partial \nu_{\mathcal{A}}} = 0, & x \in \partial\Omega,\ t \geq 0. \end{cases} \tag{14}$$

From (9), the output becomes

$$y(x,t) = -\mathcal{A} z(x,t)|_{\Gamma_0}.$$

Therefore, Theorem 1.1 is valid if and only if for some (and hence for all) $T > 0$, the solution of (14) satisfies

$$\int_0^T \int_{\Gamma_0} |\mathcal{A} z(x,t)|^2 \mathrm{d}x\mathrm{d}t \leq C_T \int_0^T \int_{\Gamma_0} |u(x,t)|^2 \mathrm{d}x\mathrm{d}t. \tag{15}$$

Proof of (15) We split the proof into eight steps.

Step 1. Let N be a vector field on $\overline{\Omega}$ of class C^2 such that (Lemma 4.1, [10])

$$N(x) = \mu(x), \quad x \in \Gamma;\ |N|_g \leq 1,\ x \in \Omega. \tag{16}$$

Multiply both sides of the first equation of (14) by $N(\overline{z})$ and integrate over $[0,T] \times \Omega$, to give

$$\int_0^T \int_\Omega z_{tt} N(\overline{z}) \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \mathcal{A}^2 z N(\overline{z}) \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega \Upsilon u_t N(\overline{z}) \mathrm{d}x\mathrm{d}t = 0. \tag{17}$$

Compute the first term of the left hand side of (17) to yield

$$\begin{aligned} \int_0^T \int_\Omega z_{tt} N(\overline{z}) \mathrm{d}x\mathrm{d}t &= \int_\Omega z_t N(\overline{z}) \mathrm{d}x \Big|_0^T - \int_0^T \int_\Omega z_t N(\overline{z_t}) \mathrm{d}x\mathrm{d}t \\ &= \int_\Omega z_t N(\overline{z}) \mathrm{d}x \Big|_0^T - \int_\Omega z N(\overline{z_t}) \mathrm{d}x \Big|_0^T + \int_0^T \int_\Omega z N(\overline{z_{tt}}) \mathrm{d}x\mathrm{d}t \end{aligned}$$

$$= \int_{\Omega} z_t N(\overline{z})\mathrm{d}x\Big|_0^T - \int_{\Omega} [\mathrm{div}_g(z\overline{z_t}N) - \overline{z_t}z\mathrm{div}_g(N) - \overline{z_t}N(z)]\mathrm{d}x\Big|_0^T$$
$$+ \int_0^T\int_{\Omega} [\mathrm{div}_g(z\overline{z_{tt}}N) - \overline{z_{tt}}z\mathrm{div}_g(N) - \overline{z_{tt}}N(z)]\mathrm{d}x\mathrm{d}t$$
$$= 2\mathrm{Re}\int_{\Omega} z_t N(\overline{z})\mathrm{d}x\Big|_0^T + \int_{\Omega} \overline{z_t}z\mathrm{div}_g(N)\mathrm{d}x\Big|_0^T$$
$$+ \int_0^T\int_{\Omega} [z\mathcal{A}^2\overline{z}\mathrm{div}_g(N) - \Upsilon\overline{u_t}z\mathrm{div}_g(N) - \overline{z_{tt}}N(z)]\mathrm{d}x\mathrm{d}t.$$

Hence

$$\mathrm{Re}\int_0^T\int_{\Omega} z_{tt}N(\overline{z})\mathrm{d}x\mathrm{d}t = \mathrm{Re}\int_{\Omega} z_t N(\overline{z})\mathrm{d}x\Big|_0^T + \frac{1}{2}\int_{\Omega}\overline{z_t}z\mathrm{div}_g(N)\mathrm{d}x\Big|_0^T$$
$$-\frac{1}{2}\int_0^T\int_{\Omega}\Upsilon\overline{u_t}z\mathrm{div}_g(N)\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^T\int_{\Omega} z\mathcal{A}^2\overline{z}\mathrm{div}_g(N)\mathrm{d}x\mathrm{d}t. \tag{18}$$

By Green's second formula in Riemannian manifolds and the fact that $z = \dfrac{\partial z}{\partial \mu} = 0$ on $\partial\Omega$, the last term of (18) is further expressed as

$$\frac{1}{2}\int_0^T\int_{\Omega} z\mathcal{A}^2\overline{z}\mathrm{div}_g(N)\mathrm{d}x\mathrm{d}t$$
$$= \frac{1}{2}\int_0^T\int_{\Omega} z[(\Delta_g + Df)(\mathcal{A}\overline{z})]\mathrm{div}_g(N)\mathrm{d}x\mathrm{d}t$$
$$= \frac{1}{2}\int_0^T\int_{\Omega} z\Delta_g(\mathcal{A}\overline{z})\mathrm{div}_g(N)\mathrm{d}x\mathrm{d}t$$
$$+ \frac{1}{2}\int_0^T\int_{\Omega} zDf(\mathcal{A}\overline{z})\mathrm{div}_g(N)\mathrm{d}x\mathrm{d}t$$
$$= \frac{1}{2}\int_0^T\int_{\Omega} \mathcal{A}\overline{z}\Delta_g(z\mathrm{div}_g(N))\mathrm{d}x\mathrm{d}t$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^T\int_{\partial\Omega} z\mathrm{div}_g(N)\frac{\partial(\mathcal{A}\overline{z})}{\partial\mu}\mathrm{d}\Gamma\mathrm{d}t\\
&-\frac{1}{2}\int_0^T\int_{\partial\Omega} \mathcal{A}\overline{z}\frac{\partial(z\mathrm{div}_g(N))}{\partial\mu}\mathrm{d}\Gamma\mathrm{d}t\\
&+\frac{1}{2}\int_0^T\int_{\Omega} zDf(\mathcal{A}\overline{z})\mathrm{div}_g(N)\mathrm{d}x\mathrm{d}t\\
=&\frac{1}{2}\int_0^T\int_{\Omega} \mathcal{A}\overline{z}[\mathcal{A}z\mathrm{div}_g(N)+2\langle Dz, D(\mathrm{div}_g(N))\rangle_g\\
&+z\mathcal{A}(\mathrm{div}_g(N))]\mathrm{d}x\mathrm{d}t-\frac{1}{2}\int_0^T\int_{\Omega}\mathcal{A}\overline{z}Df(z\mathrm{div}_g(N))\mathrm{d}x\mathrm{d}t\\
&+\frac{1}{2}\int_0^T\int_{\Omega} zDf(\mathcal{A}\overline{z})\mathrm{div}_g(N)\mathrm{d}x\mathrm{d}t. \qquad (19)
\end{aligned}
$$

Substitute (19) into (18) to obtain

$$
\begin{aligned}
\mathrm{Re}\int_0^T\int_{\Omega} z_{tt}N(\overline{z})\mathrm{d}x\mathrm{d}t =& \mathrm{Re}\int_{\Omega} z_tN(\overline{z})\mathrm{d}x\Big|_0^T+\frac{1}{2}\int_{\Omega}\overline{z_t}z\mathrm{div}_g(N)\mathrm{d}x\Big|_0^T\\
&-\frac{1}{2}\int_0^T\int_{\Omega}\Upsilon\overline{u_t}z\mathrm{div}_g(N)\mathrm{d}x\mathrm{d}t+\frac{1}{2}\int_0^T\int_{\Omega}|\mathcal{A}z|^2\,\mathrm{div}_g(N)\mathrm{d}x\mathrm{d}t\\
&+\int_0^T\int_{\Omega}\mathcal{A}\overline{z}\langle Dz, D(\mathrm{div}_g(N))\rangle_g\mathrm{d}x\mathrm{d}t\\
&+\frac{1}{2}\int_0^T\int_{\Omega} z\mathcal{A}\overline{z}\mathcal{A}(\mathrm{div}_g(N))\mathrm{d}x\mathrm{d}t\\
&-\frac{1}{2}\int_0^T\int_{\Omega}\mathcal{A}\overline{z}Df(z\mathrm{div}_g(N))\mathrm{d}x\mathrm{d}t\\
&+\frac{1}{2}\int_0^T\int_{\Omega} zDf(\mathcal{A}\overline{z})\mathrm{div}_g(N)\mathrm{d}x\mathrm{d}t. \qquad (20)
\end{aligned}
$$

Applying Green's second formula in Riemannian manifolds again, and by (11) and (12), the second term of the left hand side of (17) is expressed as

$$
\begin{aligned}
&\int_0^T\int_\Omega \mathcal{A}^2 z N(\overline{z})\mathrm{d}x\mathrm{d}t\\
&=\int_0^T\int_\Omega [(\Delta_g+Df)(\mathcal{A}z)]N(\overline{z})\mathrm{d}x\mathrm{d}t\\
&=\int_0^T\int_\Omega \Delta_g(\mathcal{A}z)N(\overline{z})\mathrm{d}x\mathrm{d}t+\int_0^T\int_\Omega Df(\mathcal{A}z)N(\overline{z})\mathrm{d}x\mathrm{d}t\\
&=\int_0^T\int_\Omega \mathcal{A}z\Delta_g(N(\overline{z}))\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\partial\Omega} N(\overline{z})\frac{\partial(\mathcal{A}z)}{\partial\mu}\mathrm{d}\Gamma\mathrm{d}t\\
&\quad-\int_0^T\int_{\partial\Omega} \mathcal{A}z\frac{\partial(N(\overline{z}))}{\partial\mu}\mathrm{d}\Gamma\mathrm{d}t+\int_0^T\int_\Omega Df(\mathcal{A}z)N(\overline{z})\mathrm{d}x\mathrm{d}t\\
&=\int_0^T\int_\Omega \mathcal{A}z\Delta_g(N(\overline{z}))\mathrm{d}x\mathrm{d}t-\int_0^T\int_{\partial\Omega} \mathcal{A}z\frac{\partial^2\overline{z}}{\partial\mu^2}\mathrm{d}\Gamma\mathrm{d}t\\
&\quad+\int_0^T\int_\Omega Df(\mathcal{A}z)N(\overline{z})\mathrm{d}x\mathrm{d}t\\
&=\int_0^T\int_\Omega \mathcal{A}z[(\triangle N)(\overline{z})+2\langle DN, D^2\overline{z}\rangle_{T^2(\mathbb{R}^n_x)}+N(\Delta_g\overline{z})\\
&\quad+\mathrm{Ric}(N, D\overline{z})]\mathrm{d}x\mathrm{d}t\\
&\quad-\int_0^T\int_{\partial\Omega} \mathcal{A}z(\mathcal{A}-Df)(\overline{z})\mathrm{d}\Gamma\mathrm{d}t+\int_0^T\int_\Omega Df(\mathcal{A}z)N(\overline{z})\mathrm{d}x\mathrm{d}t\\
&=\int_0^T\int_\Omega \mathcal{A}z[(\triangle N)(\overline{z})+2\langle DN, D^2\overline{z}\rangle_{T^2(\mathbb{R}^n_x)}+N(\mathcal{A}\overline{z})\\
&\quad-D^2f(N, D\overline{z})-D^2\overline{z}(N, Df)+\mathrm{Ric}(N, D\overline{z})]\mathrm{d}x\mathrm{d}t\\
&\quad-\int_0^T\int_{\partial\Omega} \mathcal{A}z(\mathcal{A}-Df)(\overline{z})\mathrm{d}\Gamma\mathrm{d}t+\int_0^T\int_\Omega Df(\mathcal{A}z)N(\overline{z})\mathrm{d}x\mathrm{d}t, \qquad (21)
\end{aligned}
$$

where the validity of $\dfrac{\partial^2\overline{z}}{\partial\mu^2} = \Delta_g\overline{z}$ on $\partial\Omega$ comes from

$$\overline{z}|_{[0,T]\times\partial\Omega} = \left.\frac{\partial\overline{z}}{\partial\mu}\right|_{[0,T]\times\partial\Omega} = 0 \quad \text{implies} \quad \left.\frac{\partial^2\overline{z}}{\partial\mu^2}\right|_{[0,T]\times\partial\Omega} = \Delta_g\overline{z}|_{[0,T]\times\partial\Omega}.$$

Furthermore, it comes from the divergence formula that

$$\begin{aligned}\operatorname{Re}\int_0^T\int_\Omega \mathcal{A}zN(\mathcal{A}z)\mathrm{d}x\mathrm{d}t &= \frac{1}{2}\int_0^T\int_\Omega N(|\mathcal{A}z|^2)\mathrm{d}x\mathrm{d}t\\ &= \frac{1}{2}\int_0^T\int_{\partial\Omega} |\mathcal{A}z|^2\, d\Gamma dt - \frac{1}{2}\int_0^T\int_\Omega |\mathcal{A}z|^2 \operatorname{div}_g(N)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

So (21) can be further expressed as:

$$\begin{aligned}\operatorname{Re}\int_0^T\int_\Omega \mathcal{A}^2zN(z)\mathrm{d}x\mathrm{d}t = &-\frac{1}{2}\int_0^T\int_\Omega |\mathcal{A}z|^2 \operatorname{div}_g(N)\mathrm{d}x\mathrm{d}t\\ &+\operatorname{Re}\int_0^T\int_\Omega \mathcal{A}z[(\triangle N)(z) + 2\langle DN, D^2\overline{z}\rangle_{T^2(\mathbb{R}^n_x)}\\ &- D^2f(N, D\overline{z}) - D^2\overline{z}(N, Df)\\ &+\operatorname{Ric}(N, D\overline{z})]\mathrm{d}x\mathrm{d}t - \frac{1}{2}\int_0^T\int_{\partial\Omega} |\mathcal{A}z|^2\,\mathrm{d}\Gamma\mathrm{d}t\\ &+\operatorname{Re}\int_0^T\int_\Omega N(\overline{z})Df(\mathcal{A}z)\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{22}$$

where we have used the fact that the integral of $\mathcal{A}zDf(\overline{z})$ over $(0,T)\times\partial\Omega$ is zero.

Finally, substitute (20) and (22) into (17) to obtain

$$\frac{1}{2}\int_0^T\int_{\partial\Omega} |\mathcal{A}z|^2\,\mathrm{d}\Gamma\mathrm{d}t = \mathrm{RHS}_1 + \mathrm{RHS}_2 + \mathrm{RHS}_3 + \mathrm{b}_{0,\mathrm{T}}, \tag{23}$$

where

$$\begin{aligned}\mathrm{RHS}_1 = &\operatorname{Re}\int_0^T\int_\Omega \mathcal{A}\overline{z}\langle Dz, D(\operatorname{div}_g(N)\rangle_g\mathrm{d}x\mathrm{d}t\\ &+\frac{1}{2}\operatorname{Re}\int_0^T\int_\Omega z\mathcal{A}\overline{z}A(\operatorname{div}_g(N))\mathrm{d}x\mathrm{d}t\end{aligned}$$

$$+\mathrm{Re}\int_0^T\int_\Omega \mathcal{A}z[(\triangle N)(\overline{z}) + 2\langle DN, D^2\overline{z}\rangle_{T^2(\mathbb{R}^n_x)} - D^2 f(N, D\overline{z})$$

$$- D^2\overline{z}(N, Df) + \mathrm{Ric}(N, D\overline{z})]\mathrm{d}x\mathrm{d}t,$$

$$\mathrm{RHS}_2 = -\frac{1}{2}\mathrm{Re}\int_0^T\int_\Omega \mathcal{A}\overline{z}Df(z\mathrm{div}_g(N))\mathrm{d}x\mathrm{d}t + \frac{1}{2}\mathrm{Re}\int_0^T\int_\Omega z\mathrm{div}_g(N)Df(\mathcal{A}\overline{z})\mathrm{d}x\mathrm{d}t$$

$$+\mathrm{Re}\int_0^T\int_\Omega N(\overline{z})Df(\mathcal{A}z)\mathrm{d}x\mathrm{d}t,$$

$$\mathrm{RHS}_3 = -\frac{1}{2}\mathrm{Re}\int_0^T\int_\Omega \Upsilon\overline{u_t}z\mathrm{div}_g(N)\mathrm{d}x\mathrm{d}t - \mathrm{Re}\int_0^T\int_\Omega \Upsilon u_t N(\overline{z})\mathrm{d}x\mathrm{d}t,$$

$$\mathrm{b}_{0,\mathrm{T}} = \mathrm{Re}\int_\Omega z_t N(\overline{z})\mathrm{d}x\Big|_0^T + \frac{1}{2}\mathrm{Re}\int_\Omega \overline{z_t}z\mathrm{div}_g(N)\mathrm{d}x\Big|_0^T.$$

Step 2 (estimate for RHS_1**).** Let $\Upsilon\overline{u_t} = 0$ in (23). It is noted that under the transformation $z = A^{-1}w_t \in H_0^2(\Omega)$, we have $z_t = A^{-1}w_{tt} = -w \in L^2(\Omega)$. Thus (14) is solved by a C_0-group in the space $H_0^2(\Omega) \times L^2(\Omega)$, that is to say, for any $(z_0, z_1) \in H_0^2(\Omega) \times L^2(\Omega)$, the corresponding solution to (14) satisfies $(z, z_t) \in H_0^2(\Omega) \times L^2(\Omega)$ and depends continuously on (z_0, z_1):

$$\frac{1}{2}\int_0^T\int_{\partial\Omega} |\mathcal{A}z|^2\,\mathrm{d}\Gamma\mathrm{d}t \le C_T\,\|(z_0, z_1)\|^2_{H_0^2(\Omega)\times L^2(\Omega)}.$$

This shows that B^* is admissible, hence so is B ([4]). In other words,

$$u \mapsto \{w, w_t\} \text{ is continuous from}$$
$$L^2((0,T)\times\Gamma_0) \longrightarrow C([0,T]; L^2(\Omega)\times H^{-2}(\Omega)). \tag{24}$$

By (24), $z(t) \in C([0,T]; H_0^2(\Omega))$ that is continuous in $u \in L^2((0,T)\times\Gamma_0)$, and so

$$\mathrm{RHS}_1 \le C_T\|u\|^2_{L^2((0,T)\times\Gamma_0)}, \quad \forall\, u \in L^2((0,T)\times\Gamma_0), \tag{25}$$

where we used Lemma 2.3.

Step 3 (estimate for RHS_2**).** By formulae

$$\mathrm{div}_0(z\mathrm{div}_g(N)\mathcal{A}\overline{z}Df) = z\mathrm{div}_g(N)Df(\mathcal{A}\overline{z}) + \mathcal{A}\overline{z}Df(z\mathrm{div}_g(N))$$
$$+ z\mathrm{div}_g(N)\mathcal{A}\overline{z}\mathrm{div}_0(Df)$$

and

$$\mathrm{div}_0(N(\overline{z})\mathcal{A}zDf) = N(\overline{z})Df(\mathcal{A}z) + \mathcal{A}zDf(N(\overline{z})) + N(\overline{z})\mathcal{A}z\mathrm{div}_0(Df),$$

we have

$$\begin{aligned}\frac{1}{2}\int_0^T\int_\Omega z\mathrm{div}_g(N)Df(\mathcal{A}\overline{z})\mathrm{d}x\mathrm{d}t &= \frac{1}{2}\int_0^T\int_{\partial\Omega} z\mathrm{div}_g(N)\mathcal{A}\overline{z}Df\cdot\nu\mathrm{d}\Gamma\mathrm{d}t\\ &\quad-\frac{1}{2}\int_0^T\int_\Omega \mathcal{A}\overline{z}Df(z\mathrm{div}_g(N))\mathrm{d}x\mathrm{d}t\\ &\quad-\frac{1}{2}\int_0^T\int_\Omega z\mathrm{div}_g(N)\mathcal{A}\overline{z}\mathrm{div}_0(Df)\mathrm{d}x\mathrm{d}t \qquad (26)\end{aligned}$$

and

$$\begin{aligned}\int_0^T\int_\Omega N(\overline{z})Df(\mathcal{A}z)\mathrm{d}x\mathrm{d}t &= \int_0^T\int_{\partial\Omega} N(\overline{z})\mathcal{A}zDf\cdot\nu\mathrm{d}\Gamma\mathrm{d}t - \int_0^T\int_\Omega \mathcal{A}zDf(N(\overline{z}))\mathrm{d}x\mathrm{d}t\\ &\quad-\int_0^T\int_\Omega N(\overline{z})\mathcal{A}z\mathrm{div}_0(Df)\mathrm{d}x\mathrm{d}t. \qquad (27)\end{aligned}$$

Substituting (26) and (27) into RHS_2, and noticing that both the integrals of $z\mathrm{div}_g(N)\mathcal{A}zDf\cdot\nu$ and $N(z)\mathcal{A}zDf\cdot\nu$ over $(0,T)\times\partial\Omega$ are zero, we have

$$\begin{aligned}\mathrm{RHS}_2 &= -\mathrm{Re}\int_0^T\int_\Omega \mathcal{A}\overline{z}Df(z\mathrm{div}_g(N))\mathrm{d}x\mathrm{d}t\\ &\quad-\mathrm{Re}\int_0^T\int_\Omega \mathcal{A}zDf(N(\overline{z}))\mathrm{d}x\mathrm{d}t\\ &\quad-\frac{1}{2}\mathrm{Re}\int_0^T\int_\Omega z\mathrm{div}_g(N)\mathcal{A}\overline{z}\mathrm{div}_0(Df)\mathrm{d}x\mathrm{d}t\\ &\quad-\mathrm{Re}\int_0^T\int_\Omega N(\overline{z})\mathcal{A}z\mathrm{div}_0(Df)\mathrm{d}x\mathrm{d}t. \qquad (28)\end{aligned}$$

Thus along the same line of Step 2, we get

$$\mathrm{RHS}_2 \leq C_T \|u\|^2_{L^2((0,T)\times\Gamma_0)}, \quad \forall u \ \in L^2((0,T)\times\Gamma_0), \tag{29}$$

where we used Lemma 2.3 again.

The following Steps 4–6 are the same as [12] for the constant coefficients case. But for the sake of completeness, we still list the sketch of the proof here.

Step 4 (regularity of z_t). To handle RHS_3, we need the regularity of z_t.

$$z_t = A^{-1}w_{tt} = A^{-1}(-Aw + \tilde{A}\Upsilon u) = -w + \Upsilon u \in L^2((0,T)\times\Omega). \tag{30}$$

Since $w \in C([0,T]; L^2(\Omega))$, $\Upsilon u \in L^2((0,T)\times\Gamma_0)$ continuously in $u \in L^2((0,T)\times\Gamma_0)$, it follows that

$$z_t \in L^2((0,T)\times\Omega) \ \text{ continuously in } u \in L^2((0,T)\times\Gamma_0). \tag{31}$$

Step 5 (estimates of RHS_3 and $b_{0,T}$ for smoother u). To estimate both RHS_3 and $b_{0,T}$, confine u within the smoother class that is dense in $L^2((0,T)\times\Gamma_0)$,

$$u \in C^1([0,T]\times\partial\Omega), \quad u|_{\Gamma_1} = 0, \quad u(\cdot,0) = u(\cdot,T) = 0. \tag{32}$$

We will show that

$$\mathrm{RHS}_3 \leq C_T \|u\|^2_{L^2((0,T)\times\Gamma_0)} \tag{33}$$

and

$$\mathrm{b}_{0,\mathrm{T}} \leq C_T \|u\|^2_{L^2((0,T)\times\Gamma_0)} \tag{34}$$

for all u in the class of (32). From now on, we assume $z_0 = z_1 = 0$ in (14).

Step 6 (proof of (34)). By the fact that $w_t \in C([0,T]; H^{-2}(\Omega))$ continuously in $u \in L^2((0,T)\times\Gamma_0)$, $A^{-1} \in \mathcal{L}(H^{-2}(\Omega), H_0^2(\Omega))$ and $w_t(\cdot,0) = 0$ we have

$$z(\cdot,0) = 0, \ z(\cdot,T) = A^{-1}w_t \in H_0^2(\Omega)$$
$$\text{continuously in } \ u \in L^2((0,T)\times\Gamma_0). \tag{35}$$

Next by (30), (32) and $w(\cdot,0) = 0$,

$$\begin{cases} z_t(\cdot,0) = -w(\cdot,0) + \Upsilon u(\cdot,0) = 0, \\ z_t(\cdot,T) = -w(\cdot,T) \in L^2(\Omega) \text{ continuously in } u \in L^2((0,T)\times\Gamma_0), \end{cases} \tag{36}$$

where the regularity comes from (24).

Using (24), (35) and (36), we readily obtain that

$$\mathrm{b}_{0,\mathrm{T}} = \mathrm{Re}\int_{\Omega} z_t N(z)\mathrm{d}x\bigg|_0^T + \mathrm{Re}\frac{1}{2}\int_{\Omega} z_t z \,\mathrm{div}_g(N)\mathrm{d}x\bigg|_0^T \le C_T\|u\|^2_{L^2((0,T)\times\Gamma_0)}. \tag{37}$$

Step 7 (proof of (33)). For the second term with u in the class (32), we integrate by parts with respect to t and make use of divergence theorem again to obtain

$$\begin{aligned}
-\mathrm{Re}\int_0^T\int_{\Omega} \Upsilon u_t N(z)\mathrm{d}x\mathrm{d}t &= -\mathrm{Re}\int_{\Omega}\Upsilon u N(\overline{z})\mathrm{d}x\bigg|_0^T + \mathrm{Re}\int_0^T\int_{\Omega}\Upsilon u N(\overline{z_t})\mathrm{d}x\mathrm{d}t\\
&= \mathrm{Re}\int_0^T\int_{\Omega}\Upsilon u N(\overline{z_t})\mathrm{d}x\mathrm{d}t\\
&= \mathrm{Re}\int_0^T\int_{\Omega}\mathrm{div}_0(\Upsilon u\overline{z_t}N)\mathrm{d}x\mathrm{d}t - \mathrm{Re}\int_0^T\int_{\Omega}\Upsilon u\overline{z_t}\mathrm{div}_0(N)\mathrm{d}x\mathrm{d}t\\
&\quad -\mathrm{Re}\int_0^T\int_{\Omega}\overline{z_t}N(\Upsilon u)\mathrm{d}x\mathrm{d}t\\
&= \mathrm{Re}\int_0^T\int_{\partial\Omega}\Upsilon u\overline{z_t}N\cdot\nu\mathrm{d}\Gamma\mathrm{d}t - \mathrm{Re}\int_0^T\int_{\Omega}\Upsilon u\overline{z_t}\mathrm{div}_0(N)\mathrm{d}x\mathrm{d}t\\
&\quad -\mathrm{Re}\int_0^T\int_{\Omega}\overline{z_t}N(\Upsilon u)\mathrm{d}x\mathrm{d}t\\
&= -\mathrm{Re}\int_0^T\int_{\Omega}\Upsilon u\overline{z_t}\mathrm{div}_0(N)\mathrm{d}x\mathrm{d}t - \mathrm{Re}\int_0^T\int_{\Omega}\overline{z_t}N(\Upsilon u)\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{38}$$

Noticing (30) and $\Upsilon u \in L^2(0,T;H^{3/2}(\Omega))$ hence $N(\Upsilon u)\in L^2(0,T;H^{1/2}(\Omega))$, all continuously in $u\in L^2((0,T)\times\Gamma_0)$, it follows from (38) that

$$-\mathrm{Re}\int_0^T\int_{\Omega}\Upsilon u_t N(\overline{z})\mathrm{d}x\mathrm{d}t \le C\|u\|^2_{L^2((0,T)\times\Gamma_0)}.$$

A similar estimate holds true for the first term of RHS_3 and thus we get (33).

Step 8. We can then extend estimate (33) of RHS$_3$ and (34) of $b_{0,T}$ to all $u \in L^2((0,T) \times \Gamma_0)$ by density argument, which together with (29) and (25) gives (15). The proof is complete. □

4 Proof of Theorem 1.2

In this section, we still use the multiplier method on Riemannian manifolds to generalize Theorem 1.2 of [8] to the system (1). In particular, for the proof of Lemma 4.1 below, the Riemannian geometry method seems necessary.

Now, it follows from the Appendix of [6] that the transfer function of the system (10) is

$$H(\lambda) = \lambda B^*(\lambda^2 + \tilde{A})^{-1} B, \tag{39}$$

where $\tilde{A}$, B and B^* are given by (7), (8) and (9) respectively. Moreover, from the well-posedness claimed by Theorem 1.1, it follows that there are constants $M, \beta > 0$ such that ([5])

$$\sup_{\mathrm{Re}\lambda \geq \beta} \|H(\lambda)\|_{\mathcal{L}(U)} = M < \infty. \tag{40}$$

Following the same steps in the proof of Proposition 3.1 of [8], we can assert that: Theorem 1.2 is valid if for any $u \in C_0^\infty(\Gamma_0)$, the solution w to the following equation

$$\begin{cases} \lambda^2 w(x) + \mathcal{A}^2 w(x) = 0, & x \in \Omega, \\ w(x) = 0, & x \in \partial\Omega, \\ \dfrac{\partial w(x)}{\partial \nu_{\mathcal{A}}} = 0, & x \in \Gamma_1, \\ \dfrac{\partial w(x)}{\partial \nu_{\mathcal{A}}} = u(x), & x \in \Gamma_0 \end{cases} \tag{41}$$

satisfies

$$\lim_{\lambda \in \mathbb{R}, \lambda \to +\infty} \int_{\Gamma_0} \left| \frac{1}{\lambda} \mathcal{A} w(x) \right|^2 \mathrm{d}x = 0. \tag{42}$$

In order to prove (42), we need the following lemma.

Lemma 4.1 *Let w be the solution of (41). Then there exists a function $a(x)$ independent of λ, which is continuous on $\partial\Omega$, such that*

$$\Delta_g w(x) = \frac{\partial^2 w(x)}{\partial \mu^2} + a(x) \frac{\partial w(x)}{\partial \mu}, \quad \forall\, x \in \partial\Omega. \tag{43}$$

Proof Denote by D and $\overline{D}$ the Levi-Civita connection on Ω and $\partial\Omega$ according to g, respectively. Then we have the second fundamental form ((13.9) of [19], p. 233; Theorem 8.2 of [13], p. 135)

$$S(N_1, N_2) = \overline{D}_{N_1} N_2 - D_{N_1} N_2, \quad \forall N_1, N_2 \in \mathfrak{X}(\mathbb{R}^n), \\ N_1, N_2 \text{ are tangent to } \partial\Omega. \tag{44}$$

For any $x \in \partial\Omega$, let $\{e_i\}_{i=1}^{n-1}$ be an orthonormal basis of $T_x(\partial\Omega)$, i.e. $\langle e_i, e_j\rangle_g = \delta_{ij}$, $1 \leq i, j \leq n-1$. Here $T_x(\partial\Omega)$ is the tangent space of Riemannian manifold $(\partial\Omega, g)$ at x. By parallel translation of $\{e_i\}_{i=1}^{n}$ respect to $\overline{D}$, we can get a frame field $\{E_i\}_{i=1}^{n-1}$ normal at x on the Riemannian manifold $(\partial\Omega, g)$. This means that there exists a neighborhood $U(x) \subset \partial\Omega$ of x, such that $E_i \in \mathfrak{X}(\mathbb{R}^n)$, $\langle E_i(y), E_j(y)\rangle_g = \delta_{ij}$ for all $y \in U(x)$ and $\overline{D}_{E_i}E_j(x) = 0$, $1 \leq i, j \leq n-1$. Next, for any $y \in U(x)$, there is a normal geodesic $\gamma_y(t)$ such that $\gamma_y(0) = y$, $\dot{\gamma}_y(0) = -\mu(y)$. Again by parallel translation of $\{E_i(y)\}_{i=1}^{n-1}$ respect to D along γ_y, we get an orthonormal frame field $\{\widetilde{E}_i(y)\}_{i=1}^{n} \bigcup\{-\dot{\gamma}_y\}$. In this way, we can construct a local orthonormal frame field near x on Riemannian manifold (Ω, g). And we still denote it by $\{E_i\}_{i=1}^{n}$ without confusion, where $E_n(y) := -\dot{\gamma}$ satisfying $D_{E_n}E_n(x) = 0$.

Thirdly, it follows from (44) that at each $x \in \partial\Omega$,

$$\begin{aligned}
\Delta_g w &= \sum_{i=1}^{n} D^2 w(E_i, E_i) = \sum_{i=1}^{n} D_{E_i}(dw)(E_i) \\
&= \sum_{i=1}^{n}(E_i E_i w - D_{E_i}E_i w) = E_n E_n w - \sum_{i=1}^{n-1} D_{E_i}E_i w \\
&= \frac{\partial^2 w}{\partial \mu^2} - \sum_{i=1}^{n-1}(\overline{D}_{E_i}E_i w - S(E_i, E_i))w = \frac{\partial^2 w}{\partial \mu^2} + \sum_{i=1}^{n-1} S(E_i, E_i)w \\
&= \frac{\partial^2 w}{\partial \mu^2} + \eta w,
\end{aligned}$$

where $\eta := \sum_{i=1}^{n-1} S(E_i, E_i)$ is the mean curvature normal field of $\partial\Omega$, and $|\eta|$ is the mean curvature of $\partial\Omega$ ([19], p. 230).

The proof is complete by choosing $a(x)$ with $\eta(x) = a(x)\frac{\partial}{\partial\mu}$. □

Proof of Theorem 1.2 First, multiply the both sides of the first equation of (41) by $\overline{w}$ and integrate by parts to give

$$\begin{aligned}
0 &= \int_\Omega \lambda^2 |w|^2 + \mathcal{A}^2 w \cdot \overline{w} \mathrm{d}x \\
&= \int_\Omega \lambda^2 |w|^2 \mathrm{d}x - \int_\Omega \langle D(\mathcal{A}w), D\overline{w}\rangle_g \mathrm{d}x + \int_{\partial\Omega} \overline{w}\frac{\partial(\mathcal{A}w)}{\partial\nu_{\mathcal{A}}}\mathrm{d}\Gamma \\
&= \int_\Omega \lambda^2 |w|^2 \mathrm{d}x + \int_\Omega |\mathcal{A}w|^2 \mathrm{d}x - \int_{\partial\Omega} \mathcal{A}w\frac{\partial\overline{w}}{\partial\nu_{\mathcal{A}}}\mathrm{d}\Gamma \\
&= \int_\Omega \lambda^2 |w|^2 \mathrm{d}x + \int_\Omega |\mathcal{A}w|^2 \mathrm{d}x - \int_{\Gamma_0} \overline{u}\mathcal{A}w\mathrm{d}\Gamma,
\end{aligned}$$

which implies that

$$\int_\Omega |w|^2\,\mathrm{d}x + \frac{1}{\lambda^2}\int_\Omega |\mathcal{A}w|^2\,\mathrm{d}x \le \frac{1}{\lambda}\,\|u\|_{L^2(\Gamma_0)}\left\|\frac{1}{\lambda}\mathcal{A}w\right\|_{L^2(\Gamma_0)}. \tag{45}$$

Secondly, choose the vector field N on $\overline{\Omega}$ as in (16). Now, as it was done in Sect. 3, multiply the both sides of the first equation of (41) by $N(\overline{w})$, integrate by parts and use (11), (12), (27), (43) and the divergence formula, to yield

$$\begin{aligned}
0 &= \mathrm{Re}\int_\Omega [\lambda^2 wN(w) + \mathcal{A}^2 wN(\overline{w})]\mathrm{d}x\\
&= \frac{\lambda^2}{2}\int_\Omega [\mathrm{div}_0(|w|^2 N) - |w|^2\,\mathrm{div}_0(N)]\mathrm{d}x + \mathrm{Re}\int_\Omega \mathcal{A}w\Delta_g(N(\overline{w}))\mathrm{d}x\\
&\quad + \mathrm{Re}\int_{\partial\Omega} N(\overline{w})\frac{\partial(\mathcal{A}w)}{\partial\mu}\mathrm{d}\Gamma\\
&\quad - \mathrm{Re}\int_{\partial\Omega} \mathcal{A}w\frac{\partial N((\overline{w}))}{\partial\mu}\mathrm{d}\Gamma + \mathrm{Re}\int_\Omega Df(\mathcal{A}w)N(\overline{w})\mathrm{d}x\\
&= -\frac{\lambda^2}{2}\int_\Omega \mathrm{div}_0(N)\,|w|^2\,\mathrm{d}x + \mathrm{Re}\int_\Omega \mathcal{A}w\Delta_g(N(\overline{w}))\mathrm{d}x\\
&\quad + \mathrm{Re}\int_{\Gamma_0} \frac{\overline{u}}{|\nu_{\mathcal{A}}|_g}\frac{\partial(\mathcal{A}w)}{\partial\mu}\mathrm{d}\Gamma\\
&\quad - \mathrm{Re}\int_{\partial\Omega} \mathcal{A}w\frac{\partial^2\overline{w}}{\partial\mu^2}\mathrm{d}\Gamma + \mathrm{Re}\int_{\partial\Omega} N(\overline{w})\mathcal{A}wDf\cdot\nu\mathrm{d}\Gamma\\
&\quad - \mathrm{Re}\int_\Omega \mathcal{A}wDf(N(\overline{w}))\mathrm{d}x - \mathrm{Re}\int_\Omega N(\overline{w})\mathcal{A}w\mathrm{div}_0(Df)\mathrm{d}x\\
&= -\frac{\lambda^2}{2}\int_\Omega \mathrm{div}_0(N)\,|w|^2\,\mathrm{d}x + \mathrm{Re}\int_{\Gamma_0} \frac{\overline{u}}{|\nu_{\mathcal{A}}|_g}\frac{\partial(\mathcal{A}w)}{\partial\mu}\mathrm{d}\Gamma\\
&\quad + \mathrm{Re}\int_{\Gamma_0} \frac{\overline{u}}{|\nu_{\mathcal{A}}|_g}\overline{a(x)}\mathcal{A}w\mathrm{d}\Gamma\\
&\quad + \mathrm{Re}\int_{\Gamma_0} \frac{\overline{u}}{|\nu_{\mathcal{A}}|_g}\mathcal{A}wDf\cdot\nu\mathrm{d}\Gamma - \frac{1}{2}\int_{\partial\Omega} |\mathcal{A}w|^2\,\mathrm{d}\Gamma\\
&\quad + \mathrm{Re}\int_{\partial\Omega} \mathcal{A}wDf(\overline{w})\mathrm{d}\Gamma - \frac{1}{2}\int_\Omega |\mathcal{A}w|^2\,\mathrm{div}_g(N)\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\mathrm{Re}\int_{\Omega} \mathcal{A}w[(\triangle N)(\overline{w}) + 2\langle DN, D^2\overline{w}\rangle_{T^2(\mathbb{R}_x^2)} \\
&- D^2 f(N, D\overline{w}) - D^2\overline{w}(N, Df) + \mathrm{Ric}(N, D\overline{w})]\mathrm{d}x \\
&- \mathrm{Re}\int_{\Omega} \mathcal{A}w Df(N(\overline{w}))\mathrm{d}x - \mathrm{Re}\int_{\Omega} N(\overline{w})\mathcal{A}w\mathrm{div}_0(Df)\mathrm{d}x,
\end{aligned}
$$

where we used the facts that

$$
N(\overline{w})|_{\Gamma_1} = \left.\frac{\partial \overline{w}}{\partial \mu}\right|_{\Gamma_1} = 0, \quad N(\overline{w})|_{\Gamma_0} = \left.\frac{\partial \overline{w}}{\partial \mu}\right|_{\Gamma_0} = \frac{1}{|\nu_{\mathcal{A}}|_g}\left.\frac{\partial \overline{w}}{\partial \nu_{\mathcal{A}}}\right|_{\Gamma_0} = \frac{\overline{u}}{|\nu_{\mathcal{A}}|_g}.
$$

Hence

$$
\begin{aligned}
\left\|\frac{1}{\lambda}\mathcal{A}w\right\|^2_{L^2(\partial\Omega)} &= -\int_{\Omega} \mathrm{div}_0(N)\,|w|^2\,\mathrm{d}x + \frac{2}{\lambda^2}\mathrm{Re}\int_{\Gamma_0} \frac{\overline{u}}{|\nu_{\mathcal{A}}|_g}\frac{\partial(\mathcal{A}w)}{\partial\mu}\mathrm{d}\Gamma \\
&\quad + \frac{2}{\lambda^2}\mathrm{Re}\int_{\Gamma_0} \frac{\overline{u}}{|\nu_{\mathcal{A}}|_g}\overline{a(x)}\mathcal{A}w\mathrm{d}\Gamma + \frac{2}{\lambda^2}\mathrm{Re}\int_{\Gamma_0} \frac{\overline{u}}{|\nu_{\mathcal{A}}|_g}\mathcal{A}wDf\cdot\nu\mathrm{d}\Gamma \\
&\quad + \frac{2}{\lambda^2}\mathrm{Re}\int_{\partial\Omega} \mathcal{A}wDf(\overline{w})\mathrm{d}\Gamma - \frac{1}{\lambda^2}\int_{\Omega} |\mathcal{A}w|^2\,\mathrm{div}_g(N)\mathrm{d}x \\
&\quad + \frac{2}{\lambda^2}\mathrm{Re}\int_{\Omega} \mathcal{A}w[(\triangle N)(\overline{w}) + 2\langle DN, D^2\overline{w}\rangle_{T^2(\mathbb{R}_x^2)} \\
&\quad - D^2 f(N, D\overline{w}) - D^2\overline{w}(N, Df) + \mathrm{Ric}(N, D\overline{w})]\mathrm{d}x \\
&\quad - \frac{2}{\lambda^2}\mathrm{Re}\int_{\Omega} \mathcal{A}wDf(N(\overline{w}))\mathrm{d}x - \frac{2}{\lambda^2}\mathrm{Re}\int_{\Omega} N(\overline{w})\mathcal{A}w\mathrm{div}_0(Df)\mathrm{d}x \\
&\le C_1\,\|w\|^2_{L^2(\Omega)} + \frac{C_2}{\lambda^2}\,\|u\|_{L^2(\Gamma_0)}\,\|w\|_{H^4(\Omega)} \\
&\quad + \frac{C_3}{\lambda}\,\|u\|_{L^2(\Gamma_0)}\left\|\frac{1}{\lambda}\mathcal{A}w\right\|_{L^2(\Gamma_0)} + \frac{C_4}{\lambda^2}\,\|\mathcal{A}w\|^2_{L^2(\Omega)}, \qquad (46)
\end{aligned}
$$

where $C_i > 0, i = 1, 2, 3, 4$ are constants independent of λ. Notice that in the last inequality above, we have used Lemma 2.3 and the following facts:

$$
\begin{aligned}
&\sup_{x\in\partial\Omega} |a(x)| \le C, \quad \|Df(w)\|_{L^2(\partial\Omega)} \le C\,\|u\|_{L^2(\Gamma_0)}, \\
&\|w\|_{H^2(\Omega)} \le C\,\|\mathcal{A}w\|_{L^2(\Omega)}, \quad \left\|\frac{\partial(\mathcal{A}w)}{\partial\mu}\right\|_{L^2(\Gamma_0)} \le C\,\|w\|_{H^4(\Omega)},
\end{aligned}
$$

for some constant $C > 0$ independent of λ. The last but one fact is well-known due to vanishing condition of w on $\partial\Omega$ and the last fact comes from the trace theorem in Sobolev space (see [14]).

Finally, by (7.27) of [14] on p. 189, the solution of (41) satisfies

$$\|w\|_{H^4(\Omega)} \leq C_5 \left[\left\|\lambda^2 w\right\|_{L^2(\Omega)} + \|u\|_{H^{5/2}(\Gamma_0)} \right]$$

for some constant C_5 independent of λ. This together with (45) and (46) yields

$$\begin{aligned}\left\|\frac{1}{\lambda}\mathcal{A}w\right\|^2_{L^2(\Gamma_0)} &\leq (C_1 + C_3 + C_4)\frac{1}{\lambda}\|u\|_{L^2(\Gamma_0)}\left\|\frac{1}{\lambda}\mathcal{A}w\right\|_{L^2(\Gamma_0)} \\ &\quad + C_2C_5\|u\|_{L^2(\Gamma_0)}\|w\|_{L^2(\Omega)} + C_2C_5\frac{1}{\lambda^2}\|u\|_{L^2(\Gamma_0)}\|u\|_{H^{5/2}(\Gamma_0)} \\ &\leq (C_1 + C_3 + C_4)\frac{1}{\lambda}\|u\|_{L^2(\Gamma_0)}\left\|\frac{1}{\lambda}\mathcal{A}w\right\|_{L^2(\Gamma_0)} \\ &\quad + C_2C_5\lambda^{-1/2}\|u\|^{3/2}_{L^2(\Gamma_0)}\left\|\frac{1}{\lambda}\mathcal{A}w\right\|^{1/2}_{L^2(\Gamma_0)} \\ &\quad + C_2C_5\frac{1}{\lambda^2}\|u\|_{L^2(\Gamma_0)}\|u\|_{H^{5/2}(\Gamma_0)}.\end{aligned}$$

The above inequality implies that $\limsup_{\lambda\in\mathbb{R},\lambda\to+\infty}\left\|\frac{1}{\lambda}\mathcal{A}w\right\|_{L^2(\Gamma_0)} < +\infty$. This together with above inequality yields

$$\lim_{\lambda\in\mathbb{R},\lambda\to+\infty}\left\|\frac{1}{\lambda}\mathcal{A}w\right\|_{L^2(\Gamma_0)} = 0.$$

This is (42). The proof is complete. □

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