Well-posedness for a multi-dimensional viscous liquid-gas two-phase flow model

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Abstract

In this talk, I review recent results on the gas-liquid two-phase flow model based on the joint work with Hai-Liang Li.

The Cauchy problem to a simplified version of the viscous compressible liquid-gas two-phase flow model of drift-flux type in $\mathbb{R}^d$ ($d \geqslant 2$), where the gas phase has not been taken into account in the momentum equation except the pressure term and the equal velocity of the liquid and gas flows has been assumed, reads

$$\begin{cases} \tilde{m}_t + \mathrm{div}(\tilde{m}\mathbf{u}) = 0, \\ \tilde{n}_t + \mathrm{div}(\tilde{n}\mathbf{u}) = 0, \\ (\tilde{m}\mathbf{u})_t + \mathrm{div}(\tilde{m}\mathbf{u}\otimes\mathbf{u}) + \nabla P(\tilde{m},\tilde{n}) = \tilde{\mu}\Delta\mathbf{u} + (\tilde{\mu}+\tilde{\lambda})\nabla\mathrm{div}\mathbf{u}, \end{cases} \tag{1}$$

with the initial data

$$(\tilde{m},\tilde{n},\mathbf{u})|_{t=0} = (\tilde{m}_0,\tilde{n}_0,\mathbf{u}_0)(x), \quad \text{in } \mathbb{R}^d,$$

where $\tilde{m} = \alpha_l\rho_l$ and $\tilde{n} = \alpha_g\rho_g$ denote the liquid mass and the gas mass, respectively. The unknowns α_l, $\alpha_g \in [0,1]$ denote the liquid and gas volume fractions, satisfying the fundamental relation: $\alpha_l + \alpha_g = 1$. The unknown variables ρ_l and ρ_g denote the liquid and gas densities, satisfying the equations of states $\rho_l = \rho_{l,0} + (P - P_{l,0})/a_l^2$, $\rho_g = P/a_g^2$, where a_l and a_g denote the sonic speeds of the liquid and the gas, respectively, and $P_{l,0}$ and $\rho_{l,0}$ are the reference pressure and density given as constants. $\mathbf{u}$ denotes the mixed velocity of the liquid and the gas, and P is the common pressure for both phases, which satisfies

$$P(\tilde{m},\tilde{n}) = C_0\left(-b(\tilde{m},\tilde{n}) + \sqrt{b^2(\tilde{m},\tilde{n}) + c(\tilde{m},\tilde{n})}\right),$$

with $C_0 = a_l^2/2$, $k_0 = \rho_{l,0} - P_{l,0}/a_l^2 > 0$, $a_0 = a_g^2/a_l^2$ and

$$b(\tilde{m},\tilde{n}) = k_0 - \tilde{m} - a_0\tilde{n}, \quad c(\tilde{m},\tilde{n}) = 4k_0a_0\tilde{n}.$$

$\tilde{\mu}$ and $\tilde{\lambda}$ are the viscosity constants, satisfying

$$\tilde{\mu} > 0, \quad 2\tilde{\mu} + d\tilde{\lambda} \geqslant 0.$$

Definition 1. For $T > 0$ and $s \in \mathbb{R}$, we denote

$$\begin{aligned} E_T^s = \big\{(m,n,\mathbf{u}) : \; & n \in \mathcal{C}([0,T]; B^{s-1,s}(\mathbb{R}^d)) \\ & m \in \mathcal{C}([0,T]; B^{s-1,s}(\mathbb{R}^d)) \cap L^1([0,T]; B^{s+1,s}(\mathbb{R}^d)) \\ & \mathbf{u} \in \big(\mathcal{C}([0,T]; B^{s-1}(\mathbb{R}^d)) \cap L^1([0,T]; B^{s+1}(\mathbb{R}^d))\big)^d\big\}, \end{aligned}$$

and

$$\begin{aligned} \|(m,n,\mathbf{u})\|_{E_T^s} = & \|n\|_{\tilde{L}^\infty([0,T];B^{s-1,s})} + \|m\|_{\tilde{L}^\infty([0,T];B^{s-1,s})} + \|\mathbf{u}\|_{\tilde{L}^\infty([0,T];B^{s-1})} \\ & + \|m\|_{L^1([0,T];B^{s+1,s})} + \|\mathbf{u}\|_{L^1([0,T];B^{s+1})}. \end{aligned}$$

We use the notation E^s if $T = +\infty$, changing $[0,T]$ into $[0,\infty)$ in the definition above.

Definition 2. Let $\alpha \in [0,1]$ and $T > 0$, denote

$$\begin{aligned} F_T^\alpha :=&(\tilde{\mathcal{C}}([0,T]; B^{d/2,d/2+\alpha}))^{1+1} \\ &\times (\tilde{\mathcal{C}}([0,T]; B^{d/2-1,d/2-1+\alpha}) \cap L^1([0,T]; B^{d/2+1,d/2+1+\alpha}))^d. \end{aligned}$$

Now, we state the global well-posedness results briefly as follows.

Theorem 1 (Global well-posedness for small data)**.** *Let $d \geqslant 2$, $\bar{n} \geqslant 0$, $\bar{m} > (1-\mathrm{sgn}\bar{n})k_0$, $\tilde{\mu} > 0$ and $2\tilde{\mu} + d\tilde{\lambda} \geqslant 0$, in addition, $\tilde{\mu} + \tilde{\lambda} > 0$ if $d = 2$. There exist two positive constants σ and Q such that if $\tilde{m}_0 - \bar{m}$, $\tilde{n}_0 - \bar{n} \in B^{d/2-1,d/2}$ and $\mathbf{u}_0 \in B^{d/2-1}$ satisfying*

$$\|\tilde{m}_0 - \bar{m}\|_{B^{d/2-1,d/2}} + \|\tilde{n}_0 - \bar{n}\|_{B^{d/2-1,d/2}} + \|\mathbf{u}_0\|_{B^{d/2-1}} \leqslant \sigma,$$

then the following results hold

(i) *Existence: The system* (1) *has a solution $(\tilde{m}, \tilde{n}, \mathbf{u})$ satisfying*

$$\tilde{m} - \bar{m},\ \tilde{n} - \bar{n} \in \mathcal{C}\left(\mathbb{R}^+; B^{d/2-1,d/2}\right), \quad \mathbf{u} \in \mathcal{C}\left(\mathbb{R}^+; B^{d/2-1}\right),$$

and moreover,

$$\begin{aligned} &\|(a(\tilde{m} - \bar{m}) + ba_0(\tilde{n}/\tilde{m} - \bar{n}/\bar{m}), \tilde{n}/\tilde{m} - \bar{n}/\bar{m}, \mathbf{u})\|_{E^{d/2}} \\ \leqslant& Q\big(\|\tilde{m}_0 - \bar{m}\|_{B^{d/2-1,d/2}} + \|\tilde{n}_0 - \bar{n}\|_{B^{d/2-1,d/2}} + \|\mathbf{u}_0\|_{B^{d/2-1}}\big), \end{aligned}$$

where the constants a and b are defined by

$$\begin{aligned} a =& \frac{1}{\bar{m}^2}\left(a_0\bar{n} + \bar{m} + \frac{(\bar{m} - a_0\bar{n})(\bar{m} - a_0\bar{n} - k_0)}{\sqrt{(\bar{m} + a_0\bar{n} - k_0)^2 + 4k_0a_0\bar{n}}}\right) > 0, \\ b =& 1 + \frac{(\bar{m} + a_0\bar{n} + k_0)}{\sqrt{(\bar{m} + a_0\bar{n} - k_0)^2 + 4k_0a_0\bar{n}}} > 0. \end{aligned}$$

(ii) *Uniqueness: Uniqueness holds in $\mathcal{C}\left(\mathbb{R}^+; (B^{d/2-1,d/2})^{1+1} \times (B^{d/2})^d\right)$ if $d \geqslant 3$. If $d = 2$, one should also suppose that $\tilde{m}_0 - \bar{m}$, $\tilde{n}_0 - \bar{n} \in B^{\varepsilon,1+\varepsilon}$ and $\mathbf{u}_0 \in B^\varepsilon$ for a $\varepsilon \in (0,1)$, to get uniqueness in $\mathcal{C}(\mathbb{R}^+; (B^{0,1})^{1+1} \times (B^1)^d)$.*

For the general data bounded away from the infinity and the vacuum, we have the following local well-posedness theory.

Theorem 2 (Local well-posedness for general data)**.** *Let $d \geqslant 2$, $\tilde{\mu} > 0$, $2\tilde{\mu} + d\tilde{\lambda} \geqslant 0$, the constants $\bar{m} > 0$ and $\bar{n} \geqslant 0$. Assume that $\tilde{m}_0^{-1} - \bar{m}^{-1} \in B^{d/2,d/2+1}$, $\tilde{n}_0 - \bar{n} \in B^{d/2,d/2+1}$ and $\mathbf{u}_0 \in B^{d/2-1,d/2}$. In addition, $\sup_{x\in\mathbb{R}^d} \tilde{m}_0(x) < \infty$. Then there exists a positive time T such that the system* (1) *has a unique solution $(\tilde{m}, \tilde{n}, \mathbf{u})$ on $[0,T] \times \mathbb{R}^d$ and that $(\tilde{m}^{-1} - \bar{m}^{-1}, \tilde{n} - \bar{n}, \mathbf{u})$ belongs to F_T^1 and satisfies $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \tilde{m}(t,x) < \infty$.*

We also have the following continuation criterion for the local existence of the solution.

Theorem 3 (Continuation criterion)**.** *Under the hypotheses of Theorem 2, assume that the system* (1) *has a solution $(\tilde{m}, \tilde{n}, \mathbf{u})$ on $[0,T) \times \mathbb{R}^d$ such that $(\tilde{m}^{-1} - \bar{m}^{-1}, \tilde{n} - \bar{n}, \mathbf{u})$ belongs to $F_{T'}^1$ for all $T' < T$ and satisfies*

$$\begin{gathered} \tilde{m}^{-1} - \bar{m}^{-1}, \tilde{n} - \bar{n} \in L^\infty([0,T); B^{d/2,d/2+1}), \\ \sup_{(t,x)\in[0,T)\times\mathbb{R}^d} \tilde{m}(t,x) < \infty, \quad \int_0^T \|\nabla\mathbf{u}\|_\infty dt < \infty. \end{gathered}$$

Then, there exists some $T^* > T$ such that $(\tilde{m}, \tilde{n}, \mathbf{u})$ may be continued on $[0,T^*] \times \mathbb{R}^d$ to a solution of (1) *such that $(\tilde{m}^{-1} - \bar{m}^{-1}, \tilde{n} - \bar{n}, \mathbf{u})$ belongs to $F_{T^*}^1$.*