

Well-posedness by perturbations of variational-hemivariational inequalities with perturbations

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Abstract. In this paper, we consider an extension of the notion of well-posedness by perturbations, introduced by Zolezzi for a minimization problem, to a class of variational-hemivariational inequalities with perturbations in Banach spaces, which includes as a special case the class of mixed variational inequalities. Under very mild conditions, we establish some metric characterizations for the well-posed variational-hemivariational inequality, and show that the well-posedness by perturbations of a variational-hemivariational inequality is closely related to the well-posedness by perturbations of the corresponding inclusion problem. Furthermore, in the setting of finite-dimensional spaces we also derive some conditions under which the variational-hemivariational inequality is strongly generalized well-posed-like by perturbations.

1. Introduction

It is well-known that the notion of well-posedness has played an important role in the optimization theory. Tykhonov [38] first introduced the classical notion of well-posedness for a minimization problem, which has been known as the Tykhonov well-posedness. A minimization problem is said to be Tykhonov well-posed if it has a unique solution toward which every minimizing sequence of the problem converges. It is clear that the notion of Tykhonov well-posedness is inspired by the numerical methods producing optimizing sequences for optimization problems. The notion of generalized Tykhonov well-posedness is also introduced for a minimization problem having more than one solution, which requires the existence of solutions and the convergence of some subsequence of every minimizing sequence toward some solution. Another important notion of well-posedness for a minimization problem is the well-posedness by perturbations or extended well-posedness due to Zolezzi [43, 44]. The notion of well-posedness by perturbations establishes a form of continuous dependence of the solutions upon a parameter. There are many other

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notions of well-posedness in optimization problems. For more details, see, e.g., [1, 6, 10, 15, 18, 26, 31, 38, 39, 43, 44].

On the other hand, the concept of well-posedness has been generalized to other variational problems, such as variational inequalities [5, 9, 11, 12, 23–26], saddle point problems [4], Nash equilibrium problems [25, 27–30, 32], equilibrium problems [13], inclusion problems [21, 22] and fixed point problems [21, 22, 40]. An initial notion of well-posedness for a variational inequality is due to Lucchetti and Patrone [26]. They introduced the notion of well-posedness for variational inequalities and proved some related results by means of Ekeland's variational principle. Since then, many papers have been devoted to the extensions of well-posedness of minimization problems to various variational inequalities. Lignola and Morgan [24] generalized the notion of well-posedness by perturbations to a variational inequality and established the equivalence between the well-posedness by perturbations of a variational inequality and the well-posedness by perturbations of the corresponding minimization problem. Lignola and Morgan [25] introduced the concepts of α -well-posedness for variational inequalities. Del Prete et al. [9] further proved that the α -well-posedness of variational inequalities is closely related to the well-posedness of minimization problems. Recently, Fang et al. [14] generalized the notions of well-posedness and α -well-posedness to a mixed variational inequality. In the setting of Hilbert spaces, Fang et al. [14] proved that under suitable conditions the well-posedness of a mixed variational inequality is equivalent to the existence and uniqueness of its solution. They also showed that the well-posedness of a mixed variational inequality has close links with the well-posedness of the corresponding inclusion problem and corresponding fixed point problem in the setting of Hilbert spaces. Very recently, Fang et al. [15] generalized the notion of well-posedness by perturbations to a mixed variational inequality in Banach spaces. In the setting of Banach spaces, they established some metric characterizations, and showed that the well-posedness by perturbations of a mixed variational inequality is closely related to the well-posedness by perturbations of the corresponding inclusion problem and corresponding fixed point problem. They also derived some conditions under which the well-posedness by perturbations of the mixed variational inequality is equivalent to the existence and uniqueness of its solution.

In this paper, we further extend the notion of well-posedness by perturbations to a class of variational-hemivariational inequalities with perturbations in Banach spaces, which includes as a special case the class of mixed variational inequalities in [15]. Under very mild conditions, we establish some metric characterizations for the well-posed variational-hemivariational inequality, and show that the well-posedness by perturbations of a variational-hemivariational inequality is closely related to the well-posedness by perturbations of the corresponding inclusion problem. In addition, in the setting of finite-dimensional spaces we also derive some conditions under which the variational-hemivariational inequality is strongly generalized well-posed-like by perturbations.

2. Preliminaries

Throughout this paper, unless stated otherwise, we always suppose that X is a real reflexive Banach space with its dual X^* and the duality pairing $\langle \cdot, \cdot \rangle$ between X and X^* . For convenience, we denote strong (resp., weak) convergence by \rightarrow (resp., \rightharpoonup). In what follows, let $A : X \rightarrow X^*$ be a mapping, $T : X \rightarrow X^*$ be a perturbation, and $\varphi : X \rightarrow \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional. Denote by $\text{dom}\varphi$ the domain of functional φ , i.e.,

$$\text{dom}\varphi = \{x \in X : \varphi(x) < +\infty\}.$$

Consider the following variational-hemivariational inequality of finding $x \in X$ such that

$$\text{VHVI} : \quad \langle Ax + Tx - f, y - x \rangle + J^\circ(x, y - x) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in X, \quad (2.1)$$

where $J^\circ(x, y)$ denotes the generalized directional derivative in the sense of Clarke of a locally Lipschitz functional $J : X \rightarrow \mathbf{R}$ at x in the direction y (see [1]) given by

$$J^\circ(x, y) = \limsup_{z \rightarrow x, \lambda \downarrow 0} \frac{J(z + \lambda y) - J(z)}{\lambda}.$$

A concrete example of variational-hemivariational inequality is the adhesive contact problem between a linear elastic body and a rubber support, which is subject to a nonmonotone multivalued boundary condition. See, e.g., [34], for more details. More special cases of the VHVI are presented as follows:

(i) Let Ω be an open bounded subset of \mathbf{R}^3 which is occupied by a linear elastic body, Γ be the boundary of the Ω which is assumed to be appropriately regular ($C^{0,1}$, i.e., a Lipschitzian boundary, is sufficient). If $\varphi = \delta_K$ and $J(u) = \int_{\Omega} j(v, u) d\Omega$, where δ_K denotes the indicator functional of a nonempty, convex subset K of a function space X defined on Ω and $j : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ is a locally Lipschitz continuous function, then the VHVI reduces to the following variational-hemivariational inequality:

$$\text{VHVI} : \quad \langle Ax + Tx, y - x \rangle + J^\circ(x, y - x) \geq \langle f, y - x \rangle, \quad \forall y \in K,$$

which has been considered by Goeleven and Motreanu in [17].

(ii) If $\varphi = 0$ and $T = 0$, then the VHVI reduces to finding $x \in X$ such that

$$\text{HVI} : \quad \langle Ax, y - x \rangle + J^\circ(x, y - x) \geq \langle f, y - x \rangle, \quad \forall y \in X,$$

which is known as the hemivariational inequality studied intensively by many authors (see, e.g., [3, 33–35]).

(iii) If $T = 0$ and $J = 0$, then the VHVI is equivalent to the following problem: find $x \in X$ such that

$$\text{MVI} : \quad \langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq \langle f, y - x \rangle, \quad \forall y \in X,$$

which is known as the mixed variational inequality (see, e.g., [5, 14, 37, 42] and the references therein).

(iv) If $T = 0$, $J = 0$ and $\varphi = \delta_K$, then the VHVI reduces to the classical variational inequality:

$$\text{VI} : \quad \langle Ax - f, y - x \rangle \geq 0, \quad \forall y \in K.$$

(v) If $A = 0$, $T = 0$, $J = 0$ and $f = 0$, then the VHVI reduces to the global minimization problem:

$$\text{MP} : \quad \min_{x \in X} \varphi(x).$$

Suppose that L is a parametric normed space, $P \subset L$ is a closed ball with positive radius, $p^* \in P$ is a fixed point. The perturbed problem of the VHVI (2.1) is always given by

$$\begin{aligned} \text{VHVI}_p : \quad & \text{find } x \in X \text{ such that} \\ & \langle \widetilde{A}(p, x) + \widetilde{T}(p, x) - f, x - y \rangle + (\widetilde{J}(p, \cdot))^\circ(x, y - x) + \widetilde{\varphi}(p, y) - \widetilde{\varphi}(p, x) \geq 0, \quad \forall y \in X, \end{aligned}$$

where $\widetilde{A}, \widetilde{T} : P \times X \rightarrow X^*$ is such that $\widetilde{A}(p^*, \cdot) = A$, $\widetilde{T}(p^*, \cdot) = T$, $\widetilde{J} : P \times X \rightarrow \mathbf{R}$ is such that $\widetilde{J}(p^*, \cdot) = J$ and $\widetilde{\varphi} : P \times X \rightarrow \mathbf{R} \cup \{+\infty\}$ is such that $\widetilde{\varphi}(p^*, \cdot) = \varphi$.

Let $\partial\varphi : X \rightarrow 2^{X^*} \setminus \{\emptyset\}$ and $\bar{\partial}J : X \rightarrow 2^{X^*} \setminus \{\emptyset\}$ denote the subgradient of convex functional φ in the sense of convex analysis (see [36]) and the Clarke's generalized gradient of locally Lipschitz functional J (see [8]), respectively. That is,

$$\partial\varphi(x) = \{x^* \in X^* : \varphi(y) - \varphi(x) \geq \langle x^*, y - x \rangle, \quad \forall y \in X\}$$

and

$$\bar{\partial}J(x) = \{z^* \in X^* : J^\circ(x, y) \geq \langle z^*, y \rangle, \quad \forall y \in X\}.$$

Remark 2.1. (see [3]) The Clarke's generalized gradient of a locally Lipschitz functional $J : X \rightarrow \mathbf{R}$ at a point x is given by

$$\bar{\partial}J(x) = \partial(J^\circ(x, \cdot))(0).$$

About the subgradient in the sense of convex analysis, the Clarke's generalized directional derivative and the Clarke's generalized gradient, we have the following basic properties (see, e.g., [2, 3, 34, 36]).

Proposition 2.2. Let X be a Banach space and $\varphi : X \rightarrow \mathbf{R} \cup \{+\infty\}$ be a convex and proper functional. Then we have the following properties of $\partial\varphi$:

- (i) $\partial\varphi(x)$ is convex and weak*-closed;
- (ii) If φ is continuous at $x \in \text{dom}\varphi$, then $\partial\varphi(x)$ is nonempty, convex, bounded, and weak*-compact;
- (iii) If φ is Gateaux differentiable at $x \in \text{dom}\varphi$, then $\partial\varphi(x) = \{D\varphi(x)\}$, where $D\varphi(x)$ is the Gateaux derivative of φ at x .

Proposition 2.3. Let X be a Banach space and $\varphi_1, \varphi_2 : X \rightarrow \mathbf{R} \cup \{+\infty\}$ be two convex functionals. If there is a point $x_0 \in \text{dom}\varphi_1 \cap \text{dom}\varphi_2$ at which φ_1 is continuous, then the following equation holds:

$$\partial(\varphi_1 + \varphi_2)(x) = \partial\varphi_1(x) + \partial\varphi_2(x), \quad \forall x \in X.$$

Proposition 2.4. Let X be a Banach space, $x, y \in X$ and J be a locally Lipschitz functional defined on X . Then

- (i) The function $y \mapsto J^\circ(x, y)$ is finite, positively homogeneous, subadditive and then convex on X ;
- (ii) $J^\circ(x, y)$ is upper semicontinuous as a function of (x, y) , as a function of y alone, is Lipschitz continuous on X ;
- (iii) $J^\circ(x, -y) = (-J)^\circ(x, y)$;
- (iv) $\bar{\partial}J(x)$ is a nonempty, convex, bounded, weak*-compact subset of X^* ;
- (v) For every $y \in X$, one has

$$J^\circ(x, y) = \max\{\langle \xi, y \rangle : \xi \in \bar{\partial}J(x)\}.$$

Now we recall some important definitions and useful results.

Definition 2.5. (see [41]) Let X be a real Banach space with its dual X^* and T be an operator from X to its dual space X^* . T is said to be monotone if

$$\langle Tx - Ty, x - y \rangle \geq 0, \quad \forall x, y \in X.$$

Definition 2.6. (see [41]) A mapping $T : X \rightarrow X^*$ is said to be hemicontinuous if for any $x, y \in X$, the function $t \mapsto \langle T(x + t(y - x)), y - x \rangle$ from $[0, 1]$ into \mathbf{R} is continuous at 0^+ .

Clearly, the continuity implies the hemicontinuity, but the converse is not true in general.

Theorem 2.7. (see [16]) Let $C \subset X$ be nonempty, closed and convex, $C^* \subset X^*$ be nonempty, closed, convex and bounded, $\psi : X \rightarrow \mathbf{R} \cup \{+\infty\}$ be proper, convex and lower semicontinuous and $y \in C$ be arbitrary. Assume that, for each $x \in C$, there exists $x^*(x) \in C^*$ such that

$$\langle x^*(x), x - y \rangle \geq \psi(y) - \psi(x).$$

Then, there exists $y^* \in C^*$ such that

$$\langle y^*, x - y \rangle \geq \psi(y) - \psi(x), \quad \forall x \in C.$$

Definition 2.8. (see [20]) Let S be a nonempty subset of X . The measure, say μ , of noncompactness for the set S is defined by

$$\mu(S) := \inf\{\varepsilon > 0 : S \subset \bigcup_{i=1}^n S_i, \text{diam}|S_i| < \varepsilon, i = 1, 2, \dots, n, \text{ for some integer } n \geq 1\},$$

where $\text{diam}|S_i|$ means the diameter of set S_i .

Definition 2.9. (see [20]) Let A, B be nonempty subsets of X . The Hausdorff metric $\mathcal{H}(\cdot, \cdot)$ between A and B is defined by

$$\mathcal{H}(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) := \sup_{a \in A} d(a, B)$ with $d(a, B) := \inf_{b \in B} \|a - b\|$.

Let $\{A_n\}$ be a sequence of nonempty subsets of X . We say that A_n converges to A in the sense of Hausdorff metric if $\mathcal{H}(A_n, A) \rightarrow 0$. It is easy to see that $e(A_n, A) \rightarrow 0$ if and only if $d(a_n, A) \rightarrow 0$ for all section $a_n \in A_n$. For more details on this topic, we refer the readers to [20].

Lemma 2.10. *Let $A : X \rightarrow X^*$ be monotone and hemicontinuous, and $\varphi : X \rightarrow \mathbf{R} \cup \{+\infty\}$ be proper, convex and lower semicontinuous. Then for a given $x \in X$, the following statements are equivalent:*

- (i) $\langle A(x) + T(x) - f, y - x \rangle + J^\circ(x, y - x) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in X;$
- (ii) $\langle A(y) + T(x) - f, y - x \rangle + J^\circ(x, y - x) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in X.$

Proof. (i) \Rightarrow (ii). It is easy to see that the conclusion (ii) follows from the monotonicity of mapping A .

(ii) \Rightarrow (i). Suppose that

$$\langle A(y) + T(x) - f, y - x \rangle + J^\circ(x, y - x) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in X. \quad (2.2)$$

For any $z \in X$ and $t \in [0, 1]$, letting $y = tz + (1 - t)x = x + t(z - x)$ in (3.2), we obtain

$$\langle A(tz + (1 - t)x) + T(x) - f, t(z - x) \rangle + J^\circ(x, t(z - x)) + \varphi(tz + (1 - t)x) - \varphi(x) \geq 0.$$

Since Clarke's generalized directional derivative $J^\circ(x, y)$ is positively homogeneous with respect to y and φ is convex, it follows that

$$\langle A(tz + (1 - t)x) + T(x) - f, z - x \rangle + J^\circ(x, z - x) + \varphi(z) - \varphi(x) \geq 0. \quad (2.3)$$

Taking the limit $t \rightarrow 0^+$ in (2.3), we conclude from the hemicontinuity of mapping A that

$$\langle A(x) + T(x) - f, z - x \rangle + J^\circ(x, z - x) + \varphi(z) - \varphi(x) \geq 0. \quad (2.3)$$

Thus, the conclusion (i) follows from the arbitrariness of $z \in X$. This completes the proof. \square

3. Well-posedness by perturbations and metric characterizations

In this section, we generalize the concepts of well-posedness by perturbations to the variational-hemivariational inequality and establish their metric characterizations. In the sequel we always denote by \rightarrow and \rightharpoonup the strong convergence and weak convergence, respectively. Let $\alpha \geq 0$ be a fixed number. For convenience, we write $\tilde{J}_p = \tilde{J}(p, \cdot)$ and $\tilde{J}_p^\circ(x, y) = (\tilde{J}(p, \cdot))^\circ(x, y)$ for all $(x, y) \in X \times X$ and $p \in P$. In particular, $\tilde{J}_{p^*} = J$, $\tilde{J}_{p^*}^\circ = J^\circ$ and $\partial \tilde{J}_{p^*} = \partial J$.

Definition 3.1. Let $\{p_n\} \subset P$ be with $p_n \rightarrow p^*$. A sequence $\{x_n\} \subset X$ is called an α -approximating sequence corresponding to $\{p_n\}$ for VHVI (2.1) if there exists a sequence $\{\varepsilon_n\}$ of nonnegative numbers with $\varepsilon_n \rightarrow 0$ such that $x_n \in \text{dom} \tilde{\varphi}(p_n, \cdot)$ and

$$\begin{aligned} & \tilde{A}(p_n, x_n) + \tilde{T}(p_n, x_n) - f, y - x_n \rangle + \tilde{J}_{p_n}^\circ(x_n, y - x_n) + \tilde{\varphi}(p_n, y) - \tilde{\varphi}(p_n, x_n) \\ & \geq -\frac{\alpha}{2} \|y - x_n\|^2 - \varepsilon_n, \quad \forall y \in X, n \geq 1. \end{aligned}$$

Whenever $\alpha = 0$, we say that $\{x_n\}$ is an approximating sequence corresponding to $\{p_n\}$ for VHVI (2.1). Clearly, every α_2 -approximating sequence corresponding to $\{p_n\}$ is α_1 -approximating corresponding to $\{p_n\}$ whenever $\alpha_1 > \alpha_2 \geq 0$.

Definition 3.2. We say that VHVI (2.1) is strongly (resp., weakly) α -well-posed by perturbations if VHVI (2.1) has a unique solution and for any $\{p_n\} \subset P$ with $p_n \rightarrow p^*$, every α -approximating sequence corresponding to $\{p_n\}$ converges strongly (resp., weakly) to the unique solution. In the sequel, strong (resp., weak) 0-well-posedness by perturbations is always called as strong (resp., weak) well-posedness by perturbations. If $\alpha_1 > \alpha_2 \geq 0$, then strong (resp., weak) α_1 -well-posedness by perturbations implies strong (resp., weak) α_2 -well-posedness by perturbations.

Remark 3.3. (i) If $\tilde{T} = 0$, $f = 0$ and $\tilde{J} = 0$, Definitions 3.1 and 3.2 coincide with Definitions 3.1 and 3.2 of [15], respectively. (ii) When X is a Hilbert space, $\tilde{T} = 0$, $f = 0$, $\tilde{J} = 0$ and $p_n = p^*$ ($\forall n \geq 1$), Definitions 3.1 and 3.2 coincide with Definitions 3.1 and 3.2 of [14], respectively. (iii) When $\tilde{T} = 0$, $f = 0$, $\tilde{J} = 0$ and $\varphi = \delta_K$, Definitions 3.1 and 3.2 reduce to the definitions of approximating sequences of the classical variational inequality (see [9, 24]).

Definition 3.4. We say that VHVI (2.1) is strongly (resp., weakly) generalized α -well-posed by perturbations if VHVI (2.1) has a nonempty solution set S and for any $\{p_n\} \subset P$ with $p_n \rightarrow p^*$, every α -approximating sequence corresponding to $\{p_n\}$ has some subsequence which converges strongly (resp., weakly) to some point of S . Strong (resp., weak) generalized 0-well-posedness by perturbations is always called as strong (resp., weak) generalized well-posedness by perturbations. Clearly, if $\alpha_1 > \alpha_2 \geq 0$, then strong (resp., weak) generalized α_1 -well-posedness by perturbations implies strong (resp., weak) generalized α_2 -well-posedness by perturbations.

Remark 3.5. (i) When X is a Hilbert space, $\tilde{T} = 0$, $f = 0$, $\tilde{J} = 0$ and $p_n = p^*$ ($\forall n \geq 1$), Definition 3.4 coincides with Definition 3.3 of [14]. (ii) When $\tilde{T} = 0$, $f = 0$, $\tilde{J} = 0$ and $\tilde{\varphi} = \delta_K$, Definition 3.4 reduces to the definition of strong (resp., weak) parametric α -well-posedness in the generalized sense for the classical variational inequality (see [9, 24, 25]). (iii) When $\tilde{T} = 0$, $f = 0$, $\tilde{J} = 0$, $\tilde{A} = 0$ and $\alpha = 0$, Definition 3.4 coincides with the definition of well-posedness by perturbations introduced for a minimization problem [43, 44].

To derive the metric characterizations of α -well-posedness by perturbations, we consider the following approximating solution set of VHVI (2.1):

$$\Omega_\alpha(\varepsilon) = \bigcup_{p \in B(p^*, \varepsilon)} \{x \in \text{dom} \tilde{\varphi}(p, \cdot) : \langle \tilde{A}(p, x) + \tilde{T}(p, x) - f, y - x \rangle + (\tilde{J}(p, \cdot))^\circ(x, y - x) + \tilde{\varphi}(p, y) - \tilde{\varphi}(p, x) \geq -\frac{\alpha}{2} \|y - x\|^2 - \varepsilon, \forall y \in X\}, \quad \forall \varepsilon \geq 0,$$

where $B(p^*, \varepsilon)$ denotes the closed ball centered at p^* with radius ε . In this section, we always suppose that x^* is a fixed solution of VHVI (2.1). Define

$$\theta(\varepsilon) = \sup\{\|x - x^*\| : x \in \Omega_\alpha(\varepsilon)\}, \quad \forall \varepsilon \geq 0.$$

It is easy to see that $\theta(\varepsilon)$ is the radius of the smallest closed ball centered at x^* containing $\Omega_\alpha(\varepsilon)$. Now, we give a metric characterization of strong α -well-posedness by perturbations by considering the behavior of $\theta(\varepsilon)$ when $\varepsilon \rightarrow 0$.

Theorem 3.6. VHVI (2.1) is strongly α -well-posed by perturbations if and only if $\theta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Proof. Let VHVI (2.1) be strongly α -well-posed by perturbations. Then $x^* \in X$ is the unique solution of VHVI (2.1). Assume by contradiction that $\theta(\varepsilon) \not\rightarrow 0$ as $\varepsilon \rightarrow 0$. Then there exist δ and $0 < \varepsilon_n \rightarrow 0$ such that

$$\theta(\varepsilon_n) > \delta > 0.$$

By the definition of θ , there exists $x_n \in \Omega_\alpha(\varepsilon_n)$ such that

$$\|x_n - x^*\| > \delta. \quad (3.1)$$

Being $x_n \in \Omega_\alpha(\varepsilon_n)$, there exists $p_n \in B(p^*, \varepsilon_n)$ such that

$$\langle \tilde{A}(p_n, x_n) + \tilde{T}(p_n, x_n) - f, y - x_n \rangle + (\tilde{J}(p_n, \cdot))^\circ(x_n, y - x_n) + \tilde{\varphi}(p_n, y) - \tilde{\varphi}(p_n, x_n) \geq -\frac{\alpha}{2} \|y - x_n\|^2 - \varepsilon_n,$$

for all $y \in X$ and $n \geq 1$. Clearly, $p_n \rightarrow p^*$ and $\{x_n\}$ is an α -approximating sequence corresponding to $\{p_n\}$ for VHVI (2.1). Since VHVI (2.1) is strongly α -well-posed by perturbations, we get $\|x_n - x^*\| \rightarrow 0$, a contradiction to (3.1).

Conversely, suppose that $\theta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Then $x^* \in X$ is the unique solution of VHVI (2.1). Indeed, if $\bar{x} (\neq x^*)$ is another solution of VHVI (2.1). By definition, $\theta(\varepsilon) \geq \|x^* - \bar{x}\| > 0$ for all $\varepsilon \geq 0$, a contradiction. Let $p_n \in P$ be with $p_n \rightarrow p^*$ and let $\{x_n\}$ be an α -approximating sequence corresponding to $\{p_n\}$ for VHVI (2.1). Then there exists $0 < \varepsilon_n \rightarrow 0$ such that

$$\langle \tilde{A}(p_n, x_n) + \tilde{T}(p_n, x_n) - f, y - x_n \rangle + (\tilde{J}(p_n, \cdot))^\circ(x_n, y - x_n) + \tilde{\varphi}(p_n, y) - \tilde{\varphi}(p_n, x_n) \geq -\frac{\alpha}{2}\|y - x_n\|^2 - \varepsilon_n,$$

for all $y \in X$ and $n \geq 1$. Take $\delta_n = \|p_n - p^*\|$ and $\varepsilon'_n = \max\{\delta_n, \varepsilon_n\}$. It is easy to see that $x_n \in \Omega_\alpha(\varepsilon'_n)$ with $\varepsilon'_n \rightarrow 0$. Set

$$t_n = \|x_n - x^*\|.$$

By the definition of θ ,

$$\theta(\varepsilon'_n) \geq t_n = \|x_n - x^*\|.$$

Since $\theta(\varepsilon'_n) \rightarrow 0$, we get $\|x_n - x^*\| \rightarrow 0$ as $n \rightarrow \infty$. So, VHVI (2.1) is strongly α -well-posed by perturbations. \square

Remark 3.7. Theorem 3.6 improves Proposition 2.2 of [9], Theorem 3.1 of [14] and Theorem 3.1 of [15].

Now, we give an example to illustrate Theorem 3.6.

Example 3.8. Let $X = \mathbf{R}$, $P = [-1, 1]$, $p^* = 0$, $\alpha = 2$, $\tilde{A}(p, x) = \tilde{T}(p, x) = \frac{1}{2}x(p^2 + 1)$, $f = 0$, $\tilde{J} = 0$ and $\tilde{\varphi}(p, x) = x^2$ for all $x \in X$, $p \in P$. Clearly, $x^* = 0$ is a solution of VHVI (2.1). For any $\varepsilon > 0$, it follows that

$$\begin{aligned} \Omega_\alpha^p(\varepsilon) &= \{x \in \mathbf{R} : x(p^2 + 1)(y - x) + y^2 - x^2 \geq -(y - x)^2 - \varepsilon, \forall y \in \mathbf{R}\} \\ &= \{x \in \mathbf{R} : x(p^2 + 1)(x - y) + x^2 - y^2 \leq (x - y)^2 + \varepsilon, \forall y \in \mathbf{R}\} \\ &= \{x \in \mathbf{R} : -2(y + \frac{p^2-1}{4}x)^2 + \frac{(p^2+3)^2}{8}x^2 \leq \varepsilon, \forall y \in \mathbf{R}\} \\ &= [-\frac{2\sqrt{2\varepsilon}}{(p^2+3)}, \frac{2\sqrt{2\varepsilon}}{(p^2+3)}]. \end{aligned}$$

Therefore,

$$\Omega_\alpha(\varepsilon) = \bigcup_{p \in B(0, \varepsilon)} \Omega_\alpha^p(\varepsilon) = [-\frac{2\sqrt{2\varepsilon}}{3}, \frac{2\sqrt{2\varepsilon}}{3}]$$

for sufficiently small $\varepsilon > 0$. By trivial computation, we have

$$\theta(\varepsilon) = \sup\{\|x - x^*\| : x \in \Omega_\alpha(\varepsilon)\} = \frac{2\sqrt{2\varepsilon}}{3} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

By Theorem 3.6, VHVI (2.1) is 2-well-posed by perturbations.

To derive a characterization of strong generalized α -well-posedness by perturbations, we need another function q which is defined by

$$q(\varepsilon) = e(\Omega_\alpha(\varepsilon), S), \quad \forall \varepsilon \geq 0,$$

where S is the solution set of VHVI (2.1) and e is defined as in Proposition 2.2.

Theorem 3.9. VHVI (2.1) is strongly generalized α -well-posed by perturbations if and only if S is nonempty compact and $q(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Proof. Suppose that VHVI (2.1) is strongly generalized α -well-posed by perturbations. Obviously, S is nonempty. Let $\{x_n\}$ be any sequence in S and $\{p_n\} \subset P$ be with $p_n = p^*$. Then $\{x_n\}$ is an α -approximating sequence corresponding to $\{p_n\}$ for VHVI (2.1). By the strong generalized α -well-posedness by perturbations of VHVI (2.1), $\{x_n\}$ has a subsequence which converges strongly to some point of S . Thus S is compact. If $q(\varepsilon) \not\rightarrow 0$ as $\varepsilon \rightarrow 0$, then there exist $l > 0$, $0 < \varepsilon_n \rightarrow 0$, and $x_n \in \Omega_\alpha(\varepsilon_n)$ such that

$$x_n \notin S + B(0, l), \quad \forall n \geq 1. \quad (3.2)$$

Since $x_n \in \Omega_\alpha(\varepsilon_n)$, there exists $p_n \in B(p^*, \varepsilon_n)$ such that

$$\langle \widetilde{A}(p_n, x_n) + \widetilde{T}(p_n, x_n) - f, y - x_n \rangle + (\widetilde{J}(p_n, \cdot))^\circ(x_n, y - x_n) + \widetilde{\varphi}(p_n, y) - \widetilde{\varphi}(p_n, x_n) \geq -\frac{\alpha}{2}\|y - x_n\|^2 - \varepsilon_n,$$

for all $y \in X$ and $n \geq 1$. Clearly, $p_n \rightarrow p^*$ and $\{x_n\}$ is an α -approximating sequence corresponding to $\{p_n\}$ for VHVI (2.1). Since VHVI (2.1) is strongly generalized α -well-posed by perturbations, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging strongly to some point of S . This contradicts (3.2), and so $q(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Conversely, we suppose that S is nonempty compact and $q(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Let $\{p_n\} \subset P$ be with $p_n \rightarrow p^*$ and let $\{x_n\}$ be an α -approximating sequence corresponding to $\{p_n\}$. Take $\varepsilon'_n = \max\{\varepsilon_n, \|p_n - p^*\|\}$. It is easy to see that $\varepsilon'_n \rightarrow 0$ and $x_n \in \Omega_\alpha(\varepsilon'_n)$. It follows that

$$d(x_n, S) \leq e(\Omega_\alpha(\varepsilon'_n), S) = q(\varepsilon'_n) \rightarrow 0.$$

Since S is compact, there exists $\bar{x}_n \in S$ such that

$$\|x_n - \bar{x}_n\| = d(x_n, S) \rightarrow 0.$$

Again from the compactness of S , $\{\bar{x}_n\}$ has a subsequence $\{\bar{x}_{n_k}\}$ converging strongly to $\bar{x} \in S$. Hence, the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges strongly to \bar{x} . Thus, VHVI (2.1) is strongly generalized α -well-posed by perturbations. \square

Example 3.10. Let $X = \mathbf{R}$, $P = [-1, 1]$, $p^* = 0$, $\alpha = 2$, $\widetilde{A}(p, x) = \widetilde{T}(p, x) = \frac{1}{2}x(p^2 + 1)$, $f = 0$, $\widetilde{J} = 0$ and $\widetilde{\varphi}(p, x) = x^2$ for all $x \in X$, $p \in P$. Clearly, $x^* = 0$ is a solution of VHVI (2.1). Repeating the same argument as in Example 3.8, we obtain that for any $\varepsilon > 0$,

$$\Omega_\alpha(\varepsilon) = \bigcup_{p \in B(0, \varepsilon)} \Omega_\alpha^p(\varepsilon) = \left[-\frac{2\sqrt{2\varepsilon}}{3}, \frac{2\sqrt{2\varepsilon}}{3}\right]$$

for sufficiently small $\varepsilon > 0$. By trivial computation, we have

$$q(\varepsilon) = e(\Omega_\alpha(\varepsilon), S) = \sup_{x(\varepsilon) \in \Omega_\alpha(\varepsilon)} d(x(\varepsilon), S) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

By Theorem 3.9, VHVI (2.1) is generalized α -well-posed by perturbations.

The strong generalized α -well-posedness by perturbations can be also characterized by the behavior of the noncompactness measure $\mu(\Omega_\alpha(\varepsilon))$.

Theorem 3.11. Let L be finite-dimensional, $(\widetilde{A} + \widetilde{T}) : P \times X \rightarrow X^*$ be a continuous mapping, $\widetilde{J}_p^\circ(x, y)$ be upper semicontinuous as a functional of $(p, x, y) \in P \times X \times X$ and $\widetilde{\varphi} : P \times X \rightarrow \mathbf{R} \cup \{+\infty\}$ be a continuous functional such that $\widetilde{\varphi}(p, \cdot)$ is proper and convex. Then VHVI (2.1) is strongly generalized α -well-posed by perturbations if and only if $\Omega_\alpha(\varepsilon) \neq \emptyset$, $\forall \varepsilon > 0$ and $\mu(\Omega_\alpha(\varepsilon)) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Proof. First, we shall prove that $\Omega_\alpha(\varepsilon)$ is closed for all $\varepsilon \geq 0$. Let $\{x_n\} \subset \Omega_\alpha(\varepsilon)$ with $x_n \rightarrow \bar{x}$. Then there exists $p_n \in B(p^*, \varepsilon)$ such that

$$\langle \widetilde{A}(p_n, x_n) + \widetilde{T}(p_n, x_n) - f, y - x_n \rangle + \widetilde{J}_{p_n}^\circ(x_n, y - x_n) + \widetilde{\varphi}(p_n, y) - \widetilde{\varphi}(p_n, x_n) \geq -\frac{\alpha}{2}\|y - x_n\|^2 - \varepsilon, \quad (3.3)$$

for all $y \in X$ and $n \geq 1$. Without loss of generality, we may assume $p_n \rightarrow \bar{p} \in B(p^*, \varepsilon)$ since L is finite-dimensional. Note that $\widetilde{J}_p^\circ(x, y)$ is upper semicontinuous as a functional of $(p, x, y) \in P \times X \times X$. Hence it follows from (3.3) and the continuity of $(\widetilde{A} + \widetilde{T})$ and $\widetilde{\varphi}$ that

$$\begin{aligned} & \langle \widetilde{A}(\bar{p}, \bar{x}) + \widetilde{T}(\bar{p}, \bar{x}) - f, y - \bar{x} \rangle + \widetilde{J}_{\bar{p}}^\circ(\bar{x}, y - \bar{x}) + \widetilde{\varphi}(\bar{p}, y) - \widetilde{\varphi}(\bar{p}, \bar{x}) \\ & \geq \limsup_{n \rightarrow \infty} \{ \langle \widetilde{A}(p_n, x_n) + \widetilde{T}(p_n, x_n) - f, y - x_n \rangle + \widetilde{J}_{p_n}^\circ(x_n, y - x_n) + \widetilde{\varphi}(p_n, y) - \widetilde{\varphi}(p_n, x_n) \} \\ & \geq \limsup_{n \rightarrow \infty} \{ -\frac{\alpha}{2}\|y - x_n\|^2 - \varepsilon \} \\ & = -\frac{\alpha}{2}\|y - \bar{x}\|^2 - \varepsilon, \quad \forall y \in X. \end{aligned}$$

Thus, $\bar{x} \in \Omega_\alpha(\varepsilon)$ and so $\Omega_\alpha(\varepsilon)$ is closed.

Second, we show that

$$S = \bigcap_{\varepsilon > 0} \Omega_\alpha(\varepsilon). \quad (3.4)$$

It is obvious that $S \subset \bigcap_{\varepsilon > 0} \Omega_\alpha(\varepsilon)$. Let $x^* \in \bigcap_{\varepsilon > 0} \Omega_\alpha(\varepsilon)$. Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\varepsilon_n \rightarrow 0$. Then $x^* \in \Omega_\alpha(\varepsilon_n)$ and so there exists $p_n \in B(p^*, \varepsilon_n)$ such that

$$\langle \tilde{A}(p_n, x^*) + \tilde{T}(p_n, x^*) - f, y - x^* \rangle + \tilde{J}_{p_n}^\circ(x^*, y - x^*) + \tilde{\varphi}(p_n, y) - \tilde{\varphi}(p_n, x^*) \geq -\frac{\alpha}{2} \|y - x^*\|^2 - \varepsilon_n,$$

for all $y \in X$ and $n \geq 1$. It is clear that $p_n \rightarrow p^*$. Letting $n \rightarrow \infty$ in the last inequality we get

$$\begin{aligned} & \langle A(x^*) + T(x^*) - f, y - x^* \rangle + J^\circ(x^*, y - x^*) + \varphi(y) - \varphi(x^*) \\ &= \langle \tilde{A}(p^*, x^*) + \tilde{T}(p^*, x^*) - f, y - x^* \rangle + \tilde{J}_{p^*}^\circ(x^*, y - x^*) + \tilde{\varphi}(p^*, y) - \tilde{\varphi}(p^*, x^*) \\ &\geq -\frac{\alpha}{2} \|y - x^*\|^2, \quad \forall y \in X. \end{aligned} \quad (3.5)$$

For any $z \in X$ and $t \in (0, 1)$, letting $y = x^* + t(z - x^*)$ in (3.5) we have

$$t\{\langle A(x^*) + T(x^*) - f, z - x^* \rangle + J^\circ(x^*, z - x^*) + \varphi(z) - \varphi(x^*)\} \geq -\frac{\alpha t^2}{2} \|z - x^*\|^2.$$

This implies that

$$\langle A(x^*) + T(x^*) - f, z - x^* \rangle + J^\circ(x^*, z - x^*) + \varphi(z) - \varphi(x^*) \geq -\frac{\alpha t}{2} \|z - x^*\|^2, \quad \forall z \in X.$$

Letting $t \rightarrow 0$ in the last inequality we get

$$\langle A(x^*) + T(x^*) - f, z - x^* \rangle + J^\circ(x^*, z - x^*) + \varphi(z) - \varphi(x^*) \geq 0, \quad \forall z \in X.$$

Consequently, $x^* \in S$ and so (3.4) is proved.

Now, we suppose that VHVI (2.1) is strongly generalized α -well-posed by perturbations. By Theorem 3.9, S is nonempty compact and $q(\varepsilon) \rightarrow 0$. Then $\Omega_\alpha(\varepsilon) \neq \emptyset$ since $S \subset \Omega_\alpha(\varepsilon)$ for all $\varepsilon > 0$. Observe that for all $\varepsilon > 0$,

$$\mathcal{H}(\Omega_\alpha(\varepsilon), S) = \max\{e(\Omega_\alpha(\varepsilon), S), e(S, \Omega_\alpha(\varepsilon))\} = e(\Omega_\alpha(\varepsilon), S).$$

Taking into account the compactness of S , we get

$$\mu(\Omega_\alpha(\varepsilon)) \leq 2\mathcal{H}(\Omega_\alpha(\varepsilon), S) = 2e(\Omega_\alpha(\varepsilon), S) = 2q(\varepsilon) \rightarrow 0.$$

Conversely, we suppose that $\Omega_\alpha(\varepsilon) \neq \emptyset$, $\forall \varepsilon > 0$ and $\mu(\Omega_\alpha(\varepsilon)) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Since $\Omega_\alpha(\varepsilon)$ is increasing with respect to $\varepsilon > 0$, by the Kuratowski theorem ([20, p. 318]), we have from (3.4)

$$q(\varepsilon) = \mathcal{H}(\Omega_\alpha(\varepsilon), S) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

and S is nonempty compact. By Theorem 3.9, VHVI (2.1) is strongly generalized α -well-posed by perturbations. \square

Remark 3.12. Theorem 3.11 generalizes Theorem 3.2 of [14] and Theorem 3.3 of [15].

Remark 3.13. Clearly, any solution of VHVI (2.1) is a solution of the α problem: find $x \in X$ such that

$$\langle A(x) + T(x) - f, y - x \rangle + J^\circ(x, y - x) + \varphi(y) - \varphi(x) \geq -\frac{\alpha}{2} \|y - x\|^2, \quad \forall y \in X,$$

but the converse is not true in general. To show this, let $X = \mathbf{R}$, $A(x) = T(x) = \frac{1}{2}x$, $f = 0$, $J = 0$ and $\varphi(x) = -x^2$ for all $x \in X$. It is easy to verify that the solution set of VHVI (2.1) is empty and 0 is the unique solution of the corresponding α problem with $\alpha = 2$. If φ is proper and convex, then VHVI (2.1) and α problem have the same solution (This fact has been shown in the proof of Theorem 3.11).

4. Links with the well-posedness by perturbations of inclusion problems

Lemaire et al. [22] introduced the concept of well-posedness by perturbations for an inclusion problem. In this section, we shall show that the well-posedness by perturbations of a variational-hemivariational inequality is closely related to the well-posedness by perturbations of the corresponding inclusion problem. Let us recall some concepts. Let $M : X \rightarrow 2^{X^*}$. The inclusion problem associated with M is defined by

$$\text{IP}(M) : \text{ find } x \in X \text{ such that } 0 \in M(x).$$

The perturbed problem of $\text{IP}(M)$ is given by

$$\text{IP}_p(M) : \text{ find } x \in X \text{ such that } 0 \in \widetilde{M}(p, x),$$

where $\widetilde{M} : P \times X \rightarrow 2^{X^*}$ is such that $\widetilde{M}(p^*, \cdot) = M$.

Definition 4.1. (see [22]) Let $\{p_n\} \subset P$ be with $p_n \rightarrow p^*$. A sequence $\{x_n\} \subset X$ is called an approximating sequence corresponding to $\{p_n\}$ for $\text{IP}(M)$ if $x_n \in \text{Dom} \widetilde{M}(p_n, \cdot)$ for all $n \geq 1$ and $d(0, \widetilde{M}(p_n, x_n)) \rightarrow 0$, or equivalently, there exists $\zeta_n \in \widetilde{M}(p_n, x_n)$ such that $\|\zeta_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Definition 4.2. (see [22]) We say that $\text{IP}(M)$ is strongly (resp., weakly) well-posed by perturbations if it has a unique solution and for any $\{p_n\} \subset P$ with $p_n \rightarrow p^*$, every approximating sequence corresponding to $\{p_n\}$ converges strongly (resp., weakly) to the unique solution of $\text{IP}(M)$. $\text{IP}(M)$ is said to be strongly (resp., weakly) generalized well-posed by perturbations if the solution set S of $\text{IP}(M)$ is nonempty and for any $\{p_n\} \subset P$ with $p_n \rightarrow p^*$, every approximating sequence corresponding to $\{p_n\}$ has a subsequence which converges strongly (resp., weakly) to a point of S .

Definition 4.3. Let $\{p_n\} \subset P$ be with $p_n \rightarrow p^*$. A sequence $\{x_n\} \subset X$ is called an approximating-like sequence corresponding to $\{p_n\}$ for VHVI (2.1) if there exists a nonnegative sequence $\{\varepsilon_n\}$ with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ such that

$$\langle \widetilde{A}(p_n, x_n) + \widetilde{T}(p_n, x_n) - f, y - x_n \rangle + \widetilde{J}_{p_n}^\circ(x_n, y - x_n) + \widetilde{\varphi}(p_n, y) - \widetilde{\varphi}(p_n, x_n) \geq -\varepsilon_n \|y - x_n\|, \quad \forall y \in X.$$

Definition 4.4. We say that VHVI (2.1) is strongly (resp., weakly) well-posed-like by perturbations if it has a unique solution and for any $\{p_n\} \subset P$ with $p_n \rightarrow p^*$, every approximating-like sequence corresponding to $\{p_n\}$ converges strongly (resp., weakly) to the unique solution of VHVI (2.1). VHVI (2.1) is said to be strongly (resp., weakly) generalized well-posed-like by perturbations if the solution set S of VHVI (2.1) is nonempty and for any $\{p_n\} \subset P$ with $p_n \rightarrow p^*$, every approximating-like sequence corresponding to $\{p_n\}$ has a subsequence which converges strongly (resp., weakly) to some solution of VHVI (2.1).

Let $\varphi : X \rightarrow \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional. Denote by $\partial\varphi$ and $\partial_\varepsilon\varphi$ the subdifferential and ε -subdifferential of φ respectively, i.e.,

$$\partial\varphi(x) = \{x^* \in X^* : \varphi(y) - \varphi(x) \geq \langle x^*, y - x \rangle, \quad \forall y \in X\}, \quad \forall x \in \text{dom}\varphi,$$

and

$$\partial_\varepsilon\varphi(x) = \{x^* \in X^* : \varphi(y) - \varphi(x) \geq \langle x^*, y - x \rangle - \varepsilon, \quad \forall y \in X\}, \quad \forall x \in \text{dom}\varphi.$$

It is known that $\partial\varphi$ is maximal monotone and $\partial_\varepsilon\varphi(x) \supset \partial\varphi(x) \neq \emptyset$ for all $x \in \text{dom}\varphi$ and for all $\varepsilon > 0$. In terms of $\partial\varphi$, VHVI (2.1) is equivalent to the following inclusion problem:

$$\text{IP}(A + T - f + \bar{\partial}J + \partial\varphi) : \text{ find } x \in X \text{ such that } 0 \in A(x) + T(x) - f + \bar{\partial}J(x) + \partial\varphi(x).$$

In other words, we have the following lemma.

Lemma 4.5. (see [39, Lemma 4.1]) Let A, T be two mappings from Banach space X to its dual X^* , $J : X \rightarrow \mathbf{R}$ be a locally Lipschitz functional and $\varphi : X \rightarrow \mathbf{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional. Then $x \in X$ is a solution of VHVI (2.1) if and only if x is a solution of the following inclusion problem

$$\text{IP}(A + T - f + \bar{\partial}J + \partial\varphi) : \text{ find } x \in X \text{ such that } 0 \in A(x) + T(x) - f + \bar{\partial}J(x) + \partial\varphi(x).$$

Naturally, we consider the perturbed problem of $\text{IP}(A + T - f + \bar{\partial}J + \partial\varphi)$ as follows:

$$\text{IP}_p(A + T - f + \bar{\partial}J + \partial\varphi) : \text{ find } x \in X \text{ such that } 0 \in \bar{A}(p, x) + \bar{T}(p, x) - f + \bar{\partial}\tilde{J}_p(x) + \partial\tilde{\varphi}(p, \cdot)(x),$$

where $\tilde{J} : P \times X \rightarrow \mathbf{R}$ is such that $\tilde{J}(p, \cdot)$ is a locally Lipschitz functional for each $p \in P$ and $\tilde{J}(p^*, \cdot) = J$, and $\tilde{\varphi} : P \times X \rightarrow \mathbf{R} \cup \{+\infty\}$ is such that $\tilde{\varphi}(p, \cdot)$ is proper, convex and lower semicontinuous for each $p \in P$ and $\tilde{\varphi}(p^*, \cdot) = \varphi$.

The following theorems establish the relations between the strong (resp., weak) well-posedness by perturbations of variational-hemivariational inequalities and the strong (resp., weak) well-posedness by perturbations of inclusion problems.

Theorem 4.6. Let X be a real Banach space and X^* be its dual space. Let $\bar{A}, \bar{T} : P \times X \rightarrow X^*$ be two mappings, and $\tilde{J} : P \times X \rightarrow \mathbf{R}$ be a functional such that $\tilde{J}(p, \cdot)$ is locally Lipschitz continuous for each $p \in P$. Let $\tilde{\varphi} : P \times X \rightarrow \mathbf{R} \cup \{+\infty\}$ be a functional such that $\tilde{\varphi}(p, \cdot)$ is proper, convex and lower semicontinuous for each $p \in P$. Then the following hold:

(a) $\text{IP}(A + T - f + \bar{\partial}J + \partial\varphi)$ is strongly (resp., weakly) well-posed by perturbations whenever VHVI (2.1) is strongly (resp., weakly) 1-well-posed by perturbations;

(b) VHVI (2.1) is strongly (resp., weakly) well-posed-like by perturbations whenever $\text{IP}(A + T - f + \bar{\partial}J + \partial\varphi)$ is strongly (resp., weakly) well-posed by perturbations.

Proof. (a) Suppose that VHVI (2.1) is strongly (resp., weakly) 1-well-posed by perturbations. Then VHVI (2.1) has a unique solution $x^* \in X$. Hence from Lemma 4.5 it follows that x^* is the unique solution of $\text{IP}(A + T - f + \bar{\partial}J + \partial\varphi)$. Let $\{p_n\} \subset P$ be with $p_n \rightarrow p^*$ and let $\{x_n\}$ be an approximating sequence corresponding to $\{p_n\}$ for $\text{IP}(A + T - f + \bar{\partial}J + \partial\varphi)$. Then $x_n \in \text{dom}\tilde{\varphi}(p_n, \cdot)$ for all $n \geq 1$, and there exists a sequence $\omega_n \in \bar{A}(p_n, x_n) + \bar{T}(p_n, x_n) - f + \bar{\partial}\tilde{J}_{p_n}(x_n) + \partial\tilde{\varphi}(p_n, \cdot)(x_n)$ such that $\|\omega_n\| \rightarrow 0$ as $n \rightarrow \infty$. And so there exists $\xi_n \in \bar{\partial}\tilde{J}_{p_n}(x_n)$ and $\eta_n \in \partial\tilde{\varphi}(p_n, \cdot)(x_n)$ such that

$$\omega_n = \bar{A}(p_n, x_n) + \bar{T}(p_n, x_n) - f + \xi_n + \eta_n, \quad \forall n \geq 1. \quad (4.1)$$

From the definition of the Clarke's generalized gradient for locally Lipschitz functional and the subgradient for convex functional, we obtain by multiplying $y - x_n$ at both sides of the last equation (4.1) that

$$\begin{aligned} & \langle \bar{A}(p_n, x_n) + \bar{T}(p_n, x_n) - f, y - x_n \rangle + \tilde{J}_{p_n}^\circ(x_n, y - x_n) + \tilde{\varphi}(p_n, y) - \tilde{\varphi}(p_n, x_n) \\ & \geq \langle \bar{A}(p_n, x_n) + \bar{T}(p_n, x_n) - f, y - x_n \rangle + \langle \xi_n, y - x_n \rangle + \langle \eta_n, y - x_n \rangle \\ & = \langle \omega_n, y - x_n \rangle \\ & \geq -\|\omega_n\|\|y - x_n\| \\ & \geq -\frac{1}{2}\|\omega_n\|^2 - \frac{1}{2}\|y - x_n\|^2, \quad \forall y \in X. \end{aligned} \quad (4.2)$$

Putting $\varepsilon_n = \frac{1}{2}\|\omega_n\|^2$ and $\alpha = 1$, from (4.2) with $\|\omega_n\| \rightarrow 0$ we deduce that $\{x_n\}$ is an α -approximating sequence corresponding to $\{p_n\}$ for VHVI (2.1) where $\alpha = 1$. Therefore, it follows from the strong (resp., weak) 1-well-posedness by perturbations of VHVI (2.1) that $\{x_n\}$ converges strongly (resp., weakly) to the unique solution x^* of $\text{IP}(A + T - f + \bar{\partial}J + \partial\varphi)$. Consequently, the inclusion problem $\text{IP}(A + T - f + \bar{\partial}J + \partial\varphi)$ is strongly (resp., weakly) well-posed by perturbations.

(b) Suppose that $\text{IP}(A + T - f + \bar{\partial}J + \partial\varphi)$ is strongly (resp., weakly) well-posed by perturbations. Then $\text{IP}(A + T - f + \bar{\partial}J + \partial\varphi)$ has a unique solution $x^* \in X$, which hence implies that x^* is the unique solution of VHVI (2.1) by Lemma 4.5. Let $\{p_n\} \subset P$ be with $p_n \rightarrow p^*$ and let $\{x_n\}$ be an approximating-like sequence

corresponding to $\{p_n\}$ for VHVI (2.1). Then there exists a nonnegative sequence $\{\varepsilon_n\}$ with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ such that

$$\langle \widetilde{A}(p_n, x_n) + \widetilde{T}(p_n, x_n) - f, y - x_n \rangle + \widetilde{J}_{p_n}^\circ(x_n, y - x_n) + \widetilde{\varphi}(p_n, y) - \widetilde{\varphi}(p_n, x_n) \geq -\varepsilon_n \|y - x_n\|, \quad \forall y \in X.$$

From the fact that

$$\widetilde{J}_{p_n}^\circ(x_n, y - x_n) = (\widetilde{J}(p_n, \cdot))^\circ(x_n, y - x_n) = \max\{\langle \omega, y - x_n \rangle : \omega \in \overline{\partial} \widetilde{J}_{p_n}(x_n)\},$$

we conclude that there exists a $\omega_{p_n}(x_n, y) \in \overline{\partial} \widetilde{J}_{p_n}(x_n)$ such that

$$\langle \widetilde{A}(p_n, x_n) + \widetilde{T}(p_n, x_n) - f, y - x_n \rangle + \langle \omega_{p_n}(x_n, y), y - x_n \rangle + \widetilde{\varphi}(p_n, y) - \widetilde{\varphi}(p_n, x_n) \geq -\varepsilon_n \|y - x_n\|, \quad (4.3)$$

for all $y \in X$. Putting $\Phi(p_n, y) = \widetilde{\varphi}(p_n, y) + \varepsilon_n \|y - x_n\|$, $\forall y \in X$, we can easily see that $\Phi(p_n, \cdot)$ is proper, convex and lower semicontinuous. Note that $\{\widetilde{A}(p_n, x_n) + \widetilde{T}(p_n, x_n) - f + \omega : \omega \in \overline{\partial} \widetilde{J}_{p_n}(x_n)\}$ is nonempty, convex and bounded in X^* . Thus, it follows from (4.3) and Theorem 2.7 with $\Phi(p_n, y) = \widetilde{\varphi}(p_n, y) + \varepsilon_n \|y - x_n\|$, that there exists $\omega_{p_n}(x_n) \in \overline{\partial} \widetilde{J}_{p_n}(x_n)$ such that

$$\langle \widetilde{A}(p_n, x_n) + \widetilde{T}(p_n, x_n) - f, y - x_n \rangle + \langle \omega_{p_n}(x_n), y - x_n \rangle + \widetilde{\varphi}(p_n, y) - \widetilde{\varphi}(p_n, x_n) \geq -\varepsilon_n \|y - x_n\|, \quad (4.4)$$

for all $y \in X$. For convenience, we write $\omega_n = \omega_{p_n}(x_n)$, it follows from (4.4) that

$$\widetilde{\varphi}(p_n, x_n) \leq \widetilde{\varphi}(p_n, y) + \langle \widetilde{A}(p_n, x_n) + \widetilde{T}(p_n, x_n) - f + \omega_n, y - x_n \rangle + \varepsilon_n \|y - x_n\|,$$

for all $y \in X$. Define the functional $\bar{\varphi}(p_n, \cdot) : X \rightarrow \mathbf{R} \cup \{+\infty\}$ as follows:

$$\bar{\varphi}(p_n, y) = \widetilde{\varphi}(p_n, y) + H(p_n, y) + \varepsilon_n Q_n(y),$$

where $H(p_n, \cdot)$ and Q_n are two functionals on X defined by

$$H(p_n, y) = \langle \widetilde{A}(p_n, x_n) + \widetilde{T}(p_n, x_n) - f + \omega_n, y - x_n \rangle \quad \text{and} \quad Q_n(y) = \|y - x_n\|.$$

Clearly, $\bar{\varphi}(p_n, \cdot)$ is proper, convex and lower semicontinuous and x_n is a global minimizer of $\bar{\varphi}(p_n, \cdot)$ on X . Thus, $0 \in \partial \bar{\varphi}(p_n, \cdot)(x_n)$. Since the functionals $H(p_n, \cdot)$ and Q_n are continuous on X and $\widetilde{\varphi}(p_n, \cdot)$ is proper, convex and lower semicontinuous, it follows from Proposition 2.2 that

$$\partial \bar{\varphi}(p_n, \cdot)(y) = \partial \widetilde{\varphi}(p_n, \cdot)(y) + \widetilde{A}(p_n, x_n) + \widetilde{T}(p_n, x_n) - f + \omega_n + \varepsilon_n \partial Q_n(y).$$

It is easy to calculate

$$\partial Q_n(y) = \{y^* \in X^* : \|y^*\| = 1, \langle y^*, y - x_n \rangle = \|y - x_n\|\}$$

and so there exists a $\xi_n \in \partial Q_n(x_n)$ with $\|\xi_n\| = 1$ such that

$$0 \in \partial \widetilde{\varphi}(p_n, \cdot)(x_n) + \widetilde{A}(p_n, x_n) + \widetilde{T}(p_n, x_n) - f + \omega_n + \varepsilon_n \xi_n. \quad (4.5)$$

Letting $\zeta_n = -\varepsilon_n \xi_n$, then $\|\zeta_n\| \rightarrow 0$ as $\varepsilon_n \rightarrow 0$. Moreover, since $\omega_n \in \overline{\partial} \widetilde{J}_{p_n}(x_n)$, it follows from (4.5) that

$$\zeta_n \in \widetilde{A}(p_n, x_n) + \widetilde{T}(p_n, x_n) - f + \overline{\partial} \widetilde{J}_{p_n}(x_n) + \partial \widetilde{\varphi}(p_n, \cdot)(x_n),$$

which implies that $\{x_n\}$ is an approximating sequence for $\text{IP}(A + T - f + \overline{\partial} J + \partial \varphi)$. Since $\text{IP}(A + T - f + \overline{\partial} J + \partial \varphi)$ is strongly (resp., weakly) well-posed by perturbations, it is known that $\{x_n\}$ converges strongly (resp., weakly) to the unique solution x^* . Therefore, the variational-hemivariational inequality VHVI (2.1) is strongly (resp., weakly) well-posed-like by perturbations. This completes the proof. \square

Theorem 4.7. Let X be a real Banach space and X^* be its dual space. Let $\tilde{A}, \tilde{T} : P \times X \rightarrow X^*$ be two mappings, and $\tilde{J} : P \times X \rightarrow \mathbf{R}$ be a functional such that $\tilde{J}(p, \cdot)$ is locally Lipschitz continuous for each $p \in P$. Let $\tilde{\varphi} : P \times X \rightarrow \mathbf{R} \cup \{+\infty\}$ be a functional such that $\tilde{\varphi}(p, \cdot)$ is proper, convex and lower semicontinuous for each $p \in P$. Then the following hold:

(a) $\text{IP}(A + T - f + \bar{\partial}J + \partial\varphi)$ is strongly (resp., weakly) generalized well-posed by perturbations whenever VHVI (2.1) is strongly (resp., weakly) generalized 1-well-posed by perturbations;

(b) VHVI (2.1) is strongly (resp., weakly) generalized well-posed-like by perturbations whenever $\text{IP}(A + T - f + \bar{\partial}J + \partial\varphi)$ is strongly (resp., weakly) generalized well-posed by perturbations.

Proof. The proof is similar to that in Theorem 4.6 and so we omit it here. \square

For any $\varepsilon > 0$, we define the following set:

$$\Omega(\varepsilon) = \{(p, x) \in P \times X : \langle \tilde{A}(p, x) + \tilde{T}(p, x) - f, y - x \rangle + \tilde{J}_p^\circ(x, y - x) + \tilde{\varphi}(p, y) - \tilde{\varphi}(p, x) \geq -\varepsilon \|y - x\|, \forall y \in X\}.$$

Theorem 4.8. Let $L = \mathbf{R}^m$. Suppose that $\tilde{A}, \tilde{T} : P \times \mathbf{R}^m \rightarrow \mathbf{R}^m$ be two mappings such that $\tilde{A}(\cdot, x)$ is a continuous mapping for each $x \in \mathbf{R}^m$, $\tilde{A}(p, \cdot)$ is a monotone and hemicontinuous mapping for each $p \in P$ and \tilde{T} is a continuous mapping. Let $\tilde{J} : P \times \mathbf{R}^m \rightarrow \mathbf{R}$ be a functional such that $\tilde{J}(p, \cdot)$ is locally Lipschitz continuous for each $p \in P$ and $\tilde{J}_p^\circ(x, y)$ is upper semicontinuous as a functional of (p, x, y) . Let $\tilde{\varphi} : P \times \mathbf{R}^m \rightarrow \mathbf{R} \cup \{+\infty\}$ be a continuous functional such that $\tilde{\varphi}(p, \cdot)$ is proper and convex for each $p \in P$. If there exists some $\varepsilon > 0$ such that $\Omega(\varepsilon)$ is nonempty and bounded. Then the variational-hemivariational inequality VHVI (2.1) is strongly generalized well-posed-like by perturbations.

Proof. Let $\{p_n\} \subset P$ be with $p_n \rightarrow p^*$ and let $\{x_n\}$ be an approximating-like sequence corresponding to $\{p_n\}$ for VHVI (2.1). Then there exists a nonnegative sequence $\{\varepsilon_n\}$ with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ such that

$$\langle \tilde{A}(p_n, x_n) + \tilde{T}(p_n, x_n) - f, y - x_n \rangle + \tilde{J}_{p_n}^\circ(x_n, y - x_n) + \tilde{\varphi}(p_n, y) - \tilde{\varphi}(p_n, x_n) \geq -\varepsilon_n \|y - x_n\|, \quad \forall y \in \mathbf{R}^m. \quad (4.6)$$

Let $\varepsilon_0 > 0$ be such that $\Omega(\varepsilon_0)$ is nonempty and bounded. Then there exists $n_0 \geq 1$ such that $(p_n, x_n) \in \Omega(\varepsilon_0)$ for all $n > n_0$ and this implies that $\{x_n\}$ is bounded by the boundedness of $\Omega(\varepsilon_0)$. Thus, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow \bar{x}$ as $k \rightarrow \infty$. Since $\tilde{A}(\cdot, x)$ is a continuous mapping for each $x \in \mathbf{R}^m$, $\tilde{A}(p, \cdot)$ is a monotone mapping for each $p \in P$, \tilde{T} is a continuous mapping, the Clarke generalized directional derivative $\tilde{J}_p^\circ(x, y)$ is upper semicontinuous with respect to (p, x, y) and $\tilde{\varphi}$ is continuous, it follows from (4.6) that

$$\begin{aligned} & \langle A(y) + T(\bar{x}) - f, y - \bar{x} \rangle + J^\circ(\bar{x}, y - \bar{x}) + \varphi(y) - \varphi(\bar{x}) \\ &= \langle \tilde{A}(p^*, y) + \tilde{T}(p^*, \bar{x}) - f, y - \bar{x} \rangle + \tilde{J}_{p^*}^\circ(\bar{x}, y - \bar{x}) + \tilde{\varphi}(p^*, y) - \tilde{\varphi}(p^*, \bar{x}) \\ &\geq \limsup_{n \rightarrow \infty} \{ \tilde{A}(p_{n_k}, y) + \tilde{T}(p_{n_k}, x_{n_k}) - f, y - x_{n_k} \rangle + \tilde{J}_{p_{n_k}}^\circ(x_{n_k}, y - x_{n_k}) + \tilde{\varphi}(p_{n_k}, y) - \tilde{\varphi}(p_{n_k}, x_{n_k}) \} \\ &\geq \limsup_{n \rightarrow \infty} \{ \tilde{A}(p_{n_k}, x_{n_k}) + \tilde{T}(p_{n_k}, x_{n_k}) - f, y - x_{n_k} \rangle + \tilde{J}_{p_{n_k}}^\circ(x_{n_k}, y - x_{n_k}) + \tilde{\varphi}(p_{n_k}, y) - \tilde{\varphi}(p_{n_k}, x_{n_k}) \} \\ &\geq \limsup_{n \rightarrow \infty} (-\varepsilon_{n_k} \|y - x_{n_k}\|) \\ &= 0, \quad \forall y \in \mathbf{R}^m. \end{aligned} \quad (4.7)$$

Since $\tilde{A}(p^*, \cdot) : \mathbf{R}^m \rightarrow \mathbf{R}^m$ is monotone and hemicontinuous, and $\tilde{\varphi}(p^*, \cdot) : \mathbf{R}^m \rightarrow \mathbf{R} \cup \{+\infty\}$ be proper, convex and lower semicontinuous, in terms of Lemma 2.10 we get

$$\langle A(\bar{x}) + T(\bar{x}) - f, y - \bar{x} \rangle + J^\circ(\bar{x}, y - \bar{x}) + \varphi(y) - \varphi(\bar{x}) \geq 0, \quad \forall y \in \mathbf{R}^m,$$

which implies that \bar{x} solves the VHVI (2.1). Therefore, the VHVI (2.1) is strongly generalized well-posed-like by perturbations. \square

5. Concluding remarks

In this paper, we introduce some concepts of well-posedness by perturbations for a class of variational-hemivariational inequalities with perturbations, which includes as special cases the classical variational inequalities and hemivariational inequalities. Under very mild conditions, we establish some metric characterizations for the well-posed variational-hemivariational inequality, and investigate the relation between the strong (resp., weak) well-posedness by perturbations of a variational-hemivariational inequality and the strong (resp., weak) well-posedness by perturbations of the corresponding inclusion problem. In addition, we also give some conditions under which the variational-hemivariational inequality is strongly generalized well-posed-like by perturbations in the finite-dimensional space \mathbf{R}^m .

It is worth pointing out that there are many other concepts of well-posedness for optimization problems, variational inequalities and Nash equilibrium problems, such as L -well-posedness [23], parametric well-posedness [11] and Levitin-Polyak well-posedness [19], etc. However, we wonder whether the concepts mentioned as above can be extended to the strongly mixed variational-hemivariational inequality. Beyond question, this is an interesting problem.

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