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Well-Posedness in Smooth Function Spaces for the Moving-Boundary Three-Dimensional Compressible Euler Equations in Physical Vacuum

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Abstract

We prove well-posedness for the three-dimensional compressible Euler equations with moving *physical* vacuum boundary, with an equation of state given by $p(\rho) = C_\gamma \rho^\gamma$ for $\gamma > 1$. The physical vacuum singularity requires the sound speed c to go to zero as the square-root of the distance to the moving boundary, and thus creates a degenerate and characteristic hyperbolic *free-boundary* system wherein the density vanishes on the free-boundary, the uniform Kreiss–Lopatinskii condition is violated, and manifest derivative loss ensues. Nevertheless, we are able to establish the existence of unique solutions to this system on a short time-interval, which are smooth (in Sobolev spaces) all the way to the moving boundary, and our estimates have no derivative loss with respect to initial data. Our proof is founded on an approximation of the Euler equations by a degenerate parabolic regularization obtained from a specific choice of a degenerate artificial viscosity term, chosen to preserve as much of the geometric structure of the Euler equations as possible. We first construct solutions to this degenerate parabolic regularization using a higher-order version of Hardy's inequality; we then establish estimates for solutions to this degenerate parabolic system which are independent of the artificial viscosity parameter. Solutions to the compressible Euler equations are found in the limit as the artificial viscosity tends to zero. Our regular solutions can be viewed as *degenerate viscosity solutions*. Our methodology can be applied to many other systems of degenerate and characteristic hyperbolic systems of conservation laws.

Contents

1. Introduction

1.1. The Compressible Euler Equations in Eulerian Variables

For $0 \leqq t \leqq T$, the evolution of a three-dimensional compressible gas moving inside of a dynamic vacuum boundary is modeled by the one-phase compressible Euler equations:

$$\rho[u_t + u \cdot Du] + Dp(\rho) = 0 \quad \text{in } \Omega(t), \tag{1.1a}$$
$$\rho_t + \operatorname{div}(\rho u) = 0 \quad \text{in } \Omega(t), \tag{1.1b}$$
$$p = 0 \quad \text{on } \Gamma(t), \tag{1.1c}$$
$$\mathcal{V}(\Gamma(t)) = u \cdot n(t) \tag{1.1d}$$
$$(\rho, u) = (\rho_0, u_0) \quad \text{on } \Omega(0), \tag{1.1e}$$
$$\Omega(0) = \Omega. \tag{1.1f}$$

The open, bounded subset $\Omega(t) \subset \mathbb{R}^3$ denotes the changing volume occupied by the gas, $\Gamma(t) := \partial\Omega(t)$ denotes the moving vacuum boundary, $\mathcal{V}(\Gamma(t))$ denotes the normal velocity of $\Gamma(t)$, and $n(t)$ denotes the exterior unit normal vector to $\Gamma(t)$. The vector-field $u = (u_1, u_2, u_3)$ denotes the Eulerian velocity field, p denotes the pressure function, and ρ denotes the density of the gas. The equation of state $p(\rho)$ is given by

$$p(x,t) = C_\gamma\, \rho(x,t)^\gamma \quad \text{for} \quad \gamma > 1, \tag{1.2}$$

where C_γ is the adiabatic constant which we set to unity, and

$$\rho > 0 \ \text{ in } \ \Omega(t) \quad \text{and} \quad \rho = 0 \ \text{ on } \Gamma(t).$$

Equation (1.1a) is the conservation of momentum; (1.1b) is the conservation of mass; the boundary condition (1.1c) states that the pressure (and hence the density function) vanish along the moving vacuum boundary $\Gamma(t)$; (1.1d) states that the vacuum boundary $\Gamma(t)$ is moving with speed equal to the normal component of the fluid velocity, and (1.1e)–(1.1f) are the initial conditions for the density, velocity, and domain. Using the equation of state (1.2), (1.1a) is written as

$$\rho[u_t + u \cdot Du] + D\rho^\gamma = 0 \quad \text{in } \Omega(t). \tag{1.1a'}$$

1.2. Physical Vacuum

With the sound speed given by $c := \sqrt{\partial p/\partial \rho}$ and N denoting the *outward* unit normal to the initial surface Γ, satisfaction of the condition

$$\frac{\partial c_0^2}{\partial N} < 0 \quad \text{on } \Gamma \tag{1.3}$$

defines a *physical vacuum* boundary (see [20,23–26,43]), where $c_0 = c|_{t=0}$ denotes the initial sound speed of the gas.

The physical vacuum condition (1.3) is equivalent to the requirement that

$$\frac{\partial \rho_0^{\gamma-1}}{\partial N} < 0 \quad \text{on } \Gamma, \tag{1.4}$$

a condition necessary for the gas particles on the boundary to accelerate. Since $\rho_0 > 0$ in Ω, (1.4) implies that for some positive constant C and $x \in \Omega$ near the vacuum boundary Γ,

$$\rho_0^{\gamma-1}(x) \geqq C \text{dist}(x, \Gamma). \tag{1.5}$$

Because of condition (1.5), the compressible Euler system (1.1) is a *degenerate* and *characteristic* hyperbolic system which violates the uniform Kreiss–Lopatinskii condition [17] because of resonant wave speeds at the vacuum boundary for the linearized problem; it may be that the methods which have already been developed for symmetric hyperbolic conservation laws whose linearization is only weakly well-posed would be extremely difficult to implement for this problem, wherein the degeneracy of the vacuum creates further difficulties for the linearized estimates. The moving boundary is characteristic because of the evolution law (1.1d), and the system of conservation laws is degenerate because of the appearance of the density function as a coefficient in the nonlinear wave equation which governs the dynamics of the divergence of the velocity of the gas. In turn, weighted estimates show that this wave equation indeed loses derivatives with respect to the uniformly hyperbolic non-degenerate case of a compressible liquid, wherein the density takes the value of a strictly positive constant on the moving boundary [6]. We provide a brief history of results in this area in Section 1.9 below.

We note that with a faster rate of degeneracy of the density function, such as, for example, $\text{dist}(x, \Gamma(t))^b$ for $b = 2, 3, \ldots$., the analysis becomes significantly easier; for instance, if $b = 2$, then $\frac{D\rho_0^{\gamma-1}(x,t)}{\sqrt{\rho_0^{\gamma-1}(x,t)}}$ is bounded for all $x \in \Omega$. This bound makes it possible to readily control error terms in energy estimates, and in effect removes the singular behavior associated with the physical vacuum condition (1.5). On the other hand, if $\rho^{\gamma-1}$ tends to zero like $\text{dist}(x, \Gamma(t))^b$ for $b = 2, 3, \ldots$., then the gas cannot accelerate into vacuum.

1.3. Fixing the Domain and the Lagrangian Variables on Ω

We transform the system (1.1) into Lagrangian variables. We let $\eta(x, t)$ denote the "position" of the gas particle x at time t. Thus,

$$\partial_t \eta = u \circ \eta \text{ for } t > 0 \quad \text{and} \quad \eta(x, 0) = x ,$$

where $\circ$ denotes composition so that $[u \circ \eta](x, t) := u(\eta(x, t), t)$. We set

$$\begin{aligned} v &= u \circ \eta \text{ (Lagrangian velocity)}, \\ f &= \rho \circ \eta \text{ (Lagrangian density)}, \\ A &= [D\eta]^{-1} \text{ (inverse of deformation tensor)}, \\ J &= \det D\eta \text{ (Jacobian determinant)}, \\ a &= J\, A \text{ (transpose of cofactor matrix)}. \end{aligned}$$

Using Einstein's summation convention defined in Section 2.5 below, and using the notation $F,_k$ to denote $\frac{\partial F}{\partial x_k}$, the k$th$-partial derivative of F for $k = 1, 2, 3$, the Lagrangian version of equations (1.1a)–(1.1b) can be written on the fixed reference domain Ω as

$$f v_t^i + A_i^k f^\gamma,_k = 0 \qquad \text{in } \Omega \times (0, T], \tag{1.6a}$$

$$f_t + f A_i^j v^i,_j = 0 \qquad \text{in } \Omega \times (0, T], \tag{1.6b}$$

$$f = 0 \qquad \text{in } \Omega \times (0, T], \tag{1.6c}$$

$$(f, v, \eta) = (\rho_0, u_0, e) \qquad \text{in } \Omega \times \{t = 0\}, \tag{1.6d}$$

where $e(x) = x$ denotes the identity map on Ω.

Since $J_t = J A_i^j v^i,_j$ and since $J(0) = 1$ (since we have taken $\eta(x, 0) = x$), it follows that

$$f = \rho_0 J^{-1}, \tag{1.7}$$

so that the initial density function ρ_0 can be viewed as a parameter in the Euler equations. Let $\Gamma := \partial\Omega$ denote the initial vacuum boundary. Using the fact that $A_i^k = J^{-1} a_i^k$, we write the compressible Euler equations (1.6) as

$$\rho_0 v_t^i + a_i^k (\rho_0^\gamma J^{-\gamma}),_k = 0 \qquad \text{in } \Omega \times (0, T], \tag{1.8a}$$

$$(\eta, v) = (e, u_0) \qquad \text{in } \Omega \times \{t = 0\}, \tag{1.8b}$$

$$\rho_0^{\gamma - 1} = 0 \qquad \text{on } \Gamma, \tag{1.8c}$$

with $\rho_0^{\gamma-1}(x) \geqq C \operatorname{dist}(x, \Gamma)$ for $x \in \Omega$ near Γ.

1.4. Setting $\gamma = 2$

We will begin our analysis for the case that $\gamma = 2$, and in Section 11, we will explain the modifications required for the case of general $\gamma > 1$.

With γ set to 2, we thus seek solutions $\eta(t)$ to the following system:

$$\rho_0 v_t^i + a_i^k (\rho_0^2 J^{-2}),_k = 0 \qquad \text{in } \Omega \times (0, T], \tag{1.9a}$$

$$(\eta, v) = (e, u_0) \qquad \text{on } \Omega \times \{t = 0\}, \tag{1.9b}$$

$$\rho_0 = 0 \qquad \text{on } \Gamma, \tag{1.9c}$$

with $\rho_0(x) \geqq C \operatorname{dist}(x, \Gamma)$ for $x \in \Omega$ near Γ.

The equation (1.9a) is equivalent to

$$v_t^i + 2A_i^k(\rho_0 J^{-1}),_k = 0, \tag{1.10}$$

and (1.10) can be written as

$$v_t^i + \rho_0 a_i^k J^{-2},_k + 2\rho_0,_k a_i^k J^{-2} = 0. \tag{1.11}$$

Because of the degeneracy caused by $\rho_0 = 0$ on Γ, all three equivalent forms of the compressible Euler equations are crucially used in our analysis. The equation (1.9a) is used for energy estimates, while (1.10) is used for estimates of the vorticity, and (1.11) is used for additional elliptic-type estimates used to recover the bounds for normal derivatives.

1.5. The Reference Domain Ω

To avoid the use of local coordinate charts necessary for arbitrary geometries, and to simplify our exposition, we will assume that the initial domain at time $t = 0$ is given by

$$\Omega = \mathbb{T}^2 \times (0, 1),$$

where $\mathbb{T}^2$ denotes the 2-torus and is identified with the unit square with periodic boundary conditions. This permits the use of *one* global Cartesian coordinate system. At $t = 0$, the reference *vacuum* boundary is comprised of the *bottom* and *top* of the domain Ω so that

$$\Gamma = \{x_3 = 0\} \cup \{x_3 = 1\}.$$

Then, according to the evolution law for the moving vacuum boundary $\Gamma(t)$ given by (1.1d), we have that

$$\Gamma(t) = \eta(t)(\Gamma).$$

(We will sometimes write $\eta(t, \Gamma)$ to denote $\eta(t)(\Gamma)$.) Hence, solving (1.9) for $\eta(t)$ (and $v(t) = \eta_t(t)$) completely determines the motion and regularity of the moving vacuum boundary $\Gamma(t)$.

1.6. The Higher-Order Energy Function for the Case $\gamma = 2$

The physical energy $\int_\Omega \left[\frac{1}{2}\rho_0|v|^2 + \rho_0^2 J^{-1}\right]\mathrm{d}x$ is a conserved quantity, but is far too weak for the purposes of constructing solutions; instead, we consider the higher-order energy function

$$\begin{aligned} E(t) = \sum_{a=0}^{4} \Big[\|\partial_t^{2a}\eta(t)\|_{4-a}^2 + \|\rho_0 \partial_t^{2a} \bar\partial^{4-a} D\eta(t)\|_0^2 + \|\sqrt{\rho_0}\bar\partial^{4-a}\partial_t^{2a} v(t)\|_0^2 \Big] \\ + \|\operatorname{curl}_\eta v(t)\|_3^2 + \|\rho_0 \bar\partial^4 \operatorname{curl}_\eta v(t)\|_0^2, \end{aligned} \tag{1.12}$$

where $\bar\partial = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ and $\operatorname{curl}_\eta v = [\operatorname{curl} u] \circ \eta$. Section 2 explains the notation.

We also define $M_0 = P(E(0))$, where P denotes a polynomial function of its argument.

While the higher-order energy function $E(t)$ is not conserved, we will construct solutions to (1.9) for which $\sup_{t\in[0,T]} E(t)$ remains bounded whenever $T > 0$ is taken sufficiently small; the bound depends only on $E(0)$.

1.7. Main Result

Theorem 1. (Existence and uniqueness for the case $\gamma = 2$) *Suppose that* $\rho_0 \in H^4(\Omega),\ \rho_0(x) > 0$ *for* $x \in \Omega,\ \rho_0 = 0$ *on* Γ*, and* ρ_0 *satisfies* (1.5). *Furthermore, suppose that* u_0 *is given such that* $M_0 < \infty$. *Then there exists a solution to* (1.9) (*and hence to* (1.1)) *on* $[0, T]$ *for* $T > 0$ *taken sufficiently small, such that*

$$\sup_{t\in[0,T]} E(t) \leqq 2M_0.$$

In particular, the flow map $\eta \in L^\infty(0, T; H^4(\Omega))$ *and the moving vacuum boundary* $\Gamma(t)$ *is of Sobolev class* $H^{3.5}$.

Moreover if the initial data satisfy

$$\sum_{a=0}^{5}\Big[\|\partial_t^{2a}\eta(0)\|^2_{5-a} + \|\rho_0\partial_t^{2a}D\eta(0)\|^2_{5-a} + \|\sqrt{\rho_0}\bar\partial^{5-a}\partial_t^{2a}v(0)\|^2_0\Big] + \|\operatorname{curl}_\eta v(0)\|^2_4 + \|\rho_0\bar\partial^5\operatorname{curl}_\eta v(0)\|^2_0 < \infty, \tag{1.13}$$

then the solution is unique.

Remark 1. The case of arbitrary $\gamma > 1$ is treated in Theorem 4 below.

Theorem 1 also covers the two-dimensional case that $\Omega \subset \mathbb{R}^2$. We established the analogous result in one dimension in [10]. We note that by using a collection of local coordinate charts, we could modify our proof to allow for arbitrary initial domains Ω, as long as the initial boundary is of Sobolev class $H^{3.5}$.

The multidimensional physical vacuum problem is not only a characteristic hyperbolic system, but is also degenerate because the density function vanishes on the boundary Γ. In one dimension, the two characteristic curves of the isentropic system intersect with the moving vacuum boundary $\Gamma(t)$ tangentially; this triple point of intersection is suggestive of singular behavior. While the degeneracy produces "honest" derivative loss with respect to uniformly hyperbolic systems, we develop a methodology based on *nonlinear estimates* which provides us with a priori control of smooth solutions which do not suffer from the derivative loss phenomenon (see [7] for the a priori estimates to this problem). As we will outline below, our method for constructing smooth solutions does not rely on linearization, Kreiss–Lopatinskii theory, or the Nash–Moser iteration scheme, but rather on a carefully chosen nonlinear approximation to the characteristic and degenerate Euler equations, which preserves a great deal of the nonlinear structure of the original system.

1.8. History of Prior Results on the Analysis of Multidimensional Free-Boundary Euler Problems

1.8.1. The Incompressible Setting There has been a recent explosion of interest in the analysis of the free-boundary *incompressible* Euler equations, particularly in irrotational form, that has produced a number of different methodologies for obtaining a priori estimates. The accompanying existence theories have relied mostly on the Nash–Moser iteration to deal with derivative loss in linearized equations when arbitrary domains are considered, or on complex analysis tools for the irrotational problem with infinite depth. We refer the reader to [1,9,19,21,32,34,41,42,45], and [46] for a partial list of papers on this topic.

1.8.2. The Compressible Setting The mathematical analysis of moving hypersurfaces in the multidimensional compressible Euler equations is essential for the understanding of shock waves, vortex sheets or contact discontinuities, as well as phase transitions such as the motion of gas into the vacuum state considered herein.

The stability and regularity of the multidimensional shock solution was initiated in [28] and extensively studied by [12–14], and [31] (see the references in these articles for a more extensive bibliography). The shock wave problem is non-characteristic on the boundary and, in fact, produces the so-called dissipative boundary conditions, and satisfies the uniform Kreiss–Lopatinskii condition. Even so the methodologies employed produce derivative loss with respect to initial data.

More delicate than the non-characteristic case, the characteristic boundary case is encountered in the study of vortex sheet or current vortex sheet problems. This class of problems has been studied by [2,4,5,37,39] and others, and has the relative disadvantage of violating the uniform Kreiss–Lopatinskii condition, which produces derivative loss in the linearization, similar to that experienced by many authors in the incompressible flow setting (both irrotational flows and flows with vorticity).

1.9. History of Prior Results for the Compressible Euler Equations with Vacuum Boundary

The physical vacuum free-boundary problem, also described as the physical vacuum singularity, has a rich history, as well as a great deal of renewed interest (see [40]).

Some of the early developments in the theory of vacuum states for compressible gas dynamics can be found in [20,26]. We are aware of only a handful of previous theorems pertaining to the existence of solutions to the compressible and *undamped* Euler equations[1] with a moving vacuum boundary. In [29], compactly supported initial data were considered, and the compressible Euler equations were treated as

[1] The parabolic free-boundary viscous Navier–Stokes equations do not experience the same sort of analytical difficulties as the compressible Euler equations, so we do not focus on the viscous regime in this paper. We refer the reader to [15,27,30], and [33] for the analysis of the corresponding viscous system.

a PDE set on $\mathbb{R}^3 \times (0, T]$. Unfortunately, with the methodology of [29], it is not possible to track the location of the vacuum boundary (nor is it necessary); nevertheless, an existence theory was developed in this context by a variable change that permitted the standard theory of symmetric hyperbolic systems to be employed, but the constraints on the data were too severe to allow for the evolution of the physical vacuum boundary.

Existence and uniqueness for the three-dimensional compressible Euler equations modeling a *liquid* rather than a gas were established in [22]. As discussed in [6], for a compressible liquid, the density $\rho \geqq \lambda > 0$ is assumed to be a strictly positive constant on the moving vacuum boundary $\Gamma(t)$ and ρ is thus uniformly bounded from below by a positive constant. As such, the compressible liquid provides a uniformly hyperbolic, but characteristic, system. Lagrangian variables combined with Nash–Moser iteration was used in [22] to construct solutions. More recently, [38] provided an alternative proof for the existence of a compressible liquid, employing a solution strategy based on symmetric hyperbolic systems combined with Nash–Moser iteration, but as stated in Remark 2.2 of that paper, the γ-gas law equation-of-state $p = \rho^\gamma$ cannot be used.

In the presence of damping, and with mild singularity, some existence results of smooth solutions are available, based on the adaptation of the theory of symmetric hyperbolic systems. In [24], a local existence theory was developed for the case that c^α (with $0 < \alpha \leqq 1$) is smooth across Γ, using methods that are not applicable to the local existence theory for the physical vacuum boundary. An existence theory for the small perturbation of a planar wave was developed in [43]. See also [25] and [44], for other features of the vacuum state problem.

In the one-dimensional setting, recently, the authors of [16] have established existence and uniqueness using weighted Sobolev norms for their energy estimates. From these weighted norms, the regularity of the solutions cannot be directly determined. Letting d denote the distance function to the boundary ∂I, and letting $\|\cdot\|_0$ denote the $L^2(\Omega)$-norm, an example of the type of bound that is proved for their rescaled velocity field u in [16] is the following:

$$\begin{aligned}&\|d\, u\|_0^2 + \|d\, u_x\|_0^2 + \|d\, u_{xx} + 2u_x\|_0^2 + \|d\, u_{xxx} + 2u_{xx} - 2d^{-1}\, u_x\|_0^2 \\ &\quad + \|d\, u_{xxxx} + 4u_{xxx} - 4d^{-1}\, u_{xx}\|_0^2 < \infty.\end{aligned} \tag{1.14}$$

This bound is obtained from their paper by considering the case $\gamma = 3$, $k = 1$, and making the assumption that $\phi = \xi$ (using the variable terminology of their paper), which is certainly true near the boundary. The problem with inferring the regularity of u from this bound can already be seen at the level of an $H^1(\Omega)$ estimate. In particular, the bound on the norm $\|d\, u_{xx} + 2u_x\|_0^2$ implies a bound only on $\|d\, u_{xx}\|_0^2$ and $\|u_x\|_0^2$ if the integration by parts on the cross-term,

$$4\int_I d\, u_{xx}\, u_x\, \mathrm{d}x = -2\int_I d_x\, |u_x|^2\, \mathrm{d}x,$$

can be justified, which in turn requires having better regularity for u_x than the a priori bounds provide. Any methodology which seeks regularity in (unweighted) Sobolev spaces for solutions must contend with this type of issue.

We overcame this difficulty in one dimension [10] by constructing (sufficiently) smooth solutions to a degenerate parabolic regularization and consequently avoiding this sort of integration-by-parts difficulty. Our solution strategy in [10] was based on a one-dimensional version of Hardy's inequality (in higher-order form). In this paper, we extend our ideas to the multidimensional setting.

1.10. Outline of the Paper and our Methodology

Section 2 defines the notation used throughout the paper. In Section 3, we state Hardy's inequality (in higher-order form) for functions on Ω that vanish on Γ; this inequality is of fundamental importance to our strategy for constructing solutions. In this section, we also state a lemma on κ-independent estimates for equations $\kappa f_t + f = g$, which will be of great use to us in the elliptic-type estimates that we shall employ for bounding normal derivatives. We end this section with a standard weighted embedding into standard Sobolev spaces. Section 4 defines the Lagrangian curl and divergence operators. In Section 5, we provide basic differentiation rules for the Jacobian determinant J and cofactor matrix a, and state the basic geometric and relevant analytical properties of the cofactor matrix. Section 6 provides some well-known elliptic estimates based on the Hodge decomposition of vector fields, as well as some basic trace estimates for the normal and tangential components of vectors fields in $L^2(\Omega)$.

In Section 7, we introduce the degenerate parabolic approximation (7.2) to the compressible Euler equations (1.9), which takes the form $\rho_0 v_t^i + a_i^k(\rho_0^2 J^{-2}),_k$ $+\kappa\,\partial_t\big[a_i^k(\rho_0^2 J^{-2}),_k\big] = 0$, where $\kappa > 0$ denotes the artificial viscosity parameter, and with the special choice of the degenerate parabolic operator $\kappa\,\partial_t\big[a_i^k(\rho_0^2 J^{-2}),_k\big]$, which preserves a majority of the geometric structure of the Euler equations. In particular, the structures of the energy estimates for the horizontal space derivatives, as well as time derivatives, are essentially preserved, the elliptic-type estimates for vertical (or normal) derivatives are kept intact, while the estimates for vorticity are not exactly preserved, but can still be obtained by employing some additional structural observations.

Section 8 is devoted to the construction of solutions to the degenerate parabolic κ-problem (7.2) on a time interval $[0, T_\kappa]$, where T_κ may a priori approach zero as $\kappa \to 0$. The one-dimensional version of the parabolic κ-problem has been studied by us in [10] and also in [11] in the context of the Wright-Fisher diffusion arising in mathematical biology.

The construction of solutions in the three-dimensional setting is significantly more challenging. Our approach is to (1) compute the Lagrangian divergence of the κ-problem to find a nonlinear degenerate parabolic equation for $\rho_0\,\mathrm{div}_\eta\, v$, (2) compute the Lagrangian curl of the κ-problem to find the evolution equation for $\mathrm{curl}_\eta\, v$, and (3) to consider the vertical (or normal) component of the trace of the κ-problem on the boundary Γ, and find an evolution equation for v^3. We then linearize these three evolution equations and obtain a solution to the linearized problem via an additional approximation scheme, which requires us to horizontally smooth the linearized boundary evolution PDE for v^3, using convolution operators on Γ. We find a fixed-point to this horizontally smoothed problem using the contraction mapping principle, and then perform energy estimates to find a solution on a time-interval which is independent of the horizontal convolution parameter. An additional contraction mapping argument is then made to find a solution of the nonlinear κ-problem. One of the serious subtleties of our analysis involves the solution and regularity of the degenerate parabolic equation for the $\rho_0 \operatorname{div}_\eta v$.

In Section 9, we establish κ-independent estimates for the solutions that we have constructed to the κ-problem (7.2). This is done by a combination of energy estimates for the horizontal and time-derivatives of $\eta(t)$, which rely on the determinant structure of the Euler equations in Lagrangian variables, followed by elliptic-type estimates that give bounds on the vertical derivatives of $\eta(t)$ and its time-derivatives.

Section 10 uses these κ-independent estimates to construct a solution to the compressible Euler equations as a limit of the sequence of parabolic solutions as $\kappa \to 0$. Uniqueness is proven as well.

Finally, in Section 11, we describe the modifications which are necessary for the case of general $\gamma > 1$.

The methodology developed for the multidimensional compressible Euler equations with physical vacuum singularity is somewhat general, and can be applied to a host of other degenerate and characteristic hyperbolic systems of conservation laws such as the equations of magneto-hydrodynamics.

1.11. Generalization of the Isentropic Gas Assumption

The general form of the compressible Euler equations in three space dimensions is the 5×5 system of conservation laws

$$\rho[u_t + u \cdot Du] + Dp(\rho) = 0, \tag{1.15a}$$

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1.15b}$$

$$(\rho \mathfrak{E})_t + \operatorname{div}(\rho u \mathfrak{E} + pu) = 0, \tag{1.15c}$$

where (1.15a), (1.15b) and (1.15c) represent the respective conservations of momentum, mass, and total energy. Here, the quantity $\mathfrak{E}$ is the sum of contributions from the kinetic energy $\frac{1}{2}|u|^2$, and the internal energy e, that is, $\mathfrak{E} = \frac{1}{2}|u|^2 + e$. For a single phase of compressible liquid or gas, e becomes a well-defined function of ρ and p through the theory of thermodynamics, $e = e(\rho, p)$. Other interesting and useful physical quantities, the temperature $T(\rho, p)$ and the entropy $S(\rho, p)$ are defined through the following consequence of the second law of thermodynamics

$$T \, \mathrm{d}S = de - \frac{p}{\rho^2} \, d\rho.$$

For *ideal gases*, the quantities e, T, S have the explicit formulae:

$$e(\rho, p) = \frac{p}{\rho(\gamma - 1)} = \frac{T}{\gamma - 1}$$

$$T(\rho, p) = \frac{p}{\rho}$$

$$p = e^S \rho^\gamma, \quad \gamma > 1, \quad \text{constant}.$$

In regions of smoothness, one often uses velocity and a convenient choice of two additional variables among the five quantities S, T, p, ρ, e as independent variables. For the Lagrangian formulation, the entropy S plays an important role, as it satisfies the transport equation

$$S_t + (u \cdot D)S = 0,$$

and as such, $S \circ \eta = S_0$, where $S_0(x) = S(x, 0)$ is the initial entropy function. Thus, by replacing f with $\mathrm{e}^{S\circ\eta}\rho_0^{\gamma} J^{-\gamma}$, our analysis for the isentropic case naturally generalizes to the 5×5 system of conservation laws.

2. Notation and Weighted Spaces

2.1. The Gradient and the Horizontal Derivative

The reference domain Ω is defined in Section 1.5. Throughout the paper the symbol D will be used to denote the three-dimensional gradient vector

$$D = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right),$$

and we shall let $\bar\partial$ denote the horizontal derivative $\bar\partial = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$.

2.2. Notation for Partial Differentiation

The kth partial derivative of F will be denoted by $F,_k = \frac{\partial F}{\partial x_k}$.

2.3. The Divergence and Curl Operators

We use the notation div V for the divergence of a vector field V on Ω:

$$\operatorname{div} V = V^1,_1 + V^2,_2 + V^3,_3\,,$$

and we use curl V to denote the curl of a vector V on Ω:

$$\operatorname{curl} V = \left(V^3,_2 - V^2,_3\,,\ V^1,_3 - V^3,_1\,,\ V^2,_1 - V^1,_2\right).$$

Throughout the paper, we will make use of the permutation symbol

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{even permutation of } \{1,2,3\}, \\ -1, & \text{odd permutation of } \{1,2,3\}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

This allows us to write the ith component of the curl of a vector-field V as

$$[\operatorname{curl} V]_i = \varepsilon_{ijk} V^k,_j \ \text{ or equivalently } \ \operatorname{curl} V = \varepsilon_{\cdot jk} V^k,_j$$

which agrees with our definition above, and is notationally convenient.

We will also define the Lagrangian divergence and curl operators as follows:

$$\operatorname{div}_\eta W = A^j_i W^i,_j \,, \tag{2.2}$$

$$\operatorname{curl}_\eta W = \varepsilon_{\cdot jk} A^r_j W^k,_r \,. \tag{2.3}$$

In the sequel we shall also use the notation $\operatorname{div}_{\bar\eta}$ and $\operatorname{curl}_{\bar\eta}$ to mean the operations defined by (2.2) and (2.3), respectively, with $\bar A = [D\bar\eta]^{-1}$ replacing A.

Finally, we will make use of the two-dimensional divergence operator $\operatorname{div}_\Gamma$ for vector-fields F on the two-dimensional boundary Γ:

$$\operatorname{div}_\Gamma F = F^1,_1 + F^2,_2 \,. \tag{2.4}$$

2.4. Sobolev Spaces on Ω

For integers $k \geqq 0$, we define the Sobolev space $H^k(\Omega)$ ($H^k(\Omega;\mathbb{R}^3)$) to be the completion of the functions in $C^\infty(\overline{\Omega}) := C^\infty(\mathbb{T}^2 \times [0,1])$ (namely the functions in $C^\infty(\mathbb{R}^2 \times [0,1])$ which are 1-periodic in the directions e_1 and e_2) in the norm

$$\|u\|_k = \left(\sum_{|a| \leqq k} \int_\Omega \left|D^a u(x)\right|^2 \mathrm{d}x \right)^{1/2} := \left(\sum_{|a| \leqq k} \int_{(0,1)^3} \left|D^a u(x)\right|^2 \mathrm{d}x \right)^{1/2},$$

for a multi-index $a \in \mathbb{Z}^3_+$, with the standard convention that $|a| = a_1 + a_2 + a_3$. For real numbers $s \geqq 0$, the Sobolev spaces $H^s(\Omega)$ and the norms $\|\cdot\|_s$ are defined by interpolation. We will write $H^s(\Omega)$ instead of $H^s(\Omega;\mathbb{R}^3)$ for vector-valued functions.

Our analysis will often make use of the following subspace of $H^1(\Omega)$:

$$\dot H^1_0(\Omega) = \{u \in H^1(\Omega) \ :\ u = 0 \quad \text{on } \Gamma,\ (x_1,x_2) \mapsto u(x_1,x_2,\cdot) \text{ is 1-periodic}\},$$

where, as usual, the vanishing of u on Γ is understood in the sense of trace.

We will, on occasion, also refer to the Banach space $W^{1,\infty}(\Omega)$ consisting of $L^\infty(\Omega)$ functions whose weak derivatives are also in $L^\infty(\Omega)$.

2.5. Einstein's Summation Convention

Repeated Latin indices i, j, k, etc., are summed from 1 to 3, and repeated Greek indices α, β, γ, etc., are summed from 1 to 2. For example, $F,_{ii} := \sum_{i=1,3} \frac{\partial^2 F}{\partial x_i \partial x_i}$, and $F^i,_\alpha I^{\alpha\beta} G^i,_\beta := \sum_{i=1}^3 \sum_{\alpha=1}^2 \sum_{\beta=1}^2 \frac{\partial F^i}{\partial x_\alpha} I^{\alpha\beta} \frac{\partial G^i}{\partial x_\beta}$.

2.6. Sobolev Spaces on Γ

For functions $u \in H^k(\Gamma)$, $k \geqq 0$, we set

$$|u|_k := \left(\sum_{|\alpha| \leqq k} \int_\Gamma \left|\bar\partial^\alpha u(x)\right|^2 \mathrm{d}x \right)^{1/2},$$

for a multi-index $\alpha \in \mathbb{Z}^2_+$. For real $s \geqq 0$, the Hilbert space $H^s(\Gamma)$ and the boundary norm $|\cdot|_s$ is defined by interpolation. The negative-order Sobolev spaces $H^{-s}(\Gamma)$ are defined via duality: for real $s \geqq 0$, $H^{-s}(\Gamma) := [H^s(\Gamma)]'$.

2.7. Notation for Derivatives and Norms

Throughout the paper, we will use the following notation:

$$
\begin{aligned}
D &= \text{ three-dimensional gradient vector } = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right),\\
\bar{\partial} &= \text{ two-dimensional gradient, } \textit{horizontal derivative } = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right),\\
\operatorname{div} &= \text{ three-dimensional divergence operator,}\\
\operatorname{div}_\eta &= \text{ three-dimensional Lagrangian divergence operator,}\\
\operatorname{curl} &= \text{ three-dimensional curl operator,}\\
\operatorname{curl}_\eta &= \text{ three-dimensional Lagrangian curl operator,}\\
\operatorname{div}_\Gamma &= \text{ two-dimensional divergence operator,}\\
\|\cdot\|_s &= H^s(\Omega)\text{interior norm,}\\
|\cdot|_s &= H^s(\Gamma) \text{ boundary norm.}
\end{aligned}
$$

2.8. The Outward Unit Normal to Γ

We set $N = (0, 0, 1)$ on $\{x_3 = 1\}$ and $N = (0, 0, -1)$ on $\{x_3 = 0\}$. We use the standard basis on $\mathbb{R}^3$: $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

3. A Higher-Order Version of Hardy's Inequality and Some Useful Lemmas

We will make fundamental use of the classical Hardy inequality, which we employ in the context of higher-order derivatives.

Lemma 1. (Hardy's inequality in higher-order form) *Let $s \geqq 1$ be a given integer, and suppose that*

$$u \in H^s(\Omega) \cap \dot{H}^1_0(\Omega).$$

If $d(x) > 0$ for $x \in \Omega$, $d \in H^r(\Omega)$, $r = \max(s-1, 3)$, and d is the distance function to Γ near Γ, then $\frac{u}{d} \in H^{s-1}(\Omega)$ and

$$\left\| \frac{u}{d} \right\|_{s-1} \leqq C\|u\|_s. \tag{3.1}$$

Proof. Given the assumptions on $d(x)$, it is clear that (3.1) holds on all interior regions, and so on all open subsets $\omega \subset \Omega$. We thus need to prove only that this inequality holds near the boundary Γ, wherein the function d coincides with x_3 near $\{x_3 = 0\}$ and with $1 - x_3$ near $\{x_3 = 1\}$. The proof is identical to that given for Lemma 3.1 in [10].

3.1. κ-Independent Elliptic Estimates

In order to obtain estimates for solutions of our approximate κ-problem (7.2) defined below in Section 7, which are independent of the regularization parameter κ , we will need the following Lemma, whose proof can be found in Lemma 1, Section 6 of [8]:

Lemma 2. *Let $\kappa > 0$ and $g \in L^\infty(0, T; H^s(\Omega)))$ be given, and let $f \in H^1(0, T; H^s(\Omega))$ be such that*

$$f + \kappa f_t = g \quad in\ (0, T) \times \Omega .$$

Then,

$$\| f \|_{L^\infty(0,T;H^s(\Omega))} \leqq C \ \max\{\| f(0)\|_s, \|g\|_{L^\infty(0,T;H^s(\Omega))}\}.$$

In practice, f will usually denote $L(V)$, where L is some nonlinear (possibly degenerate) elliptic-type operator and V is some combination of space and time derivatives of $\eta(t)$.

3.2. The Embedding of a Weighted Sobolev Space

The derivative loss inherent to this degenerate problem is a consequence of the weighted embedding we now describe.

Using d to denote the distance function to the boundary Γ, and letting $p = 1$ or 2, the weighted Sobolev space $H^1_{d^p}(\Omega)$, with norm given by $\left[\int_\Omega d(x)^p(|F(x)|^2 + |DF(x)|^2)\right]^{\frac{1}{2}}$ for any $F \in H^1_{d^p}(\Omega)$, satisfies the following embedding:

$$H^1_{d^p}(\Omega) \hookrightarrow H^{1-\frac{p}{2}}(\Omega).$$

Therefore, there is a constant $C > 0$ depending only on Ω and p, such that

$$\|F\|^2_{1-p/2} \leqq C \int_\Omega d(x)^p \left(|F(x)|^2 + |DF(x)|^2\right) \mathrm{d}x. \tag{3.2}$$

See, for example, Section 8.8 in KUFNER [18].

4. The Lagrangian Vorticity and Divergence

We use the permutation symbol (2.1) to write the basic identity regarding the ith component of the curl of a vector field u:

$$(\operatorname{curl} u)_i = \varepsilon_{ijk} u^k,_j .$$

The chain rule shows that

$$(\operatorname{curl} u)_i(\eta) = \varepsilon_{ijk} A^s_j v^k,_s .$$

Using our definition (2.3) of the Lagrangian curl operator curl_η, we write

$$[\text{curl}_\eta\, v]_i := \varepsilon_{ijk} A^s_j v^k,_s\, . \tag{4.1}$$

Taking the Lagrangian curl of (1.10) yields the Lagrangian vorticity equation

$$\varepsilon_{kji} A^s_j v^i_t,_s = 0, \quad \text{or} \quad \text{curl}_\eta\, v_t = 0. \tag{4.2}$$

Similarly, the chain-rule shows that $\text{div}\, u(\eta) = A^j_i v^i,_j$, and according to (2.2),

$$\text{div}_\eta\, v = A^j_i v^i,_j\, . \tag{4.3}$$

5. Properties of the Determinant J, Cofactor Matrix a, Unit Normal n, and a Polynomial-Type Inequality

5.1. Differentiating the Jacobian Determinant

The following identities will be useful to us:

$$\bar\partial J = a^s_r \bar\partial \frac{\partial \eta^r}{\partial x^s} \text{ (horizontal differentiation)}, \tag{5.1}$$

$$\partial_t J = a^s_r \frac{\partial v^r}{\partial x^s} \quad \text{(time differentiation using } v = \eta_t). \tag{5.2}$$

5.2. Differentiating the Cofactor Matrix

Using (5.1) and (5.2) and the fact that $a = J\, A$, we find that

$$\bar\partial a^k_i = \bar\partial \frac{\partial \eta^r}{\partial x^s} J^{-1}[a^s_r a^k_i - a^s_i a^k_r] \text{ (horizontal differentiation)}, \tag{5.3}$$

$$\partial_t a^k_i = \frac{\partial v^r}{\partial x^s} J^{-1}[a^s_r a^k_i - a^s_i a^k_r] \quad \text{(time differentiation using} v = \eta_t). \tag{5.4}$$

5.3. The Piola Identity

It is a fact that the columns of every cofactor matrix are divergence-free and satisfy

$$a^k_i,_k = 0. \tag{5.5}$$

The identity (5.5) will play a vital role in our energy estimates. (Note that we use the notation cofactor for what is commonly termed the *adjugate matrix*, or the transpose of the cofactor.)

5.4. A Geometric Identity Involving the Curl Operator

Lemma 3.

$$\partial_t a_i^k,_j\, a_i^j = [\text{curl}\,\text{curl}\, v]^k + v^r,_{sj} \left(J^{-1}[a_r^s a_i^k - a_i^s a_r^k] a_i^j - [\delta_r^s \delta_i^k - \delta_i^s \delta_r^k]\delta_i^j \right)$$
$$+ v^r,_s \left(J^{-1}[a_r^s a_i^k - a_i^s a_r^k] \right),_j a_i^j .$$

The structure of the right-hand side will be very important to us: the curl structure of the first term will be crucially used in order to construct solutions; the second term can be made small by virtue of the fact that $J^{-1}[a_r^s a_i^k - a_i^s a_r^k]a_i^j - [\delta_r^s\delta_i^k - \delta_i^s\delta_r^k]\delta_i^j$ can be made small for short time; the third term is lower-order with respect to the derivative count on v and can be made small using the fundamental theorem of calculus.

Proof of Lemma 3. Using the identity (5.4), we see that

$$\partial_t a_i^k,_j\, a_i^j = v^r,_{sj}\, J^{-1}[a_r^s a_i^k - a_i^s a_r^k] a_i^j + v^r,_s \left(J^{-1}[a_r^s a_i^k - a_i^s a_r^k] \right),_j a_i^j .$$

Adding and subtracting $[\delta_r^s \delta_i^k - \delta_i^s \delta_r^k]\delta_i^j$, and using the identity curl curl $= D\,\text{div} - \Delta$ yields the result. □

5.5. Geometric Dentities for the Surface $\eta(t)(\Gamma)$

The vectors $\eta,_\alpha$ for $\alpha = 1, 2$ span the tangent plane to the surface $\Gamma(t) = \eta(t)(\Gamma)$ in $\mathbb{R}^3$, and

$$\tau_1 := \frac{\eta,_1}{|\eta,_1|}, \quad \tau_2 := \frac{\eta,_2}{|\eta,_2|}, \quad \text{and} \quad n := \frac{\eta,_1 \times \eta,_2}{|\eta,_1 \times \eta,_2|}$$

are the unit tangent and normal vectors, respectively, to Γ.

Let $g_{\alpha\beta} = \eta,_\alpha \cdot \eta,_\beta$ denote the induced metric on the surface Γ; then det $g = |\eta,_1 \times \eta,_2|^2$ so that

$$\sqrt{g}\, n := \eta,_1 \times \eta,_2 ,$$

where we will use the notation $\sqrt{g}$ to mean $\sqrt{\det g}$.

By definition of the cofactor matrix, the row vector

$$a_i^3 = \begin{bmatrix} \eta^2,_1 \eta^3,_2 - \eta^3,_1 \eta^2,_2 \\ \eta^3,_1 \eta^1,_2 - \eta^1,_1 \eta^3,_2 \\ \eta^1,_1 \eta^2,_2 - \eta^1,_2 \eta^2,_1 \end{bmatrix}, \text{ and } \sqrt{g} = |a_i^3|. \tag{5.6}$$

It follows that

$$n = a_i^3 / \sqrt{g}. \tag{5.7}$$

5.6. A Polynomial-Type Inequality

For a constant $M_0 \geqq 0$, suppose that $f(t) \geqq 0$, $t \mapsto f(t)$ is continuous, and for $\alpha = 1$ or $\frac{1}{2}$,

$$f(t) \leqq M_0 + t^{\alpha} P(f(t)), \tag{5.8}$$

where P denotes a polynomial function. Then for $t \geqq 0$ taken sufficiently small (independently of the function $f \geqq 0$ satisfying (5.8)), we have the bound

$$f(t) \leqq 2M_0.$$

This type of inequality arises in a natural way in the analysis of quasilinear hyperbolic systems (see for instance [8]), and can be viewed as a generalization of standard nonlinear Gronwall inequalities. We will make use of this inequality often in our subsequent analysis.

6. Trace Estimates and the Hodge Decomposition Elliptic Estimates

The normal trace theorem provides the existence of the normal trace $w \cdot N$ of a velocity field $w \in L^2(\Omega)$ with $\operatorname{div} w \in L^2(\Omega)$ (see, for example, [36]). For our purposes, the following form is most useful: if $\bar\partial w \in L^2(\Omega)$ with $\operatorname{div} w \in L^2(\Omega)$, then $\bar\partial w \cdot N$ exists in $H^{-0.5}(\Gamma)$ and

$$\|\bar\partial w \cdot N\|^2_{H^{-0.5}(\Gamma)} \leqq C\Big[\|\bar\partial w\|^2_{L^2(\Omega)} + \|\operatorname{div} w\|^2_{L^2(\Omega)}\Big] \tag{6.1}$$

for some constant C independent of w. In addition to the normal trace theorem, we have the following

Lemma 4. *Let $\bar\partial w \in L^2(\Omega)$ so that* $\operatorname{curl} w \in L^2(\Omega)$*, and let T_1, T_2 denote the unit tangent vectors on Γ, so that any vector field u on Γ can be uniquely written as $u^\alpha T_\alpha$. Then*

$$\|\bar\partial w \cdot T_\alpha\|^2_{H^{-0.5}(\Gamma)} \leqq C\Big[\|\bar\partial w\|^2_{L^2(\Omega)} + \|\operatorname{curl} w\|^2_{L^2(\Omega)}\Big], \qquad \alpha = 1, 2 \tag{6.2}$$

for some constant C independent of w.

See [3] for the proof. Combining (6.1) and (6.2),

$$\|\bar\partial w\|_{H^{-0.5}(\Gamma)} \leqq C\Big[\|\bar\partial w\|_{L^2(\Omega)} + \|\operatorname{div} w\|_{L^2(\Omega)} + \|\operatorname{curl} w\|_{L^2(\Omega)}\Big] \tag{6.3}$$

for some constant C independent of w.

The construction of our higher-order energy function is based on the following Hodge-type elliptic estimate:

Proposition 1. *For an H^r domain Ω, $r \geqq 3$, if $F \in L^2(\Omega;\mathbb{R}^3)$ with* $\operatorname{curl} F \in H^{s-1}(\Omega;\mathbb{R}^3)$, $\operatorname{div} F \in H^{s-1}(\Omega)$, *and* $F \cdot N|_\Gamma \in H^{s-\frac{1}{2}}(\Gamma)$ *for* $1 \leqq s \leqq r$, *then there exists a constant* $\bar{C} > 0$ *depending only on* Ω *such that*

$$
\begin{aligned}
&\|F\|_s \leqq \bar{C}\left(\|F\|_0 + \|\operatorname{curl} F\|_{s-1} + \|\operatorname{div} F\|_{s-1} + |\bar{\partial} F \cdot N|_{s-\frac{3}{2}}\right),\\
&\|F\|_s \leqq \bar{C}\left(\|F\|_0 + \|\operatorname{curl} F\|_{s-1} + \|\operatorname{div} F\|_{s-1} + \textstyle\sum_{\alpha=1}^2 |\bar{\partial} F \cdot T_\alpha|_{s-\frac{3}{2}}\right),
\end{aligned}
\tag{6.4}
$$

where N denotes the outward unit-normal to Γ, and T_α are tangent vectors for $\alpha = 1, 2$.

These estimates are well-known and follows from the identity $-\Delta F = \operatorname{curl}\operatorname{curl} F - D\operatorname{div} F$; a convenient reference is TAYLOR [35].

7. An Asymptotically Consistent Degenerate Parabolic κ-Approximation of the Compressible Euler Equations in Vacuum

In order to construct solutions to (1.9), we will add a specific artificial viscosity term to the Euler equations that preserves much of the geometric structure of the Euler equations, which is so important for our estimates, and which produces a degenerate parabolic approximation, which we term the approximate κ-problem.

7.1. Smoothing the Initial Data

For the purpose of constructing solutions, we will smooth the initial velocity field u_0. We will also smooth the initial density field ρ_0 while preserving the conditions that $\rho(x) > 0$ for $x \in \Omega$, $\rho_0 = 0$ on Γ, and that ρ_0 satisfies (1.5) near Γ.

For $\vartheta > 0$, let $0 \leqq \varrho_\vartheta \in C_0^\infty(\mathbb{R}^3)$ denote the standard family of mollifiers with $\operatorname{spt}(\varrho_\vartheta) \subset \overline{B(0,\vartheta)}$, and let $\mathcal{E}_\Omega$ denote a Sobolev extension operator mapping from $H^s(\Omega)$ to $H^s(\mathbb{T}^2 \times \mathbb{R})$ for $s \geqq 0$.

We set $u_0^\vartheta = \varrho_\vartheta * \mathcal{E}_\Omega(u_0)$, so that for $\vartheta > 0$, $u_0^\vartheta \in C^\infty(\overline{\Omega})$. The smoothed initial density function ρ_0^ϑ is defined as the solution of the fourth-order elliptic equation

$$\Delta^2 \rho_0^\vartheta = \varrho_\vartheta * \mathcal{E}_\Omega(\Delta^2 \rho_0) \qquad \text{in } \Omega, \tag{7.1a}$$

$$\rho_0^\vartheta = 0 \qquad \text{on } \Gamma, \tag{7.1b}$$

$$\frac{\partial \rho_0^\vartheta}{\partial N} = \Lambda_\vartheta \frac{\partial \rho_0}{\partial N} \qquad \text{on } \Gamma, \tag{7.1c}$$

$$(x_1, x_2) \mapsto \rho_0(x_1, x_2, x_3) \text{ is 1-periodic.} \tag{7.1d}$$

Λ_ϑ is the boundary convolution operator defined in Section 8.5.1. By elliptic regularity, $\rho_0^\vartheta \in C^\infty(\overline{\Omega})$, and by choosing $\vartheta > 0$ sufficiently small, we see that $\rho_0^\vartheta(x) > 0$ for $x \in \Omega$, and that the physical vacuum condition (1.5) is satisfied near Γ. This follows from the fact that $\frac{\partial \rho_0^\vartheta}{\partial N} < 0$ on Γ for $\vartheta > 0$ taken sufficiently small, which implies that $\rho_0^\vartheta(x) > 0$ for $x \in \Omega$ very close to Γ. On the other hand, by simple variational principle, $\rho_0^\vartheta \to \rho_0$ in $H^2(\Omega)$ as $\vartheta \to 0$. This implies that $\rho_0^\vartheta(x) > 0$ for all $x \in \omega$ for any open subset $\omega \subset \Omega$ by taking ϑ sufficiently small.

Until Section 10.4, for notational convenience, we will denote u_0^ϑ by u_0 and ρ_0^ϑ by ρ_0. In Section 10.4, we will show that Theorem 1 holds with the optimal regularity stated therein.

7.2. The Degenerate Parabolic Approximation to the Compressible Euler Equations: The κ-Problem

Definition 1. (*The approximate κ-problem*) For $\kappa > 0$, we consider the following sequence of degenerate parabolic approximate κ-problems:

$$\rho_0 v_t^i + a_i^k(\rho_0^2 J^{-2}),_k + \kappa \partial_t \big[a_i^k(\rho_0^2 J^{-2}),_k\big] = 0 \quad \text{in } \Omega \times (0, T_\kappa], \tag{7.2a}$$

$$(\eta, v) = (e, u_0) \quad \text{on } \Omega \times \{t = 0\}, \tag{7.2b}$$

$$\rho_0 = 0 \quad \text{on } \Gamma. \tag{7.2c}$$

Solutions to (1.9) will be found in the limit as $\kappa \to 0$.

Note that (7.2a) can be equivalently written in a form that is essential for the curl estimates that we shall present below:

$$v_t^i + 2A_i^k(\rho_0 J^{-1}),_k + 2\kappa \partial_t \big[A_i^k(\rho_0 J^{-1}),_k\big] = 0. \tag{7.2a'}$$

Remark 2. There appear to be few other possible choices for the artificial viscosity term given in (7.2a). Our choice, $\kappa \partial_t \big[a_i^k({\rho_0}^2 J^{-2}),_k\big]$, preserves the structure of the energy estimates and also, thanks to Lemma 2, the structure of the elliptic-type estimates that we use to bound normal derivatives. On the other hand, the addition of this artificial parabolic term does not exactly preserve the transport structure of vorticity, but instead produces error terms that we can nevertheless control.

Remark 3. Note that we do not require any compatibility conditions on the initial data in order to solve the Euler equations (1.9) (or in Eulerian form (1.1)), and the same remains true for our approximate κ-problem (7.2). The lack of compatibility conditions stems from the degeneracy condition (1.5) which allows us to solve for η and v without prescribing any boundary conditions on displacements or velocities.

7.3. Time-Differentiated Velocity Fields at $t = 0$

Given u_0 and ρ_0, and using the fact that $\eta(x, 0) = x$, the quantity $v_t|_{t=0}$ for the degenerate parabolic κ-problem is computed using (7.2a'):

$$v_t^i|_{t=0} = -\Big(2\kappa \partial_t[A_i^k(\rho_0 J^{-1}),_k] + 2A_i^k(\rho_0 J^{-1}),_k\Big)\Big|_{t=0}$$
$$= (2\kappa\rho_0 \operatorname{div} u_0 - 2\rho_0),_i + 2\kappa u^k_0,_i\, \rho_0,_k\,.$$

Similarly, for all $k \geqq 1$,

$$\partial_t^k v^i|_{t=0} = \frac{\partial^{k-1}}{\partial t^{k-1}}\Big(-2\kappa \partial_t[A_i^k(\rho_0 J),_k] - 2A_i^k(\rho_0 J^{-1}),_k\Big)\Big|_{t=0}.$$

These formulae make it clear that each $\partial_t^k v|_{t=0}$ is a function of space-derivatives of u_0 and ρ_0.

7.4. Introduction of the X Variable and the κ-Problem as a Function of X

We consider a heat-type equation which arises by letting $a_i^j \partial_{x_j}$ act upon equation (7.2a), and using the Piola identity (5.5):

$$a_i^j v_t^i,_j + \kappa \Big[a_i^j a_i^k \frac{1}{\rho_0} (\rho_0^2 \partial_t J^{-2}),_k \Big],_j = -\kappa [a_i^j \partial_t a_i^k \frac{1}{\rho_0} (\rho_0^2 J^{-2}),_k],_j - 2[a_i^j A_i^k (\rho_0 J^{-1}),_k],_j \,. \tag{7.3}$$

Since $\partial_t J^{-2} = -2J^{-3} J_t$, we write (7.3) as

$$a_i^j v_t^i,_j - 2\kappa \Big[a_i^j a_i^k \frac{1}{\rho_0} (\rho_0^2 J^{-3} J_t),_k \Big],_j = -\kappa [a_i^j \partial_t a_i^k \frac{1}{\rho_0} (\rho_0^2 J^{-2}),_k],_j - 2[a_i^j A_i^k (\rho_0 J^{-1}),_k],_j \,. \tag{7.4}$$

Definition 2. (*The X variable*) We set

$$X = \rho_0 J^{-3} J_t = \rho_0 J^{-3} a_r^s v^r,_s = \rho_0 J^{-2} \operatorname{div}_\eta v. \tag{7.5}$$

Using (7.5), we see that

$$a_i^j v_t^i,_j = J_{tt} - \partial_t a_i^j \, v^i,_j = \tfrac{J^3 X_t}{\rho_0} + 3J^{-1}(J_t)^2 - \partial_t a_i^j \, v^i,_j \,,$$

so that we can rewrite (7.4) as the following nonlinear heat-type equation for X:

$$\frac{J^3 X_t}{\rho_0} - 2\kappa \Big[a_i^j a_i^k \frac{1}{\rho_0} (\rho_0 X),_k \Big],_j = -\kappa [a_i^j \partial_t a_i^k \frac{1}{\rho_0} (\rho_0^2 J^{-2}),_k],_j - 3J^{-1}(J_t)^2 + \partial_t a_i^j \, v^i,_j - 2[a_i^j A_i^k (\rho_0 J^{-1}),_k],_j \,. \tag{7.6}$$

It follows from (7.5) that

$$\operatorname{div}_\eta v = \frac{(XJ^2)}{\rho_0}, \tag{7.7}$$

so that time-differentiating (7.7), we see that

$$\operatorname{div}_\eta v_t = \frac{(XJ^2)_t}{\rho_0} - \partial_t A_i^j v^i,_j \,. \tag{7.8}$$

7.5. The Nonlinear Lagrangian Vorticity Equation

The analogue of (4.1) for our approximate κ-problem takes the form, with $f = \rho_0 J^{-1}$,

$$\begin{aligned} \operatorname{curl}_\eta v_t &= 2\kappa \varepsilon_{\cdot ji} v^r,_s A_i^s \big[f,_l A_r^l \big],_m A_j^m \\ &= 2\kappa \varepsilon_{\cdot ji} v^r,_s A_i^s \big[\rho,_{rj} (\eta) \big], \end{aligned} \tag{7.9}$$

where $f = \rho(\eta)$.

We now explain how the formula (7.9) is obtained. We have that the kth component of the Lagrangian curl is

$$\begin{aligned}[\text{curl}_\eta v_t]^k &= -2\kappa\varepsilon_{kji}[\partial_t A^l_i f,_l + A^l_i \partial_t f,_l],_r A^r_j \\ &= -2\kappa\varepsilon_{kji}[\partial_t A^l_i f,_l],_r A^r_j,\end{aligned}$$

where we have used the Lagrangian version of the fact that the curl operator annihilates the gradient operator; namely $\varepsilon_{kji}(A^r_i F,_r),_j = 0$ for all differentiable F.

It is now convenient to switch back to Eulerian variables. We expand $\partial_t A^l_i$, and write

$$\partial_t A^l_i f,_l = -v^r,_s A^s_i f,_l A^l_r = -[u^r,_i \rho,_r] \circ \eta.$$

Now we can compute the standard curl operator of this quantity to find that

$$\begin{aligned}\varepsilon_{kji}[u^r,_i \rho,_r],_j &= \varepsilon_{kji}u^r,_{ij}\rho,_r + \varepsilon_{kji}u^r,_i \rho,_{rj} \\ &= \varepsilon_{kji}u^r,_i \rho,_{rj}\,.\end{aligned}$$

Reverting back to Lagrangian variables yields the identity (7.9).

7.6. A Boundary Identity for the Approximate κ-Problem

For the purposes of constructing solutions to (7.2) we will need the formula for the normal (or vertical) component of v_t on Γ:

$$\begin{aligned}v^3_t &= -2J^{-2}a^3_3\rho_0,_3 - 2\kappa\partial_t[J^{-2}a^3_3]\rho_0,_3 \\ &= -2J^{-2}a^3_3\rho_0,_3 - 2\kappa J^{-2}\partial_t a^3_3\rho_0,_3 - 2\kappa\partial_t J^{-2}\, a^3_3\rho_0,_3\,, \end{aligned} \tag{7.10}$$

where

$$a^3_3 = (\eta,_1 \times \eta,_2)\cdot e_3, \tag{7.11}$$

$$\partial_t a^3_3 = (v,_1 \times \eta,_2 + \eta,_1 \times v,_2)\cdot e_3. \tag{7.12}$$

We note for later use that linearizing (7.12) about $\eta = e$ produces $\text{div}_\Gamma\, v$ as the linearized analogue of $\partial_t a^3_3$.

8. Solving the Parabolic κ-Problem (7.2) by a Fixed-Point Method

8.1. Functional Framework for the Fixed-Point Scheme and Some Notational Conventions

For $T > 0$, we shall denote by $\boldsymbol{X}_T$ and $\boldsymbol{Y}_T$ the following Hilbert spaces:

$$\boldsymbol{X}_T = \left\{v \in L^2(0,T;H^4(\Omega)) |\ \partial^a_t v \in L^2(0,T;H^{4-a}(\Omega)),\ \ a=1,2,3\right\},$$

$$\boldsymbol{Y}_T = \left\{y \in L^2(0,T;H^3(\Omega)) |\ \partial^a_t y \in L^2(0,T;H^{3-a}(\Omega)),\ \ a=1,2,3\right\},$$

$$\boldsymbol{Z}_T = \left\{v \in \boldsymbol{X}_T \mid \rho_0 D v \in \boldsymbol{X}_T\right\},$$

endowed with their natural Hilbert norms:

$$\|v\|^2_{X_T} = \sum_{a=0}^{3} \|\partial_t^a v\|^2_{L^2(0,T;H^{4-a}(\Omega))}, \quad \|y\|^2_{Y_T} = \sum_{a=0}^{3} \|\partial_t^a y\|^2_{L^2(0,T;H^{3-a}(\Omega))},$$
$$\text{and } \|v\|^2_{Z_T} = \|v\|^2_{X_T} + \|\rho_0 Dv\|^2_{X_T}. \tag{8.1}$$

For $M > 0$, we define the following closed, bounded, convex subset of X_T:

$$\mathcal{C}_T(M) = \{w \in Z_T \ : \ \|w\|^2_{Z_T} \leqq M, w(0) = u_0, \partial_t^k w(0) = \partial_t^k v|_{t=0} \ (k = 1, 2)\}, \tag{8.2}$$

where we define the polynomial function $\mathcal{N}_0$ of norms of the initial data as follows:

$$\mathcal{N}_0 = P(\|u_0\|_{100}, \|\rho_0\|_{100}). \tag{8.3}$$

Since we have smoothed the initial data u_0 and ρ_0, we can use the artificially high $H^{100}(\Omega)$-norm in $\mathcal{N}_0$. Later, in Section 10.4, we produce the optimal regularity for this initial data.

Henceforth, we assume that $T > 0$ is given such that independently of the choice of $v \in \mathcal{C}_T(M)$,

$$\eta(x, t) = x + \int_0^t v(x, s)\mathrm{d}s$$

is injective for $t \in [0, T]$, and that

$$\frac{1}{2} \leqq J(x, t) \leqq \frac{3}{2} \quad \text{for } t \in [0, T] \quad \text{and } x \in \overline{\Omega}.$$

This can be achieved by taking $T > 0$ sufficiently small: with $e(x) = x$, notice that

$$\|J(\cdot, t) - 1\|_{L^\infty(\Omega)} \leqq C\|J(\cdot, t) - 1\|_2 = \|\int_0^t a_r^s(\cdot, s)v^r,_s(\cdot, s)\mathrm{d}s\|_2 \leqq C\sqrt{T}M.$$

In the same fashion, we can take $T > 0$ small enough to ensure that on $[0, T]$ and for some $\lambda > 0$,

$$2\lambda|\xi|^2 \leqq a_i^j(x, t)a_i^k(x, t)\xi_j\xi_k \ \ \forall\, \xi \in \mathbb{R}^3, x \in \Omega. \tag{8.4}$$

The space Z_T will be appropriate for our fixed-point methodology to prove existence of a solution to our degenerate parabolic κ problem (7.2).

Theorem 2. (Solutions to the κ-problem) *Given smooth initial data with ρ_0 satisfying $\rho_0(x) > 0$ for $x \in \Omega$ and verifying the physical vacuum condition* (1.5) *near Γ, for $T_\kappa > 0$ sufficiently small, there exists a unique solution $v \in Z_{T_\kappa}$ to the degenerate parabolic κ-problem* (7.2).

The remainder of Section 8 will be devoted to the proof of Theorem 2.

8.2. Implementation of the Fixed-Point Scheme for the κ-Problem (7.2)

Given $\bar{v} \in \mathcal{C}_T(M)$, we define $\bar{\eta}(t) = e + \int_0^t \bar{v}(t')\mathrm{d}t'$, and set

$$\bar{A} = [D\bar{\eta}]^{-1}, \quad \bar{J} = \det D\bar{\eta}, \quad \text{and } \bar{a} = \bar{J}\bar{A}.$$

Next, we set

$$\bar{B}^{jk} = \bar{a}_i^j \bar{a}_i^k \text{ the positive definite, symmetric coefficient matrix.}$$

Linearizing (7.6), we define $\bar{X}$ to be the solution of the following linear and degenerate parabolic problem:

$$\frac{\bar{J}^3 \bar{X}_t}{\rho_0} - 2\kappa\Big[\bar{B}^{jk}\frac{1}{\rho_0}(\rho_0\bar{X}),_k\Big],_j = \bar{G} \qquad \text{in } \Omega \times (0, T_\kappa], \tag{8.5a}$$

$$\bar{X} = 0 \qquad \text{on } \Gamma \times (0, T_\kappa], \tag{8.5b}$$

$$(x_1, x_2) \mapsto \bar{X}(x_1, x_2, x_3, t) \text{ is 1-periodic,} \tag{8.5c}$$

$$\bar{X} = X_0 := \rho_0 \operatorname{div} u_0 \quad \text{on } \Omega \times \{0\}, \tag{8.5d}$$

where the forcing function $\bar{G}$ is defined as

$$\bar{G} = -\kappa\Big[\bar{a}_i^j\,\partial_t\bar{a}_i^k\frac{1}{\rho_0}(\rho_0^2\bar{J}^{-2}),_k\Big],_j - 2\Big[\bar{a}_i^j\,\bar{A}_i^k(\rho_0\bar{J}^{-1}),_k\Big],_j - 3\frac{(\bar{J}_t)^2}{\bar{J}} + \partial_t\bar{a}_i^j\,\bar{v}^i,_j\,. \tag{8.6}$$

We shall establish the following

Proposition 2. *For $T > 0$ taken sufficiently small, there exists a unique solution to (8.5) satisfying*

$$\|\bar{X}\|^2_{\boldsymbol{X}_T} \leqq \mathcal{N}_0 + T\,P(\|\bar{v}\|^2_{\boldsymbol{Z}_T}) + P(\|\operatorname{curl}\bar{v}\|^2_{\boldsymbol{Y}_T}),$$

with the norms $\boldsymbol{X}_T$, $\boldsymbol{Y}_T$, and $\boldsymbol{Z}_T$ defined in (8.1), and once again P denotes a generic polynomial function of its arguments. (Generic constants are absorbed by the constants in our generic polynomial function P.)

The proof of Proposition 2 will be given in Sections 8.4.2–8.4.7.

8.3. The Definition of the Velocity Field v

We will define, later on, a linear elliptic system of equations for v which should be viewed as the linear analogue of equations (7.8), (7.9), and (7.10).

Definition 3. (*The linear system for the velocity-field $v(t)$*) With $\bar{v} \in \mathcal{C}_T(M)$ given, and $\bar{X}$ obtained by solving the linear problem (8.5), we will show in Section 8.5 that we can define $v(t)$ on $[0, T_\kappa]$ by specifying the divergence and curl of its time derivative in Ω, as well as the trace of its normal component on the boundary Γ in the following way:

$$v(0) = u_0 \quad \text{in } \Omega, \tag{8.7a}$$

$$\operatorname{div} v_t = \operatorname{div} \bar{v}_t - \operatorname{div}_{\bar{\eta}} \bar{v}_t + \frac{[\bar{X}\bar{J}^2]_t}{\rho_0} - \partial_t \bar{A}_i^j \bar{v}^i,_j \quad \text{in } \Omega, \tag{8.7b}$$

$$\operatorname{curl} v_t = \operatorname{curl} \bar{v}_t - \operatorname{curl}_{\bar{\eta}} \bar{v}_t + 2\kappa \varepsilon_{\cdot ji} \bar{v},_s^r \bar{A}_i^s \bar{\Xi},_r^j (\bar{\eta}) + \bar{\mathfrak{C}} \quad \text{in } \Omega, \tag{8.7c}$$

$$\begin{aligned} v_t^3 + 2\kappa\rho_0,_3 \operatorname{div}_\Gamma v &= 2\kappa\rho_0,_3 \operatorname{div}_\Gamma \bar{v} - 2\rho_0,_3 \bar{J}^{-2}\bar{a}_3^3 - 2\kappa\rho_0,_3 \bar{J}^{-2}\partial_t \bar{a}_3^3 \\ &\quad -2\kappa\rho_0,_3 \bar{a}_3^3 \partial_t \bar{J}^{-2} + \bar{c}(t) N^3 \quad \text{on } \Gamma, \end{aligned} \tag{8.7d}$$

$$\int_\Omega v_t^\alpha \mathrm{d}x = -2\int_\Omega \bar{A}_\alpha^k (\frac{\rho_0}{\bar{J}}),_k \mathrm{d}x - 2\kappa \int_\Omega \partial_t \Big[\bar{A}_\alpha^k (\frac{\rho_0}{\bar{J}}),_k \Big] \mathrm{d}x, \tag{8.7e}$$

$$(x_1, x_2) \mapsto v_t(x_1, x_2, x_3, t) \text{ is 1-periodic}, \tag{8.7f}$$

where the presence of $\operatorname{div}_\Gamma \bar{v}$ (defined in (2.4)) in (8.7d) represents the linearization of $\partial_t \bar{a}_3^3$ about $\bar{\eta} = e$, and where

$$\begin{aligned} \bar{a}_3^3 &= e_3 \cdot (\bar{\eta},_1 \times \bar{\eta},_2), \\ \partial_t \bar{a}_3^3 &= e_3 \cdot (\bar{v},_1 \times \bar{\eta},_2 + \bar{\eta},_1 \times \bar{v},_2), \end{aligned} \tag{8.8}$$

the function $\bar{c}(t)$ (a constant in x) on the right-hand side of (8.7d) is defined by

$$\begin{aligned} \bar{c}(t) = &\frac{1}{2}\int_\Omega (\operatorname{div} \bar{v}_t - \operatorname{div}_{\bar{\eta}} \bar{v}_t)\mathrm{d}x + \frac{1}{2}\int_\Omega \frac{[\bar{X}\bar{J}^2]_t}{\rho_0}\mathrm{d}x - \frac{1}{2}\int_\Omega \partial_t \bar{A}_i^j \bar{v}^i,_j \mathrm{d}x \\ &+ \int_\Gamma \bar{J}^{-2}\bar{a}_3^3 \rho_0,_3 N^3 \mathrm{d}S + \kappa \int_\Gamma \bar{J}^{-2}\partial_t \bar{a}_3^3 \rho_0,_3 N^3 \mathrm{d}S \\ &+ \kappa \int_\Gamma \partial_t \bar{J}^{-2}\bar{a}_3^3 \rho_0,_3 N^3 \mathrm{d}S + \kappa \int_\Gamma \operatorname{div}_\Gamma (v - \bar{v})\rho_0,_3 N^3 \mathrm{d}S, \end{aligned} \tag{8.9}$$

and where the vector field $\bar{\Xi}(\bar{\eta})$ on the right-hand side of (8.7c) is defined on $[0, T] \times \Omega$ as the solution of the ODE

$$\bar{v}_t + 2\bar{\Xi}(\bar{\eta}) + 2\kappa[\bar{\Xi}(\bar{\eta})]_t = 0, \tag{8.10a}$$

$$\bar{\Xi}(0) = D\rho_0. \tag{8.10b}$$

The vector field $\bar{\mathfrak{C}}$ on the right-hand side of (8.7c) is then defined on $[0, T] \times \Omega$ by

$$\bar{\mathfrak{C}}^i = 2\bar{A}_i^j \psi,_j + 2\kappa[2\bar{A}_i^j \psi,_j]_t, \tag{8.11}$$

where ψ is solution of the following time-dependent elliptic-type problem for $t \in [0, T]$:

$$2[\bar{A}_i^j \psi,_j],_i + 2\kappa \partial_t [\bar{A}_i^j \psi,_j],_i = \operatorname{div}\big(\operatorname{curl}_{\bar{\eta}} \bar{v}_t - 2\kappa \varepsilon_{\cdot ji} \bar{v},_s^r \bar{A}_i^s \bar{\Xi},_r^j (\bar{\eta})\big) \text{ in } \Omega, \tag{8.12a}$$

$$\psi = 0 \text{ on } \Gamma, \tag{8.12b}$$

$$(x_1, x_2) \mapsto \psi(x_1, x_2, x_3, t) \text{ is 1-periodic}, \tag{8.12c}$$

$$\psi|_{t=0} = 0 \text{ in } \Omega, \tag{8.12d}$$

so that we have the compatibility condition for (8.7c)

$$\operatorname{div}\big(-\operatorname{curl}_{\bar{\eta}} \bar{v}_t + 2\kappa \varepsilon_{\cdot ji} \bar{v},_s^r \bar{A}_i^s \bar{\Xi},_r^j (\bar{\eta})\big) + \bar{\mathfrak{C}}\big) = 0 \text{ in } \Omega \times [0, T]. \tag{8.13}$$

An integrating factor provides us with a closed-form solution to the ODE (8.10), and by employing integration-by-parts in the time integral, we find that

$$\begin{aligned} \bar{\Xi}(\bar{\eta})(t, \cdot) &= \mathrm{e}^{-\frac{t}{\kappa}} D\rho_0(\cdot) - \int_0^t \frac{\mathrm{e}^{\frac{t'-t}{\kappa}}}{2\kappa} \bar{v}_t(t', \cdot) \mathrm{d}t', \\ &= \mathrm{e}^{-\frac{t}{2\kappa}} D\rho_0(\cdot) + \int_0^t \frac{\mathrm{e}^{\frac{t'-t}{\kappa}}}{2\kappa^2} \bar{v}(t', \cdot) \mathrm{d}t' - \frac{1}{2\kappa} \bar{v}(t, \cdot) + \frac{\mathrm{e}^{-\frac{t}{\kappa}}}{2\kappa} u_0(\cdot). \end{aligned} \tag{8.14}$$

The formula (8.14) shows that $\bar{\Xi}(\bar{\eta})$ has the same regularity as $\bar{v}$. This gain in regularity is remarkable and should be viewed as one of the key reasons that permit us to construct solutions to (7.2) using the linearization (8.7) with a fixed-point argument.

Similarly, we notice that

$$\begin{aligned} 2[\bar{A}_i^j \psi,_j],_i (t, \cdot) &= \int_0^t \frac{\mathrm{e}^{\frac{t'-t}{\kappa}}}{2\kappa} \operatorname{div}\big(\operatorname{curl}_{\bar{\eta}} \bar{v}_t - 2\kappa \varepsilon_{\cdot ji} \bar{v},_s^r \bar{A}_i^s \bar{\Xi},_r^j (\bar{\eta})\big)(t', \cdot)\, \mathrm{d}t' \\ &= \int_0^t \frac{\mathrm{e}^{\frac{t'-t}{\kappa}}}{2\kappa} \varepsilon_{kji} \big[\bar{v}^i,_r \bar{A}_s^r \bar{v}^s,_l \bar{A}_j^l - \frac{1}{\kappa} \bar{v}^i,_r \bar{A}_j^r\big],_k (t', \cdot)\, \mathrm{d}t' \\ &\quad - \int_0^t \frac{\mathrm{e}^{\frac{t'-t}{\kappa}}}{2\kappa} \varepsilon_{kji} \big[2\kappa \bar{v},_s^r \bar{A}_i^s \bar{\Xi},_r^j (\bar{\eta})\big],_k (t', \cdot)\, \mathrm{d}t' \\ &\quad + \frac{\mathrm{e}^{\frac{t'-t}{\kappa}}}{2\kappa} \operatorname{div} \operatorname{curl}_{\bar{\eta}} \bar{v}. \end{aligned} \tag{8.15}$$

Since we can rewrite the left-hand side of (8.15) as $2\Delta\psi + 2[(\bar{A}_i^j - \delta_i^j)\psi,_j],_i$, and with $\bar{v} \in \mathcal{C}_T(M)$, the elliptic problem (8.15) is well-defined and together with the boundary condition (8.12.b) provides the following estimates for any $t \in [0, T]$:

$$\|\psi(t)\|_4 \leqq \mathcal{N}_0 + C\Big(T\|\bar{v}(t)\|_4 + \int_0^t \|\bar{v}\|_4\Big), \tag{8.16a}$$

$$\|\psi_t(t)\|_3 \leqq \mathcal{N}_0 + C\Big(T\|\bar{v}_t(t)\|_3 + \|\bar{v}(t)\|_3 + \int_0^t \|\bar{v}\|_3\Big). \tag{8.16b}$$

Using (8.11), the estimates (8.16) lead to

$$\|\bar{\mathfrak{C}}\|_2 \leqq \mathcal{N}_0 + C\Big(T\|\bar{v}_t(t)\|_3 + \|\bar{v}(t)\|_3 + \int_0^t \|\bar{v}\|_3\Big), \tag{8.17a}$$

$$\Big\|\int_0^t \bar{\mathfrak{C}}\Big\|_3 \leqq \mathcal{N}_0 + C\Big(T\|\bar{v}(t)\|_4 + \int_0^t \|\bar{v}\|_4\Big). \tag{8.17b}$$

Remark 4. The function $\bar{c}(t)$ is added to the right-hand side of (8.7d) to ensure that the solvability condition for the elliptic system (8.7) is satisfied; in particular, the solvability condition is obtained from an application of the divergence theorem to equation (8.7b).

Remark 5. Condition (8.7e) is necessary only because of the periodicity of our domain in the directions e_1 and e_2. In particular, our elliptic system is defined modulo a constant vector, and the addition of $\bar{c}(t)N^3$ to the right-hand side of (8.7d) fixes the constant in the vertical direction, while the condition (8.7e) fixes the two constants in the tangential directions. The particular choice for the average of v_t^α, $\alpha = 1, 2$, permits us to close the fixed-point argument, and obtain the unique solution of (7.2).

8.4. Construction of Solutions and Regularity Theory for $\bar{X}$ and its Time Derivatives

This section will be devoted to the proof of Proposition 2.

8.4.1. Smoothing $\bar{v}$ We will proceed with a two stage process. First, we smooth $\bar{v}$ and obtain strong solutions to the linear equation (8.5) in the case in which the forcing function $\bar{G}$ and the coefficient matrix $\bar{B}^{jk}$ are $C^\infty(\overline{\Omega})$-functions. Second, having strong solutions to (8.5), with bounds that depend on the smoothing parameter of $\bar{v}$, we use interpolation estimates (together with the Sobolev embedding theorem) to conclude the proof of Proposition 2.

Using the notation of Section 7.1, for each $t \in [0, T_\kappa]$ and for $\nu > 0$, we define

$$\bar{v}^\nu(\cdot, t) = \varrho_\nu * \mathcal{E}_\Omega(\bar{v}(\cdot, t)),$$

so that for each $\nu > 0$, $\bar{v}^\nu(\cdot, t) \in C^\infty(\overline{\Omega})$. We define $\bar{G}^\nu$ by replacing $\bar{A}$, $\bar{a}$, $\bar{J}$, and $\bar{v}$ in (8.6) with $\bar{A}^\nu$, $\bar{a}^\nu$, $\bar{J}^\nu$, and $\bar{v}^\nu$, respectively. The quantities $\bar{A}^\nu$, $\bar{a}^\nu$, $\bar{J}^\nu$ are defined just as their unsmoothed analogues from the map $\bar{\eta}^\nu = e + \int_0^t \bar{v}^\nu$. We also define $[\bar{B}^\nu]^{jk} = (\bar{a}^\nu)_i^j (\bar{a}^\nu)_i^k$; according to (8.4), we can choose $\nu > 0$ sufficiently small so that for $t \in [0, T_\kappa]$,

$$\lambda|\xi|^2 \leqq [\bar{B}^\nu]^{jk}(x, t)\xi_j\xi_k \quad \forall\, \xi \in \mathbb{R}^3,\ x \in \Omega. \tag{8.18}$$

Until Section 8.4.7, we will use $\bar{B}^\nu$ and $\bar{G}^\nu$ as the coefficient matrix and forcing function, respectively, but for notational convenience we will not explicitly write the superscript ν.

8.4.2. $L^2(0,T;\dot H^1_0(\Omega))$ Regularity for $\bar X_{ttt}$ We will use the definition of the constant $\lambda > 0$ given in (8.4).

Definition 4. (*Weak Solutions of* (8.5)) $\bar X \in L^2(0,T;\dot H^1_0(\Omega))$ with $\frac{\bar X_t}{\rho_0} \in L^2(0,T;H^{-1}(\Omega))$ is a weak solution of (8.5) if

(i) for all $\mathcal{W} \in \dot H^1_0(\Omega)$,

$$\langle \frac{\bar J^3 \bar X_t}{\rho_0}, \mathcal{W}\rangle + 2\kappa \int_\Omega \frac{\bar B^{jk}}{\rho_0}(\rho_0 \bar X),_k \mathcal{W},_j \,\mathrm{d}x = \langle \bar G, \mathcal{W}\rangle \quad \text{a.e. } [0,T], \tag{8.19}$$

(ii) $\bar X(0) = X_0$.

The duality pairing between $\dot H^1_0(\Omega)$ and $H^{-1}(\Omega)$ is denoted by $\langle\cdot,\cdot\rangle$, and $\bar G \in L^2(0,T;H^{-1}(\Omega))$.

Recall that if $\bar G \in H^{-1}(\Omega)$, then $\|\bar G\|_{H^{-1}(\Omega)} = \sup\{\langle \bar G, \mathcal{W}\rangle \mid \mathcal{W} \in \dot H^1_0(\Omega), \|\mathcal{W}\|_{\dot H^1_0(\Omega)} = 1\}$. Furthermore, there exist functions $\bar G_0, \bar G_1, \bar G_2, \bar G_3$ in $L^2(\Omega)$ such that $\langle \bar G, \mathcal{W}\rangle = \int_\Omega \bar G_0 \mathcal{W} + \bar G_i \mathcal{W},_i \,\mathrm{d}x$, so that $\|\bar G\|^2_{H^{-1}(\Omega)} = \inf \sum_{a=0}^3 \|\bar G_a\|_0^2$, the infimum being taken over all such functions $\bar G_a$.

Lemma 5. *If $\bar G \in L^2(0,T;H^{-1}(\Omega))$ and $\frac{X_0}{\sqrt{\rho_0}} \in L^2(\Omega)$, then for $T > 0$ taken sufficiently small so that* (8.4) *holds, there exists a unique weak solution to* (8.5) *such that for constants $C_p > 0$ and $C_{\kappa\lambda} > 0$,*

$$\left\| \frac{\bar X_t}{\rho_0} \right\|^2_{L^2(0,T;H^{-1}(\Omega))} + \sup_{t\in[0,T]} C \left\| \frac{\bar X(t)}{\sqrt{\rho_0}} \right\|_0^2 + C_p \|\bar X\|^2_{L^2(0,T;\dot H^1_0(\Omega))} \leqq \left\| \frac{X_0}{\sqrt{\rho_0}} \right\|_0^2 + C_{\kappa\lambda} \|\bar G\|^2_{L^2(0,T;H^{-1}(\Omega))}.$$

Proof. Let $(e_n)_{n\in\mathbb{N}}$ denote a Hilbert basis of $\dot H^1_0(\Omega)$, with each e_n being smooth. Such a choice of basis is indeed possible as we can take, for instance, the eigenfunctions of the Laplace operator on Ω with vanishing Dirichlet boundary conditions on Γ and 1-periodic in e_1 and e_2. We then define the Galerkin approximation at order $n \geqq 1$ of (8.19) as being under the form $X_n = \sum_{i=0}^n \lambda_i^n(t) e_i$ such that: $\forall \ell \in \{0,\dots,n\}$,

$$\left(\bar J^3 \frac{X_{nt}}{\rho_0}, e_\ell\right)_{L^2(\Omega)} + 2\kappa \left(\frac{\bar B^{jk}}{\rho_0}(\rho_0 X_n),_k, e_\ell,_j\right)_{L^2(\Omega)} = (\bar G_0, e_\ell)_{L^2(\Omega)} - \left(\bar G_i, \frac{\partial e_\ell}{\partial x_i}\right)_{L^2(\Omega)} \text{ in } [0,T], \tag{8.20a}$$

$$\lambda_\ell^n(0) = (X_0, e_\ell)_{L^2(\Omega)}. \tag{8.20b}$$

Since each e_ℓ is in $H^{k+1}(\Omega) \cap \dot H^1_0(\Omega)$ for every $k \geqq 1$, we have by Hardy's inequality (1) that

$$\frac{e_\ell}{\rho_0} \in H^k(\Omega) \ \text{ for } k \geqq 1;$$

therefore, each integral written in (8.20) is well-defined.

Furthermore, as the e_ℓ are linearly independent, so are the $\frac{e_\ell}{\sqrt{\rho_0}}$ and therefore the determinant of the matrix

$$\Big[\Big(\frac{e_i}{\sqrt{\rho}_0}, \frac{e_j}{\sqrt{\rho}_0}\Big)_{L^2(\Omega)}\Big]_{(i,j)\in\mathbb{N}_n=\{1,\dots,n\}}$$

is nonzero. This implies that our finite-dimensional Galerkin approximation (8.20) is a well-defined first-order differential system of order $n+1$, which therefore has a solution on a time interval $[0, T_n]$, where T_n a priori depends on the rank n of the Galerkin approximation. In order to prove that $T_n = T$, with T independent of n, we notice that since X_n is a linear combination of the e_ℓ ($\ell \in \{1, \dots, n\}$), we have that on $[0, T_n]$,

$$\left(\frac{\bar{J}^3 X_{nt}}{\rho_0}, X_n\right)_{L^2(\Omega)} + 2\,\kappa\,\left(\frac{\bar{B}^{jk}}{\rho_0}\frac{\partial(\rho_0 X_n)}{\partial x_k}, \frac{\partial X_n}{\partial x_j}\right)_{L^2(\Omega)} = (\bar{G}_0,\ X_n)_{L^2(\Omega)} - \left(\bar{G}_i,\ \frac{\partial X_n}{\partial x_i}\right)_{L^2(\Omega)}.$$

Since

$$\int_\Omega \frac{\bar{B}^{jk}}{\rho_0}(\rho_0 X_n),_k\ X_n,_j\ \mathrm{d}x = \int_\Omega \bar{B}^{jk} X_n,_k\ X_n,_j\ \mathrm{d}x + \int_\Omega \frac{\bar{B}^{jk}}{\rho_0}\rho_0,_k X_n\, X_n,_j\ \mathrm{d}x$$

and

$$2\int_\Omega \frac{\bar{B}^{jk}}{\rho_0}\rho_0,_k X_n\, X_n,_j\ \mathrm{d}x = -\int_\Omega \frac{\rho_0,_{jk}}{\rho_0}\bar{B}^{jk}|X_n|^2 + \frac{\rho_0,_k}{\rho_0}\bar{B}^{jk},_j|X_n|^2\ \mathrm{d}x + \int_\Omega \frac{\rho_0,_k\,\rho_0,_j}{\rho_0^2}\bar{B}^{jk}|X_n|^2\ \mathrm{d}x,$$

it follows that on $[0, T_n]$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \bar{J}^3\frac{|X_n|^2}{\rho_0}\mathrm{d}x + 2\kappa\int_\Omega \bar{B}^{jk}X_n,_k\ X_n,_j\ \mathrm{d}x + \kappa\int_\Omega \frac{\rho_0,_k\,\rho_0,_j}{\rho_0^2}\bar{B}^{jk}|X_n|^2\ \mathrm{d}x$$
$$= \frac{1}{2}\int_\Omega (\bar{J}^3)_t\frac{|X_n|^2}{\rho_0}\mathrm{d}x + \kappa\int_\Omega \frac{\rho_0,_{jk}}{\rho_0}\bar{B}^{jk}|X_n|^2 + \frac{\rho_0,_k}{\rho_0}\bar{B}^{jk},_j|X_n|^2\ \mathrm{d}x + \int_\Omega \bar{G}_0\,X_n\,\mathrm{d}x - \int_\Omega \bar{G}_i\ X_n,_i\ \mathrm{d}x.$$

Using (8.4), we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \bar{J}^3\frac{|X_n|^2}{\rho_0}\mathrm{d}x + 2\kappa\lambda\int_\Omega |DX_n|^2\,\mathrm{d}x + \kappa\lambda\int_\Omega \frac{|D\rho_0|^2}{\rho_0^2}|X_n|^2\ \mathrm{d}x$$
$$\leqq \|\frac{1}{2}(\bar{J}^3)_t + \kappa\rho_0,_{jk}\,\bar{B}^{jk} + \kappa\rho_0,_k\,\bar{B}^{jk},_j\|_{L^\infty(\Omega)}\int_\Omega \frac{1}{\rho_0}|X_n|^2\ \mathrm{d}x + C\|\bar{G}\|_{H^{-1}(\Omega)}\|DX_n\|_0. \tag{8.21}$$

Using the Sobolev embedding theorem and the Cauchy-Young inequality, we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \bar{J}^3\frac{|X_n|^2}{\rho_0}\mathrm{d}x + \kappa\lambda\int_\Omega |DX_n|^2\,\mathrm{d}x + \kappa\lambda\int_\Omega \frac{|D\rho_0|^2}{\rho_0^2}|X_n|^2\,\mathrm{d}x$$
$$\leqq C\|\frac{(\bar{J}^3)_t}{2} + \kappa\rho_0,_{jk}\bar{B}^{jk} + \kappa\rho_0,_k\bar{B},_j^{jk}\|_2\int_\Omega \frac{|X_n|^2}{\rho_0}\,\mathrm{d}x + C_{\kappa\lambda}\|\bar{G}\|^2_{H^{-1}(\Omega)},$$

where the constant $C_{\kappa\lambda}$ depends inversely on $\kappa\lambda$. Since $\bar{v}\in\mathcal{C}_T(M)$, we have that on $[0,T]$,

$$\int_0^t \|\frac{1}{2}(\bar{J}^3)_t + \kappa\rho_0,_{jk}\bar{B}^{jk} + \kappa\rho_0,_k\bar{B}^{jk},_j\|_2\mathrm{d}t \leqq C_M\sqrt{t}$$

for a constant C_M depending on M, so that Gronwall's inequality shows that $T_n = T$ (with T independent of $n\in\mathbb{N}$), and with $\frac{1}{2}\leqq\bar{J}$ for all $\bar{v}\in\mathcal{C}_T(M)$, we see that

$$\sup_{t\in[0,T]} C\left\|\frac{X_n(t)}{\sqrt{\rho_0}}\right\|_0^2 + \kappa\lambda\int_0^T \|DX_n(t)\|_0^2 \leqq \left\|\frac{X(0)}{\sqrt{\rho_0}}\right\|_0^2 + C_{\kappa\lambda}\int_0^T \|\bar{G}(t)\|^2_{H^{-1}(\Omega)}.$$

Setting $C_p = \frac{\kappa\lambda}{\text{Poincaré constant}}$, we see that

$$\sup_{t\in[0,T]} C\left\|\frac{X_n(t)}{\sqrt{\rho_0}}\right\|_0^2 + C_p\int_0^T \|X_n(t)\|_1^2 \leqq \left\|\frac{X(0)}{\sqrt{\rho_0}}\right\|_0^2 + C_{\kappa\lambda}\int_0^T \|\bar{G}(t)\|^2_{H^{-1}(\Omega)}.$$

Thus, there exists a subsequence $\{X_{nm}\}\subset\{X_n\}$ which converges weakly to some $\bar{X}$ in $L^2(0,T;\dot{H}_0^1(\Omega))$, which satisfies

$$\sup_{t\in[0,T]} C\left\|\frac{\bar{X}(t)}{\sqrt{\rho_0}}\right\|_0^2 + C_p\int_0^T \|\bar{X}(t)\|_1^2 \leqq \left\|\frac{X(0)}{\sqrt{\rho_0}}\right\|_0^2 + C_{\kappa\lambda}\int_0^T \|\bar{G}(t)\|^2_{H^{-1}(\Omega)}.$$

Furthermore, it can also be shown from the previous estimates, by using standard arguments for weak solutions of linear parabolic systems, that

$$\frac{\bar{X}_t}{\rho_0}\in L^2(0,T;H^{-1}(\Omega)),$$

and that $\bar{X}(0) = X_0$ and that this $\bar{X}$ verifies the identity (8.19). Uniqueness follows by letting $\mathcal{W} = \bar{X}$ in (8.19). □

Since $\|\bar{G}\|^2_{L^2(0,T;L^2(\Omega))}\leqq P(\|\bar{v}\|^2_{X_T})$, it thus follows from Lemma 5 and (8.3) that

$$\left\|\frac{\bar{X}_t}{\rho_0}\right\|^2_{L^2(0,T;H^{-1}(\Omega))} + \sup_{t\in[0,T]}\left\|\frac{\bar{X}(t)}{\sqrt{\rho_0}}\right\|_0^2 + C_p\|\bar{X}\|^2_{L^2(0,T;\dot{H}_0^1(\Omega))}\leqq C. \quad (8.22)$$

In order to build regularity for X, we construct weak solutions for the time-differentiated version of (8.5). It is convenient to proceed from the first to third time-differentiated problems. We begin with the first time-differentiated version of (8.5):

$$\frac{\bar{J}^3 \bar{X}_{tt}}{\rho_0} - 2\kappa\Big[\frac{\bar{B}^{jk}}{\rho_0}(\rho_0 \bar{X}_t),_k\Big],_j = \bar{G}_t + \mathcal{G}_1 \text{ in } \Omega \times (0, T_\kappa], \tag{8.23a}$$

$$\bar{X}_t = 0 \quad \text{on } \Gamma \times (0, T_\kappa], \tag{8.23b}$$

$$\bar{X}_t = X_1 \quad \text{on } \Omega \times \{t = 0\}, \tag{8.23c}$$

where the initial condition X_1 is given as

$$X_1 = 2\kappa\rho_0\Big[\frac{(\rho_0 X_0),_i}{\rho_0}\Big],_i + \rho_0 \bar{G}(0), \tag{8.24}$$

the additional forcing term $\mathcal{G}_1$ is defined by

$$\mathcal{G}_1 = 2\kappa\Big[\bar{B}_t^{jk}\frac{1}{\rho_0}(\rho_0 \bar{X}),_k\Big],_j - \frac{(\bar{J}^3)_t \bar{X}_t}{\rho_0}, \tag{8.25}$$

$X_0 = \rho_0 \operatorname{div} u_0$, and

$$\bar{G}(0) = -2\kappa \operatorname{curl}\operatorname{curl} u_0 \cdot D\rho_0 - 2\kappa \operatorname{div} u_0\, \Delta\rho_0 + 2\kappa u_0^j,_i\, \rho_0,_{ij} - 2\Delta\rho_0$$
$$-2(\operatorname{div} u_0)^2 - u_0^i,_j\, u_0^j,_i\, .$$

According to the estimate (8.22), $\|\bar{G}_t + \mathcal{G}_1\|^2_{L^2(0,T;H^{-1}(\Omega))} \leqq C$; hence by Lemma 5 (with $\bar{X}_t,\ \bar{G}_t + \mathcal{G}_1,\ X_1$ replacing $\bar{X},\ \bar{G},\ X_0$, respectively),

$$\left\|\frac{\bar{X}_{tt}}{\rho_0}\right\|^2_{L^2(0,T;H^{-1}(\Omega))} + \sup_{t\in[0,T]}\left\|\frac{\bar{X}_t(t)}{\sqrt{\rho_0}}\right\|^2_0 + C_p\, \|\bar{X}_t\|^2_{L^2(0,T;\dot{H}^1_0(\Omega))} \leqq C. \tag{8.26}$$

Next, we consider the second time-differentiated version of (8.5):

$$\frac{\bar{J}^3 \bar{X}_{ttt}}{\rho_0} - 2\kappa\Big[\frac{\bar{B}^{jk}}{\rho_0}(\rho_0 \bar{X}_{tt}),_k\Big],_j = \bar{G}_{tt} + \mathcal{G}_2 \text{ in } \Omega \times (0, T_\kappa], \tag{8.27a}$$

$$\bar{X}_{tt} = 0 \quad \text{on } \Gamma \times (0, T_\kappa], \tag{8.27b}$$

$$\bar{X}_{tt} = X_2 \quad \text{on } \Omega \times \{t = 0\}, \tag{8.27c}$$

where the initial condition X_2 is given as

$$X_2 = 2\kappa\rho_0\Big[\frac{(\rho_0 X_1),_i}{\rho_0}\Big],_i + \rho_0\mathcal{G}_1(0) + \rho_0 \bar{G}_t(0), \tag{8.28}$$

and the forcing function $\mathcal{G}_2$ is defined by

$$\mathcal{G}_2 = \partial_t \mathcal{G}_1 + 2\kappa\Big[\bar{B}_t^{jk}\frac{1}{\rho_0}(\rho_0 \bar{X}_t),_k\Big],_j - \frac{(\bar{J}^3)_t \bar{X}_{tt}}{\rho_0}. \tag{8.29}$$

We do not precisely define $\mathcal{G}_1(0)$, but note that its highest-order terms scale like either $D^3 u_0$ or $\rho_0 D^4 u_0$ or $D^3\rho_0$, so that $\|\sqrt{\rho_0}\mathcal{G}_1(0)\|^2_0 \leqq \mathcal{N}_0$. Using the estimate

(8.26), we see that $\|\bar{G}_{tt} + \mathcal{G}_2\|^2_{L^2(0,T;H^{-1}(\Omega))} \leqq C$. It thus follows from Lemma 5 (with $\bar{X}_{tt},\ \bar{G}_{tt} + \mathcal{G}_2,\ X_2$ replacing $X,\ \bar{G},\ X_0$, respectively),

$$\left\|\frac{\bar{X}_{ttt}}{\rho_0}\right\|^2_{L^2(0,T;H^{-1}(\Omega))} + \sup_{t\in[0,T]}\left\|\frac{\bar{X}_{tt}(t)}{\sqrt{\rho_0}}\right\|^2_0 + C_p\,\|\bar{X}_{tt}\|^2_{L^2(0,T;\dot{H}^1_0(\Omega))} \leqq C. \tag{8.30}$$

Finally, we consider the third time-differentiated version of (8.5):

$$\frac{\bar{J}^3\bar{X}_{tttt}}{\rho_0} - 2\kappa\Big[\frac{\bar{B}^{jk}}{\rho_0}(\rho_0\bar{X}_{ttt}),_k\Big],_j = \bar{G}_{ttt} + \mathcal{G}_3 \text{ in } \Omega \times (0, T_\kappa], \tag{8.31a}$$

$$\bar{X}_{ttt} = 0 \qquad \text{on } \Gamma \times (0, T_\kappa], \tag{8.31b}$$

$$\bar{X}_{ttt} = X_3 \qquad \text{on } \Omega \times \{t = 0\}, \tag{8.31c}$$

where the initial condition X_3 is given as

$$X_3 = 2\kappa\rho_0\Big[\frac{(\rho_0 X_2),_i}{\rho_0}\Big],_i + \rho_0\mathcal{G}_2(0) + \rho_0\bar{G}_{tt}(0), \tag{8.32}$$

and the forcing function $\mathcal{G}_3$ is defined by

$$\mathcal{G}_3 = \partial_t\mathcal{G}_2 + 2\kappa\Big[\bar{B}^{jk}_t\frac{1}{\rho_0}(\rho_0\bar{X}_{tt}),_k\Big],_j - \frac{(\bar{J}^3)_t\bar{X}_{ttt}}{\rho_0}. \tag{8.33}$$

Once again, we do not precisely define $\mathcal{G}_2(0)$, but note that its highest-order terms scale like either D^4u_0 or $\rho_0 D^5 u_0$ or $D^4\rho_0$, so that $\|\sqrt{\rho_0}\mathcal{G}_2(0)\|^2_0 \leqq \mathcal{N}_0$. Using the estimate (8.30), we see that $\|\bar{G}_{ttt} + \mathcal{G}_3\|^2_{L^2(0,T;H^{-1}(\Omega))}$ is bounded by a constant C. (Note that this constant C crucially depends on $\nu > 0$.) We see that Lemma 5 (with $X_{ttt},\ \bar{G}_{ttt} + \mathcal{G}_3,\ X_3$ replacing $X,\ \bar{G},\ X_0$, respectively) yields the following estimate:

$$\left\|\frac{\bar{X}_{tttt}}{\rho_0}\right\|^2_{L^2(0,T;H^{-1}(\Omega))} + \sup_{t\in[0,T]}\left\|\frac{\bar{X}_{ttt}(t)}{\sqrt{\rho_0}}\right\|^2_0 + C_p\,\|\bar{X}_{ttt}\|^2_{L^2(0,T;\dot{H}^1_0(\Omega))} \leqq C. \tag{8.34}$$

8.4.3. $L^2(0, T; H^2(\Omega))$ **Regularity for** $\bar{X}_{tt}$**.** We expand the time-derivative in the definition of $\mathcal{G}_2$ in (8.29) and write

$$\mathcal{G}_2 = 4\kappa\Big[\bar{B}^{jk}_t\frac{1}{\rho_0}(\rho_0\bar{X}_t),_k\Big],_j + 2\kappa\Big[\bar{B}^{jk}_{tt}\frac{1}{\rho_0}(\rho_0\bar{X}),_k\Big],_j - 2\frac{(\bar{J})^3_t\bar{X}_{tt}}{\rho_0} - \frac{(\bar{J})^3_{tt}\bar{X}_t}{\rho_0}. \tag{8.35}$$

According to the estimates (8.34), together with the Hardy inequality and the smoothness of $\bar{v}$,

$$\int_0^T\left(\|\bar{G}_{tt}\|^2_0 + \left\|\frac{(\bar{J})^3_t\bar{X}_{ttt}}{\rho_0}\right\|^2_0 + \left\|2\frac{(\bar{J})^3_t\bar{X}_{tt}}{\rho_0}\right\|^2_0 + \left\|\frac{(\bar{J})^3_{tt}\bar{X}_t}{\rho_0}\right\|^2_0\right)\mathrm{d}t \leqq C\,;$$

hence, (8.27a) and (8.35) show that

$$\kappa^2 \left\| \partial_{tt} \left[\frac{\bar{B}^{jk}}{\rho_0} (\rho_0 \bar{X}),_k \right],_j \right\|^2_{L^2(0,T;L^2(\Omega))} \leqq C. \tag{8.36}$$

The bound (8.36) together with the fundamental theorem of calculus then provides the bound on $[0, T]$:

$$\left\| \left[\frac{\bar{B}^{jk}}{\rho_0} (\rho_0 \bar{X}),_k \right],_j \right\|^2_0 + \left\| \partial_t \left[\frac{\bar{B}^{jk}}{\rho_0} (\rho_0 \bar{X}),_k \right],_j \right\|^2_0 \leqq C. \tag{8.37}$$

From this bound, we will infer that $\|\bar{X}\|_2^2 \leqq C$ and that $\|\bar{X}_t\|_2^2 \leqq C$, and finally that $\int_0^T \|\bar{X}_{tt}\|_2^2 \mathrm{d}t \leqq C$. We will begin this analysis by estimating horizontal derivatives of $D\bar{X}$.

Definition 5. (*Horizontal difference quotients*) For $h > 0$, we set

$$\bar{\partial}^h_\alpha u(x) = \frac{u(x + he_\alpha) - u(x)}{h} \quad (\alpha = 1, 2),$$

and $\bar{\partial}^h = (\bar{\partial}^h_1, \bar{\partial}^h_2)$.

The variational form of the fact that $\|[\frac{\bar{B}^{jk}}{\rho_0}(\rho_0 \bar{X}),_k],_j\|_0$ is bounded takes the following form: almost everywhere on $[0, T]$ and for $f(t)$ bounded in $L^2(\Omega)$,

$$2\kappa \int_\Omega \bar{B}^{jk} \bar{X},_k \, \phi,_j \, \mathrm{d}x + 2\kappa \int_\Omega \bar{B}^{jk} \frac{\rho_0,_k}{\rho_0} \bar{X} \, \phi,_j \, \mathrm{d}x = \int_\Omega f \, \phi \, \mathrm{d}x \quad \forall \phi \in \dot{H}^1_0(\Omega). \tag{8.38}$$

We substitute $\phi = -\bar{\partial}^{-h} \bar{\partial}^h \bar{X}$ into (8.38), and using the discrete product rule

$$\bar{\partial}^h_\alpha (pq) = p^h \bar{\partial}^h_\alpha q + \bar{\partial}^h_\alpha p \, q, \quad p^h(x) = p(x + he_\alpha),$$

we find that

$$\underbrace{2\kappa \int_\Omega \bar{B}^{jk,h} \, \bar{\partial}^h_\alpha \bar{X},_k \, \bar{\partial}^h_\alpha \bar{X},_j \, \mathrm{d}x}_{\mathrm{i}_1} + \underbrace{2\kappa \int_\Omega \bar{\partial}^h_\alpha \bar{B}^{jk} \, \bar{X},_k \, \bar{\partial}^h_\alpha \bar{X},_j \, \mathrm{d}x}_{\mathrm{i}_2}$$
$$+ \underbrace{2\kappa \int_\Omega \bar{B}^{jk,h} \rho_0,^h_k \, \bar{\partial}^h_\alpha \left(\frac{\bar{X}}{\rho_0} \right) \bar{\partial}^h_\alpha \bar{X},_j \, \mathrm{d}x}_{\mathrm{i}_3} + \underbrace{2\kappa \int_\Omega \bar{\partial}^h_\alpha [\bar{B}^{jk} \rho_0,_k] \, \frac{\bar{X}}{\rho_0} \, \bar{\partial}^h_\alpha \bar{X},_j \, \mathrm{d}x}_{\mathrm{i}_4}$$
$$= \underbrace{- \int_\Omega f \, \bar{\partial}^{-h} \bar{\partial}^h \bar{X} \, \mathrm{d}x}_{\mathrm{i}_5}.$$

We proceed to the analysis of the integrals i_a, $a = 1, \ldots, 5$. By the uniform ellipticity condition (8.18) obtained on our time interval $[0, T]$, we see that

$$2\kappa\lambda \|\bar{\partial}^h D\bar{X}\|_0^2 \leqq \mathrm{i}_1. \tag{8.39}$$

The term $\mathfrak{i}_2$ can be estimated by the L^∞–L^2–L^2 Hölder's inequality:

$$|\mathfrak{i}_2| \leqq 2\kappa \|\bar\partial^h_\alpha \bar B^{jk}\|_{L^\infty(\Omega)} \|\bar X,_k\|_0 \|\bar\partial^h_\alpha \bar X,_j\|_0.$$

We then see that using the Cauchy-Young inequality for $\epsilon > 0$ we obtain

$$|\mathfrak{i}_2| \leqq C\|\bar X\|_1^2 + \epsilon \|\bar\partial^h_\alpha \bar X,_j\|_0^2. \tag{8.40}$$

The integral $\mathfrak{i}_3$ requires us to form an exact derivative and integrate by parts. With

$$\bar\partial^h_\alpha\Big(\frac{\bar X}{\rho_0}\Big) = \bar\partial^h_\alpha(\rho_0^{-1})\bar X^h + \rho_0^{-1}\bar\partial^h_\alpha \bar X = -\tfrac{1}{\rho_0}\bar\partial^h_\alpha \rho_0 \tfrac{\bar X^h}{\rho_0^h} + \tfrac{\bar\partial^h_\alpha \bar X}{\rho_0}, \tag{8.41}$$

we write $\mathfrak{i}_3$ as

$$\mathfrak{i}_3 = \underbrace{2\kappa\int_\Omega \frac{\bar B^{jk,h}\rho_0,^h_k}{\rho_0}\bar\partial^h_\alpha \bar X\, \bar\partial^h_\alpha \bar X,_j \,\mathrm{d}x}_{\mathfrak{i}_{3a}} - \underbrace{2\kappa\int_\Omega \bar B^{jk,h}\rho_0,^h_k \frac{\bar\partial^h_\alpha \rho_0}{\rho_0}\frac{\bar X^h}{\rho_0^h}\bar\partial^h_\alpha \bar X,_j \,\mathrm{d}x}_{\mathfrak{i}_{3b}}.$$

Integration by parts with respect to x_j shows that

$$\mathfrak{i}_{3a} = \underbrace{\kappa\int_\Omega \bar B^{jk,h}\rho_0,^h_k\, \rho_0,_j \left|\frac{\bar\partial^h \bar X}{\rho_0}\right|^2 \mathrm{d}x}_{\mathfrak{i}_{3ai}} - \underbrace{\kappa\int_\Omega (\bar B^{jk,h}\rho_0,^h_k),_j \frac{\bar\partial^h \bar X}{\rho_0}\bar\partial^h \bar X\,\mathrm{d}x}_{\mathfrak{i}_{3aii}}.$$

Next, we notice that

$$\mathfrak{i}_{3ai} = \kappa\int_\Omega \bar B^{jk,h}\big(\rho_0,_k \rho_0,_j + (\rho_0,^h_k - \rho_0,_k)\rho_0,_j\big)\left|\frac{\bar\partial^h \bar X}{\rho_0}\right|^2 \mathrm{d}x$$

and therefore, according to (8.18),

$$\begin{aligned}
\mathfrak{i}_{3ai} &\geqq -C\,h\,\kappa\|D^2\rho_0\|_{L^\infty} C_M \int_\Omega \left|\frac{\bar\partial^h \bar X}{\rho_0}\right|^2 \mathrm{d}x + \kappa\lambda\int_\Omega |D\rho_0|^2 \left|\frac{\bar\partial^h \bar X}{\rho_0}\right|^2 \mathrm{d}x \\
&\geqq -C\,h\,\kappa C\left\|\bar\partial^h \bar X\right\|_1^2 + \kappa\lambda\int_\Omega |D\rho_0|^2 \left|\frac{\bar\partial^h \bar X}{\rho_0}\right|^2 \mathrm{d}x.
\end{aligned} \tag{8.42}$$

On the other hand, for $\epsilon > 0$,

$$\begin{aligned}
|\mathfrak{i}_{3aii}| &\leqq \kappa\|(\bar B^{jk,h}\rho_0,^h_k),_j\|_{L^\infty}\left\|\frac{\bar\partial^h \bar X}{\rho_0}\right\|_0 \|\bar\partial^h \bar X\|_0 \\
&\leqq C\|\bar\partial^h \bar X\|_0^2 + \epsilon\left\|\frac{\bar\partial^h \bar X}{\rho_0}\right\|_0^2 \\
&\leqq C\|\bar\partial^h \bar X\|_0^2 + \epsilon C\|\bar\partial^h D\bar X\|_0^2,
\end{aligned} \tag{8.43}$$

where we have used the Hardy and Poincaré inequalities for the last inequality in (8.43). Furthermore,

$$\begin{aligned}|\mathfrak{i}_{3b}| &\leqq C \Big\| \bar{B}^{jk,h} \rho_0,_k^h \Big(\frac{1}{\rho_0} \bar{\partial}_\alpha^h \rho_0 \Big) \Big\|_{L^\infty(\Omega)} \Big\| \frac{\bar{X}^h}{\rho_0^h} \Big\|_0 \| \bar{\partial}_\alpha^h X,_j \|_0 \\ &\leqq C \Big\| \frac{\bar{\partial}_\alpha^h \rho_0}{\rho_0} \Big\|_2 \| \bar{X} \|_1 \| \bar{\partial}_\alpha^h \bar{X},_j \|_0 \\ &\leqq C \| \rho_0 \|_4 \| \bar{X} \|_1 \| \bar{\partial}_\alpha^h \bar{X},_j \|_0 \\ &\leqq C \| X \|_1^2 + \epsilon \| \bar{\partial}_\alpha^h \bar{X},_j \|_0^2. \end{aligned} \tag{8.44}$$

Similarly, we have that

$$\begin{aligned}|\mathfrak{i}_4| &\leqq C \| \bar{\partial}_\alpha^h [\bar{B}^{jk} \rho_0,_k] \|_{L^\infty(\Omega)} \Big\| \frac{\bar{X}}{\rho_0} \Big\|_0 \| \bar{\partial}_\alpha^h \bar{X},_j \|_0 \\ &\leqq C \| \bar{X} \|_1^2 + \epsilon \| \bar{\partial}_\alpha^h \bar{X},_j \|_0^2, \end{aligned} \tag{8.45}$$

and finally

$$|\mathfrak{i}_5| \leqq C \| f \|_0^2 + \epsilon \| \bar{\partial}_\alpha^h \bar{X},_j \|_0^2. \tag{8.46}$$

Combining the estimates (8.39)–(8.46), and taking $\epsilon > 0$ sufficiently small, we find that for $h > 0$ small enough

$$\| \bar{\partial}^h D \bar{X} \|_0^2 \leqq C (\| \bar{X} \|_1^2 + \| f \|_0^2) \leqq C,$$

with C independent of h. It thus follows that

$$\| \bar{\partial} \bar{X} \|_1^2 \leqq C. \tag{8.47}$$

Since ρ_0 is strictly positive on any open interior subdomain of Ω, standard regularity theory shows that the solution $\bar{X}$ and its time derivatives are smooth in the interior, and hence the equation (8.5) holds in the classical sense in the interior of Ω. It remains to estimate $\| \bar{X},_3 \|^2_{L^2(0,T;H^1(\Omega))}$; for this purpose we expand $(\frac{\bar{B}^{jk}}{\rho_0} (\rho_0 \bar{X}),_k),_j$ to find that

$$\begin{aligned}\bar{X},_{33} + \rho_0,_3 \Big(\frac{\bar{X}}{\rho_0} \Big),_3 = \frac{1}{\bar{B}^{33}} \Big(\Big[\frac{\bar{B}^{jk}}{\rho_0} (\rho_0 \bar{X}),_k \Big],_j - \bar{B}^{\alpha 3} \bar{X},_{3\alpha} - \bar{B}^{3j},_j \bar{X},_3 \\ - (\bar{B}^{j\alpha} \bar{X},_\alpha),_j - (\bar{B}^{jk} \rho_0,_k),_j \frac{\bar{X}}{\rho_0} \\ - \bar{B}^{j\alpha} \rho_0,_j \Big(\frac{\bar{X}}{\rho_0} \Big),_\alpha - \bar{B}^{\alpha 3} \rho_0,_\alpha \Big(\frac{\bar{X}}{\rho_0} \Big),_3 \Big). \end{aligned} \tag{8.48}$$

Since

$$\bar{B}^{\alpha 3} \rho_0,_\alpha \Big(\frac{\bar{X}}{\rho_0} \Big),_3 = \bar{B}^{\alpha 3} \frac{\rho_0,_\alpha}{\rho_0} \Big(\bar{X},_3 - \frac{\bar{X}}{\rho_0} \rho_0,_3 \Big),$$

and since $\|\rho_{0},_{\alpha}/\rho_0\|_{L^\infty(\Omega)}$ is bounded by a constant thanks to Hardy's inequality in higher-order form (Lemma 1), we see from (8.47) and (8.37) that

$$\|\bar{X},_{33}+\rho_0,_3\Big(\frac{\bar{X}}{\rho_0}\Big),_3\|_0^2 \leqq C. \tag{8.49}$$

We introduce the variable Y defined by

$$Y(t,x_1,x_2,x_3)=\int_0^{x_3}\frac{\bar{X}(t,x_1,x_2,y_3)}{\rho_0(x_1,x_2,y_3)}dy_3, \tag{8.50}$$

so that Y vanishes at $x_3=0$, and will allow us to employ the Poincaré inequality with this variable. It is easy to see that

$$\bar{X}=\rho_0 Y,_3\,. \tag{8.51}$$

Thanks to the standard Hardy inequality, we thus have that for all $t\in[0,T]$

$$\|Y,_3\|_0^2 \leqq C\|\bar{X}\|_1^2 \leqq C.$$

The estimate (8.47) then shows that

$$\|DY\|_0^2 \leqq C, \tag{8.52}$$

and hence by Poincaré's inequality,

$$\|Y\|_0^2 \leqq C. \tag{8.53}$$

We notice that

$$\begin{aligned}\bar{X},_{33}+\rho_0,_3\Big(\frac{\bar{X}}{\rho_0}\Big),_3 &= (\rho_0 Y,_3),_{33}+\rho_0,_3 Y,_{33}\\ &= \rho_0 Y,_{333}+3\rho_0,_3 Y,_{33}+\rho_0,_{33} Y,_3\,,\end{aligned}$$

so that

$$\|\rho_0 Y,_{333}+3\rho_0,_3 Y,_{33}\|_0^2 \leqq C.$$

The product rule then implies that $\|(\rho_0 Y),_{333}\|_0^2 \leqq C$. The same $L^2(\Omega)$ bound can easily be established for the lower-order terms $\rho_0 Y$, $(\rho_0 Y),_3$, and $(\rho_0 Y),_{33}$; for instance the identity (8.51) shows that

$$\|(\rho_0 Y,_3),_3\|_0^2 \leqq C.$$

By (8.52), we see that $\|\rho_0 Y,_{33}\|_0^2$ enjoys the same bound. Since

$$(\rho_0 Y),_{33}=\rho_0 Y,_{33}+2\rho_0,_3 Y,_3+\rho_0,_{33} Y,$$

the estimates (8.52) and (8.53) prove that $\|(\rho_0 Y),_{33}\|_0^2 \leqq C$. By definition of the $H^3(0,1)$-norm, we then see that

$$\int_{\mathbb{T}^2}\|\rho_0 Y(x_1,x_2,\cdot)\|_{H^3(0,1)}^2 \mathrm{d}x_1\mathrm{d}x_2 \leqq C. \tag{8.54}$$

Now, thanks to the high-order Hardy's inequality set on the one-dimensional domain (0, 1), we infer from (8.54) that

$$\int_{\mathbb{T}^2} \|Y(x_1, x_2, \cdot)\|^2_{H^2(0,1)} \mathrm{d}x_1 \mathrm{d}x_2 \leqq C. \tag{8.55}$$

From (8.54),

$$\int_{\mathbb{T}^2} \|\rho_0 Y,_3(x_1, x_2, \cdot) + \rho_0,_3 Y(x_1, x_2, \cdot)\|^2_{H^2(0,1)} \mathrm{d}x_1 \mathrm{d}x_2 \leqq C,$$

from which it follows, with (8.51), that

$$\int_{\mathbb{T}^2} \|\bar{X}(x_1, x_2, \cdot)\|^2_{H^2(0,1)} \mathrm{d}x_1 \mathrm{d}x_2 \leqq C,$$

hence,

$$\|\bar{X},_{33}\|^2_0 \leqq C. \tag{8.56}$$

Combining the inequalities (8.56) and (8.47), we see that for all $t \in [0, T]$

$$\|\bar{X}\|^2_2 \leqq C. \tag{8.57}$$

The estimate (8.57) together with (8.37) then shows that

$$\left\| \left[\frac{\bar{B}^{jk}}{\rho_0} (\rho_0 \bar{X}_t),_k \right],_j \right\|^2_0 \text{ is bounded.}$$

By identically repeating for $\bar{X}_t$ the $H^2(\Omega)$-regularity estimates that we just detailed for $\bar{X}$, we obtain that

$$\|\bar{X}_t\|^2_2 \leqq C. \tag{8.58}$$

The estimates (8.57) and (8.58) together with (8.36) then prove that

$$\|[\frac{\bar{B}^{jk}}{\rho_0} (\rho_0 \bar{X}_{tt}),_k],_j\|^2_{L^2(0,T;L^2(\Omega))} \leqq C.$$

Once again we repeat the estimates for $\bar{X}_{tt}$ which we just explained for $\bar{X}$, this time L^2-in-time, and we obtain the desired result; namely,

$$\|\bar{X}_{tt}\|^2_{L^2(0,T;H^2(\Omega))} \leqq C. \tag{8.59}$$

It follows that (8.23) holds almost everywhere.

8.4.4. $L^2(0,T;H^3(\Omega))$ **Regularity for** $\bar{X}_t$**.** Using (8.25), we write (8.23a) as

$$-2\kappa\,\partial_t\Big[\frac{\bar{B}^{jk}}{\rho_0}(\rho_0\bar{X}),_k\Big],_j=\bar{G}_t-\frac{\bar{J}^3\bar{X}_{tt}}{\rho_0}-\frac{(\bar{J}^3)_t\bar{X}_t}{\rho_0}.$$

The estimates (8.58) and (8.59) together with the higher-order Hardy inequality shows that

$$\Big\|\partial_t\Big[\frac{\bar{B}^{jk}}{\rho_0}(\rho_0\bar{X}),_k\Big],_j\Big\|^2_{L^2(0,T;H^1(\Omega))}\leqq C, \tag{8.60}$$

and hence by the fundamental theorem of calculus,

$$\Big\|\Big[\frac{\bar{B}^{jk}}{\rho_0}(\rho_0\bar{X}),_k\Big],_j\Big\|^2_1\leqq C.$$

We employ the identity (8.41) to find that

$$\begin{aligned}\bar{\partial}^h\Big[\frac{\bar{B}^{jk}}{\rho_0}(\rho_0\bar{X}),_k\Big],_j&=\Big[\frac{\bar{B}^{jk,h}}{\rho_0}(\rho_0\bar{\partial}^h\bar{X}),_k\Big],_j+\Big[\bar{\partial}^h\bar{B}^{jk}\Big(\bar{X},_k+\rho_0,_k\frac{\bar{X}}{\rho_0}\Big)\Big],_j\\&\quad+\Big[\bar{B}^{jk,h}\bar{\partial}^h\rho_0,_k\frac{\bar{X}}{\rho_0}\Big],_j-\Big[\bar{B}^{jk,h}\rho_0,_k\frac{\bar{\partial}^h\rho_0}{\rho_0}\frac{\bar{X}^h}{\rho_0^h}\Big],_j.\end{aligned}$$

Since the last three term on the right-hand side are bounded in $L^2(\Omega)$ thanks to the higher-order Hardy inequality and (8.57), we see that

$$\Big\|\Big[\frac{\bar{B}^{jk,h}}{\rho_0}(\rho_0\bar{\partial}^h\bar{X}),_k\Big],_j\Big\|^2_0\leqq C.$$

Now, repeating the argument which led to (8.57) with $\bar{\partial}^h\bar{X}$ replacing $\bar{X}$, we find that

$$\|\bar{\partial}\bar{X}\|^2_2\leqq C. \tag{8.61}$$

By differentiating the relation (8.48) with respect to x_3 and using the estimate (8.61), we see that $\|\bar{X},_{333}+\rho_0,_3(\frac{\bar{X}}{\rho_0}),_{33}\|^2_0\leqq C$. Now, by using the variable Y defined by (8.50), we can repeat our argument to find that $\|\bar{X},_{333}\|^2_0\leqq C$ and hence that

$$\|\bar{X}\|^2_3\leqq C. \tag{8.62}$$

From (8.60), we then easily infer that $\int_0^T\|[\frac{\bar{B}^{jk}}{\rho_0}(\rho_0\bar{X}_t),_k],_j\|^2_1\mathrm{d}t\leqq C$, so that the argument just given allows us to conclude that

$$\|\bar{X}_t\|^2_{L^2(0,T;H^3(\Omega))}\leqq C. \tag{8.63}$$

8.4.5. $L^2(0,T;H^4(\Omega))$ Regularity for $\bar{X}$. By repeating the argument of Section 8.4.4, we find that

$$\|\bar{X}\|^2_{L^2(0,T;H^4(\Omega))} \leqq C. \tag{8.64}$$

We have thus established existence and regularity of our solution $\bar{X}$; however, the bounds and time-interval of existence depend on $\nu > 0$. We next turn to better Sobolev-type estimates to establish bounds for $\bar{X}$ and its time-derivatives which are independent of ν and are useful for our fixed-point scheme.

8.4.6. Estimates for $\|\bar{X}\|^2_{\mathbf{X}_T}$ Independent of ν

Step 1 We begin this section by getting ν-independent energy estimates for the third time-differentiated problem (8.31).

Lemma 6. *For $T > 0$ taken sufficiently small and $\delta > 0$,*

$$\begin{aligned}\left\|\frac{\bar{X}_{tttt}}{\rho_0}\right\|^2_{L^2(0,T;H^{-1}(\Omega))} &+ \sup_{t\in[0,T]}\left\|\frac{\bar{X}_{ttt}(t)}{\sqrt{\rho_0}}\right\|^2_0 + C_p\,\|\bar{X}_{ttt}\|^2_{L^2(0,T;\dot{H}^1_0(\Omega))}\\ &\leqq \mathcal{N}_0 + T P(\|\bar{X}\|^2_{\mathbf{X}_T}) + C\delta\|\bar{X}\|^2_{\mathbf{X}_T} + T\,P(\|\bar{v}\|^2_{\mathbf{Z}_T}) + P(\|\operatorname{curl}\bar{v}\|^2_{\mathbf{Y}_T}).\end{aligned} \tag{8.65}$$

Proof. We write the forcing function $\bar{G}_{ttt} + \mathcal{G}_3$ as

$$\bar{G}_{ttt} + \mathcal{G}_3 = \underbrace{\bar{G}_{ttt} + \partial_t\mathcal{G}_2}_{\mathcal{T}_1} + \underbrace{2\kappa\Big[\bar{B}^{jk}_t\frac{1}{\rho_0}(\rho_0\bar{X}_{tt}),_k\Big],_j}_{\mathcal{T}_2} - \underbrace{\frac{(\bar{J}^3)_t\bar{X}_{ttt}}{\rho_0}}_{\mathcal{T}_3}. \tag{8.66}$$

We test (8.31a) with $\bar{X}_{ttt}$. In the identical fashion that we obtained (8.21), we see that

$$\begin{aligned}\frac{1}{32}\;\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{|\bar{X}_{ttt}|^2}{\rho_0}\mathrm{d}x &+ 2\kappa\lambda\int_\Omega |D\bar{X}_{ttt}|^2\,\mathrm{d}x + \kappa\lambda\int_\Omega \frac{|D\rho_0|^2}{\rho_0^2}|\bar{X}_{ttt}|^2\,\mathrm{d}x\\ &\leqq \Big\|\frac{1}{2}(\bar{J}^3)_t + \kappa\rho_0,_{jk}\bar{B}^{jk} + \kappa\rho_0,_k\bar{B}^{jk},_j\Big\|_{L^\infty(\Omega)}\int_\Omega\frac{1}{\rho_0}|\bar{X}_{ttt}|^2\,\mathrm{d}x\\ &\quad + \langle\bar{G}_{ttt} + \mathcal{G}_3\,,\;\bar{X}_{ttt}\rangle.\end{aligned}$$

Integrating this inequality from 0 to $t \in (0,T]$, we see that

$$\begin{aligned}\frac{1}{32}\sup_{t\in[0,T]}\int_\Omega\frac{1}{\rho_0}|\bar{X}_{ttt}(t)|^2\mathrm{d}x &+ 2\kappa\lambda\int_0^T\int_\Omega |D\bar{X}_{ttt}|^2\,\mathrm{d}x\mathrm{d}t\\ \leqq \mathcal{N}_0 + T\sup_{t\in[0,T]}&\Big\|\frac{(\bar{J}^3)_t}{2} + \kappa\rho_0,_{jk}\bar{B}^{jk} + \kappa\rho_0,_k\bar{B}^{jk},_j\Big\|_{L^\infty(\Omega)}\int_\Omega\frac{|\bar{X}_{ttt}(t)|^2}{\rho_0}\,\mathrm{d}x\\ &+ \int_0^T\langle\bar{G}_{ttt} + \mathcal{G}_3\,,\;\bar{X}_{ttt}\rangle\mathrm{d}t.\end{aligned}$$

By the Sobolev embedding theorem $\| \frac{(\bar{J}^3)_t}{2} + \kappa\rho_0,_{jk} \bar{B}^{jk} + \kappa\rho_0,_k \bar{B}^{jk},_j \|_{L^\infty(\Omega)}$ is less or equal than $C\| \frac{(\bar{J}^3)_t}{2} + \kappa\rho_0,_{jk} \bar{B}^{jk} + \kappa\rho_0,_k \bar{B}^{jk},_j \|_2$. The highest-order derivative in the term $\frac{1}{2}(\bar{J}^3)_t$ scales like Dv, while the highest-order derivative in $\bar{B}^{jk},_j$ scales like $D^2\eta$, which means that we have to be able to bound $\sup_{t\in[0,T]} \|v(t)\|_3$ as well as $\sup_{t\in[0,T]} \|\eta(t)\|_4$, and these are clearly bounded by $\mathcal{N}_0 + C\sqrt{t}\|\bar{v}\|_{X_T}$. Therefore, by choosing T sufficiently small and invoking the Poincaré inequality, we see that

$$C \sup_{t\in[0,T]} \int_\Omega \frac{1}{\rho_0}|\bar{X}_{ttt}(t)|^2 \mathrm{d}x + C_{\kappa\lambda} \int_0^T \int_\Omega |D\bar{X}_{ttt}|^2 \,\mathrm{d}x\mathrm{d}t \leqq \mathcal{N}_0 + \int_0^T \langle \bar{G}_{ttt} + \mathcal{G}_3\ ,\ \bar{X}_{ttt}\rangle \mathrm{d}t.$$

We proceed to the analysis of the terms in $\int_0^T \langle \bar{G}_{ttt} + \mathcal{G}_3\ ,\ \bar{X}_{ttt}\rangle \mathrm{d}t$, and we begin with the term $\mathcal{T}_3$ in (8.66). We have that

$$\int_0^T \langle \mathcal{T}_3, \bar{X}_{ttt}\rangle \mathrm{d}t \leqq \sup_{t\in[0,T]} \|(\bar{J}^3)_t\|_{L^\infty(\Omega)} \int_0^T \int_\Omega \frac{1}{\rho_0}|\bar{X}_{ttt}|^2 \,\mathrm{d}x\mathrm{d}t$$
$$\leqq (\mathcal{N}_0 + \sqrt{T}C_M)T \sup_{t\in[0,T]} \int_\Omega \frac{1}{\rho_0}|\bar{X}_{ttt}|^2 \,\mathrm{d}x,$$

where we have made use of the Sobolev embedding theorem giving the inequality $\|(\bar{J}^3)_t\|_{L^\infty(\Omega)} \leqq C\|(\bar{J}^3)_t\|_2 \leqq \mathcal{N}_0 + \sqrt{t}C_M$, where C_M depends on M.

To estimate the term $\mathcal{T}_2$ in (8.66), notice that

$$\begin{aligned}\langle \mathcal{T}_2, \bar{X}_{ttt}\rangle &= -2\kappa \int_\Omega \bar{B}_t^{jk}(\bar{X}_{tt},_k + \rho_0,_k \frac{\bar{X}_{tt}}{\rho_0})\bar{X}_{ttt},_j \,\mathrm{d}x \leqq C\|\bar{B}_t\|_2\|\bar{X}_{tt}\|_1\|\bar{X}_{ttt}\|_1 \\ &\leqq \delta\|\bar{X}_{ttt}\|_1^2 + C\|\bar{B}_t\|_2^2\|\bar{X}_{tt}\|_1^2 \\ &\leqq \delta\|\bar{X}_{ttt}\|_1^2 + C\|\bar{B}_t\|_2^2(\|\bar{X}_{tt}(0)\|_1^2 + t\|\bar{X}_{ttt}(t)\|_1^2),\end{aligned}$$

and thus

$$\int_0^T \langle \mathcal{T}_2, \bar{X}_{ttt}\rangle \mathrm{d}t \leqq \mathcal{N}_0 + \delta\|\bar{X}\|_{\mathbf{X}_T}^2 + T\ P(\|\bar{v}\|_{\mathbf{Z}_T}^2) + T\,P(\|\bar{X}\|_{\mathbf{X}_T}^2).$$

It remains to estimate $\langle \mathcal{T}_1, \bar{X}_{ttt}\rangle$; we use the identity (8.29) defining $\mathcal{G}_2$ to expand $\mathcal{T}_1$ as

$$\mathcal{T}_1 = \bar{G}_{ttt} + \partial_t\mathcal{G}_2 = \bar{G}_{ttt} + \partial_{tt}\mathcal{G}_1 + \partial_t\Big(2\kappa\Big[\bar{B}_t^{jk}\frac{1}{\rho_0}(\rho_0\bar{X}_t),_k\Big],_j - \frac{(\bar{J}^3)_t\bar{X}_{tt}}{\rho_0}\Big).$$

The terms $\langle \partial_t(2\kappa[\bar{B}_t^{jk}\frac{1}{\rho_0}(\rho_0\bar{X}_t),_k],_j - \frac{(\bar{J}^3)_t\bar{X}_{tt}}{\rho_0}), \bar{X}_{ttt}\rangle$ are estimated in the same way and have the same bounds as $\langle \mathcal{T}_2, \bar{X}_{ttt}\rangle$ and $\langle \mathcal{T}_3, \bar{X}_{ttt}\rangle$ above, so we focus on estimating $\langle \bar{G}_{ttt} + \partial_{tt}\mathcal{G}_1, \bar{X}_{ttt}\rangle$. To do so, we use the identity (8.25) defining $\mathcal{G}_1$ and write

$$\bar{G}_{ttt} + \partial_{tt}\mathcal{G}_1 = \bar{G}_{ttt} + \underbrace{\partial_{tt}\Big(2\kappa\Big[\bar{B}_t^{jk}\frac{1}{\rho_0}(\rho_0\bar{X}),_k\Big],_j\Big)}_{\mathcal{S}_1} - \underbrace{\partial_{tt}\Big(\frac{(\bar{J}^3)_t\bar{X}_t}{\rho_0}\Big)}_{\mathcal{S}_2}.$$

Expanding $\mathcal{S}_1$ as

$$\mathcal{S}_1 = \underbrace{2\kappa\Big[\bar{B}^{jk}_{ttt}\frac{(\rho_0\bar{X}),_k}{\rho_0}\Big],_j}_{\mathcal{S}_{1a}} + \underbrace{4\kappa\Big[\bar{B}^{jk}_{tt}\frac{(\rho_0\bar{X}_t),_k}{\rho_0}\Big],_j}_{\mathcal{S}_{1b}} + \underbrace{2\kappa\Big[\bar{B}^{jk}_{t}\frac{(\rho_0\bar{X}_{tt}),_k}{\rho_0}\Big],_j}_{\mathcal{S}_{1c}},$$

we see that for $\delta > 0$,

$$\begin{aligned}\langle \mathcal{S}_{1a}, \bar{X}_{ttt}\rangle &= -2\kappa\int_\Omega \bar{B}^{jk}_{ttt}\Big(\bar{X},_k + \rho_0,_k\frac{\bar{X}}{\rho_0}\Big)\bar{X}_{ttt},_j\,\mathrm{d}x \\ &\leqq C\|\bar{B}^{jk}_{ttt}\|_0\,(\|\bar{X},_k\|_2 + \|\frac{\bar{X}}{\rho_0}\|_2)\,\|D\bar{X}_{ttt}\|_0 \\ &\leqq C\|\bar{B}^{jk}_{ttt}\|_0\,\|\bar{X}\|_3\,\|D\bar{X}_{ttt}\|_0 \\ &\leqq C\|\bar{B}^{jk}_{ttt}\|_0^2\,(\|\bar{X}(0)\|_3^2 + t\|\bar{X}_t\|_3^2) + \delta\|\bar{X}_{ttt}\|_1^2.\end{aligned}$$

where we have used the Sobolev embedding theorem for the first inequality, the higher-order Hardy inequality Lemma 1 for the second inequality, and the Cauchy-Young inequality together with the fundamental theorem of calculus for the third inequality. We see that

$$\int_0^T \langle \mathcal{S}_{1a}, \bar{X}_{ttt}\rangle \mathrm{d}t \leqq \mathcal{N}_0 + \delta\|\bar{X}\|^2_{\mathbf{X}_T} + T\,P(\|\bar{v}\|^2_{\mathbf{Z}_T}) + T\,P(\|\bar{X}\|^2_{\mathbf{X}_T}).$$

The duality pairing involving $\mathcal{S}_{1b}$ and $\mathcal{S}_{1c}$ can be estimated in the same way to provide the estimate

$$\int_0^T \langle \mathcal{S}_{1}, \bar{X}_{ttt}\rangle \mathrm{d}t \leqq \mathcal{N}_0 + \delta\|\bar{X}\|^2_{\mathbf{X}_T} + T\,P(\|\bar{v}\|^2_{\mathbf{Z}_T}) + T\,P(\|\bar{X}\|^2_{\mathbf{X}_T}).$$

The duality pairing involving $\mathcal{S}_2$ is estimated in the same manner as $\mathcal{T}_3$ and $\mathcal{S}_1$ to yield

$$\begin{aligned}\int_0^T \langle \mathcal{S}_{2}, \bar{X}_{ttt}\rangle \mathrm{d}t \leqq{}& \mathcal{N}_0 + \delta\|\bar{X}\|^2_{\mathbf{X}_T} + T\,P(\|\bar{v}\|^2_{\mathbf{Z}_T}) + T\,P(\|\bar{X}\|^2_{\mathbf{X}_T}) \\ &+ (\mathcal{N}_0 + \sqrt{T}C_M)T\sup_{t\in[0,T]}\int_\Omega \frac{1}{\rho_0}|\bar{X}_{ttt}|^2\,\mathrm{d}x.\end{aligned}$$

It thus remains to estimate the duality pairing $\int_0^T \langle \bar{G}_{ttt}, \bar{X}_{ttt}\rangle \mathrm{d}t$. We write

$$\begin{aligned}\bar{G}_{ttt} ={}& -\underbrace{\partial_{ttt}\Big(3\bar{J}^{-1}(\bar{J}_t)^2 - \partial_t\bar{a}^j_i\,\bar{v}^i,_j\Big)}_{\mathcal{L}_1} - \underbrace{2\partial_{ttt}[\bar{a}^j_i\,\bar{A}^k_i(\rho_0\bar{J}^{-1}),_k],_j}_{\mathcal{L}_2} \\ &+ \underbrace{\kappa\,\partial_{ttt}[\bar{a}^j_i\,\partial_t\bar{a}^k_i\frac{1}{\rho_0}(\rho_0^2\bar{J}^{-2}),_k],_j}_{\mathcal{L}_3}.\end{aligned}$$

Notice that by the Cauchy–Schwarz inequality

$$\int_0^T \langle \mathcal{L}_1\ ,\ \bar{X}_{ttt}\rangle \mathrm{d}t \leqq \int_0^T \left\| \sqrt{\rho_0}\partial_{ttt}\Big(3\bar{J}^{-1}(\bar{J}_t)^2 - \partial_t \bar{a}_i^j\, \bar{v}^i,_j\Big)\right\|_0 \left\|\frac{\bar{X}_{ttt}}{\sqrt{\rho_0}}\right\|_0 \mathrm{d}t$$

$$\leqq T\ P(\|\bar{v}\|^2_{\mathbf{Z}_T}) + CT \sup_{t\in[0,T]} \int_\Omega \frac{1}{\rho_0}|\bar{X}_{ttt}|^2\ \mathrm{d}x.$$

Next, we write $\langle \mathcal{L}_2, \bar{X}_{ttt}\rangle = 2\int_\Omega \partial_{ttt}[\bar{a}_i^j\, \bar{A}_i^k(\rho_0\bar{J}^{-1}),_k]\ \bar{X}_{ttt},_j\ \mathrm{d}x$.

We notice that the higher-order derivatives in $\partial_{ttt}[\bar{a}_i^j\,\bar{A}_i^k(\rho_0\bar{J}^{-1}),_k]$ scale like either $D(\rho_0 D\bar{v}_{tt})$ or $D\bar{v}_{tt}$ so the fundamental theorem of calculus and the Cauchy-Young inequality once again shows that for $\delta > 0$,

$$\int_0^T \langle \mathcal{L}_2, \bar{X}_{ttt}\rangle \mathrm{d}t = T\ P(\|\bar{v}\|^2_{\mathbf{Z}_T}) + \delta\|\bar{X}\|^2_{\mathbf{X}_T}.$$

A good estimate for $\mathcal{L}_3$ requires the curl structure of Lemma 3. We write the highest-order term in

$$\kappa\,\partial_{ttt}\Big[\bar{a}_i^j\,\partial_t\bar{a}_i^k\frac{1}{\rho_0}(\rho_0^2\bar{J}^{-2}),_k\Big],_j = \kappa\,\partial_{ttt}[2\bar{a}_i^j\,\partial_t\bar{a}_i^k\,\rho_0,_k\ \bar{J}^{-2} + \rho_0\bar{a}_i^j\,\partial_t\bar{a}_i^k\,\bar{J}^{-2},_k)],_j$$

as

$$2\kappa\,\partial_{ttt}[(\bar{a}_i^j\,\partial_t\bar{a}_i^k)\rho_0,_k\ \bar{J}^{-2}],_j\ ; \tag{8.67}$$

all of the other terms arising from the distribution of ∂_{ttt} are lower-order and can be estimated in the same way as $\mathcal{L}_2$. Now, using Lemma 3, the highest-order term in (8.67) is written as

$$\partial_{ttt}[(\bar{a}_i^j\,\partial_t\bar{a}_i^k)\rho_0,_k\ \bar{J}^{-2}],_j = \underbrace{[\operatorname{curl}\operatorname{curl}\bar{v}_{ttt}]^k\rho_0,_k\ \bar{J}^{-2}}_{\mathrm{t}_1}$$
$$+\underbrace{\bar{v}^r_{ttt},_{sj}\left(\frac{[\bar{a}_r^s\bar{a}_i^k - \bar{a}_i^s\bar{a}_r^k]\bar{a}_i^j}{\bar{J}} - [\delta_r^s\delta_i^k - \delta_i^s\delta_r^k]\delta_i^j\right)\frac{\rho_0,_k}{\bar{J}^2}}_{\mathrm{t}_2}$$
$$+\underbrace{\bar{v}^r_{ttt},_s\Big(\bar{J}^{-1}[\bar{a}_r^s\bar{a}_i^k - \bar{a}_i^s\bar{a}_r^k]\Big),_j\,\bar{a}_i^j\,\rho_0,_k\ \bar{J}^{-2}}_{\mathrm{t}_3} + \mathcal{R},$$

with $\mathcal{R}$ being lower-order and once again estimated as $\mathcal{L}_2$. Integration by parts with respect to the curl operator in the term t_1, we see that

$$\langle \mathrm{t}_1, \bar{X}_{ttt}\rangle = \int_\Omega \operatorname{curl}\bar{v}_{ttt}\cdot D\times(D\rho_0\bar{J}^{-2}\bar{X}_{ttt})\mathrm{d}x$$
$$\leqq \|\operatorname{curl}\bar{v}_{ttt}(t)\|_0\Big(\|D\rho_0\bar{J}^{-2}\|_{L^\infty(\Omega)}\ \|D\bar{X}_{ttt}\|_0$$
$$+\|\operatorname{curl}(D\rho_0\bar{J}^{-2})\|_{L^3(\Omega)}\ \|\bar{X}_{ttt}\|_{L^6(\Omega)}\Big)$$
$$\leqq \|\operatorname{curl}\bar{v}_{ttt}(t)\|_0\Big(\|D\rho_0\bar{J}^{-2}\|_2 + \|\operatorname{curl}(D\rho_0\bar{J}^{-2})\|_1\Big)\|\bar{X}_{ttt}\|_1$$
$$\leqq C\|\operatorname{curl}\bar{v}_{ttt}(t)\|_0^2\Big(\|D\rho_0\bar{J}^{-2}\|_2^2 + \|\operatorname{curl}(D\rho_0\bar{J}^{-2})\|_1^2\Big) + \delta\|\bar{X}_{ttt}\|_1^2.$$

It follows that

$$\int_0^T \langle \mathfrak{t}_1, \bar{X}_{ttt}\rangle \mathrm{d}t \leqq P(\|\operatorname{curl}\bar{v}\|^2_{\mathbf{Y}_T}) + T\, P(\|\bar{v}\|^2_{\mathbf{Z}_T}) + \delta \|\bar{X}\|^2_{\mathbf{X}_T}.$$

For $\mathfrak{t}_2$,

$$\langle \mathfrak{t}_2, \bar{X}_{ttt}\rangle = -\int_\Omega \bar{v}^r_{ttt},_s \left[\left(\frac{[\bar{a}^s_r\bar{a}^k_i - \bar{a}^s_i\bar{a}^k_r]\bar{a}^j_i}{\bar{J}} - [\delta^s_r\delta^k_i - \delta^s_i\delta^k_r]\delta^j_i\right)\frac{\rho_0,_k}{\bar{J}^2}\bar{X}_{ttt}\right],_j .$$

Given that $\|a(t) - \mathrm{Id}\,\|_3 = \|\int_0^t a_t(t')\mathrm{d}t'\|_3 \leqq \sqrt{t}\, P(\|\bar{v}\|_{\mathbf{Z}_T})$, we see that

$$\int_0^T \langle \mathfrak{t}_2, \bar{X}_{ttt}\rangle \mathrm{d}t \leqq T\, P(\|\bar{v}\|^2_{\mathbf{Z}_T}) + \delta \|\bar{X}\|^2_{\mathbf{X}_T}.$$

The duality pairing involving $\mathfrak{t}_3$ can be estimated in the same way.

Summing together the above inequalities and taking $T > 0$ sufficiently small concludes the proof. □

It is easy to see that we have the same estimates for the weak solutions $\bar{X}$, $\bar{X}_t$, and $\bar{X}_{tt}$ solving (8.5), (8.23), and (8.27), respectively:

$$\begin{aligned}\sum_{a=0}^{3} \left\|\partial_t^a \frac{\bar{X}_t}{\rho_0}\right\|^2_{L^2(0,T;H^{-1}(\Omega))} + \sup_{t\in[0,T]} \left\|\frac{\partial_t^a \bar{X}(t)}{\sqrt{\rho_0}}\right\|^2_0 + C_p \left\|\partial_t^a \bar{X}\right\|^2_{L^2(0,T;\dot{H}^1_0(\Omega))} \\ \leqq \mathcal{N}_0 + T P(\|\bar{X}\|^2_{\mathbf{X}_T}) + C\delta\|\bar{X}\|^2_{\mathbf{X}_T} + T\, P(\|\bar{v}\|^2_{\mathbf{Z}_T}) + P(\|\operatorname{curl}\bar{v}\|^2_{\mathbf{Y}_T}).\end{aligned} \tag{8.68}$$

Step 2 Returning to the definition of $\mathcal{G}_2$ in (8.29), by using the estimate (8.68) together with the Hardy inequality, we see that

$$\begin{aligned}\|\bar{G}_{tt} + \mathcal{G}_2\|^2_{L^2(0,T;L^2(\Omega))} \leqq \mathcal{N}_0 + T P(\|\bar{X}\|^2_{\mathbf{X}_T}) + C\delta\|\bar{X}\|^2_{\mathbf{X}_T} \\ + T\, P(\|\bar{v}\|^2_{\mathbf{Z}_T}) + P(\|\operatorname{curl}\bar{v}\|^2_{\mathbf{Y}_T}).\end{aligned}$$

Combining this with the estimate (8.65), the equation (8.27a) shows that

$$\begin{aligned}4\kappa^2 \left\|\left[\frac{\bar{B}^{jk}}{\rho_0}(\rho_0\bar{X}_{tt}),_k\right],_j\right\|^2_{L^2(0,T;L^2(\Omega))} = \left\|-\frac{\bar{J}^3\bar{X}_{ttt}}{\rho_0} + \mathcal{G}_2\right\|^2_{L^2(0,T;L^2(\Omega))} \\ \leqq \mathcal{N}_0 + T P(\|\bar{X}\|^2_{\mathbf{X}_T}) + C\delta\|\bar{X}\|^2_{\mathbf{X}_T} + T\, P(\|\bar{v}\|^2_{\mathbf{Z}_T}) + P(\|\operatorname{curl}\bar{v}\|^2_{\mathbf{Y}_T}).\end{aligned}$$

By repeating our argument of Section 8.4.3 but this time using Sobolev-type estimates for the horizontal-derivative estimates (replacing the difference quotient estimates as we already have regularity), we obtain the desired bound:

$$\begin{aligned}\|\bar{X}_{tt}\|^2_{L^2(0,T;L^2(\Omega))} \leqq \mathcal{N}_0 + T P(\|\bar{X}\|^2_{\mathbf{X}_T}) + C\delta\|\bar{X}\|^2_{\mathbf{X}_T} \\ + T\, P(\|\bar{v}\|^2_{\mathbf{Z}_T}) + P(\|\operatorname{curl}\bar{v}\|^2_{\mathbf{Y}_T}).\end{aligned} \tag{8.69}$$

Step 3 From the definition of $\mathcal{G}_1$ in (8.25), we similarly see that

$$\|\bar{G}_t + \mathcal{G}_1\|^2_{L^2(0,T;H^1(\Omega))} \leqq \mathcal{N}_0 + T P(\|\bar{X}\|^2_{\boldsymbol{X}_T}) + C\delta\|\bar{X}\|^2_{\boldsymbol{X}_T} + T\ P(\|\bar{v}\|^2_{\boldsymbol{Z}_T}) + P(\|\operatorname{curl} \bar{v}\|^2_{\boldsymbol{Y}_T}).$$

Following our argument for the regularity of $\bar{X}_{tt}$, we obtain the estimate

$$\|\bar{X}_t\|^2_{L^2(0,T;H^3(\Omega))} \leqq \mathcal{N}_0 + T P(\|\bar{X}\|^2_{\boldsymbol{X}_T}) + C\delta\|\bar{X}\|^2_{\boldsymbol{X}_T} + T\ P(\|\bar{v}\|^2_{\boldsymbol{Z}_T}) + P(\|\operatorname{curl} \bar{v}\|^2_{\boldsymbol{Y}_T}). \tag{8.70}$$

Step 4 Finally,

$$\|\bar{G}\|^2_{L^2(0,T;H^2(\Omega))} \leqq \mathcal{N}_0 + T P(\|\bar{X}\|^2_{\boldsymbol{X}_T}) + C\delta\|\bar{X}\|^2_{\boldsymbol{X}_T} + T\ P(\|\bar{v}\|^2_{\boldsymbol{Z}_T}) + P(\|\operatorname{curl} \bar{v}\|^2_{\boldsymbol{Y}_T}).$$

Following the argument of Step 3 and using Sobolev-type estimates for the horizontal-derivative bounds (which replace the horizontal difference-quotient estimates), we finally conclude that

$$\|\bar{X}\|^2_{L^2(0,T;H^4(\Omega))} \leqq \mathcal{N}_0 + T P(\|\bar{X}\|^2_{\boldsymbol{X}_T}) + C\delta\|\bar{X}\|^2_{\boldsymbol{X}_T} + T\ P(\|\bar{v}\|^2_{\boldsymbol{Z}_T}) + P(\|\operatorname{curl} \bar{v}\|^2_{\boldsymbol{Y}_T}). \tag{8.71}$$

8.4.7. The Proof of Proposition 2. Summing the inequalities (8.65), (8.69), (8.70), and (8.71), we obtain the estimate

$$\|\bar{X}\|^2_{\boldsymbol{X}_T} \leqq \mathcal{N}_0 + T P(\|\bar{X}\|^2_{\boldsymbol{X}_T}) + C\delta\|\bar{X}\|^2_{\boldsymbol{X}_T} + T\ P(\|\bar{v}\|^2_{\boldsymbol{Z}_T}) + P(\|\operatorname{curl} \bar{v}\|^2_{\boldsymbol{Y}_T}).$$

Choosing $\delta > 0$ and $T > 0$ sufficiently small (and readjusting the constants), we see that

$$\|\bar{X}\|^2_{\boldsymbol{X}_T} \leqq \mathcal{N}_0 + T\ P(\|\bar{v}\|^2_{\boldsymbol{Z}_T}) + P(\|\operatorname{curl} \bar{v}\|^2_{\boldsymbol{Y}_T}).$$

As the right-hand side does not depend on $\nu > 0$, we can pass to the limit as $\nu \to 0$ in (8.5). This completes the proof of Proposition 2.

Remark 6. Suppose that $\bar{v}_1$ and $\bar{v}_2$ are both elements of $\mathcal{C}_T(M)$. For $a = 1, 2$, let $\bar{X}_a$ denote the solution of (8.5) with coefficient matrix $\bar{B}_a$ and forcing function $\bar{G}_a$ formed from $\bar{v}_a$, rather than $\bar{v}$. Our proof of Proposition 2 then shows that

$$\|\bar{X}_1 - \bar{X}_2\|^2_{\boldsymbol{X}_T} \leqq T P(\|\bar{v}_1 - \bar{v}_2\|^2_{\boldsymbol{Z}_T}) + P(\|\operatorname{curl} \bar{v}_1 - \operatorname{curl} \bar{v}_2\|^2_{\boldsymbol{Y}_T}). \tag{8.72}$$

We will make use of this inequality in our iteration scheme below.

8.5. Existence of the Fixed-Point and the Proof of Theorem 3

The purpose of this section is to construct smooth unique solutions to (8.7), and to show that the map $\bar{v} \mapsto v$ has a unique fixed-point. This fixed-point is a solution to our approximate κ-problem (7.2).

8.5.1. The Boundary Convolution Operator Λ_ϵ on Γ. For $\epsilon > 0$, let $0 \leqq \rho_\epsilon \in C_0^\infty(\mathbb{R}^2)$ with $\text{spt}(\rho_\epsilon) \subset \overline{B(0,\epsilon)}$ denote a standard family of mollifiers on $\mathbb{R}^2$. With $x_h = (x_1, x_2)$, we define the operation of *convolution on the boundary* as follows:

$$\Lambda_\epsilon f(x_h) = \int_{\mathbb{R}^2} \rho_\epsilon(x_h - y_h) f(y_h) \mathrm{d}x_h \;\text{ for } f \in L^1_{loc}(\mathbb{R}^2).$$

By standard properties of convolution, there exists a constant C which is independent of ϵ, such that for $s \geqq 0$,

$$|\Lambda_\epsilon F|_s \leqq C|F|_s \quad \forall\, F \in H^s(\Gamma).$$

Furthermore,

$$\epsilon|\bar\partial \Lambda_\epsilon F|_0 \leqq C|F|_0 \quad \forall\, F \in L^2(\Omega). \tag{8.73}$$

8.5.2. Solutions to (8.7) Via Intermediate ϵ-Regularization We shall establish the existence of a solution v to (8.7) by first considering, for any $\epsilon > 0$, the ϵ-regularized system, where the higher-in-space order term in (8.7d) is smoothed via two boundary convolution operators on Γ:

$$\operatorname{div} v^\epsilon_t = \operatorname{div} \bar v_t - \operatorname{div}_{\bar\eta} \bar v_t + \frac{[\bar X \bar J^2]_t}{\rho_0} - \partial_t \bar A^j_i \bar v^i,_j \quad \text{in } \Omega, \tag{8.74a}$$

$$\operatorname{curl} v^\epsilon_t = \operatorname{curl} \bar v_t - \operatorname{curl}_{\bar\eta} \bar v_t + 2\kappa \varepsilon_{\cdot ji} \bar v,^r_s \bar A^s_i \,\bar\Xi,^j_r(\bar\eta) + \bar{\mathfrak{C}} \quad \text{in } \Omega, \tag{8.74b}$$

$$\begin{aligned} &(v^\epsilon)^3_t + 2\kappa\rho_0,_3 \Lambda^2_\epsilon \operatorname{div}_\Gamma v^\epsilon \\ &= 2\kappa\rho_0,_3 \Lambda_\epsilon \operatorname{div}_\Gamma \bar v - 2\rho_0,_3 \Lambda_\epsilon[\bar J^{-2}\bar a^3_3] - 2\kappa\rho_0,_3 \Lambda_\epsilon[\bar J^{-2}\partial_t \bar a^3_3] \\ &\quad -2\kappa\rho_0,_3 \Lambda_\epsilon[\bar a^3_3 \partial_t \bar J^{-2}] + \bar c_\epsilon(t) N^3 \qquad \text{on } \Gamma, \end{aligned} \tag{8.74c}$$

$$\int_\Omega (v^\epsilon)^\alpha_t \mathrm{d}x = -2\int_\Omega \bar A^k_\alpha(\rho_0 \bar J^{-1}),_k \mathrm{d}x - 2\kappa \int_\Omega \partial_t[\bar A^k_\alpha(\rho_0\bar J^{-1}),_k]\mathrm{d}x, \tag{8.74d}$$

$$(x_1, x_2) \mapsto v^\epsilon_t(x_1, x_2, x_3, t) \text{ is 1-periodic}, \tag{8.74e}$$

where the vector $\bar\Xi(\eta)$ is defined in (8.10) and the function $\bar c_\epsilon(t)$ (a constant in x) on the right-hand side of (8.74c) is defined by

$$\begin{aligned} \bar c_\epsilon(t) &= \frac12 \int_\Omega (\operatorname{div} \bar v_t - \operatorname{div}_{\bar\eta} \bar v_t)\mathrm{d}x + \frac12 \int_\Omega \frac{[\bar X\bar J^2]_t}{\rho_0}\mathrm{d}x - \frac12 \int_\Omega \partial_t \bar A^j_i \bar v^i,_j \mathrm{d}x \\ &\quad + \int_\Gamma \Lambda_\epsilon[\bar J^{-2}\bar a^3_3]\rho_0,_3 N^3 \mathrm{d}S + \kappa \int_\Gamma \Lambda_\epsilon[\bar J^{-2}\partial_t \bar a^3_3]\rho_0,_3 N^3 \mathrm{d}S \\ &\quad + \kappa \int_\Gamma \Lambda_\epsilon[\partial_t \bar J^{-2}\bar a^3_3]\rho_0,_3 N^3\mathrm{d}S + \kappa \int_\Gamma \operatorname{div}_\Gamma(\Lambda^2_\epsilon v^\epsilon - \Lambda_\epsilon \bar v)\rho_0,_3 N^3 \mathrm{d}S. \end{aligned} \tag{8.75}$$

We now outline the steps remaining in this section. We shall first prove, by a fixed-point approach, that for a small time $T_\epsilon > 0$ depending a priori on ϵ, we have the existence of a solution to this problem. We shall then prove, via ϵ-independent energy estimates on the solutions of (8.74), that $T_\epsilon = T$, with T independent of ϵ, and that the sequence v^ϵ converges in an appropriate space to a solution v of

(8.7), which also satisfies the same energy estimates. These estimates will allow us to conclude the existence of a fixed-point $v = \bar{v}$.

Step 1: Solutions to (8.74) via the contraction mapping principle. For

$$w \in \mathcal{X}_T^3 = \{w \in L^2(0,T;H^3(\Omega)) \ : \ \partial_t^s w \in L^2(0,T;H^{4-s}(\Omega)), 1 \leqq s \leqq 3, \quad (x_1,x_2) \mapsto w \text{ is 1-periodic}\}, \tag{8.76}$$

with norm $\|w\|^2_{\mathcal{X}_T^3} = \|w\|^2_{L^2(0,T;H^3(\Omega))} + \sum_{s=1}^3 \|\partial_t^s w\|^2_{L^2(0,T;H^{4-s}(\Omega))}$, we set $\Phi(w) = u_0 + \int_0^t \partial_t \Phi(w)$, where $\partial_t \Phi(w)$ is defined by the elliptic system which specifies the divergence, curl, and normal trace of the vector field $\partial_t \Phi(w)$:

$$\operatorname{div} \partial_t \Phi(w) = \operatorname{div} \bar{v}_t - \operatorname{div}_{\bar{\eta}} \bar{v}_t + \frac{[\bar{X}\bar{J}^2]_t}{\rho_0} - \partial_t \bar{A}_i^j \bar{v},_j^i \quad \text{in } \Omega, \tag{8.77a}$$

$$\operatorname{curl} \partial_t \Phi(w) = \operatorname{curl} \partial_t \bar{v} - \operatorname{curl}_{\bar{\eta}} \partial_t \bar{v} + 2\kappa \varepsilon_{\cdot ji} \bar{v},_s^r \bar{A}_i^s \, \bar{\Xi},_r^j (\bar{\eta}) + \bar{\mathfrak{C}} \quad \text{in } \Omega, \tag{8.77b}$$

$$\begin{aligned}\partial_t \Phi(w) \cdot e_3 = & -2\kappa \rho_0,_3 \Lambda_\epsilon^2 \operatorname{div}_\Gamma w + 2\kappa \rho_0,_3 \Lambda_\epsilon \operatorname{div}_\Gamma \bar{v} \\ & -2\kappa \rho_0,_3 \Lambda_\epsilon [\bar{J}^{-2} \partial_t \bar{a}_3^3] - 2\rho_0,_3 \Lambda_\epsilon [\bar{J}^{-2} \bar{a}_3^3] \\ & -2\kappa \rho_0,_3 \Lambda_\epsilon [\bar{a}_3^3 \partial_t \bar{J}^{-2}] + \bar{c}(w) N^3 \qquad \text{on } \Gamma,\end{aligned} \tag{8.77c}$$

$$\int_\Omega \partial_t \Phi(w)^\alpha \mathrm{d}x = -2 \int_\Omega \bar{A}_\alpha^k (\frac{\rho_0}{\bar{J}}),_k \mathrm{d}x - 2\kappa \int_\Omega \partial_t [\bar{A}_\alpha^k (\frac{\rho_0}{\bar{J}}),_k] \mathrm{d}x, \tag{8.77d}$$

$$\forall t \in [0,T], \qquad \partial_t \Phi(w)(t) \text{ is 1-periodic in the directions } e_1 \text{ and } e_2. \tag{8.77e}$$

The function $\bar{c}(w)(t)$ in (8.77c) is defined by

$$\begin{aligned}[\bar{c}(w)](t) = & \frac{1}{2} \int_\Omega (\operatorname{div} \bar{v}_t - \operatorname{div}_{\bar{\eta}} \bar{v}_t) \mathrm{d}x + \frac{1}{2} \int_\Omega \frac{[\bar{X}\bar{J}^2]_t}{\rho_0} \mathrm{d}x - \frac{1}{2} \int_\Omega \partial_t \bar{A}_i^j \bar{v}^i,_j \mathrm{d}x \\ & + \int_\Gamma \Lambda_\epsilon [\bar{J}^{-2} \bar{a}_3^3] \rho_0,_3 N^3 \mathrm{d}S + \kappa \int_\Gamma \Lambda_\epsilon [\bar{J}^{-2} \partial_t \bar{a}_3^3] \rho_0,_3 N^3 \mathrm{d}S \\ & + \kappa \int_\Gamma \Lambda_\epsilon [\partial_t \bar{J}^{-2} \bar{a}_3^3] \rho_0,_3 N^3 \mathrm{d}S \\ & + \kappa \int_\Gamma \operatorname{div}_\Gamma (\Lambda_\epsilon^2 w - \Lambda_\epsilon \bar{v}) \rho_0,_3 N^3 \mathrm{d}S,\end{aligned} \tag{8.78}$$

and is introduced so that the elliptic system (8.77) satisfies all of the solvability conditions. Thus, due to the definition (8.78) , the problem (8.77) defining $\partial_t \Phi(w)$ is perfectly well-posed. Applying Proposition 1 to (8.77) and its first, second, and third time-differentiated versions, we find that

$$\|\partial_t \Phi(w) - \partial_t \Phi(\tilde{w})\|_{\mathcal{X}^3_{T_\epsilon}} \leqq C(M,\epsilon) \|w - \tilde{w}\|_{\mathcal{X}^3_{T_\epsilon}} + C_M T_\epsilon \|w - \tilde{w}\|_{\mathcal{X}^3_{T_\epsilon}}, \tag{8.79}$$

the ϵ dependence in the constant $C(M,\epsilon)$ coming from repeated use of (8.73). Note that the lack of w on the right-hand sides of (8.77a) and (8.77b) implies that both the divergence and curl of $\partial_t \Phi(w) - \partial_t \Phi(\tilde{w})$ vanish, and that on Γ,

$$[\partial_t \Phi(w) - \partial_t \Phi(\tilde{w})] \cdot e_3 = 2\kappa \rho_0,_3 \Lambda_\epsilon^2 \operatorname{div}_\Gamma (\tilde{w} - w) + [\bar{c}(w) - \bar{c}(\tilde{w})] N^3.$$

It follows from (8.79) that

$$\|\Phi(w) - \Phi(\tilde{w})\|_{\mathcal{X}^3_{T_\epsilon}} \leqq T_\epsilon C(M,\epsilon)\|w - \tilde{w}\|_{\mathcal{X}^3_{T_\epsilon}},$$

and therefore the mapping $\Phi : \mathcal{X}^3_{T_\epsilon} \to \mathcal{X}^3_{T_\epsilon}$ is a contraction if T_ϵ is taken sufficiently small, leading to the existence and uniqueness of a fixed-point $v^\epsilon = \Phi(v^\epsilon)$, which is therefore a solution of (8.74) on $[0, T_\epsilon]$.

Step 2: ϵ-independent energy estimates for v^ϵ. Having obtained a unique solution to (8.74), we now proceed with $\epsilon-$independent estimates on this system. We integrate the divergence (8.74a) and curl (8.74b) relations in time, and we now view the PDE for the normal trace (8.74c) as a parabolic equation for v^ϵ on Γ:

$$\operatorname{div} v^\epsilon = \operatorname{div} \bar{v} - \operatorname{div}_{\bar{\eta}} \bar{v} + \frac{\bar{X}\bar{J}^2}{\rho_0} \quad \text{in } \Omega\,, \tag{8.80a}$$

$$\begin{aligned}\operatorname{curl} v^\epsilon &= \operatorname{curl} u_0 + \operatorname{curl} \bar{v} - \operatorname{curl}_{\bar{\eta}} \bar{v} + 2\kappa \int_0^t \varepsilon_{\cdot ji}\, \bar{v},^r_s\, \bar{A}^s_i\, \bar{\Xi},^j_r(\bar{\eta}) \\ &\quad + \int_0^t (\varepsilon_{\cdot ji}\, \bar{v},^i_s\, \partial_t \bar{A}^s_j + \bar{\mathfrak{C}}) \ \text{ in } \Omega,\end{aligned} \tag{8.80b}$$

$$\begin{aligned}&(v^\epsilon)^3_t + 2\kappa\rho_0,_3\, \Lambda^2_\epsilon \operatorname{div}_\Gamma v^\epsilon \\ &\quad = +2\kappa\rho_0,_3\, \Lambda_\epsilon \operatorname{div}_\Gamma \bar{v} - 2\rho_0,_3\, \Lambda_\epsilon[\bar{J}^{-2}\bar{a}^3_3] - 2\kappa\rho_0,_3\, \Lambda_\epsilon[\bar{J}^{-2}\partial_t\bar{a}^3_3] \\ &\qquad -2\kappa\rho_0,_3\, \Lambda_\epsilon[\bar{a}^3_3\partial_t\bar{J}^{-2}] + \bar{c}_\epsilon(t)N^3 \quad \text{on } \Gamma,\end{aligned} \tag{8.80c}$$

$$v^\epsilon(0) = u_0, \tag{8.80d}$$

$$\int_\Omega (v^\epsilon)^\alpha \mathrm{d}x = \int_\Omega u^\alpha_0 \mathrm{d}x - 2\int_0^t \int_\Omega \bar{A}^k_\alpha (\frac{\rho_0}{\bar{J}}),_k \mathrm{d}x - 2\kappa \int_\Omega [\bar{A}^k_\alpha (\frac{\rho_0}{\bar{J}}),_k]\mathrm{d}x, \tag{8.80e}$$

$$\forall t \in [0,T], \qquad v^\epsilon(t) \text{ is 1-periodic in the directions } e_1 \text{ and } e_2, \tag{8.80f}$$

where $\bar{c}^\epsilon(t)$ is defined in (8.75).

We will establish the existence of a fixed-point in $\mathcal{C}_{T_\kappa}(M)$ defined in (8.2), but to do so we will first make use of the space (depending on ϵ)

$$\begin{aligned}\mathcal{X}^4_T = \{&w \in L^\infty(0,T; H^{\frac{7}{2}}(\Omega)) \cap L^2(0,T; \dot{H}^1_0(\Omega)) : \partial^s_t w \in L^2(0,T; H^{4-s}(\Omega)) \\ &1 \leqq s \leqq 3,\ \Lambda_\epsilon w \in L^2(0,T; H^4(\Omega)),\ w(0) = u_0\},\end{aligned}$$

with norm

$$\|w\|^2_{\mathcal{X}^4_T} = \|\Lambda_\epsilon w\|^2_{L^2(0,T;H^4(\Omega))} + \sum_{s=1}^3 \|\partial^s_t w\|^2_{L^2(0,T;H^{4-s}(\Omega))} + \sup_{[0,T]} \|w\|^2_{3.5}.$$

Since $\bar{v} \in \mathcal{C}_{T_\kappa}(M)$, equations (8.80a) and (8.80b) show that both $\operatorname{div} v^\epsilon$ and $\operatorname{curl} v^\epsilon$ are in $L^2(0,T_\epsilon; H^3(\Omega))$; additionally, from (8.80c) and (8.73), we see that $(v^\epsilon)^3$ is in $L^\infty(0,T_\epsilon; H^{3.5}(\Gamma))$, and hence according to Proposition 1, $v^\epsilon \in L^2(0,T_\epsilon; H^4(\Omega))$, with a bound that a priori depends on ϵ. We next show that, in fact, we can control $\Lambda_\epsilon v^\epsilon$ in $\boldsymbol{Z}_T$ independently of ϵ, on a time interval $[0,T]$ with $T > 0$ independent of ϵ.

We proceed by letting $\bar\partial^3$ act on each side of (8.80c), multiplying this equation by $-\frac{N^3}{\rho_{0,3}}\bar\partial^3(v^\epsilon)^3$, and then integrating over Γ. This yields the following identity:

$$
\begin{aligned}
-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \frac{N^3}{\rho_{0,3}}|\bar\partial^3(v^\epsilon)^3|^2\mathrm{d}S + 2\kappa\int_\Gamma \bar\partial^3\Lambda_\epsilon^2\,\mathrm{div}_\Gamma\, v^\epsilon\,\bar\partial^3(v^\epsilon)^3 N^3\mathrm{d}S \\
= -\int_\Gamma G\,\bar\partial^3(v^\epsilon)^3\frac{N^3}{\rho_{0,3}}\mathrm{d}S - \int_\Gamma \bar\partial^3\big[\rho_{0,3}\,\Lambda_\epsilon F\big]\bar\partial^3(v^\epsilon)^3\frac{N^3}{\rho_{0,3}}\mathrm{d}S, \qquad (8.81)
\end{aligned}
$$

where

$$
G = -2\kappa\Big[\bar\partial^3\rho_{0,3}\;\Lambda_\epsilon^2\,\mathrm{div}_\Gamma\, v^\epsilon + 3\bar\partial^2\rho_{0,3}\;\bar\partial\Lambda_\epsilon^2\,\mathrm{div}_\Gamma\, v^\epsilon + 3\bar\partial\rho_{0,3}\;\bar\partial^2\Lambda_\epsilon^2\,\mathrm{div}_\Gamma\, v^\epsilon\Big], \qquad (8.82)
$$

$$
F = 2\kappa\,\mathrm{div}_\Gamma\,\bar v - 2\bar J^{-2}\bar a_3^3 - 2\kappa\,\bar J^{-2}\partial_t\bar a_3^3 - 2\kappa\bar a_3^3\partial_t\bar J^{-2}. \qquad (8.83)
$$

Since G contains lower-order terms, we see that for any $t\in[0,T_\epsilon]$:

$$
-\int_\Gamma G\bar\partial^3(v^\epsilon)^3\frac{N^3}{\rho_{0,3}}\,\mathrm{d}S \leqq C|\bar\partial^3(v^\epsilon)|_0^2. \qquad (8.84)
$$

We then write

$$
\bar\partial^3\big[\rho_{0,3}\,\Lambda_\epsilon F\big] = \rho_{0,3}\,\bar\partial^3\Lambda_\epsilon F + \bar\partial^3\rho_{0,3}\,\Lambda_\epsilon F + 3\bar\partial^2\rho_{0,3}\,\bar\partial\Lambda_\epsilon F + 3\bar\partial\rho_{0,3}\,\bar\partial^2\Lambda_\epsilon F, \qquad (8.85)
$$

and notice that since the last three terms on the right-hand side are lower-order, we easily obtain the estimate

$$
\begin{aligned}
\Big|\int_\Gamma\Big[(\bar\partial^3\rho_{0,3}\,\Lambda_\epsilon F + 3\bar\partial^2\rho_{0,3}\,\bar\partial\Lambda_\epsilon F + 3\bar\partial\rho_{0,3}\,\bar\partial^2\Lambda_\epsilon F)\Big]\bar\partial^3(v^\epsilon)^3\frac{N^3}{\rho_{0,3}}\mathrm{d}S\Big| \\
\leqq C|\bar\partial^3(v^\epsilon)^3|_0|\bar\partial^2 D\bar v|_0. \qquad (8.86)
\end{aligned}
$$

Next, to estimate the highest-order term $\int_\Gamma \bar\partial^3\Lambda_\epsilon F\,\bar\partial^3(v^\epsilon)^3N^3\mathrm{d}S$, we notice that by the standard properties of the boundary convolution operator Λ_ϵ, we have that

$$
\begin{aligned}
&\int_\Gamma \bar\partial^3\Lambda_\epsilon F\,\bar\partial^3(v^\epsilon)^3N^3\mathrm{d}S = \int_\Gamma \bar\partial^3 F\,\bar\partial^3\Lambda_\epsilon(v^\epsilon)^3N^3\mathrm{d}S \\
&= \underbrace{-2\kappa\int_\Gamma \bar\partial^3\Big(\frac{\bar\partial_t a_3^3}{\bar J^2}\Big)\,\bar\partial^3\Lambda_\epsilon(v^\epsilon)^3N^3\mathrm{d}S}_{\mathcal{J}_1} \underbrace{-2\kappa\int_\Gamma \bar\partial^3(\bar a_3^3\partial_t\bar J^{-2})\,\bar\partial^3\Lambda_\epsilon(v^\epsilon)^3N^3\mathrm{d}S}_{\mathcal{J}_2} \\
&\quad \underbrace{+2\kappa\int_\Gamma \bar\partial^3\,\mathrm{div}_\Gamma\,\bar v\,\bar\partial^3\Lambda_\epsilon(v^\epsilon)^3N^3\mathrm{d}S}_{\mathcal{J}_3} \underbrace{-2\int_\Gamma \bar\partial^3\big[\bar J^{-2}\bar a_3^3\big]\,\bar\partial^3\Lambda_\epsilon(v^\epsilon)^3N^3\mathrm{d}S}_{\mathcal{J}_4}.
\end{aligned}
\qquad (8.87)
$$

In order to estimate the integral $\mathcal{J}_1$, we recall the formula for $\partial_t \bar{a}_3^3$ given in (8.8), and write

$$\begin{aligned}\mathcal{J}_1 &= -2\kappa \int_\Gamma \bar\partial^3(\bar{J}^{-2}[\bar v,_1 \times \bar\eta,_2 + \bar\eta,_1 \times \bar v,_2]^3)\, \bar\partial^3 \Lambda_\epsilon (v^\epsilon)^3 N^3 \mathrm{d}S \\ &= \underbrace{-2\kappa \int_\Gamma \big[\bar\partial^3 \bar v,_1 \times \big(\frac{\bar\eta,_2}{\bar J^2} - \bar\eta,_2(0)\big)\big]^3 \bar\partial^3 \Lambda_\epsilon (v^\epsilon)^3 N^3 \mathrm{d}S}_{\mathcal{J}_{1a}} \\ &\quad \underbrace{- 2\kappa \int_\Gamma \bar\partial^3 \bar v^1,_1 \; \bar\partial^3 \Lambda_\epsilon (v^\epsilon)^3 N^3 \mathrm{d}S}_{\mathcal{J}_{1b}} \underbrace{- 2\kappa \int_\Gamma \bar\partial^3 \bar v^2,_2 \; \bar\partial^3 \Lambda_\epsilon (v^\epsilon)^3 N^3 \mathrm{d}S}_{\mathcal{J}_{1d}} \\ &\quad \underbrace{-2\kappa \int_\Gamma \big[\big(\frac{\bar\eta,_1}{\bar J^2} - \bar\eta,_1(0)\big) \times \bar\partial^3 \bar v,_2\big]^3 \bar\partial^3 \Lambda_\epsilon (v^\epsilon)^3 N^3 \mathrm{d}S}_{\mathcal{J}_{1c}} + \mathcal{R}_1, \end{aligned} \tag{8.88}$$

where $\mathcal{R}_1$ is a lower-order integral over Γ that contains all of the remaining terms from the action of $\bar\partial^3$, so that there are at most three space derivatives on $\bar v$ on Γ. The trace theorem combined with the Cauchy–Schwarz inequality easily show that $|\mathcal{R}_1| \leqq C|(v^\epsilon)^3|_3 \|\bar v\|_4$. The next crucial observation is that

$$\mathcal{J}_{1b} + \mathcal{J}_{1d} + \mathcal{J}_3 = 0. \tag{8.89}$$

We use the fundamental theorem of calculus,

$$\frac{\bar\eta,_2}{\bar J^2}(t) - \bar\eta,_2(0) = \int_0^t \partial_t \frac{\bar\eta,_2}{\bar J^2} \quad \text{and} \quad \frac{\bar\eta,_1}{\bar J^2}(t) - \bar\eta,_1(0) = \int_0^t \partial_t \frac{\bar\eta,_1}{\bar J^2},$$

to estimate the integrals $\mathcal{J}_{1a}$ and $\mathcal{J}_{1c}$ so that (8.88) and (8.89) show that

$$|\mathcal{J}_1 + \mathcal{J}_3| \leqq Ct\|\Lambda_\epsilon (v^\epsilon)^3\|_4 \|\bar v\|_4 + C|(v^\epsilon)^3|_3 \|\bar v\|_4.$$

Next, we write the integral $\mathcal{J}_2$ as

$$\mathcal{J}_2 = 4\kappa \int_\Gamma \bar a_3^3 \bar J^{-3} \bar a_r^s \bar\partial^3 \bar v^r,_s \; \bar\partial^3 \Lambda_\epsilon (v^\epsilon)^3 N^3 \,\mathrm{d}S + \mathcal{R}_2,$$

with $\mathcal{R}_2 \sim \int_\Gamma \bar\partial^3 D\bar\eta \, \bar\partial^3 \Lambda_\epsilon (v^\epsilon)^3 N^3 \mathrm{d}S$, where the symbol $\sim$ is used here to mean that $\mathcal{R}_2$ is comprised of integrands which have the derivative count of the integrand $\bar\partial^3 D\bar\eta \, \bar\partial^3 \Lambda_\epsilon (v^\epsilon)^3$. It follows that

$$|\mathcal{R}_2| \leqq |D\bar\eta|_{2.5} |\Lambda_\epsilon (v^\epsilon)^3|_{3.5} \leqq C\|\bar\eta\|_4 \|\Lambda_\epsilon v^\epsilon\|_4,$$

the last inequality following from the trace theorem. Since $\bar\eta(t) = e + \int_0^t \bar v$, we see that for some $\delta > 0$,

$$|\mathcal{R}_2| \leqq \mathcal{N}_0 + (\delta + Ct^2)\|\Lambda_\epsilon v^\epsilon\|_4^2 + Ct^2 \|\bar v\|_4^2.$$

Returning to the remaining term in $\mathcal{J}_2$, we write

$$4\kappa \int_\Gamma \bar{a}_3^3 \bar{J}^{-3} \bar{a}_r^s \bar\partial^3 \bar{v}^r,_s \; \bar\partial^3 \Lambda_\epsilon (v^\epsilon)^3 N^3 \, \mathrm{d}S = \underbrace{4\kappa \int_\Gamma \bar\partial^3 \operatorname{div} \bar{v} \, \bar\partial^3 \Lambda_\epsilon (v^\epsilon)^3 N^3 \, \mathrm{d}S}_{\mathcal{J}_{2a}}$$
$$+ \underbrace{4\kappa \int_\Gamma (\bar{a}_3^3 \bar{J}^{-3} \bar{a}_r^s - \delta_r^s) \bar\partial^3 \bar{v}^r,_s \; \bar\partial^3 \Lambda_\epsilon (v^\epsilon)^3 N^3 \, \mathrm{d}S}_{\mathcal{J}_{2b}} .$$

We estimate the integral $\mathcal{J}_{2a}$ with the Cauchy–Schwarz inequality. The integral $\mathcal{J}_{2b}$ can be estimated using the fundamental theorem of calculus:

$$|\mathcal{J}_{2b}| \leqq Ct \|\Lambda_\epsilon (v^\epsilon)^3\|_4 \|\bar{v}\|_4,$$

so that we have established the following estimate:

$$|\mathcal{J}_2| + |\mathcal{J}_1 + \mathcal{J}_3| \leqq \mathcal{N}_0 + TC_M + (\delta + Ct^2)\|\Lambda_\epsilon v^\epsilon\|_4^2 + Ct^2 \|\bar{v}\|_4^2$$
$$+ Ct \|\Lambda_\epsilon (v^\epsilon)^3\|_4 \|\bar{v}\|_4 + C|(v^\epsilon)^3|_3 \|\bar{v}\|_4 + C\|\bar{X}\|_4^2 + C\| \operatorname{div} \bar{v}\|_3 \|\Lambda_\epsilon v^\epsilon\|_4,$$

where the bound on the integral $\mathcal{J}_{2a}$ in contained in the right-hand side. Finally, the integral $\mathcal{J}_4$ can be estimated in the same way as $\mathcal{R}_2$ above, so that with the identities (8.87), (8.86), and (8.85) we have shown that

$$\left| \int_\Gamma \bar\partial^3 \;\; (\rho_0,_3 \Lambda_\epsilon F) \bar\partial^3 (v^\epsilon)^3 \frac{N^3}{\rho_0,_3} \mathrm{d}S \right|$$
$$\leqq \mathcal{N}_0 + TC_M + (\delta + Ct^2)\|\Lambda_\epsilon v^\epsilon\|_4^2 + Ct^2 \|\bar{v}\|_4^2 + Ct \|\Lambda_\epsilon (v^\epsilon)^3\|_4 \|\bar{v}\|_4$$
$$+ C|(v^\epsilon)^3|_3 \|\bar{v}\|_4 + C\|\bar{X}\|_4^2 + C\| \operatorname{div} \bar{v}\|_3 \|\Lambda_\epsilon v^\epsilon\|_4. \tag{8.90}$$

We now turn our attention to the second term on the left-hand side of (8.81), which will give us as a sign-definite energy term plus a small perturbation. We first see that by the properties of the boundary convolution Λ_ϵ,

$$\int_\Gamma \bar\partial^3 \left[\Lambda_\epsilon^2 (v^\epsilon,_1^1 + v^\epsilon,_2^2)\right] \bar\partial^3 (v^\epsilon)^3 N^3 \mathrm{d}S = \underbrace{\int_\Gamma \bar\partial^3 \Lambda_\epsilon (v^\epsilon,_1^1 + v^\epsilon,_2^2) \, \bar\partial^3 \Lambda_\epsilon v^\epsilon \cdot N \mathrm{d}S}_{\mathcal{I}} . \tag{8.91}$$

The divergence theorem applied to the integral $\mathcal{I}$ (as our domain $\Omega = \mathbb{T}^2 \times (0, 1)$) implies that

$$\mathcal{I} = \underbrace{\int_\Omega \bar\partial^3 \Lambda_\epsilon (v^\epsilon,_1^1 + v^\epsilon,_2^2) \, \bar\partial^3 \Lambda_\epsilon (v^\epsilon)^3,_3 \, \mathrm{d}x}_{\mathcal{I}_1}$$
$$+ \underbrace{\int_\Omega \bar\partial^3 \left[\Lambda_\epsilon (v^\epsilon,_{13}^1 + v^\epsilon,_{23}^2)\right] \bar\partial^3 \Lambda_\epsilon (v^\epsilon)^3 \mathrm{d}x}_{\mathcal{I}_2} .$$

Now, $\mathcal{I}_1 = -\int_\Omega \left|\bar\partial^3 \Lambda_\epsilon (v^\epsilon)^3,_3\right|^2 \mathrm{d}x + \int_\Omega \Lambda_\epsilon \bar\partial^3 \operatorname{div} v^\epsilon \, \bar\partial^3 \Lambda_\epsilon v^\epsilon,_3^3 \, \mathrm{d}x$, and

$\mathcal{I}_2 = -\int_\Omega \bar\partial^3 \Lambda_\epsilon (v^\epsilon)^1,_3 \; \bar\partial^3 \Lambda_\epsilon (v^\epsilon)^3,_1 \, \mathrm{d}x - \int_\Omega \bar\partial^3 \Lambda (v^\epsilon)^2,_3 \; \bar\partial^3 \Lambda_\epsilon (v^\epsilon)^3,_2 \, \mathrm{d}x,$

from which it follows that

$$\begin{aligned}\mathcal{I} = &-\int_\Omega \left|\bar\partial^3 \Lambda_\epsilon D(v^\epsilon)^3\right|^2 \mathrm{d}x + \int_\Omega \Lambda_\epsilon \bar\partial^3 \operatorname{div} v^\epsilon \, \bar\partial^3 \Lambda_\epsilon (v^\epsilon)^3,_3 \, \mathrm{d}x \\ &+ \int_\Omega \Lambda_\epsilon \bar\partial^3 [\operatorname{curl} v^\epsilon \cdot e_2] \bar\partial^3 \Lambda_\epsilon (v^\epsilon)^3,_1 \, \mathrm{d}x \\ &- \int_\Omega \Lambda_\epsilon \bar\partial^3 [\operatorname{curl} v^\epsilon \cdot e_1] \bar\partial^3 \Lambda_\epsilon (v^\epsilon)^3,_2 \, \mathrm{d}x. \end{aligned} \tag{8.92}$$

Now, thanks to (8.80a), we have for all $t \in [0, T_\epsilon]$

$$\begin{aligned}\|\operatorname{div} v^\epsilon\|_3^2 &\leqq Ct^2 \|\bar v\|_4^2 + C\|\bar v\|_3^2 + C\left\|\frac{\bar X}{\rho_0}\right\|_3^2 + Ct\|\bar v\|^2_{L^2(0,t;H^4(\Omega))} \\ &\leqq Ct^2\|\bar v\|_4^2 + C\|\bar v\|_3^2 + Ct\|\bar v\|^2_{L^2(0,t;H^4(\Omega))} + C\|\bar X\|_4^2, \end{aligned} \tag{8.93}$$

where we have used the higher-order Hardy inequality Lemma 1 for the second inequality.

Next, with (8.80b), we see that for all $t \in [0, T_\epsilon]$

$$\begin{aligned}\|\operatorname{curl} v^\epsilon\|_3 \leqq\; & Ct\|\bar v\|_4 + C\|\bar v\|_3 + C\|u_0\|_4 + C\sqrt{t}\|D(\bar\Xi(\bar\eta))\|_{L^2(0,t;H^3(\Omega))} \\ & + C\sqrt{t}\|\bar v\|_{L^2(0,t;H^4(\Omega))} \\ \leqq\; & Ct\|\bar v\|_4 + C_\kappa\|u_0\|_4 + C\sqrt{t}\|\bar v\|_{L^2(0,t;H^4(\Omega))}\,, \end{aligned} \tag{8.94}$$

where we have used (8.17) and the identity (8.14), relating $\Xi(\bar\eta)$ to $\bar v$ and where we have relied crucially on the chain-rule which shows that

$$\Xi^j,_r(\bar\eta) = \bar A^l,_r \, [\Xi(\bar\eta)]^j,_l \,.$$

Note that (8.14) provides us with a bound which is ϵ-independent, but which indeed depends on κ.

The action of the boundary convolution operator Λ_ϵ does not affect these estimates; thus, using Proposition 2, we see that

$$\int_0^T \|\operatorname{div} \Lambda_\epsilon v^\epsilon\|_3^2 \mathrm{d}t \leqq \mathcal{N}_0 + T\, P(\|\bar v\|^2_{Z_T}) + P(\|\operatorname{curl} \bar v\|^2_{Y_T}), \tag{8.95}$$

$$\int_0^T \|\operatorname{curl} \Lambda_\epsilon v^\epsilon\|_3^2 \mathrm{d}t \leqq \mathcal{N}_0 + T\, P(\|\bar v\|^2_{Z_T}). \tag{8.96}$$

We integrate the inequality (8.81) from 0 to t, and we use the estimates (8.90), (8.84), (8.92), (8.95), (8.96) together with the fact that $\frac{N_3}{\rho_{0,3}} \geqq C > 0$ on Γ, to obtain that for any $t \in [0, T]$,

$$
\begin{aligned}
&|\bar\partial^3 (v^\epsilon)^3(t)|_0^2 + \int_0^t \|\bar\partial^3 D\Lambda_\epsilon (v^\epsilon)^3\|_0^2 \\
&\qquad \leqq \mathcal{N}_0 + Ct|(v^\epsilon)^3|_3^2 + Ct\|\bar v\|_{\boldsymbol{Z}_T}^2 + Ct\int_0^t \|\Lambda_\epsilon (v^\epsilon)^3\|_4^2 \\
&\qquad\quad + C\sqrt{t}\int_0^t \|\bar v\|_4^2 + \frac{C}{\sqrt{t}}\int_0^t |(v^\epsilon)^3|_3^2 + \delta\|\Lambda_\epsilon v^\epsilon\|_4^2 \\
&\qquad\quad + T\,P(\|\bar v\|_{\boldsymbol{Z}_T}^2) + P(\|\operatorname{curl}\bar v\|_{\boldsymbol{Y}_T}^2) + C\|\operatorname{div}\bar v\|_{\boldsymbol{Y}_T}^2,
\end{aligned}
$$

where we have used the Cauchy-Young inequality $ab \leqq \frac{C}{\sqrt{t}}a^2 + \sqrt{t}b^2$ for $a, b \geqq 0$. By taking $\delta > 0$ sufficiently small, and using the relations (8.95) and (8.96), this implies the following inequality:

$$
\begin{aligned}
&|v^\epsilon(t)|_3^2 + \int_0^t \|\Lambda_\epsilon v^\epsilon\|_4^2 \\
&\quad \leqq \mathcal{N}_0 + Ct|(v^\epsilon)^3|_3^2 + Ct\|\bar v\|_{\boldsymbol{Z}_T}^2 + Ct\int_0^t \|\Lambda_\epsilon (v^\epsilon)^3\|_4^2 + C\sqrt{t}\int_0^t \|\bar v\|_4^2 \\
&\qquad + \frac{C}{\sqrt{t}}\int_0^t |(v^\epsilon)^3|_3^2 + T\,P(\|\bar v\|_{\boldsymbol{Z}_T}^2) + P(\|\operatorname{curl}\bar v\|_{\boldsymbol{Y}_T}^2) + C\|\operatorname{div}\bar v\|_{\boldsymbol{Y}_T}^2 \\
&\quad \leqq \mathcal{N}_0 + C(t+\sqrt{t})\sup_{t\in[0,T]}|v^\epsilon|_3^2 + C(t+\sqrt{t})\|\bar v\|_{\boldsymbol{Z}_T}^2 + Ct\int_0^t \|\Lambda_\epsilon v^\epsilon\|_4^2 \\
&\qquad + T\,P(\|\bar v\|_{\boldsymbol{Z}_T}^2) + P(\|\operatorname{curl}\bar v\|_{\boldsymbol{Y}_T}^2) + C\|\operatorname{div}\bar v\|_{\boldsymbol{Y}_T}^2.
\end{aligned}
$$

(Note that due to the presence of the convolution operators Λ_ϵ in the definition of $(v_t^\epsilon)^3$ on Γ in formula (8.80c), it is clear that $\sup_{[0,t]}|v^\epsilon|_3^2$ is bounded by some finite number, which a priori depends on ϵ.)

Since we are considering a bounded time interval, $0 \leqq t \leqq C\sqrt{t}$, and we will henceforth make use of this fact. The previous estimate then implies that

$$
\begin{aligned}
\sup_{t\in[0,T]}|v^\epsilon(t)|_3^2 + \int_0^T \|\Lambda_\epsilon v^\epsilon\|_4^2 \mathrm{d}t &\leqq \mathcal{N}_0 + \sqrt{T}P(\|v^\epsilon\|_{\mathcal{X}_T^4}^2) + \sqrt{T}\,P(\|\bar v\|_{\boldsymbol{Z}_T}^2) \\
&\quad + P(\|\operatorname{curl}\bar v\|_{\boldsymbol{Y}_T}^2) + C\|\operatorname{div}\bar v\|_{\boldsymbol{Y}_T}^2. \qquad (8.97)
\end{aligned}
$$

Estimates for the time-differentiated quantities are, in fact, very straightforward at this stage. By using the expression (8.80c), we see that thanks to estimate (8.97),

$$
\begin{aligned}
\int_0^T |(v_t^\epsilon)^3|_{2.5}^2 \mathrm{d}t &\leqq \mathcal{N}_0 + C\int_0^T |\Lambda_\epsilon v^\epsilon|_{3.5}^2 \mathrm{d}t + \sqrt{T}\,P(\|\bar v\|_{\boldsymbol{Z}_T}^2) \\
&\leqq \mathcal{N}_0 + \sqrt{T}P(\|v^\epsilon\|_{\mathcal{X}_T^4}^2) + \sqrt{T}\,P(\|\bar v\|_{\boldsymbol{Z}_T}^2) + P(\|\operatorname{curl}\bar v\|_{\boldsymbol{Y}_T}^2) \\
&\quad + C\|\operatorname{div}\bar v\|_{\boldsymbol{Y}_T}^2. \qquad (8.98)
\end{aligned}
$$

Now, just as we estimated the divergence and curl of v^ϵ in (8.93) and (8.94), we can repeat this procedure to estimate the divergence and curl of v_t^ϵ. By using (8.74a) and (8.74b), we also have that

$$\int_0^T \|\operatorname{div} v_t^\epsilon\|_2^2 dt \leqq \mathcal{N}_0 + \sqrt{T}\, P(\|\bar v\|_{\mathbf{Z}_T}^2) + P(\|\operatorname{curl} \bar v\|_{\mathbf{Y}_T}^2) + C\|\operatorname{div} \bar v\|_{\mathbf{Y}_T}^2,$$

$$\int_0^T \|\operatorname{curl} v_t^\epsilon\|_2^2 dt \leqq \mathcal{N}_0 + \sqrt{T}\, P(\|\bar v\|_{\mathbf{Z}_T}^2) + P(\|\operatorname{curl} \bar v\|_{\mathbf{Y}_T}^2) + C\|\operatorname{div} \bar v\|_{\mathbf{Y}_T}^2,$$

which in addition to the normal trace estimate (8.98) provides the estimate:

$$\int_0^T \|v_t^\epsilon\|_3^2 dt \leqq \mathcal{N}_0 + \sqrt{T} P(\|v^\epsilon\|_{\mathcal{X}_T^4}^2) + \sqrt{T}\, P(\|\bar v\|_{\mathbf{Z}_T}^2) + P(\|\operatorname{curl} \bar v\|_{\mathbf{Y}_T}^2) + C\|\operatorname{div} \bar v\|_{\mathbf{Y}_T}^2. \tag{8.99}$$

We proceed in a similar fashion to estimate v_{tt}^ϵ, by considering the time-differentiated version of (8.74), and using (8.97) and (8.99). This yields the following inequality:

$$\int_0^T \|v_{tt}^\epsilon\|_2^2 dt \leqq \mathcal{N}_0 + \sqrt{T} P(\|v^\epsilon\|_{\mathcal{X}_T^4}^2) + \sqrt{T}\, P(\|\bar v\|_{\mathbf{Z}_T}^2) + P(\|\operatorname{curl} \bar v\|_{\mathbf{Y}_T}^2) + C\|\operatorname{div} \bar v\|_{\mathbf{Y}_T}^2. \tag{8.100}$$

Finally, by using the second time-differentiated version of (8.74) and using (8.97), (8.99) and (8.100), we also have that

$$\int_0^T \|v_{ttt}^\epsilon\|_1^2 dt \leqq \mathcal{N}_0 + \sqrt{T} P(\|v^\epsilon\|_{\mathcal{X}_T^4}^2) + \sqrt{T}\, P(\|\bar v\|_{\mathbf{Z}_T}^2) + P(\|\operatorname{curl} \bar v\|_{\mathbf{Y}_T}^2) + C\|\operatorname{div} \bar v\|_{\mathbf{Y}_T}^2. \tag{8.101}$$

The estimate (8.97), together with (8.99), (8.100), and (8.101), provides us with

$$\|v^\epsilon\|_{\mathcal{X}_T^4}^2 \leqq \mathcal{N}_0 + \sqrt{T} P(\|v^\epsilon\|_{\mathcal{X}_T^4}^2) + \sqrt{T}\, P(\|\bar v\|_{\mathbf{Z}_T}^2) + P(\|\operatorname{curl} \bar v\|_{\mathbf{Y}_T}^2) + C\|\operatorname{div} \bar v\|_{\mathbf{Y}_T}^2, \tag{8.102}$$

where the polynomial functions P on the right-hand side are independent of ϵ.

Thanks to our polynomial estimates in Section 5.6, we infer from (8.102) the existence of $T > 0$ (which is independent of ϵ) such that $v^\epsilon \in \mathcal{X}_T^4$ and satisfies the estimate:

$$\|v^\epsilon\|_{\mathcal{X}_T^4}^2 \leqq 2\mathcal{N}_0 + 2\sqrt{T}\, P(\|\bar v\|_{\mathbf{Z}_T}^2) + P(\|\operatorname{curl} \bar v\|_{\mathbf{Y}_T}^2) + 2C\|\operatorname{div} \bar v\|_{\mathbf{Y}_T}^2. \tag{8.103}$$

(We will readjust the constant C and the polynomial functions P to absorb the multiplication by 2.)

Step 3: The Limit as $\epsilon \to 0$ and the Fixed-point of the Map $\bar v \mapsto v$

We set $\epsilon = \frac{1}{n}$, $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. From (8.103), there exists a subsequence (still denoted by ϵ) and a vector field $v \in \mathcal{X}_T^3$, $V \in L^2(0, T; H^4(\Omega)) \cap L^\infty(0, T; H^3(\Omega))$ such that

$$v^\epsilon \rightharpoonup v \ \text{ in } \mathcal{X}_T^3\,, \tag{8.104a}$$

$$v^\epsilon \to v \ \text{ in } \mathcal{X}_T^2\,, \tag{8.104b}$$

$$\Lambda_\epsilon v^\epsilon \rightharpoonup V \text{ in } L^2(0, T; H^4(\Omega)), \tag{8.104c}$$

where the space $\mathcal{X}_T^3$ is defined in (8.76) and

$$\mathcal{X}_T^2 = \{w \in L^2(0,T;H^2(\Omega)) : \; w_t \in L^2(0,T;H^2(\Omega)), \\ w_{tt} \in L^2(0,T;H^1(\Omega)), \; (x_1,x_2) \mapsto w \text{ is 1-periodic}\}.$$

Next, we notice that for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$, the space of smooth functions with compact support in Ω, we have for each $i = 1, 2, 3$ and $t \in [0, T]$, T still depending on $\kappa > 0$, that

$$\lim_{\epsilon\to 0}\int_\Omega \Lambda_\epsilon (v^\epsilon)^i \cdot \varphi \mathrm{d}x = \lim_{\epsilon\to 0}\int_\Omega (v^\epsilon)^i \cdot \Lambda_\epsilon \varphi \mathrm{d}x = \int_\Omega v^i \varphi \mathrm{d}x, \tag{8.105}$$

where we used the fact that $\Lambda_\epsilon\varphi \to \varphi$ in $L^2(\Omega)$. This shows us that $v = V$, and that

$$\|v\|^2_{\boldsymbol{X}_T} \leqq \mathcal{N}_0 + \sqrt{T}\, P(\|\bar{v}\|^2_{\boldsymbol{Z}_T}) + P(\|\operatorname{curl}\bar{v}\|^2_{\boldsymbol{Y}_T}) + C\|\operatorname{div}\bar{v}\|^2_{\boldsymbol{Y}_T}.$$

The estimates weighted by ρ_0 in the definition of the $\boldsymbol{Z}_T$-norm follow immediately from multiplication by ρ_0 of the equations (8.7b) and (8.7c); because ρ_0 vanishes on Γ and using the (unweighted) estimates already obtained, there is no need to consider the parabolic equation (8.7d), so that

$$\|v\|^2_{\boldsymbol{Z}_T} \leqq \mathcal{N}_0 + \sqrt{T}\, P(\|\bar{v}\|^2_{\boldsymbol{Z}_T}) + P(\|\operatorname{curl}\bar{v}\|^2_{\boldsymbol{Y}_T}) + C\|\operatorname{div}\bar{v}\|^2_{\boldsymbol{Y}_T}. \tag{8.106}$$

Moreover, the convergence in (8.104) and the definition of the sequence of problems (8.80) easily show us that v is a solution of the problem (8.7); furthermore, we see that we can obtain for the system (8.7) the same type of energy estimates as in Step 2 above. This shows the uniqueness of the solution v of (8.7), and hence allows us to define a mapping $\Theta : \bar{v} \in \boldsymbol{Z}_T \to v \in \boldsymbol{Z}_T$.

We next launch an iteration scheme. We choose any $v^{(1)} \in \mathcal{C}_T(M)$ and define for $n \in \mathbb{N}$,

$$v^{(n+1)} = \Theta(v^{(n)}), \quad v^{(n)}|_{t=0} = u_0.$$

For each $n \in \mathbb{N}$ we set $\eta^{(n)}(x,t) = x + \int_0^t v^{(n)}(x,t')\mathrm{d}t'$, $A^{(n)} = [D\eta^{(n)}]^{-1}$, $J^{(n)} = \det D\eta^{(n)}$, $a^{(n)} = J^{(n)}A^{(n)}$, $X^{(n)}$ is the solution to (8.5) with $v^{(n)}$, $a^{(n)}$, $J^{(n)}$, and $A^{(n)}$ replacing $\bar{v}$, $\bar{a}$, $\bar{J}$, and $\bar{A}$, respectively. Similarly, we define $\Xi^{(n)}(\eta^{(n)})$ via equation (8.14) with $v^{(n)}$ replacing $\bar{v}$; we define $\mathfrak{C}^{(n)}$ via equations (8.11) and (8.15) with $v^{(n)}$ replacing $\bar{v}$.

According to (8.106),

$$\|v^{(n+1)}\|^2_{\boldsymbol{Z}_T} \leqq \mathcal{N}_0 + \sqrt{T}P(\|v^{(n)}\|^2_{\boldsymbol{Z}_T}) + P(\|\operatorname{curl} v^{(n)}\|^2_{\boldsymbol{Y}_T}) + C\|\operatorname{div} v^{(n)}\|^2_{\boldsymbol{Y}_T}. \tag{8.107}$$

From (8.7b),

$$\operatorname{div} v^{(n)} = \operatorname{div} v^{(n-1)} - \operatorname{div}_{\eta^{(n-1)}} v^{(n-1)} + \frac{J^{(n-1)^2} X^{(n-1)}}{\rho_0}$$

so that

$$\| \operatorname{div} v^{(n)} \|^2_{Y_T} \leqq \| \operatorname{div} v^{(n-1)} - \operatorname{div}_{\eta^{(n-1)}} v^{(n-1)} \|^2_{Y_T} + \| \frac{J^{(n-1)^2} X^{(n-1)}}{\rho_0} \|^2_{Y_T}$$
$$\leqq \mathcal{N}_0 + \sqrt{T} P(\| v^{(n-1)} \|^2_{Z_T}) + P(\| \operatorname{curl} v^{(n-1)} \|^2_{Y_T}), \qquad (8.108)$$

where we have used the higher-order Hardy inequality Lemma 1 and Proposition 2 for the second inequality. Next, we use (8.7c) and write

$$\operatorname{curl} v^{(n)} = \operatorname{curl} u_0 + \operatorname{curl} v^{(n-1)} - \operatorname{curl}_{\eta^{(n-1)}} v^{(n-1)}$$
$$+2\kappa \int_0^t \varepsilon_{\cdot ji} v^{(n-1)},^r_s A^{(n-1)s}_i \, \Xi^{(n-1)},^j_r (\eta^{(n-1)})$$
$$+ \int_0^t \varepsilon_{\cdot ji} v^{(n-1)},^i_s \partial_t A^{(n-1)s}_j + \int_0^t \mathfrak{C}^{(n-1)}.$$

It then follows, using (8.14) and (8.17), that

$$\| \operatorname{curl} v^{(n)} \|^2_{Y_T} \leqq \mathcal{N}_0 + \sqrt{T} P(\| v^{(n-1)} \|^2_{Z_T}). \qquad (8.109)$$

Combining the estimates (8.107), (8.108), and (8.109), we obtain the inequality

$$\| v^{(n+1)} \|^2_{Z_T} \leqq \mathcal{N}_0 + \sqrt{T} P(\| v^{(n)} \|^2_{Z_T}) + \sqrt{T} P(\| v^{(n-1)} \|^2_{Z_T}) + \sqrt{T} P(\| v^{(n-2)} \|^2_{Z_T}).$$

This shows that by choosing $T > 0$ sufficiently small and $M >> \mathcal{N}_0$ sufficiently large, the convex set $\mathcal{C}_T(M)$ is stable under the action of Θ.

In order to see that Θ has a fixed-point, we simply notice that by proceeding in a similar fashion as in **Step 2** above (for the ϵ-independent energy estimates), and by using the inequality (8.72)

$$\| v^{(n+1)} - v^{(n)} \|^2_{Z_T} \leqq \sqrt{T} (P(\| v^{(n)} - v^{(n-1)} \|^2_{Z_T}) + P(\| v^{(n-1)} - v^{(n-2)} \|^2_{Z_T}))$$
$$+ \sqrt{T} \; P(\| v^{(n-2)} - v^{(n-3)} \|^2_{Z_T}), \qquad (8.110)$$

where the polynomial function P can be chosen under the form $P(z) = \sum_{j=1}^m a_j z^j$ for some integer $m \geqq 1$ ($a_j \geqq 0$).

Although, the inequality (8.110) is not exactly the usual hypothesis of the contraction mapping theorem, the identical argument shows that for $T = T_\kappa$ taken sufficiently small, the map Θ is a contraction, and possesses a unique fixed-point v satisfying $v = \Theta(v)$. We will next prove that this unique fixed-point v is the unique solution of the κ-problem (7.2).

8.5.3. The Fixed-Point of the Map $\bar{v} \mapsto v$ is a Solution of the κ-Problem In a straightforward manner, we deduce from (8.7) the following relations for our fixed-point $v = \bar{v}$:

$$\operatorname{div}_\eta v_t = \frac{[X J^2]_t}{\rho_0} - \partial_t A^j_i v^i,_j \qquad \text{in } \Omega \,, \qquad (8.111a)$$

$$\operatorname{curl}_\eta v_t = 2\kappa \varepsilon_{\cdot ji} v,^r_s A^s_i \, \Xi,^j_r (\eta) + \mathfrak{C} \qquad \text{in } \Omega \,, \qquad (8.111b)$$

$$v_t^3 = -2J^{-2}a_3^3\rho_0,_3 - 2\kappa[J^{-2}a_3^3]_t\rho_0,_3 + c(t)N^3 \quad \text{on } \Gamma\,, \tag{8.111c}$$

$$\int_\Omega v_t^\alpha \mathrm{d}x = -2\int_\Omega A_\alpha^k(\rho_0 J^{-1}),_k \,\mathrm{d}x - 2\kappa \int_\Omega \partial_t[A_\alpha^k(\rho_0 J^{-1}),_k\,]\mathrm{d}x, \tag{8.111d}$$

$$(x_1, x_2) \mapsto v_t(x_1, x_2, x_3, t) \text{ is 1-periodic}\,, \tag{8.111e}$$

where X is a solution of (7.6) and where the function $c(t)$ in (8.111c) is defined by

$$\begin{aligned}
-2c(t) &= \int_\Omega (-\operatorname{div} v_t + \operatorname{div}_\eta v_t)\mathrm{d}x - \int_\Omega \frac{[XJ^2]_t}{\rho_0}\mathrm{d}x + \int_\Omega \partial_t A_i^j v,_j^i \,\mathrm{d}x \\
&\quad -2\int_\Gamma J^{-2}a_3^3 N^3 \rho_0,_3 \,\mathrm{d}S - 2\kappa \int_\Gamma J^{-2}\partial_t a_3^3 N_3 \rho_0,_3 \mathrm{d}S \\
&\quad -2\kappa \int_\Gamma a_3^3 \partial_t J^{-2} N_3 \rho_0,_3 \,\mathrm{d}S = \int_\Omega (-\operatorname{div} v_t + \operatorname{div}_\eta v_t)\mathrm{d}x \\
&\quad - \int_\Omega \frac{[XJ^2]_t}{\rho_0}\mathrm{d}x + \int_\Omega \partial_t A_i^j v,_j^i \,\mathrm{d}x - 2\int_\Gamma D\rho(\eta)\cdot N \,\mathrm{d}S \\
&\quad -2\kappa \int_\Gamma [D\rho(\eta)]_t \cdot N \,\mathrm{d}S,
\end{aligned}$$

where $D\rho(\eta) = A_\cdot^k(\rho_0 J^{-1}),_k$, and $\mathrm{d}S = \mathrm{d}x_1\mathrm{d}x_2$. By using (8.111a) and the divergence theorem, we therefore obtain the identity (since the volume of Ω is equal to 1)

$$c(t) = \frac{1}{2}\int_\Gamma [v_t + 2D\rho(\eta) + 2\kappa[D\rho(\eta)]_t]\cdot N \,\mathrm{d}S. \tag{8.112}$$

The fixed-point of the map $\bar{v} \mapsto v$ (which we are labeling v as well) also satisfies the equation

$$v_t + 2\Xi(\eta) + 2\kappa[\Xi(\eta)]_t = 0, \tag{8.113a}$$

$$\Xi(0) = D\rho_0. \tag{8.113b}$$

It is thus clear from (8.111c) and (8.113) that the fixed-point is a solution to the κ-problem (7.2), **if** we can prove that

$$c(t) = 0 \text{ and } \Xi = D\rho, \tag{8.114}$$

where we remind the reader that

$$\rho(\eta) = \rho_0 J^{-1}. \tag{8.115}$$

This is in fact the case, and we now explain why (8.114) holds.

Step 1 Using $\Xi(\eta)_t = \Xi_t(\eta) + u^r(\eta)\Xi,_r(\eta)$, we apply curl_η to each term of (8.113a) and compare the resulting equation with (8.111b). This implies that

$$\kappa\varepsilon_{\cdot ji}v,_s^r A_i^s\ \Xi,_r^j(\eta)+\text{curl}_\eta[\Xi(\eta)]+\kappa[\text{curl}_\eta[\Xi_t(\eta)]+\text{curl}_\eta[(u^r\partial_r\Xi)(\eta)]]=-\frac{1}{2}\mathfrak{C}. \tag{8.116}$$

Now, by definition of curl_η,

$$\begin{aligned}\text{curl}_\eta[(u^r\partial_r\Xi)(\eta)] &= \varepsilon_{\cdot jk}A_j^s\big(v^r\partial_r\Xi^k(\eta)\big),_s\\ &=\varepsilon_{\cdot jk}A_j^s v,_s^r\partial_r\Xi^k(\eta)+u^r(\eta)\,\text{curl}_\eta[(\partial_r\Xi)(\eta)],\end{aligned} \tag{8.117}$$

which by substitution in (8.116) implies

$$\text{curl}_\eta[\Xi(\eta)]+\kappa[\text{curl}_\eta[\Xi_t(\eta)]+u^r(\eta)\,\text{curl}_\eta[(\partial_r\Xi)(\eta)]]=-\frac{1}{2}\mathfrak{C}. \tag{8.118}$$

Thanks to the fact that by definition of curl_η,

$$\text{curl}_\eta[\Theta(\eta)]=[\text{curl}\,\Theta](\eta), \tag{8.119}$$

for any vector field Θ, this provides us with

$$[\text{curl}\,\Xi](\eta)+\kappa[[\text{curl}\,\Xi_t](\eta)+u_i(\eta)[\text{curl}(\partial_i\Xi)](\eta)]=-\frac{1}{2}\mathfrak{C}, \tag{8.120}$$

which shows that

$$[\text{curl}\,\Xi](\eta)+\kappa[\text{curl}\,\Xi(\eta)]_t=-D\psi_e(\eta)-\kappa\big[D\psi_e(\eta)\big]_t, \tag{8.121}$$

where ψ_e denote the Eulerian version of ψ, given by

$$\psi_e\circ\eta=\psi. \tag{8.122}$$

According to (8.113b), $\Xi(0)=D\rho_0$; thus, we have that $[\text{curl}\,\Xi](\eta)(0)=0$. Furthermore, by our definition (8.12), we have $D\psi_e(\eta)|_{t=0}=0$ in Ω, which with (8.121) allows us to conclude that for $t\in[0,T]$,

$$[\text{curl}\,\Xi+D\psi_e](\eta)(t)=0.$$

We may therefore consider the following elliptic problem:

$$\begin{aligned}\Delta\psi_e=-\operatorname{div}(\text{curl}\,\Xi)=0 \ \text{ in } \ \eta(t)(\Omega),\\ \psi_e=0 \ \text{ on } \ \eta(t)(\Gamma),\\ (x_1,x_2)\mapsto\psi_e(x_1,x_2,x_3,t) \text{ is 1-periodic,}\end{aligned}$$

which shows that $\psi_e=0$ and hence $\mathfrak{C}=0$. Therefore, $\text{curl}\,\Xi=0$ in $\eta(t,\Omega)$ and there exists a scalar function $Y(t,\cdot)$ defined on $\eta(t,\Omega)$ such that

$$\Xi=DY. \tag{8.123}$$

It remains to establish that $DY=D\rho$. We will first prove that a Neumann-type boundary condition plus a small tangential perturbation holds for $Y-\rho$; namely, we will show that $(Y-\rho),_3N_3$ is a function $k(t)$ of the time variable only on $\eta(t,\Gamma)$.

Step 2 We take the scalar product of (8.113a) with e_3 to find that

$$v_t^3 + 2DY(\eta)\cdot e_3 + 2\kappa[DY)(\eta)]_t\cdot e_3 = 0,$$

which, by comparison with (8.111c), yields the following identity on Γ:

$$\begin{aligned} 2\Big[D(Y-\rho)(\eta) + \kappa[D(Y-\rho)(\eta)]_t\Big]\cdot e_3 &= -c(t)N^3 \\ &= \int_\Gamma \Big[D(-\rho+Y)(\eta) + \kappa[D(-\rho+Y)(\eta)]_t\Big]\cdot N\,\mathrm{d}S\, N^3, \end{aligned} \tag{8.124}$$

where we have used the expression (8.112) for $c(t)$. By denoting

$$q = \rho - Y, \tag{8.125}$$

since $N = (0, 0, N^3)$ on Γ, this implies:

$$Dq(\eta)\cdot N + \kappa[\, Dq(\eta)\cdot N\,]_t = \frac{c(t)}{2},$$

and thus by integration, and taking into account that $D\rho(0) = DY(0)$,

$$Dq(\eta)(t,\cdot)\cdot N = \frac{1}{2}\int_0^t \frac{c(s)}{\kappa}\,\mathrm{e}^{\frac{s}{\kappa}}\,\mathrm{d}s,$$

which is indeed a function depending only on time, which we denote $k(t)$. By integrating the previous relation over Γ, we finally obtain that on Γ:

$$Dq(\eta)(t,\cdot)\cdot N = k(t) = \frac{1}{2}\int_\Gamma Dq(\eta)(t,\cdot)\cdot N\,\mathrm{d}S. \tag{8.126}$$

Step 3 We now apply div_η to (8.113a), and compare the resulting equation with (8.5a). Using (8.111a) and the fact that $X(0) = \rho_0 \operatorname{div} u_0$, we have that

$$X = \rho_0 J^{-2}\operatorname{div}_\eta v. \tag{8.127}$$

This leads us to:

$$\operatorname{div}_\eta[Dq(\eta) + \kappa Dq_t(\eta) + \kappa u_i(\eta)(Dq),_i(\eta)] = 0 \ \text{ in } \Omega. \tag{8.128}$$

This is equivalent in $[0, T]\times\Omega$ to:

$$\Delta q(\eta) + \kappa\Delta q_t(\eta) + \kappa u_i(\eta)\Delta q,_i(\eta) + A_l^j v,_j^i\, q,_{li}(\eta) = 0,$$

or equivalently,

$$\Delta q(\eta) + \kappa[\Delta q(\eta)]_t + A_l^j v,_j^i\, q,_{li}(\eta) = 0. \tag{8.129}$$

Now, since $\rho_0 = Y(0)$, we have that

$$\Delta q(0) = 0 \ \text{ in } \Omega. \tag{8.130}$$

Also, from (8.126), we have the perturbed Neumann boundary condition

$$q,_3(\eta)N_3 = k(t), \quad \text{on } \Gamma. \tag{8.131}$$

By (8.111d), (8.113), we obtain that for $\alpha = 1, 2$,

$$\int_\Omega q,_\alpha(\eta)\mathrm{d}x + \kappa \int_\Omega [q,_\alpha(\eta)]_t \mathrm{d}x = 0\,,$$

or equivalently, $\int_\Omega q,_\alpha(\eta)\mathrm{d}x + \kappa\partial_t \int_\Omega q,_\alpha(\eta)\mathrm{d}x = 0$, which together with the initial condition $\int_\Omega q,_\alpha(0)\mathrm{d}x = 0$ implies that

$$\int_\Omega q,_\alpha(\eta)\mathrm{d}x = 0. \tag{8.132}$$

Therefore, by setting $f = \Delta q$, we have for all $t \in [0, T]$ the system:

$$\Delta q = f \text{ in } \eta(\Omega)\,, \tag{8.133a}$$

$$\int_{\eta(\Omega)} J^{-1} q,_\alpha \,\mathrm{d}x = 0\,, \tag{8.133b}$$

$$q,_3(\eta)\, N_3 = k(t) \text{ on } \Gamma\,, \tag{8.133c}$$

$$Dq \text{ is 1-periodic in the directions } e_1 \text{ and } e_2\,. \tag{8.133d}$$

Note that because of the periodicity of v, the domain $\eta(\Omega)$ is such that $\eta(1, x_2, x_3) = \eta(0, x_2, x_3) + (1, 0, 0)$, and $\eta(x_1, 1, x_3) = \eta(x_1, 0, x_3) + (0, 1, 0)$, which explains why condition (8.133d) holds.

We now take the vertical derivative ∂_3 of (8.133a), multiply the resulting equation by $q,_3$ and integrate by parts in $\eta(\Omega)$, using the condition (8.133d). This yields:

$$\int_{\eta(\Omega)} |Dq,_3|^2 \mathrm{d}x - \int_{\eta(\Gamma)} q,_{i3}\, q,_3\, n_i(t)\,\mathrm{d}S(t)$$
$$= \int_{\eta(\Omega)} f\, q,_{33}\,\mathrm{d}x - \int_{\eta(\Gamma)} f q,_3\, n_3(t)\,\mathrm{d}S(t),$$

where the notation $\mathrm{d}S(t)$ denotes the naturally induced surface measure on $\eta(t, \Gamma)$. Note that due to the fact that $\eta(t, \Gamma)$ is no longer necessarily horizontal for $t > 0$, integration by parts in purely horizontal directions also produces boundary contributions. Therefore, with (8.133c) we obtain:

$$\int_{\eta(\Omega)} |Dq,_3|^2 \mathrm{d}x - k(t) \int_{\eta(\Gamma)} \frac{q_{i3}}{N_3(\eta^{-1})} n_i(t)\,\mathrm{d}S(t)$$
$$= \int_{\eta(\Omega)} f\, q,_{33}\,\mathrm{d}x - k(t) \int_{\eta(\Gamma)} \frac{f}{N_3(\eta^{-1})} n_3(t)\,\mathrm{d}S(t). \tag{8.134}$$

We now denote by ϕ a smooth function in $\eta(\Omega)$ such that $\phi = \frac{1}{N_3(\eta^{-1})}$ on $\eta(\Gamma)$. An integration by parts with respect to the variable x_i provides for the boundary integral on the left-hand side of (8.134):

$$\int_{\eta(\Gamma)} \frac{q,_{i3}}{N_3(\eta^{-1})} n_i(t)\,\mathrm{d}S(t) = \int_{\eta(\Omega)} \Delta q,_3\, \phi\,\mathrm{d}x + \int_{\eta(\Omega)} q,_{i3}\, \phi,_i\,\mathrm{d}x. \tag{8.135}$$

Similarly, an integration by parts with respect to the variable x_3 provides for the boundary integral on the right-hand side of (8.134):

$$\int_{\eta(\Gamma)} \frac{f}{N_3(\eta^{-1})} n_3(t)\, \mathrm{d}S(t) = \int_{\eta(\Omega)} f,_3 \phi \, \mathrm{d}x + \int_{\eta(\Omega)} f \phi,_3 \, \mathrm{d}x. \tag{8.136}$$

The use of (8.135), (8.136), (8.133a) in (8.134) then yields:

$$\int_{\eta(\Omega)} |Dq,_3|^2 \mathrm{d}x = \int_{\eta(\Omega)} f\, q,_{33} \, \mathrm{d}x + k(t) \int_{\eta(\Omega)} q,_{i3}\, \phi,_i \, \mathrm{d}x \\ -k(t) \int_{\eta(\Omega)} f \phi,_3 \, \mathrm{d}x,$$

which provides us with the estimate

$$\|Dq,_3\|^2_{0,\eta(\Omega)} \leqq C \|f\|^2_{0,\eta(\Omega)} + C\, k(t)^2, \tag{8.137}$$

where we are using the notation $\|\cdot\|_{s,\eta(\Omega)} = \|\cdot\|_{H^s(\eta(\Omega))}$. From (8.126), we have by the divergence theorem:

$$k(t) = \frac{1}{2} \int_{\Omega} [q,_i(\eta)],_i \, \mathrm{d}x = \frac{1}{2} \int_{\Omega} [q,_{ij}(\eta)] \eta^j,_i \, \mathrm{d}x,$$

and thus since $|\eta^j,_i(t) - \delta^j_i| \leqq Ct$, we obtain,

$$|k(t)| \leqq C \|\Delta q\|_{0,\eta(\Omega)} + Ct \|D^2 q\|_{0,\eta(\Omega)},$$

which inserted in (8.137) provides:

$$\|Dq,_3\|^2_{0,\eta(\Omega)} \leqq C \|f\|^2_{0,\eta(\Omega)} + Ct \|D^2 q\|^2_{0,\eta(\Omega)}. \tag{8.138}$$

We now write (8.133a) under the form

$$q,_{11} + q,_{22} = g \text{ where } g := -q,_{33} + f. \tag{8.139}$$

It follows from (8.139) that

$$\int_{\eta(\Omega)} (q,_{11} + q,_{22})\, (q,_{11} + q,_{22}) \mathrm{d}x = \int_{\eta(\Omega)} |g|^2 \mathrm{d}x.$$

Integration by parts on the left-hand side of this equation, together with the periodicity of Dq and its derivatives, shows that

$$\int_{\eta(\Omega)} q,_{\alpha\beta}\, q,_{\alpha\beta} \, \mathrm{d}x + 2 \int_{\eta(\Gamma)} q,_{11}\, q,_2\, n_2(t) \mathrm{d}S(t) \\ - 2 \int_{\eta(\Gamma)} q,_{12}\, q,_2\, n_1(t) \mathrm{d}S(t) = \int_{\eta(\Omega)} |g|^2 \mathrm{d}x. \tag{8.140}$$

We now notice that

$$\int_{\eta(\Gamma)} q,_{11}\, q,_2\, n_2(t) \mathrm{d}S(t) = \int_{\eta(\Gamma)} q,_{11}\, q,_2 \frac{n_2(t)}{n_3(t)} n_3(t) \mathrm{d}S(t),$$

where division by $n_3(t)$ is bounded since for t taken sufficiently small, $|n_3(t)|$ is very close to 1 on $\eta(\Gamma)$.

Next, for $i = 1, 2, 3$, we smoothly extend $n_i(t,\cdot)$ into $\eta(t,\Omega)$ and denote by φ a smooth extension into $\eta(t,\Omega)$ of $\frac{1}{n_3(t,\cdot)}$, and note the integration-by-parts (with respect to x_3) identity

$$\int_{\eta(\Gamma)} q,_{11}\, q,_2\, n_2(t)\mathrm{d}S(t) = \int_{\eta(\Omega)} q,_{113}\, q,_2\, n_2(t)\varphi \mathrm{d}x + \int_{\eta(\Omega)} q,_{11}\, \partial_3\big[q,_2\, n_2(t)\varphi\big]\mathrm{d}x. \tag{8.141}$$

An integration by parts with respect to the variable x_1 for the first integral on the right-hand side of (8.141) then yields:

$$\begin{aligned}\int_{\eta(\Gamma)} q,_{11}\, q,_2\, n_2(t)\mathrm{d}S(t) = &-\int_{\eta(\Omega)} q,_{13}\, \partial_1\big[q,_2\, n_2(t)\varphi\big]\mathrm{d}x \\ &+\int_{\eta(\Gamma)} q,_{13}\, q,_2\, n_2(t)\varphi n_1(t)\mathrm{d}S(t) \\ &+\int_{\eta(\Omega)} q,_{11}\, \partial_3\big[q,_2\, n_2(t)\varphi\big]\mathrm{d}x.\end{aligned} \tag{8.142}$$

Similarly, by integrating by parts first with respect to x_3 and then with respect to x_2, we see that the third integral on the left-hand side of (8.140) can be written as

$$\begin{aligned}\int_{\eta(\Gamma)} q,_{12}\, q,_2\, n_1(t)\mathrm{d}S(t) = &-\int_{\eta(\Omega)} q,_{13}\, \partial_2\big[q,_2\, n_1(t)\varphi\big]\mathrm{d}x \\ &+\int_{\eta(\Gamma)} q,_{13}\, q,_2\, n_1(t)\varphi n_2(t)\mathrm{d}S(t) \\ &+\int_{\eta(\Omega)} q,_{12}\, \partial_3\big[q,_2\, n_1(t)\varphi\big]\mathrm{d}x,\end{aligned} \tag{8.143}$$

which shows that the boundary integrals over $\eta(\Gamma)$ cancel each other when we substitute (8.143) and (8.142) into (8.140); thus, (8.140) takes the following form:

$$\begin{aligned}&\int_{\eta(\Omega)} q,_{\alpha\beta}\, q,_{\alpha\beta}\, \mathrm{d}x \\ &= \int_{\eta(\Omega)} |g|^2\mathrm{d}x + 2\int_{\eta(\Omega)} q,_{13}\, \partial_1\big[q,_2\, n_2(t)\varphi\big]\mathrm{d}x - 2\int_{\eta(\Omega)} q,_{11}\, \partial_3\big[q,_2\, n_2(t)\varphi\big] \\ &\quad -2\int_{\eta(\Omega)} q,_{13}\, \partial_2\big[q,_2\, n_1(t)\varphi\big]\mathrm{d}x + 2\int_{\eta(\Omega)} q,_{12}\, \partial_3\big[q,_2\, n_1(t)\varphi\big]\mathrm{d}x,\end{aligned}$$

which thanks to the estimate (8.138) and the relations $|n_\alpha(t)|_{W^{1,\infty}(\Omega)} \leqq Ct$, implies that

$$\int_{\eta(\Omega)} q,_{\alpha\beta}\, q,_{\alpha\beta}\, \mathrm{d}x \leqq C\|f\|^2_{0,\eta(\Omega)} + Ct\|D^2q\|^2_{0,\eta(\Omega)} + Ct\|D^2q\|_{0,\eta(\Omega)}\|Dq\|_{0,\eta(\Omega)}. \tag{8.144}$$

Combining this estimate with (8.138), we obtain that

$$\|D^2 q\|^2_{0,\eta(\Omega)} \leqq C\|f\|^2_{0,\eta(\Omega)} + Ct\|D^2 q\|^2_{0,\eta(\Omega)} + Ct\|D^2 q\|_{0,\eta(\Omega)}\|Dq\|_{0,\eta(\Omega)}. \tag{8.145}$$

Now, we notice that the conditions (8.133c),(8.133d), and (8.132) yield Poincaré inequalities for $q,_\alpha$ and $q,_3$, so that

$$\|q,_3\|_{0,\eta(\Omega)} \leqq C\|Dq,_3\|_{0,\eta(\Omega)} + |k(t)| \leqq C\|Dq,_3\|_{0,\eta(\Omega)} + C\|\Delta q\|_{0,\eta(\Omega)} + Ct\|D^2 q\|_{0,\eta(\Omega)}, \tag{8.146a}$$

$$\|q,_\alpha\|_{0,\eta(\Omega)} \leqq C\|Dq,_\alpha\|_{0,\eta(\Omega)},\ \alpha = 1, 2, \tag{8.146b}$$

where we also used our estimate for $k(t)$ obtained just before (8.138). Therefore, (8.145) and (8.146) provide us with

$$\|Dq\|^2_{1,\eta(\Omega)} \leqq C\|f\|^2_{0,\eta(\Omega)} + Ct\|Dq\|^2_{1,\eta(\Omega)},$$

which, by taking $T > 0$ small enough, yields:

$$\|Dq\|^2_{1,\eta(\Omega)} \leqq C\|\Delta q\|^2_{0,\eta(\Omega)}. \tag{8.147}$$

We thus have proved that

$$q,_{li}(\eta) = F^{li}(t, \Delta q(\eta)), \tag{8.148}$$

where $F^{li}(t, \cdot)$ denotes a linear and continuous operator from $L^2(\Omega)$ into itself, whose norm depends in a smooth manner on v in $L^2(0, T; H^4(\Omega))$.

Therefore, the ODE

$$\Delta q(\eta) + \kappa[\Delta q(\eta)]_t + A^j_l v,^i_j F^{li}(t, \Delta q(\eta)) = 0, \tag{8.149}$$

with the initial condition (8.130) allow us to conclude by the Gronwall inequality that on $[0, T] \times \Omega$,

$$\Delta q(\eta) = 0. \tag{8.150}$$

From (8.150) and (8.147), we infer that

$$Dq = 0 \ \text{ in } [0, T] \times \Omega,$$

which finally proves that $DY = D\rho$, and therefore that $\Xi = D\rho$. Therefore

$$c(t) = 0,$$

which finally establishes that v is a solution (with the regularity of the functional framework Z_T) of the κ-problem (7.2) on a time interval $[0, T_\kappa]$. This concludes the proof of Theorem 2.

By considering more time-derivatives in our analysis, it is easy to see that as long as the initial data are smooth, we can construct solutions which are arbitrarily smooth in both space and time. We state this as the following

Theorem 3. (Smooth solutions to the κ-problem) *Given smooth initial data with ρ_0 satisfying $\rho_0(x) > 0$ for $x \in \Omega$ and verifying the physical vacuum condition* (1.5) *near Γ, for $T_\kappa > 0$ sufficiently small, there exists a unique smooth solution to the degenerate parabolic κ-problem* (7.2).

9. κ-Independent Estimates for (7.2) and Solutions to the Compressible Euler Equations (1.9)

In this section, we obtain estimates for the smooth solutions to (7.2), provided by Theorem 3, whose bounds and time interval of existence are independent of the artificial viscosity parameter κ. This permits us to consider the limit of this sequence of solutions as $\kappa \to 0$. We prove that this limit exists, and that it is the unique solution of (1.9).

Notation. For the remainder of Section 9, $\eta(t)$ denotes the solution of the κ-problem (7.2) on the time interval $[0, T_\kappa]$. In particular, $\eta(t)$ is an element of a sequence of solutions parameterized by $\kappa > 0$, but in order to reduce the number of subscripts and superscripts that appear, we will not make this sequential dependence explicit. The reader should bear in mind that η is really $\eta(\kappa)$.

9.1. A Continuous-in-Time Energy Function Appropriate for the Asymptotic Process $\kappa \to 0$

Definition 6. We set on $[0, T_\kappa]$

$$\begin{aligned}\tilde{E}(t) =& 1 + \sum_{a=0}^{4}\big[\|\partial_t^{2a}\eta(t)\|_{4-a}^2 + \|\rho_0\, \partial_t^{2a}\bar{\partial}^{4-a}D\eta(t)\|_0^2\big] \\ &+ \sum_{a=0}^{4}\|\sqrt{\rho_0}\,\bar{\partial}^{4-a}\partial_t^{2a}v(t)\|_0^2 + \sum_{a=0}^{4}\int_0^t \left[\|\sqrt{\kappa}\rho_0\,\partial_t^{2a}\bar{\partial}^{4-a}Dv(s)\|_0^2\right] ds \\ &+ \|\operatorname{curl}_\eta v(t)\|_3^2 + \|\rho_0\,\bar{\partial}^4\operatorname{curl}_\eta v(t)\|_0^2. \end{aligned} \tag{9.1}$$

The function $\tilde{E}(t)$ is the higher-order energy function appropriate for obtaining κ-independent estimates for the degenerate parabolic approximation (7.2).

According to Theorem 3, solutions to our approximate κ-problem (7.2) are smooth, and hence $T \mapsto \sup_{t\in[0,T]} \tilde{E}(t)$ is a continuous function on $[0, T_\kappa]$ to which the polynomial-type inequality of Section 5.6 can be applied.

Definition 7. For the remainder of the paper, we will use the constant $\tilde{M}_0$ to be a polynomial function of $\tilde{E}(0)$ so that

$$\tilde{M}_0 = P(\tilde{E}(0)). \tag{9.2}$$

Remark 7. Note the presence of κ-dependent coefficients in $\tilde{E}(t)$ that indeed arise as a necessity for our asymptotic study. The corresponding terms, without the κ, would of course not be asymptotically controlled.

Remark 8. The 1 is added to the norm to ensure that $\tilde{E}(t) \geqq 1$, which will sometimes be convenient, though not necessary.

Remark 9. Among all the terms on the right-hand side of (9.1), the sum $\sum_{a=0}^{4}\big[\|\partial_t^{2a}\eta(t)\|_{4-a}^2$ is the fundamental contribution, providing the basic regularity for the solution. Notice that only the even time derivatives of $\eta(t)$ appear in this norm. While it is possible to also obtain estimates for the odd time derivatives of $\eta(t)$, we will instead rely on the following interpolation estimate: For $k \geqq 1$, given a vector field $r \in L^\infty([0,T]; H^k(\Omega))$ with $r_{tt} \in L^\infty([0,T]; H^{k-1}(\Omega))$, it follows that $r_t \in L^2(0,T; H^{k-\frac{1}{2}}(\Omega))$ and

$$
\begin{aligned}
\|r_t\|^2_{L^2(0,T;H^{k-\frac{1}{2}}(\Omega))} &\leqq C\big(\|r_{tt}(t)\|_{k-1}\|r(t)\|_k\big)\Big|_0^T \\
&\quad + C\|r_{tt}\|_{L^2(0,T;H^{k-1}(\Omega))}\|r\|_{L^2(0,T;H^k(\Omega))} \\
&\leqq P(\|r(0)\|_k, \|r_t(0)\|_{k-1}) + \delta \sup_{t\in[0,T]} \|r(t)\|_k^2 \\
&\quad + C\,T \sup_{t\in[0,T]}\Big(\|r(t)\|_k^2 + \|r_{tt}(t)\|_{k-1}^2\Big).
\end{aligned}
$$

Thus, with L^2-in-time control, we see that the odd time derivatives of η verify the estimate

$$
\sum_{a=0}^{3}\|\partial_t^{2a}v\|^2_{L^2(0,T;H^{3.5-a}(\Omega))} \leqq M_0(\delta) + \delta \sup_{t\in[0,T]} E(t) + C\,T\,P(\sup_{t\in[0,T]} E(t)).
$$

See the interpolation inequality (9.27) below for further details.

9.2. Assumptions on a Priori Bounds on $[0, T_\kappa]$

For the remainder of this section, we assume that we have solutions $(\eta_\kappa) \in \boldsymbol{X}_{T_\kappa}$ on a time interval $[0, T_\kappa]$, and that for all such solutions, the time $T_\kappa > 0$ is taken sufficiently small so that for $t \in [0, T_\kappa]$ and $\xi \in \mathbb{R}^3$,

$$
\left.
\begin{array}{ll}
\frac{1}{2} \leqq J(t) \leqq \frac{3}{2}, & \lambda|\xi|^2 \leqq a_l^j a_l^k \xi_j \xi_k, \\
\det g(\eta(t))^{-1/2} \leqq 2\det g(\eta_0)^{-1/2} = 2, & \|J^{-1}A_k^r A_k^s - \delta_k^r\delta_k^s\|_{L^\infty(\Omega)} < \frac{1}{2}.
\end{array}
\right\}
\tag{9.3}
$$

We further assume that our solutions satisfy the bounds

$$
\left.
\begin{array}{l}
\|\eta(t)\|^2_{H^{3.5}(\Omega)} \leqq 2|e|^2_{3.5} + 1, \\
\|\partial_t^a v(t)\|^2_{H^{3-a/2}(\Omega)} \leqq 2\|\partial_t^a v(0)\|^2_{H^{3-a/2}(\Omega)} + 1 \quad \text{for } a = 0, 1, \ldots, 6, \\
\|\rho_0\partial_t^a \eta(t)\|^2_{H^{4.5-a/2}(\Omega)} \leqq 2\|\rho_0\partial_t^a\eta(0)\|^2_{H^{4.5-a/2}(\Omega)} + 1 \quad \text{for } a = 0, 1, \ldots, 7, \\
\|\sqrt{\kappa}\partial_t^{2a+1} v(t)\|^2_{H^{3-a}(\Omega)} \leqq 2\|\partial_t^a v(0)\|^2_{H^{3-a}(\Omega)} + 1 \quad \text{for } a = 0, 1, 2, 3.
\end{array}
\right\}
\tag{9.4}
$$

The right-hand sides appearing in the inequalities (9.4) shall be denoted by a generic constant C in the estimates appearing below. In what follows, we will prove that this can be achieved in a time interval independent of κ.

We continue to assume that ρ_0 is smooth coming from our approximation (7.1).

9.3. Curl Estimates

Proposition 3. *For all $t \in (0, T)$, where we take $T \in (0, T_\kappa)$,*

$$\sum_{a=0}^{3} \|\text{curl}\, \partial_t^{2a} \eta(t)\|_{3-a}^2 + \sum_{l=0}^{4} \|\rho_0\, \bar\partial^{4-l} \text{curl}\, \partial_t^{2l} \eta(t)\|_0^2 + \sum_{l=0}^{4} \int_0^t \|\sqrt{\kappa}\rho_0 \text{curl}_\eta\, \bar\partial^{4-l} \partial_t^{2l} v(s)\|_0^2 ds \leqq \tilde{M}_0 + C\, T\, P(\sup_{t\in[0,T]} \tilde{E}(t)). \tag{9.5}$$

Proof. Using the definition of the Lagrangian curl operator curl_η given by (4.1), we let curl_η act on (7.2a') to obtain the identity

$$(\text{curl}_\eta\, v_t)^k = -\kappa \varepsilon_{kji} v^r,_s A^s_j \big[(\rho_0 J^{-1}),_l A^l_r\big],_m A^m_i. \tag{9.6}$$

As described above, in the absence of the artificial viscosity term, the right-hand side is identically zero; we will have to make additional estimates to control *error terms* arising from κ-right-hand side forcing.

It follows from (9.6) that

$$\partial_t (\text{curl}_\eta\, v)^k = \varepsilon_{kji} A_t{}^s_j v^i,_s - \kappa \varepsilon_{kji} v^r,_s A^s_j \big[(\rho_0 J^{-1}),_l A^l_r\big],_m A^m_i.$$

Defining the kth-component of the vector field $B(A, Dv)$ by

$$B^k(A, Dv) = -\varepsilon_{kji} A^s_r v^r,_l A^l_j v^i,_s$$

and defining the k-component of the vector field F by

$$F^k = -\kappa \varepsilon_{kji} v^r,_s A^s_j \big[(\rho_0 J^{-1}),_l A^l_r\big],_m A^m_i,$$

we may write

$$\text{curl}_\eta\, v(t) = \text{curl}\, u_0 + \int_0^t [B(A(t'), Dv(t')) + F(t')] \text{d}t'. \tag{9.7}$$

Computing the gradient of this relation yields

$$\text{curl}_\eta\, Dv(t) = D\, \text{curl}\, u_0 - \varepsilon_{\cdot ji} DA^s_j v^i,_s + \int_0^t [DB(A(t'), Dv(t')) + DF(t')] \text{d}t'.$$

Applying the fundamental theorem of calculus once again, shows that

$$\text{curl}_\eta\, D\eta(t) = t\, \text{curl}\, Du_0 + \varepsilon_{\cdot ji} \int_0^t [A_t{}^s_j D\eta^i,_s - DA^s_j v^i,_s] \text{d}t' + \int_0^t \int_0^{t'} [DB(A(t''), Dv(t'')) + DF(t'')] \text{d}t'' \text{d}t',$$

and finally that

$$
\begin{aligned}
D\operatorname{curl}\eta(t) =& tD\operatorname{curl}u_0 - \varepsilon_{\cdot ji}\int_0^t A_t{}^s_j(t')\mathrm{d}t'\, D\eta^i{},_s \\
&+ \varepsilon_{\cdot ji}\int_0^t [A_t{}^s_j D\eta^i{},_s - DA^s_j v^i{},_s]\mathrm{d}t' \\
&+ \int_0^t\int_0^{t'} [DB(A(t''), Dv(t'')) + DF(t'')]\mathrm{d}t''\mathrm{d}t'. \qquad (9.8)
\end{aligned}
$$

Step 1. Estimate for curl η. To obtain an estimate for $\|\operatorname{curl}\eta(t)\|_3^2$, we let D^2 act on (9.8). With $\partial_t A^s_j = -A^s_l v^l{},_p A^p_j$ and $DA^s_j = -A^s_l D\eta^l{},_p A^p_j$, we see that the first three terms on the right-hand side of (9.8) are bounded by $\tilde{M}_0 + C\,T\,P(\sup_{t\in[0,T]}\tilde{E}(t))$, where we remind the reader that $\tilde{M}_0 = P(\tilde{E}(0))$ is a polynomial function of $\tilde{E}$ at time $t = 0$. Since

$$
DB_k(A, Dv) = -\varepsilon_{kji}[Dv^i{},_s A^s_l v^l{},_p A^p_j + v^i{},_s A^s_l Dv^l{},_p A^p_j + v^i{},_s v^l{},_p D(A^s_l A^p_j)],
$$

the highest-order term arising from the action of D^2 on $DB(A, Dv)$ is written as

$$
-\varepsilon_{kji}\int_0^t\int_0^{t'} [D^3 v^i{},_s A^s_l v^l{},_p A^p_j + v^i{},_s A^s_l D^3 v^l{},_p A^p_j]\mathrm{d}t''\mathrm{d}t'.
$$

Both summands in the integrand scale like $D^4 v\, Dv\, A\, A$. The precise structure of this summand is not very important; rather, the derivative count is the focus. Integrating by parts in time,

$$
\begin{aligned}
\int_0^t\int_0^{t'} D^4 v\, Dv\, A\, A\,\mathrm{d}t''\mathrm{d}t' = &-\int_0^t\int_0^{t'} D^4\eta\,(Dv\, A\, A)_t\mathrm{d}t''\mathrm{d}t' \\
&+ \int_0^t D^4\eta\, Dv\, A\, A\,\mathrm{d}t',
\end{aligned}
$$

from which it follows that

$$
\left\|\int_0^t\int_0^{t'} D^3 B(A(t''), Dv(t''))\mathrm{d}t''\mathrm{d}t'\right\|_0^2 \leqq C\,T\,P(\sup_{t\in[0,T]}\tilde{E}(t)).
$$

We next estimate the term associated to F. Since

$$
\begin{aligned}
DF^k = -\kappa\, \varepsilon_{kji}\Big[&Dv^r{},_s A^s_j\big[(\frac{\rho_0}{J}),_l A^l_r\big],_m A^m_i + v^r{},_s A^s_j D\big[(\frac{\rho_0}{J}),_l A^l_r\big],_m A^m_i \\
&+ v^r{},_s \big[(\rho_0 J^{-1}),_l A^l_r\big],_m D\big(A^s_j A^m_i\big)\Big],
\end{aligned}
$$

the highest-order term arising from the action of D^2 on DF is written as

$$
\kappa\varepsilon_{kji}\int_0^t\int_0^{t'} [D^3 v,^r_s A^s_j\big[(\frac{\rho_0}{J}),_l A^l_r\big],_m A^m_i + v,^r_s A^s_j D^3\big[(\frac{\rho_0}{J}),_l A^l_r\big],_m A^m_i]\mathrm{d}t''\mathrm{d}t'.
$$

The first summand in the integrand scales like $D^4 v\, D^2(\rho_0 J^{-1})\, A\, A$, and can be estimated by integrating by parts in time in a similar way as for the terms associated to $D^3 B(A, Dv)$.

The a priori more problematic term is the second one, as it seems to call for five space derivatives on $\int_0^t \eta$ that we do not have at our disposal. We first notice that since $(\rho_0 J^{-1}),_l A^l_r = \rho,_r \circ\eta = [D\rho]_r \circ \eta$, this integral is under the form $\kappa \int_0^t \int_0^{t'} D^4(D\rho \circ \eta)\, Dv\, A\, A\, \mathrm{d}t''\mathrm{d}t'$. Integrating by parts in time (in the integral from 0 to t'),

$$\kappa \int_0^t \int_0^{t'} D^4(D\rho \circ \eta)\, Dv\, A\, A\, \mathrm{d}t''\mathrm{d}t'$$
$$= -\kappa \int_0^t \int_0^{t'} (Dv\, A\, A)_t D^4 \int_0^{t''} D\rho(\eta)\, \mathrm{d}t'''\mathrm{d}t''\mathrm{d}t'$$
$$+\kappa \int_0^t Dv\, A\, A\, D^4 \int_0^{t'} D\rho(\eta)\, \mathrm{d}t''\, \mathrm{d}t'.$$

We now explain why we have control of four space derivatives of the antiderivative (with respect to time) of $D\rho(\eta)$. By definition of the κ-problem (7.2a'), we have

$$v_t + 2D\rho \circ \eta + 2\kappa[D\rho \circ \eta]_t = 0, \tag{9.9}$$

which implies by integrating (9.9) in time twice that

$$2\int_0^t \int_0^{t'} D^4(D\rho \circ \eta)\mathrm{d}t''\mathrm{d}t' + 2\kappa \int_0^t D^4(D\rho \circ \eta)\mathrm{d}t' = -D^4\eta(t) + t D^4 u_0, \tag{9.10}$$

where we have used the fact that $D^4\eta(0) = 0$ since $\eta(0) = e$. We can now use our Lemma 2 which first yields, independently of κ,

$$\left\| \int_0^t \int_0^{t'} D^4(D\rho \circ \eta)\mathrm{d}t''\mathrm{d}t' \right\|_0^2 \leqq \tilde{M}_0 + C\tilde{E}(t),$$

and then by using (9.10),

$$\left\| \int_0^t \kappa D^4(D\rho \circ \eta)\mathrm{d}t' \right\|_0^2 \leqq \tilde{M}_0 + C\tilde{E}(t). \tag{9.11}$$

Thanks to (9.11), we get the estimate

$$\left\| \int_0^t \int_0^{t'} D^3 F \mathrm{d}t''\mathrm{d}t' \right\|_0^2 \leqq C\, T\, P(\sup_{t\in[0,T]} \tilde{E}(t)),$$

and hence

$$\sup_{t\in[0,T]} \|\operatorname{curl} \eta(t)\|_3^2 \leqq \tilde{M}_0 + C\, T\, P(\sup_{t\in[0,T]} \tilde{E}(t)).$$

Step 2. Estimate for curl v_t. From (9.6),

$$\operatorname{curl} v_t = -\varepsilon_{\cdot ji} \int_0^t A_t{}^s_j(t')\mathrm{d}t'\, v^i_t,_s + F. \tag{9.12}$$

Since

$$2\kappa\,\partial_t[D\rho\circ\eta] + 2D\rho\circ\eta = -v_t,$$

by Lemma 2, we see that

$$\|[D\rho\circ\eta](t)\|_3^2 \leqq M_0 + \|v_t(t)\|_3^2, \tag{9.13}$$

from which it immediately follows that $\|F\|_2^2 \leqq \tilde{M}_0 + C\,T\,P(\sup_{t\in[0,T]}\tilde{E}(t))$. For later use, we note from equation (7.2a) that together with (9.13), we have that

$$\|\kappa\,\partial_t[D\rho\circ\eta](t)\|_3^2 \leqq M_0 + \|v_t(t)\|_3^2. \tag{9.14}$$

Since the highest-order term in $D^2B(A, Dv)$ is D^3v, we then see that $\|B(A, Dv)\|_2^2 \leqq \tilde{M}_0 + C\,T\,P(\sup_{t\in[0,T]}\tilde{E}(t))$ so that

$$\|\operatorname{curl} v_t(t)\|_2^2 \leqq \tilde{M}_0 + C\,T\,P\left(\sup_{t\in[0,T]}\tilde{E}(t)\right). \tag{9.15}$$

Step 3. Estimates for curl v_{ttt} and curl $\partial_t^5 v$. By time-differentiating (9.12), estimating in the same way as **Step 2**, we find that

$$\|\operatorname{curl} v_{ttt}(t)\|_1^2 \leqq \tilde{M}_0 + C\,T\,P\left(\sup_{t\in[0,T]}\tilde{E}(t)\right),$$

and

$$\|\operatorname{curl}\partial_t^5 v(t)\|_0^2 \leqq \tilde{M}_0 + C\,T\,P\left(\sup_{t\in[0,T]}\tilde{E}(t)\right).$$

Step 4. Estimate for $\rho_0\bar{\partial}^4$ curl η. To prove this weighted estimate, we write (9.7) as

$$\operatorname{curl} v(t) = \varepsilon_{jki} v^i{,_s}\int_0^t {A_t}^s_j(t')\mathrm{d}t' + \operatorname{curl} u_0 + \int_0^t [B(A, Dv) + F](t')\mathrm{d}t',$$

and integrate in time to find that

$$\begin{aligned}\operatorname{curl}\eta(t) = t\operatorname{curl} u_0 &+ \underbrace{\int_0^t \varepsilon_{jki} v^i{,_s}\int_0^{t'} {A_t}^s_j(t'')\mathrm{d}t''\mathrm{d}t'}_{\mathcal{I}_1} \\ &+ \underbrace{\int_0^t\int_0^{t'} B(A, Dv)(t'')\mathrm{d}t''\mathrm{d}t'}_{\mathcal{I}_2} + \underbrace{\int_0^t\int_0^{t'} F(t'')\mathrm{d}t''\mathrm{d}t'}_{\mathcal{I}_3}.\end{aligned} \tag{9.16}$$

It follows that

$$\rho_0\bar{\partial}^4\operatorname{curl}\eta(t) = t\rho_0\bar{\partial}^4\operatorname{curl} u_0 + \rho_0\bar{\partial}^4\mathcal{I}_1 + \rho_0\bar{\partial}^4\mathcal{I}_2 + \rho_0\bar{\partial}^4\mathcal{I}_3. \tag{9.17}$$

Notice that by definition, $\|t\rho_0\bar{\partial}^4 \operatorname{curl} u_0\|_0^2 \leqq \tilde{M}_0$, so we must estimate the $L^2(\Omega)$-norm of $\rho_0\bar{\partial}^4\mathcal{I}_1 + \rho_0\bar{\partial}^4\mathcal{I}_2 + \rho_0\bar{\partial}^4\mathcal{I}_3$. We first estimate $\rho_0\bar{\partial}^4\mathcal{I}_2$. We write

$$\rho_0\bar{\partial}^4\mathcal{I}_2(t) = \underbrace{\int_0^t\int_0^{t'} \varepsilon_{kji} A_t{}_j^s \rho_0\bar{\partial}^4 v,_s^i \, \mathrm{d}t''\mathrm{d}t'}_{\mathcal{K}_1} + \underbrace{\int_0^t\int_0^{t'} \varepsilon_{kji}\rho_0\bar{\partial}^4 A_t{}_j^s v,_s^i \, \mathrm{d}t''\mathrm{d}t'}_{\mathcal{K}_2} + \mathcal{R},$$

where $\mathcal{R}$ denotes *remainder terms* which are lower-order in the derivative count; in particular the terms with the highest derivative count in $\mathcal{R}$ scale like $\rho\bar{\partial}^3 Dv$ or $\rho\bar{\partial}^4\eta$, and hence satisfy the inequality $\|\mathcal{R}(t)\|_0^2 \leqq \tilde{M}_0 + C\,T\,P(\sup_{t\in[0,T]}\tilde{E}(t))$. We focus on the integral $\mathcal{K}_1$; integrating by parts in time, we find that

$$\mathcal{K}_1(t) = -\int_0^t\int_0^{t'} \varepsilon_{kji}\partial_t^2 A_j^s \rho_0\bar{\partial}^4\eta^i,_s \, \mathrm{d}t''\mathrm{d}t' + \int_0^t \varepsilon_{kji} A_t{}_j^s \rho_0\bar{\partial}^4\eta^i,_s \, \mathrm{d}t'$$

and hence

$$\|\mathcal{K}_1(t)\|_0^2 \leqq \tilde{M}_0 + C\,T\,P\left(\sup_{t\in[0,T]}\tilde{E}(t)\right).$$

Using the identity $\partial_t A_j^s = -A_p^s v_b^p A_j^b$, we see that $\mathcal{K}_2(t)$ can be estimated in the same fashion to yield the inequality

$$\left\|\rho_0\bar{\partial}^4\mathcal{I}_2(t)\right\|_0^2 \leqq \tilde{M}_0 + C\,T\,P\left(\sup_{t\in[0,T]}\tilde{E}(t)\right). \tag{9.18}$$

Using the same integration-by-parts argument, we have similarly that

$$\left\|\rho_0\bar{\partial}^4\mathcal{I}_1(t)\right\|_0^2 \leqq \tilde{M}_0 + C\,T\,P\left(\sup_{t\in[0,T]}\tilde{E}(t)\right). \tag{9.19}$$

It thus remains to estimate $\rho_0\bar{\partial}^4\mathcal{I}_3$ in (9.16). Now

$$\rho_0\bar{\partial}^4\mathcal{I}_3 = \int_0^t\int_0^{t'} \rho_0\bar{\partial}^4 F(t'')\mathrm{d}t''\mathrm{d}t',$$

which can be written under the form

$$\underbrace{\kappa\int_0^t\int_0^{t'} \rho_0\bar{\partial}^4 Dv D[D\rho(\eta)]A\mathrm{d}t''\mathrm{d}t'}_{\mathcal{T}_1} + \underbrace{\kappa\int_0^t\int_0^{t'} \rho_0\bar{\partial}^4 D[D\rho(\eta)]ADv\mathrm{d}t''\mathrm{d}t'}_{\mathcal{T}_2} + \mathcal{R},$$

where $\mathcal{R}$ once again denotes a lower-order remainder term which satisfies $\|\mathcal{R}(t)\|_0^2 \leqq \tilde{M}_0 + C\,T\,P(\sup_{t\in[0,T]}\tilde{E}(t))$.

Since

$$2\kappa\partial_t D[D\rho\circ\eta] + 2D[D\rho\circ\eta] = -Dv_t,$$

by Lemma 2, we see that independently of κ,

$$\|D[D\rho\circ\eta](t)\|_2^2 \leqq \tilde{M}_0 + \|v_t(t)\|_3^2 \leqq \tilde{M}_0 + C\tilde{E}(t)\ ,$$

and that

$$\|\kappa\partial_t D[D\rho\circ\eta](t)\|_2^2 \leqq \tilde{M}_0 + C\tilde{E}(t).$$

Thus, the Sobolev embedding theorem shows that

$$\|\kappa\partial_t D[D\rho\circ\eta](t)\|_{L^\infty(\Omega)}^2 \leqq \tilde{M}_0 + C\tilde{E}(t).$$

Hence, by using the same integration-by-parts in-time argument that we used to estimate the integral $\mathcal{K}_1$ above, we immediately obtain the inequality

$$\|\mathcal{T}_1(t)\|_0^2 \leqq \tilde{M}_0 + C\,T\,P\left(\sup_{t\in[0,T]}\tilde{E}(t)\right).$$

In order to estimate the integral $\mathcal{T}_2$, we must rely on the structure of the Euler equations (9.9) once again. Integrating in time twice, we see that

$$2\int_0^t\int_0^{t'}\rho_0\bar{\partial}^4 D(D\rho\circ\eta)+2\kappa\int_0^t\rho_0\bar{\partial}^4 D(D\rho\circ\eta)=-\rho_0\bar{\partial}^4 D\eta(t)+t\rho_0\bar{\partial}^4 Du_0. \tag{9.20}$$

According to Lemma 2, independently of κ,

$$\|\int_0^t\int_0^{t'}\rho_0\bar{\partial}^4 D(D\rho\circ\eta)\|_{L^\infty(0,T;L^2)}^2 \leqq \tilde{M}_0 + C\|\rho_0\bar{\partial}^4 D\eta(t)\|_0^2 \leqq \tilde{M}_0 + C\tilde{E}(t),$$

and then by using (9.20),

$$\left\|\kappa\int_0^t\rho_0\bar{\partial}^4 D(D\rho\circ\eta)\right\|_{L^\infty(0,T;L^2)}^2 \leqq \tilde{M}_0 + C\tilde{E}(t). \tag{9.21}$$

Returning to the estimate of $\mathcal{T}_2$, we integrate-by-parts in time (with respect to the integral from 0 to t') to find that

$$\begin{aligned}\mathcal{T}_2 &= \int_0^t\int_0^{t'}\kappa\int_0^{t''}\rho_0\bar{\partial}^4 D[D\rho(\eta)](s)\mathrm{d}s\ [A\ Dv]_t(t'')\ \mathrm{d}t''\mathrm{d}t'\\ &\quad+\int_0^t\kappa\int_0^{t'}\rho_0\bar{\partial}^4 D[D\rho(\eta)](t'')\mathrm{d}t''\ A\ Dv(t')\ \mathrm{d}t'.\end{aligned}$$

Inequality (9.21) then shows that

$$\|\mathcal{T}_2(t)\|_0^2 \leqq \tilde{M}_0 + C\,T\,P\left(\sup_{t\in[0,T]}\tilde{E}(t)\right),$$

so that $\|\rho_0\bar\partial^4\mathcal{I}_3(t)\|_0^2 \leqq \tilde M_0 + C\,T\,P(\sup_{t\in[0,T]}\tilde E(t))$, and with (9.18) and (9.19), we see that (9.17) shows that

$$\|\rho_0\bar\partial^4 \operatorname{curl}\eta(t)\|_0^2 \leqq \tilde M_0 + C\,T\,P\left(\sup_{t\in[0,T]}\tilde E(t)\right). \tag{9.22}$$

Step 5. Estimate for $\rho_0\bar\partial^3$ curl v_t. From (9.12),

$$\begin{aligned}\|\rho_0\bar\partial^3 \operatorname{curl} v_t(t)\|_0^2 &\leqq \left\|\varepsilon_{\cdot ji}\rho_0\bar\partial^3\left(\int_0^t A_t{}_j^s(t')\mathrm{d}t'\, v_t^i{}_{,s}(t)\right)\right\|_0^2 + \|\rho_0\bar\partial^3 F(t)\|_0^2\\ &\leqq C\,T\,P\left(\sup_{t\in[0,T]}\tilde E(t)\right) + \|\rho_0\bar\partial^3 F(t)\|_0^2,\end{aligned}$$

and using (9.4),

$$\begin{aligned}\|\rho_0\bar\partial^3 F(t)\|_0^2 \leqq{}& C\|\rho_0\bar\partial^3 Dv(t)\|_0^2\|\kappa D[D\rho(\eta)]\|_{L^\infty(\Omega)}^2 + C\|\kappa\rho_0\bar\partial^3 D[D\rho(\eta)]\|_0^2\\ &+C\,T\,P\left(\sup_{t\in[0,T]}\tilde E(t)\right).\end{aligned}$$

First,

$$\|\kappa D[D\rho(\eta(t))]\|_{L^\infty(\Omega)}^2 \leqq C\|\kappa D\rho(\eta(t))\|_3^2 \leqq C\tilde M_0 + C\,T\sup_{t\in[0,T]}\tilde E(t),$$

where we have used (9.14) and the fundamental theorem of calculus. Once again employing the fundamental theorem of calculus,

$$\|\rho_0\bar\partial^3 Dv(t)\|_0^2 \leqq \tilde M_0 + C\,T\sup_{t\in[0,T]}\tilde E(t),$$

and hence $\|\rho_0\bar\partial^3 Dv(t)\|_0^2\|D[D\rho(\eta)]\|_{L^\infty(\Omega)}^2 \leqq \tilde M_0 + C\,T\,P(\sup_{t\in[0,T]}\tilde E(t))$.

On the other hand, since

$$2\kappa\rho_0\partial_t\bar\partial^3 D[D\rho\circ\eta] + 2\rho_0\bar\partial^3 D[D\rho\circ\eta] = -\rho_0\bar\partial^3 Dv_t,$$

by Lemma 2, we see that independently of κ,

$$\|\rho_0\bar\partial^3 D[D\rho\circ\eta](t)\|_0^2 \leqq \tilde M_0 + \|\rho_0\bar\partial^3 Dv_t(t)\|_0^2 \leqq \tilde M_0 + C\tilde E(t),$$

and, in turn,

$$\|\kappa\rho_0\bar\partial^3 D\partial_t[D\rho\circ\eta](t)\|_0^2 \leqq \tilde M_0 + \|\rho_0\bar\partial^3 Dv_t(t)\|_0^2 \leqq \tilde M_0 + C\tilde E(t).$$

By the fundamental theorem of calculus, we thus see that

$$\|\kappa\rho_0\bar\partial^3 D[D\rho(\eta)]\|_0^2 \leqq \tilde M_0 + C\,T\,P\left(\sup_{t\in[0,T]}\tilde E(t)\right),$$

which shows that $\|\rho_0\bar\partial^3 \operatorname{curl} v_t(t)\|_0^2 \leqq \tilde M_0 + C\,T\,P(\sup_{t\in[0,T]}\tilde E(t))$.

Step 6. Estimates for $\rho_0 \bar\partial^2$ curl v_{ttt}, $\rho_0 \bar\partial$ curl $\partial_t^5 v$, and ρ_0 curl $\partial_t^7 v$. By time-differentiating (9.12) and estimating as in **Step 5**, we immediately obtain the inequality

$$\|\rho_0 \bar\partial^2 \operatorname{curl} v_{ttt}(t)\|_0^2 + \|\rho_0 \bar\partial \operatorname{curl} \partial_t^5 v(t)\|_0^2 + \|\rho_0 \operatorname{curl} \partial_t^7 v(t)\|_0^2 \leqq \tilde{M}_0 + C\, T\, P\left(\sup_{t\in[0,T]} \tilde{E}(t)\right).$$

Step 7. Estimate for $\sqrt{\kappa}\rho_0 \operatorname{curl}_\eta \bar\partial^4 v$. From (9.7)

$$\sqrt{\kappa}\rho_0 \operatorname{curl}_\eta \bar\partial^4 v(t) = \underbrace{\sqrt{\kappa}\rho_0 \bar\partial^4 \operatorname{curl} u_0}_{\mathcal{S}_1} + \underbrace{\sqrt{\kappa}\rho_0 \varepsilon_{\cdot ij} v^i{}_{,r}\, \bar\partial^4 A^r_j(t)}_{\mathcal{S}_2} + \underbrace{\int_0^t \sqrt{\kappa}\rho_0 \bar\partial^4 B(A, Dv) \mathrm{d}t'}_{\mathcal{S}_3} + \underbrace{\int_0^t \sqrt{\kappa}\rho_0 \bar\partial^4 F \mathrm{d}t'}_{\mathcal{S}_4} + \mathcal{R}(t),$$

where $\mathcal{R}(t)$ is a lower-order remainder term satisfying an inequality of the type $\int_0^T |\mathcal{R}(t)|^2 \mathrm{d}t \leqq C\, T\, P(\sup_{t\in[0,T]} \tilde{E}(t))$. We see that

$$\int_0^t \|\mathcal{S}_1\|_0^2 \mathrm{d}t' \leqq t\tilde{M}_0,$$

and since $\|\rho_0 \bar\partial^4 D\eta(t)\|_0^2$ is contained in the energy function $\tilde{E}(t)$,

$$\int_0^t \|\mathcal{S}_2\|_0^2 \mathrm{d}t' \leqq C\, T\, P\left(\sup_{t\in[0,T]} \tilde{E}(t)\right).$$

Jensen's inequality shows that

$$\int_0^t \|\mathcal{S}_3\|_0^2 \mathrm{d}t' \leqq C\, T\, P\left(\sup_{t\in[0,T]} \tilde{E}(t)\right).$$

The highest-order terms in $\mathcal{S}_4$ can be written under the form

$$\underbrace{\int_0^t \kappa^{\frac{3}{2}} \rho_0 \bar\partial^4 Dv\; A\; D[D\rho(\eta)]\; A \mathrm{d}t'}_{\mathcal{S}_{4a}} + \underbrace{\int_0^t \kappa^{\frac{3}{2}} \rho_0 \bar\partial^4 D[D\rho(\eta)]\; A\; Dv\; A \mathrm{d}t'}_{\mathcal{S}_{4b}},$$

with all other terms being lower-order and easily estimated. By Jensen's inequality and using (9.4),

$$\int_0^t \|\mathcal{S}_{4a}(t')\|_0^2 \mathrm{d}t' \leqq C\kappa \int_0^t t' \int_0^{t'} \|\sqrt{\kappa}\rho_0 \bar\partial^4 Dv(t'')\|_0^2 \mathrm{d}t'' \mathrm{d}t' \leqq C\, T \sup_{t\in[0,T]} \tilde{E}(t).$$

In order to estimate the term $\mathcal{S}_{4b}$, we use the identity

$$2\,\kappa^{\frac{3}{2}} \rho_0 \bar\partial^4 D[D\rho(\eta)](t) + 2\sqrt{\kappa} \int_0^t \rho_0 \bar\partial^4 D[D\rho(\eta)] \mathrm{d}t' = -\sqrt{\kappa}\rho_0 \bar\partial^4 Dv(t) + \sqrt{\kappa}\rho_0 \bar\partial^4 Du_0 + 2\kappa^{\frac{3}{2}} \rho_0 \bar\partial^4 D^2 \rho_0,$$

which follows from differentiating the Euler equations. Taking the $L^2(\Omega)$-inner-product of this equation with $\kappa^{\frac{3}{2}}\rho_0\bar\partial^4 D[D\rho(\eta)](t)$ and integrating in time, we deduce that

$$\int_0^t \|\kappa^{\frac{3}{2}}\rho_0\bar\partial^4 D[D\rho(\eta)](t')\|_0^2 \mathrm{d}t' \leqq \tilde M_0 + \sup_{t\in[0,T]} \tilde E(t),$$

from which it follows, using Jensen's inequality, that

$$\int_0^t \|\mathcal{S}_{4b}(t)\|_0^2 \mathrm{d}t' \leqq C\,T\,P\left(\sup_{t\in[0,T]} \tilde E(t)\right),$$

and thus $\int_0^t \|\sqrt{\kappa}\rho_0 \operatorname{curl}_\eta \bar\partial^4 v(t')\|_0^2 \mathrm{d}t' \leqq C\,T\,P(\sup_{t\in[0,T]} \tilde E(t))$.

Step 8. Estimates for $\sqrt{\kappa}\rho_0 \operatorname{curl}_\eta \bar\partial^{4-l}\partial_t^{2l} v$ for $l = 1, 2, 3, 4$. Following the identical methodology as we used for **Step 7**, we obtain the desired inequality

$$\sum_{l=1}^4 \int_0^t \|\sqrt{\kappa}\rho_0 \operatorname{curl}_\eta \bar\partial^{4-l}\partial_t^{2l} v(t')\|_0^2 \mathrm{d}t' \leqq C\,T\,P\left(\sup_{t\in[0,T]} \tilde E(t)\right).$$

□

9.4. κ-Independent Energy Estimates for Horizontal and Time Derivatives

We take $T \in (0, T_\kappa)$. In the estimates below, we will show how derivatives of the cofactor matrix $\bar\partial^4 a_i^k$ combine with derivatives of the velocity gradient $\bar\partial^4 v^i,_k$ to produce energy terms. We provide a detailed explanation as to how this energy is formed and how the error terms that arise in the process are controlled by the higher-order energy function. We will show that all of the estimates do not depend on the parameter κ.

9.4.1. The $\bar\partial^4$-Problem

Proposition 4. *For $\delta > 0$ and letting the constant $\tilde M_0$ depend on $1/\delta$,*

$$\sup_{t\in[0,T]}\left(\|\sqrt{\rho_0}\bar\partial^4 v(t)\|_0^2 + \|\rho_0\bar\partial^4 D\eta(t)\|_0^2 + \int_0^t \|\sqrt{\kappa}\rho_0\bar\partial^4 Dv(s)\|_0^2 \mathrm{d}s\right) \leqq \tilde M_0 + \delta \sup_{t\in[0,T]} \tilde E(t) + C\sqrt{T}\,P\left(\sup_{t\in[0,T]} \tilde E(t)\right). \tag{9.23}$$

Proof. Letting $\bar\partial^4$ act on (7.2a), and taking the $L^2(\Omega)$-inner product of this with $\bar\partial^4 v^i$ yields

$$\underbrace{\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho_0|\bar\partial^4 v|^2 \mathrm{d}x}_{\mathcal{I}_0} + \underbrace{\int_\Omega \bar\partial^4 a_i^k (\rho_0^2 J^{-2}),_k \bar\partial^4 v^i \mathrm{d}x}_{\mathcal{I}_1} + \underbrace{\int_\Omega a_i^k (\rho_0^2 \bar\partial^4 J^{-2}),_k \bar\partial^4 v^i \mathrm{d}x}_{\mathcal{I}_2}$$
$$+ \underbrace{\kappa\int_\Omega \bar\partial^4\partial_t a_i^k (\rho_0^2 J^{-2}),_k \bar\partial^4 v^i \mathrm{d}x}_{\mathcal{I}_3} + \underbrace{\kappa\int_\Omega a_i^k (\rho_0^2 \bar\partial^4\partial_t J^{-2}),_k \bar\partial^4 v^i \mathrm{d}x}_{\mathcal{I}_4}$$

$$+\underbrace{\kappa\int_\Omega \bar\partial^4 a_i^k(\rho_0^2\partial_t J^{-2}),_k\,\bar\partial^4 v^i\mathrm{d}x}_{\mathcal{I}_5}+\underbrace{\kappa\int_\Omega \partial_t a_i^k(\rho_0^2\bar\partial^4 J^{-2}),_k\,\bar\partial^4 v^i\mathrm{d}x}_{\mathcal{I}_6}$$

$$=\underbrace{\sum_{l=1}^3 c_l\int_\Omega \bar\partial^{4-l}a_i^k\,(\rho_0^2\bar\partial^l J^{-2}),_k\;\bar\partial^4 v^i}_{\mathcal{R}_1}+\underbrace{\kappa\sum_{l=1}^3 c_l\int_\Omega \bar\partial^{4-l}\partial_t a_i^k(\rho_0^2\bar\partial^l J^{-2}),_k\;\bar\partial^4 v^i}_{\mathcal{R}_2}$$

$$+\underbrace{\kappa\sum_{l=1}^3 c_l\int_\Omega \bar\partial^{4-l}a_i^k(\rho_0^2\bar\partial^l\partial_t J^{-2}),_k\,\bar\partial^4 v^i}_{\mathcal{R}_3}+\underbrace{\int_\Omega a_i^k\,(\bar\partial^4\rho_0^2\;J^{-2}),_k\,\bar\partial^4 v^i}_{\mathcal{R}_4}+\mathcal{R}_5. \tag{9.24}$$

The integrals $\mathcal{I}_a,\ a=1,\ldots,6$ denote the highest-order terms, while the integrals $\mathcal{R}_a,\ a=1,\ldots,5$ denote lower-order remainder terms, which throughout the paper will consist of integrals which can be shown, via elementary inequalities together with our basic assumptions (9.4), to satisfy the following estimate:

$$\int_0^T \mathcal{R}_a(t)\mathrm{d}t \leqq \tilde{M}_0+\delta\sup_{t\in[0,T]}\tilde{E}(t)+C\sqrt{T}\,P\left(\sup_{t\in[0,T]}\tilde{E}(t)\right). \tag{9.25}$$

The remainder integral $\mathcal{R}_5$ is comprised of the lower-order terms that are obtained when at most three horizontal derivatives are distributed onto ρ_0^2, and although we do not explicitly write this term, we will explain its bound directly after the analysis of the remainder term $\mathcal{R}_4$, below.

We proceed to systematically estimate each of these integrals, and we begin with the lower-order remainder terms.

Analysis of $\int_0^T \mathcal{R}_1(t)\mathrm{d}t$. We integrate by parts with respect to x_k and then with respect to the time derivative ∂_t, and use (5.5) to obtain that

$$\mathcal{R}_1=-\sum_{l=1}^3 c_l\int_0^T\int_\Omega \bar\partial^{4-l}a_i^k\;\rho_0^2\bar\partial^l J^{-2}\;\bar\partial^4 v^i,_k\;\mathrm{d}x\mathrm{d}t,$$

and thus

$$\mathcal{R}_1=\sum_{l=1}^3 c_l\int_0^T\int_\Omega \rho_0\left(\bar\partial^{4-l}a_i^k\bar\partial^l J^{-2}\right)_t\rho_0\bar\partial^4\eta^i,_k\,\mathrm{d}x\mathrm{d}t$$
$$-\sum_{l=1}^3 c_l\int_\Omega \rho_0\bar\partial^{4-l}a_i^k\bar\partial^l J^{-2}\rho_0\bar\partial^4\eta^i,_k\,\mathrm{d}x\Big|_0^T.$$

Notice that when $l=3$, the integrand in the spacetime integral on the right-hand side scales like $\ell\,[\bar\partial D\eta\,\rho_0\bar\partial^3\partial_t J^{-2}+\bar\partial Dv\,\rho_0\bar\partial^3 J^{-2}]\,\rho_0\bar\partial^4 D\eta$ where ℓ denotes an $L^\infty(\Omega)$ function. Since for any $t\in[0,T]$, $\|\rho_0\partial_t^2 J^{-2}(t)\|_3^2$ and $\|\rho_0\bar\partial^4 D\eta(t)\|_0^2$ are contained in the energy function $\tilde{E}(t)$ and $\|\bar\partial D\eta(t)\|_{L^\infty(\Omega)}$ is bounded by

$C\|\bar\partial D\eta(t)\|_2$, with $\|\bar\partial D\eta(t)\|_2^2$ being part of the energy function $\tilde E(t)$ as well, the first summand is estimated by an L^∞–L^2–L^2 Hölder's inequality, leading to a majoration by a term $C\,T\ \sup_{t\in[0,T]}\tilde E(t)^{\frac32}$, which indeed leads to the type indicated in (9.25). Similarly, for the second spacetime summand, we use that $\|\rho_0 J^{-2}(t)\|_4^2$ is contained in $\tilde E(t)$ together with an L^4–L^4–L^2 Hölder's inequality.

When $l = 1$, the integrand in the spacetime integral on the right-hand side scales like $\ell\ [\bar\partial D\eta\ \rho_0\bar\partial^3 {a_t}_i^k + \bar\partial Dv\ \rho_0\bar\partial^3 a_i^k]\ \rho_0\bar\partial^4\eta^i{,_k}$. Since $\|\rho_0\bar\partial^3 Dv_t(t)\|_0^2$ is contained in the energy function $\tilde E(t)$ and since $\bar\partial D\eta \in L^\infty(\Omega)$, the first summand is estimated using an L^∞–L^2–L^2 Hölder's inequality. We write the second summand as $\bar\partial Dv\ \rho_0\bar\partial^3 a_i^\beta\ \rho_0\bar\partial^4\eta^i{,_\beta} + \bar\partial Dv\ \rho_0\bar\partial^3 a_i^3\ \rho_0\bar\partial^4\eta^i{,_3}$. We estimate

$$
\begin{aligned}
&\int_0^T\int_\Omega \bar\partial Dv\ \rho_0\bar\partial^3 a_i^\beta\ \rho_0\bar\partial^4\eta^i{,_\beta}\ \mathrm{d}x\mathrm{d}t\\
&\quad= -\int_0^T\int_\Omega [\bar\partial Dv\ \rho_0\bar\partial^3 a_i^\beta{,_\beta}\ \rho_0\bar\partial^4\eta^i + \bar\partial Dv{,_\beta}\ \rho_0\bar\partial^3 a_i^\beta\ \rho_0\bar\partial^4\eta^i]\mathrm{d}x\mathrm{d}t\\
&\quad\leqq C\int_0^T \big(\|\bar\partial Dv(t)\|_{L^3(\Omega)}\|\rho_0\bar\partial^4 a(t)\|_0\ \|\rho_0\bar\partial^4\eta(t)\|_{L^6(\Omega)}\\
&\qquad + \|\bar\partial^2 Dv(t)\|_{L^3(\Omega)}\|\rho_0\bar\partial^4\eta(t)\|_{L^6(\Omega)}\|\bar\partial^3 a\|_0\big)\mathrm{d}t,
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\int_0^T\int_\Omega \bar\partial Dv\ \rho_0\bar\partial^3 a_i^\beta\ \rho_0\bar\partial^4\eta^i{,_\beta}\ \mathrm{d}x\mathrm{d}t\\
&\quad\leqq C\int_0^T \big(\|\bar\partial Dv(t)\|_{H^{0.5}(\Omega)}\|\rho_0\bar\partial^4 a(t)\|_0\ \|\rho_0\bar\partial^4\eta(t)\|_1\\
&\qquad + \|\bar\partial^2 Dv(t)\|_{H^{0.5}(\Omega)}\|\rho_0\bar\partial^4\eta(t)\|_1\ \|\bar\partial^3 a\|_0\big)\mathrm{d}t\\
&\quad\leqq C\int_0^T \|v(t)\|_{H^{2.5}(\Omega)}\big(\|\rho_0\bar\partial^4 D\eta(t)\|_0^2 + \|\rho_0\bar\partial^4 D\eta(t)\|_0\|\eta(t)\|_4\big)\ \mathrm{d}t\\
&\qquad + C\int_0^T \|v(t)\|_{H^{3.5}(\Omega)}\big(\|\eta(t)\|_4^2 + \|\rho_0\bar\partial^4 D\eta(t)\|_0\|\eta(t)\|_4\big)\ \mathrm{d}t,
\end{aligned}
\tag{9.26}
$$

where we have used Hölder's inequality, followed by the Sobolev embeddings $H^{0.5}(\Omega) \hookrightarrow L^3(\Omega)$ and $H^1(\Omega) \hookrightarrow L^6(\Omega)$. We also rely on the interpolation estimate

$$
\begin{aligned}
\|v\|^2_{L^2(0,T;H^{\frac72}(\Omega))} &\leqq C\big(\|v(t)\|_3\|\eta\|_4\big)\Big|_T^0 + C\|v_t\|_{L^2(0,T;H^3(\Omega))}\|\eta\|_{L^2(0,T;H^4(\Omega))}\\
&\leqq M_0 + \delta\sup_{[0,T]}\|\eta\|_4^2 + CT\sup_{[0,T]}\Big(\|\eta\|_4^2 + \|v_t\|_3^2\Big),
\end{aligned}
\tag{9.27}
$$

where the last inequality follows from Young's and Jensen's inequalities. Using this together with the Cauchy–Schwarz inequality, (9.26) is bounded by $CT\,P(\sup_{t\in[0,T]}\tilde E(t))$. Next, since (5.6) shows that each component of a_i^3 is quadratic in $\bar\partial\eta$, we see that the same analysis shows the spacetime integral of $\bar\partial Dv\ \rho_0\bar\partial^3 a_i^3\ \rho_0\bar\partial^4\eta^i{,_3}$ has the same bound, and so we have estimated the case $l = 1$.

For the case that $l = 2$, the integrand in the spacetime integral on the right-hand side of the expression for $\mathcal{R}_1$ scales like $\ell\ \bar{\partial}^2 D\eta\ \bar{\partial}^2 Dv\ \rho_0 \bar{\partial}^4 D\eta$, so that an $L^6 - L^3 - L^2$ Hölder's inequality, followed by the same analysis as for the case $l = 1$ provides the same bound as for the case $l = 1$.

To deal with the space integral on the right-hand side of the expression for $\mathcal{R}_1$, the integral at time $t = 0$ is equal to zero since $\eta(x, 0) = x$, whereas the integral evaluated at $t = T$ is written, using the fundamental theorem of calculus, as

$$-\sum_{l=1}^{3} c_l \int_\Omega \rho_0 \bar{\partial}^{4-l} a_i^k \bar{\partial}^l J^{-2} \rho_0 \bar{\partial}^4 \eta^i,_k \, dx\Big|_{t=T}$$
$$= -\sum_{l=1}^{3} c_l \int_\Omega \rho_0 \int_0^T (\bar{\partial}^{4-l} a_i^k \bar{\partial}^l J^{-2})_t \, dt\, \rho_0 \bar{\partial}^4 \eta^i,_k (T) dx,$$

which can be estimated in the identical fashion as the corresponding spacetime integral. As such, we have shown that $\mathcal{R}_1$ has the claimed bound (9.25).

Analysis of $\int_0^T \mathcal{R}_2(t)dt$. Using (5.5), we integrate by parts, to find that

$$\int_0^T \mathcal{R}_2(t)dt = -\sum_{l=1}^{3} c_l \int_0^T \int_\Omega \kappa \bar{\partial}^{4-l} a_i^k\, \rho_0^2 \bar{\partial}^l J^{-2}\, \bar{\partial}^4 v^i,_k \, dxdt$$
$$\leqq \sum_{l=1}^{3} c_l \sqrt{\kappa T} \sup_{[0,T]} \|\bar{\partial}^{4-l} a_i^k\, \rho_0 \bar{\partial}^l J^{-2}\|_0 \|\sqrt{\kappa}\rho_0 \bar{\partial}^4 v^i,_k \|_{L^2(0,T;L^2(\Omega))}$$
$$\leqq C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t)\right),$$

the last inequality following from the fact that $\|\sqrt{\kappa}\rho_0 \bar{\partial}^4 Dv\|^2_{L^2(0,T;L^2(\Omega))}$ is contained in the energy function and that for $l = 1, 2, 3$, $\bar{\partial}^{4-l} a_i^k\, \rho_0 \bar{\partial}^l J^{-2}$ contains at most four space derivatives of $\eta(t)$, and is controlled L^∞-in-time.

Analysis of $\int_0^T \mathcal{R}_3(t)dt$. This remainder integral is estimated in the same way as $\int_0^T \mathcal{R}_2(t)dt$.

Analysis of $\int_0^T \mathcal{R}_4(t)dt$. Integration by parts using (5.5) shows that

$$\int_0^T \mathcal{R}_4(t)dt = \int_0^T \int_\Omega a_i^k (\bar{\partial}^4 \rho_0^2\, J^{-2}),_k\, \bar{\partial}^4 v^i dxdt$$
$$= \int_0^T \int_\Omega \bar{\partial}^4 \rho_0^2 \, (\frac{a_i^k}{J^2})_t \bar{\partial}^4 \eta^i,_k \, dxdt - \int_\Omega \bar{\partial}^4 \rho_0^2\, \frac{a_i^k}{J^2} \bar{\partial}^4 \eta^i,_k \, dx\Big|_{t=T}$$
$$= \int_0^T \int_\Omega \bar{\partial}^4 \rho_0^2\, (J^{-2} a_i^k)_t \bar{\partial}^4 \eta^i,_k \, dxdt - \int_\Omega \bar{\partial}^4 \rho_0^2\, \bar{\partial}^4 \operatorname{div} \eta(T) dx$$
$$- \int_\Omega \bar{\partial}^4 \rho_0^2 \int_0^T \partial_t (J^{-2} a_i^k) dt\, \bar{\partial}^4 \eta^i,_k (T) dx,$$

so that by the Cauchy–Schwarz inequality and Young's inequality,

$$\int_0^T \mathcal{R}_4(t)\mathrm{d}t \leqq M_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\,T\,P\left(\sup_{t\in[0,T]} \tilde{E}(t)\right).$$

Analysis of $\int_0^T \mathcal{R}_5(t)\mathrm{d}t$. The highest-order term is

$$\int_0^T \int_\Omega a_i^k (\bar\partial^3 \rho_0^2\, \bar\partial J^{-2}),_k\, \bar\partial^4 v^i \,\mathrm{d}x\mathrm{d}t,$$

which can be estimated directly using the Cauchy–Schwarz inequality to yield

$$\int_0^T \mathcal{R}_5(t)\mathrm{d}t \leqq M_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\,T\,P\left(\sup_{t\in[0,T]} \tilde{E}(t)\right).$$

Analysis of the integral $\int_0^T \mathcal{I}_0(t)\mathrm{d}t$. Integrating $\mathcal{I}_0$ from 0 to T, we see that

$$\int_0^T \mathcal{I}_0(t)\mathrm{d}t = \frac{1}{2}\int_\Omega \rho_0 |\bar\partial^4 v(T)|^2 \mathrm{d}x - \tilde{M}_0.$$

Analysis of the integral $\int_0^T \mathcal{I}_1(t)\mathrm{d}t$. To estimate $\mathcal{I}_1$, we first integrate by parts using (5.5), to obtain

$$\mathcal{I}_1 = -\int_\Omega \bar\partial^4 a_i^k \rho_0^2 J^{-2}\, \bar\partial^4 v^i,_k \,\mathrm{d}x.$$

We then use the formula (5.3) for horizontally differentiating the cofactor matrix:

$$\mathcal{I}_1 = \int_\Omega {\rho_0}^2 J^{-3}\, \bar\partial^4 \eta^r,_s\, [a_i^s a_r^k - a_r^s a_i^k]\, \bar\partial^4 v^i,_k \,\mathrm{d}x + \mathcal{R},$$

where the remainder $\mathcal{R}$ satisfies (9.25). We decompose the highest-order term in $\mathcal{I}_1$ as the sum of the following two integrals:

$$\mathcal{I}_{1a} = \int_\Omega {\rho_0}^2 J^{-3}\, (\bar\partial^4 \eta^r,_s\, a_i^s)(\bar\partial^4 v^i,_k\, a_r^k)\mathrm{d}x,$$
$$\mathcal{I}_{1b} = -\int_\Omega {\rho_0}^2 J^{-3}\, (\bar\partial^4 \eta^r,_s\, a_r^s)(\bar\partial^4 v^i,_k\, a_i^k)\mathrm{d}x.$$

Since $v = \eta_t$, $\mathcal{I}_{1a}$ is an exact derivative modulo an antisymmetric commutation with respect to the free indices i and r; namely,

$$\bar\partial^4 \eta^r,_s\, a_i^s \bar\partial^4 v^i,_k\, a_r^k = \bar\partial^4 \eta^i,_s\, a_r^s \bar\partial^4 v^i,_k\, a_r^k + (\bar\partial^4 \eta^r,_s\, a_i^s - \bar\partial^4 \eta^i,_s\, a_r^s)\bar\partial^4 v^i,_k\, a_r^k. \tag{9.28}$$

and

$$\bar\partial^4 \eta^i,_s\, a_r^s \bar\partial^4 v^i,_k\, a_r^k = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\bar\partial^4 \eta^i,_r\, a_k^r\, \bar\partial^4 \eta^i,_s\, a_k^s\big) - \frac{1}{2}\bar\partial^4 \eta^r,_s\; \bar\partial^4 \eta^i,_k\; (a_r^s a_i^k)_t, \tag{9.29}$$

so the first term on the right-hand side of (9.28) produces an exact time derivative of a positive energy contribution.

For the second term on the right-hand side of (9.28), note the identity

$$(\bar\partial^4\eta^r,_s a_i^s - \bar\partial^4\eta^i,_s a_r^s)\bar\partial^4 v^i,_k a_r^k = -J^2\varepsilon_{ijk}\bar\partial^4\eta^k,_r A_j^r \varepsilon_{imn}\bar\partial^4 v^n,_s A_m^s. \quad (9.30)$$

We have used the permutation symbol ε to encode the anti-symmetry in this relation, and the basic fact that the trace of the product of symmetric and antisymmetric matrices is equal to zero.

Recalling our notation $[\text{curl}_\eta F]^i = \varepsilon_{ijk}F^k,_r A_j^r$ for a vector-field F, (9.30) can be written as

$$(\bar\partial^4\eta^r,_s a_i^s - \bar\partial^4\eta^i,_s a_r^s)\bar\partial^4 v^i,_k a_r^k = -J^2 \,\text{curl}_\eta \bar\partial^4\eta \cdot \text{curl}_\eta \bar\partial^4 v, \quad (9.31)$$

which can also be written as an exact derivative in time:

$$\begin{aligned}\text{curl}_\eta \bar\partial^4\eta \cdot \text{curl}_\eta \bar\partial^4 v &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\,\text{curl}_\eta \bar\partial^4\eta|^2 - \bar\partial^4\eta^k,_r \bar\partial^4\eta^k,_s (A_j^r A_j^s)_t \\ &\quad + \bar\partial^4\eta^k,_r \bar\partial^4\eta^j,_s (A_j^r A_k^s)_t. \end{aligned} \quad (9.32)$$

The terms in (9.29) and (9.32) which are not the exact time derivatives are quadratic in $\rho_0\bar\partial^4 D\eta$ with coefficients in $L^\infty([0,T]\times\Omega)$ and can thus be absorbed into remainder integrals $\mathcal{R}$ satisfying the inequality (9.25). Letting

$$D_\eta\bar\partial^4\eta = \bar\partial^4 D\eta\; A \quad (\text{matrix multiplication of } \bar\partial^4 D\eta \text{ with } A),$$

we have that

$$\mathcal{I}_{1a} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega {\rho_0}^2 J^{-1}|D_\eta\bar\partial^4\eta|^2 \mathrm{d}x - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega {\rho_0}^2 J^{-1}|\,\text{curl}_\eta \bar\partial^4\eta|^2 \mathrm{d}x + \mathcal{R}.$$

where, once again, the remainder $\mathcal{R}$ satisfies (9.25).

With the notation $\text{div}_\eta F = A_i^j F^i,_j$, the differentiation formula (5.1) shows that $\mathcal{I}_{1b}$ can be written as

$$\mathcal{I}_{1b} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega {\rho_0}^2 J^{-1}|\,\text{div}_\eta \bar\partial^4\eta|^2 \mathrm{d}x + \mathcal{R}.$$

It follows that

$$\begin{aligned}\mathcal{I}_1 &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega {\rho_0}^2 J^{-1}\left(|D_\eta\bar\partial^4\eta|^2 - |\,\text{curl}_\eta \bar\partial^4\eta|^2 - |\,\text{div}_\eta \bar\partial^4\eta|^2\right)\mathrm{d}x + \mathcal{R} \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega {\rho_0}^2 \left(|D\bar\partial^4\eta|^2 - J^{-1}|\,\text{curl}_\eta \bar\partial^4\eta|^2 - J^{-1}|\,\text{div}_\eta \bar\partial^4\eta|^2\right)\mathrm{d}x + \mathcal{R},\end{aligned}$$

where we have used the fundamental theorem of calculus for the second equality on the term $D_\eta\bar\partial^4\eta$ as well as the fact that T_κ was chosen sufficiently small so that $\frac{1}{2} < J(t) < \frac{3}{2}$; in particular, we write

$$D_\eta\bar\partial^4\eta = \bar\partial^4 D\eta\; A = \bar\partial^4 D\eta\;\text{Id} + \bar\partial^4 D\eta \int_0^t A_t(s)\mathrm{d}s.$$

It is thus clear that $\int_\Omega \rho_0^2 J^{-1}|D_\eta \bar\partial^4\eta|^2 dx$ differs from $\int_\Omega \rho_0^2|D\bar\partial^4\eta|^2 dx$ by $\mathcal{R}$. Hence,

$$\int_0^T \mathcal{I}_1(t)ds \geqq \frac{1}{2}\int_\Omega {\rho_0}^2\left(\frac{2}{3}|D\bar\partial^4\eta(T)|^2 - J^{-1}|\operatorname{curl}_\eta \bar\partial^4\eta(T)|^2\right)dx$$
$$-\frac{1}{2}\int_\Omega {\rho_0}^2\left(J^{-1}|\operatorname{div}_\eta \bar\partial^4\eta(T)|^2\right)dx - \tilde{M}_0 + \int_0^T \mathcal{R}(t)dt.$$

Analysis of the integral $\int_0^T \mathcal{I}_2(t)dt$. $\bar\partial^4 J^{-2} = -2J^{-3}\bar\partial^4 J$ plus lower-order terms, which have at most three horizontal derivatives acting on J. For such lower-order terms, we integrate by parts with respect to ∂_t, and estimate the resulting integrals in the same manner as we estimated the remainder term $\mathcal{R}_1$, and obtain the same bound.

Thus,

$$\begin{aligned}\mathcal{I}_2 &= 2\int_\Omega \rho_0^2 J^{-3} a_s^r \bar\partial^4\eta^s,_r\ a_i^k\bar\partial^4 v^i,_k\, dx + \mathcal{R}\\ &= \frac{d}{dt}\int_\Omega \frac{\rho_0^2}{J^3} a_s^r\bar\partial^4\eta^s,_r\ a_i^k\bar\partial^4\eta^i,_k\, dx - \int_\Omega \rho_0^2\Big(\frac{a_s^r a_i^k}{J^3}\Big)_t \bar\partial^4\eta^s,_r\,\bar\partial^4\eta^i,_k\, dx + \mathcal{R}\\ &= \frac{d}{dt}\int_\Omega \rho_0^2 J^{-1}|\operatorname{div}_\eta \bar\partial^4\eta|^2 dx + \mathcal{R}\end{aligned}$$

so that $\int_0^T \mathcal{I}_2(t)dt = \int_\Omega \rho_0^2 J^{-1}|\operatorname{div}_\eta \bar\partial^4\eta(T)|^2 dx - \tilde{M}_0 + \int_0^T \mathcal{R}(t)dt$.
Analysis of the integral $\int_0^T \mathcal{I}_3(t)dt$. This follows closely our analysis of the integral $\mathcal{I}_1$. We first integrate by parts, using (5.5), to obtain

$$\mathcal{I}_3 = -\kappa\int_\Omega \bar\partial^4\partial_t a_i^k\,\rho_0^2 J^{-2}\,\bar\partial^4 v^i,_k\, dx.$$

We then use the formula (5.4) for horizontally differentiating the cofactor matrix:

$$\mathcal{I}_3 = \kappa\int_\Omega {\rho_0}^2 J^{-3}\,\bar\partial^4 v^r,_s\ [a_i^s a_r^k - a_r^s a_i^k]\,\bar\partial^4 v^i,_k\ dx + \mathcal{R},$$

where the remainder $\mathcal{R}$ satisfies (9.25). We decompose the highest-order term in $\mathcal{I}_3$ as the sum of the following two integrals:

$$\mathcal{I}_{3a} = \kappa\int_\Omega \rho_0^2 J^{-3}\,(\bar\partial^4 v^r,_s\, a_i^s)(\bar\partial^4 v^i,_k\, a_r^k)dx,$$
$$\mathcal{I}_{3b} = -\kappa\int_\Omega {\rho_0}^2 J^{-1}\,|\operatorname{div}_\eta \bar\partial^4 v|^2 dx.$$

Since

$$\begin{aligned}\mathcal{I}_{3a} &= \kappa\int_\Omega \rho_0^2 J^{-3}\Big[\bar\partial^4 v^i,_s\, a_r^s\bar\partial^4 v^i,_k\, a_r^k + (\bar\partial^4 v^r,_s\, a_i^s - \bar\partial^4 v^i,_s\, a_r^s)\bar\partial^4 v^i,_k\, a_r^k\Big]dx\\ &= \kappa\int_\Omega \rho_0 J^{-1}\bar\partial^4 v^i,_s\, A_r^s\bar\partial^4 v^i,_k\, A_r^k dx - \kappa\int_\Omega \rho_0^2 J^{-1}|\operatorname{curl}_\eta \bar\partial^4 v|^2 dx,\end{aligned}$$

and letting $D_\eta \bar\partial^4 v = \bar\partial^4 Dv\ A$ (matrix multiplication of $\bar\partial^4 Dv$ with$A$), we thus have that

$$\int_0^T \mathcal{I}_3(t)\mathrm{d}t = \kappa \int_0^T \int_\Omega \frac{\rho_0^2}{J} |D_\eta \bar\partial^4 v|^2\, \mathrm{d}x\mathrm{d}t - \kappa \int_0^T \int_\Omega \frac{\rho_0^2}{J} |\operatorname{curl}_\eta \bar\partial^4 v|^2 \mathrm{d}x\mathrm{d}t$$
$$-\kappa \int_0^T \int_\Omega {\rho_0}^2 J^{-1}\, |\operatorname{div}_\eta \bar\partial^4 v|^2 \mathrm{d}x\mathrm{d}t + \int_0^T \mathcal{R}\mathrm{d}t.$$

Analysis of the integral $\int_0^T \mathcal{I}_4(t)\mathrm{d}t$. Integrating by parts, and using (5.2),

$$\int_0^T \mathcal{I}_4(t)\mathrm{d}t = 2\kappa \int_0^T \int_\Omega \rho_0^2 J^{-3} a_s^r \bar\partial^4 v^s,_r\ a_i^k \bar\partial^4 v^i,_k\, \mathrm{d}x\mathrm{d}t + \int_0^T \mathcal{R}\mathrm{d}t$$
$$= 2\kappa \int_0^T \int_\Omega {\rho_0}^2 J^{-1}\, |\operatorname{div}_\eta \bar\partial^4 v|^2 \mathrm{d}x\mathrm{d}t + \int_0^T \mathcal{R}\mathrm{d}t.$$

Analysis of the integral $\int_0^T \mathcal{I}_5(t)\mathrm{d}t$. Integrating by parts, and using the Cauchy–Schwarz inequality, we see that

$$\int_0^T \mathcal{I}_5\mathrm{d}t = -\kappa \int_0^T \int_\Omega \rho_0^2 \partial_t J^{-2}\, \bar\partial^4 a_i^k\, \bar\partial^4 v^i,_k\, \mathrm{d}x\mathrm{d}t$$
$$\leqq \sqrt{\kappa T} \sup_{[0,T]} \left(\|\partial_t J^{-2}\|_{L^\infty(\Omega)} \|\rho_0 \bar\partial^4 a_i^k\|_0 \right) \|\sqrt{\kappa}\rho_0 \bar\partial^4 v^i,_k\, \|_{L^2(0,T;L^2(\Omega))}$$
$$\leqq C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t) \right),$$

the last inequality following from the Sobolev embedding theorem and the $L^\infty(0,T)$ control of $\|\rho_0 \bar\partial^4 a_i^k(t)\|_0$.

Analysis of the integral $\int_0^T \mathcal{I}_6(t)\mathrm{d}t$. Estimating in the same fashion as for $\mathcal{I}_5$ shows that $\int_0^T \mathcal{I}_6(t)\mathrm{d}t \leqq C\sqrt{T}\, P(\sup_{t\in[0,T]} \tilde{E}(t))$.

The sum $\sum_{a=0}^6 \int_0^T \mathcal{I}_a(t)\mathrm{d}t$. By considering the sum of all the integrals $\int_0^T \mathcal{I}_a(t)\mathrm{d}t$ for $a = 0, \ldots, 6$, we obtain the inequality

$$\sup_{[0,T]} \frac{1}{2} \left(\int_\Omega \rho_0 |\bar\partial^4 v|^2 \mathrm{d}x + \int_\Omega \frac{\rho_0^2}{J} |\bar\partial^4 D\eta|^2 \mathrm{d}x \right) + \kappa \int_0^T \int_\Omega \frac{\rho_0^2}{J} |D_\eta \bar\partial^4 v(t)|^2 \mathrm{d}x\mathrm{d}t$$
$$\leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t) \right)$$
$$+ \sup_{[0,T]} \int_\Omega \frac{\rho_0^2}{J} |\bar\partial^4 \operatorname{curl} \eta|^2 \mathrm{d}x + \kappa \int_0^T \int_\Omega \frac{\rho_0^2}{J} |\operatorname{curl}_\eta \bar\partial^4 v|^2 \mathrm{d}x\mathrm{d}t.$$

Next we notice that,

$$\kappa \int_0^T \int_\Omega \frac{\rho_0^2}{J} |D_\eta \bar\partial^4 v(t)|^2 \mathrm{d}x\mathrm{d}t = \kappa \int_0^T \int_\Omega \frac{\rho_0^2}{J} \bar\partial^4 v^i{}_{,r} A_k^r \bar\partial^4 v^i{}_{,s} A_k^s \mathrm{d}x\mathrm{d}t$$
$$= \kappa \int_0^T \int_\Omega \rho_0^2 \bar\partial^4 v^i{}_{,k} \bar\partial^4 v^i{}_{,k} \,\mathrm{d}x\mathrm{d}t$$
$$+ \kappa \int_0^T \int_\Omega \rho_0^2 \left[\frac{A_k^r A_k^s}{J} - \delta_k^r \delta_k^s \right] \bar\partial^4 v,_r^i \bar\partial^4 v,_s^i \,\mathrm{d}x\mathrm{d}t.$$

It thus follows from (9.3) that

$$\sup_{[0,T]} \frac{1}{2} \left(\int_\Omega \rho_0 |\bar\partial^4 v|^2 \mathrm{d}x + \int_\Omega \frac{\rho_0^2}{J} |\bar\partial^4 D\eta|^2 \mathrm{d}x \right) + \frac{\kappa}{2} \int_0^T \int_\Omega \rho_0^2 |D\bar\partial^4 v|^2 \mathrm{d}x\mathrm{d}t$$
$$\leqq \tilde M_0 + \delta \sup_{t\in[0,T]} \tilde E(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde E(t) \right)$$
$$+ \sup_{[0,T]} \int_\Omega \frac{\rho_0^2}{J} |\bar\partial^4 \operatorname{curl} \eta|^2 \mathrm{d}x + \kappa \int_0^T \int_\Omega \rho_0^2 J^{-1} |\operatorname{curl}_\eta \bar\partial^4 v|^2 \mathrm{d}x\mathrm{d}t.$$

The curl estimates (9.5) provide the bound for the last two integrals from which the desired result is obtained and the proof of the proposition is completed. □

Corollary 1. (Estimates for the trace of the tangential components of $\eta(t)$) *For* $\alpha = 1, 2$, *and* $\delta > 0$,

$$\sup_{t\in[0,T]} |\eta^\alpha(t)|_{3.5}^2 \leqq \tilde M_0 + \delta \sup_{t\in[0,T]} \tilde E + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde E(t) \right).$$

Proof. The weighted embedding estimate (3.2) shows that

$$\|\bar\partial^4 \eta(t)\|_0^2 \leqq C \int_\Omega \rho_0^2 \left(|\bar\partial^4 \eta|^2 + |\bar\partial^4 D\eta|^2 \right) \mathrm{d}x.$$

Now

$$\sup_{t\in[0,T]} \int_\Omega \rho_0^2 |\bar\partial^4 \eta|^2 \mathrm{d}x = \sup_{t\in[0,T]} \int_\Omega \rho_0^2 \left| \int_0^t \bar\partial^4 v \mathrm{d}t' \right|^2 \mathrm{d}x \leqq T^2 \sup_{t\in[0,T]} \|\sqrt{\rho_0} \bar\partial^4 v\|_0^2.$$

It follows from Proposition 4 that

$$\sup_{t\in[0,T]} \|\bar\partial^4 \eta(t)\|_0^2 \leqq \tilde M_0 + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde E(t) \right).$$

According to our curl estimates (9.5), $\sup_{[0,T]} \|\operatorname{curl} \eta\|_3^2 \leqq \tilde M_0 + CT\, P(\sup_{[0,T]} \tilde E)$, from which it follows that

$$\sup_{t\in[0,T]} \|\bar\partial^4 \operatorname{curl} \eta(t)\|_{H^1(\Omega)'}^2 \leqq \tilde M_0 + C\,T\, P\left(\sup_{t\in[0,T]} \tilde E(t) \right),$$

since $\bar\partial$ is a horizontal derivative, and integration by parts with respect to $\bar\partial$ does not produce any boundary contributions. From the tangential trace inequality (6.2), we find that

$$\sup_{t\in[0,T]} |\bar\partial^4 \eta^\alpha(t)|^2_{-0.5} \leqq \tilde M_0 + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde E(t)\right),$$

from which it follows that

$$\sup_{t\in[0,T]} |\eta^\alpha(t)|^2_{3.5} \leqq \tilde M_0 + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde E(t)\right).$$

□

9.5. The ∂_t^8-Problem

Proposition 5. *For $\delta > 0$ and letting the constant $\tilde M_0$ depend on $1/\delta$,*

$$\sup_{t\in[0,T]}\left(\|\sqrt{\rho_0}\partial_t^8 v(t)\|_0^2 + \|\rho_0\partial_t^7 Dv(t)\|_0^2 + \int_0^t \|\sqrt{\kappa}\rho_0 D\partial_t^8 v(s)\|_0^2 \mathrm{d}s\right) \leqq \tilde M_0 + \delta \sup_{[0,T]} \tilde E + C\sqrt{T}\, P(\sup_{[0,T]} \tilde E). \tag{9.33}$$

Proof. Letting ∂_t^8 act on (7.2a), and taking the $L^2(\Omega)$-inner product of this with $\partial_t^8 v^i$ yields

$$\begin{aligned}
&\underbrace{\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho_0|\partial_t^8 v|^2\mathrm{d}x}_{\mathcal{I}_0} + \underbrace{\int_\Omega \partial_t^8 a_i^k(\rho_0^2 J^{-2}),_k\, \partial_t^8 v^i\,\mathrm{d}x}_{\mathcal{I}_1} + \underbrace{\int_\Omega a_i^k(\rho_0^2\partial_t^8 J^{-2}),_k\, \partial_t^8 v^i\,\mathrm{d}x}_{\mathcal{I}_2}\\
&\quad + \underbrace{\kappa\int_\Omega \partial_t^8\partial_t a_i^k(\rho_0^2 J^{-2}),_k\, \partial_t^8 v^i\,\mathrm{d}x}_{\mathcal{I}_3} + \underbrace{\kappa\int_\Omega a_i^k(\rho_0^2\partial_t^8\partial_t J^{-2}),_k\, \partial_t^8 v^i\,\mathrm{d}x}_{\mathcal{I}_4}\\
&\quad + \underbrace{\kappa\int_\Omega \partial_t^8 a_i^k(\rho_0^2\partial_t J^{-2}),_k\, \partial_t^8 v^i\,\mathrm{d}x}_{\mathcal{I}_5} + \underbrace{\kappa\int_\Omega \partial_t a_i^k(\rho_0^2\partial_t^8 J^{-2}),_k\, \partial_t^8 v^i\,\mathrm{d}x}_{\mathcal{I}_6}\\
&= \underbrace{\sum_{l=1}^7 c_l\int_\Omega \partial_t^{8-l} a_i^k(\rho_0^2\partial_t^l J^{-2}),_k\ \partial_t^8 v^i}_{\mathcal{R}_1} + \underbrace{\kappa\sum_{l=1}^7 c_l\int_\Omega \partial_t^{8-l}\partial_t a_i^k(\rho_0^2\partial_t^l J^{-2}),_k\ \partial_t^8 v^i}_{\mathcal{R}_2}\\
&\quad + \underbrace{\kappa\sum_{l=1}^7 c_l\int_\Omega \partial_t^{8-l} a_i^k\ (\rho_0^2\partial_t^l\partial_t J^{-2}),_k\ \partial_t^8 v^i\,\mathrm{d}x}_{\mathcal{R}_3}\,.
\end{aligned} \tag{9.34}$$

The integrals $\mathcal{I}_a$, $a = 1,\ldots,6$ denote the highest-order terms, while the integrals $\mathcal{R}_a$, $a = 1, 2, 3$ denote lower-order remainder terms, which we will once again prove satisfy

$$\int_0^T \mathcal{R}_a(t)\mathrm{d}t \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\,P\left(\sup_{t\in[0,T]} \tilde{E}(t)\right). \tag{9.35}$$

The analysis of the lower-order remainder term $\mathcal{R}_1(t)$ differs slightly from the corresponding remainder term in the $\bar{\partial}^4$ energy estimates, so we proceed with the details of this analysis.

Analysis of $\int_0^T \mathcal{R}_1(t)\mathrm{d}t$. Using (5.5), we integrate by parts with respect to x_k and then with respect to the time derivative ∂_t to obtain that

$$\begin{aligned}
\mathcal{R}_1 &= -\sum_{l=1}^{7} c_l \int_0^T \int_\Omega \partial_t^{8-l} a_i^k\, \rho_0^2 \partial_t^l J^{-2}\, \partial_t^8 v^i,_k\ \mathrm{d}x\mathrm{d}t \\
&= \sum_{l=1}^{7} c_l \int_0^T \int_\Omega \rho_0 \left(\partial_t^{8-l} a_i^k \partial_t^l J^{-2}\right)_t \rho_0 \partial_t^7 v^i,_k\, \mathrm{d}x\mathrm{d}t \\
&\quad - \sum_{l=1}^{7} c_l \int_\Omega \rho_0 \partial_t^{8-l} a_i^k \bar{\partial}^l J^{-2} \rho_0 \partial_t^7 v^i,_k\, \mathrm{d}x \Big|_0^T.
\end{aligned}$$

Notice that when $l = 7$, the integrand in the spacetime integral on the right-hand side scales like $\ell\,[Dv_t\, \rho_0 \partial_t^6 Dv + Dv\, \rho_0 \partial_t^7 Dv]\, \rho_0 \partial_t^7 Dv$, where ℓ denotes an $L^\infty(\Omega)$ function. Since $\|\rho_0 \partial_t^7 Dv(t)\|_0^2$ is contained in the energy function $\tilde{E}(t)$, and $\|Dv_t(t)\|_{L^\infty(\Omega)} \leqq C\|Dv_t(t)\|_2$, with $\|Dv_t(t)\|_2^2$ being a part of $\tilde{E}(t)$, and since we can write $\rho_0\partial_t^6 Dv(t) = \rho_0 \partial_t^6 Dv(0) + \int_0^t \rho_0 \partial_t^7 Dv(t')\mathrm{d}t'$, the first and second summands are both estimated using an L^∞–L^2–L^2 Hölder's inequality, leading to a bound similar to (9.35).

The case $l = 6$ is estimated exactly the same way as the case $l = 3$ in the proof of Proposition 4. For the case $l = 5$, the integrand in the spacetime integral scales like $\ell[Dv_{tt}\rho_0\partial_t^6 J^{-2} + Dv_{ttt}\rho_0 Dv_{tttt}]\rho_0\partial_t^7 Dv$. Both summands can be estimated using an L^3–L^6–L^2 Hölder's inequality. The case $l = 4$ is treated as the case $l = 5$. The case $l = 3$ is also treated in the same way as $l = 5$. The case $l = 2$ is estimated exactly the same way as the case $l = 1$ in the proof of Proposition 4. The case $l = 1$ is treated in the same way as the case $l = 7$.

To deal with the space integral on the right-hand side of the expression for $\mathcal{R}_1$, the integral at time $t = 0$ is bounded by $\tilde{M}_0$, whereas the integral evaluated at $t = T$ is written, using the fundamental theorem of calculus, as

$$\begin{aligned}
&\sum_{l=1}^{7} c_l \int_\Omega \rho_0 \partial_t^{8-l} a_i^k \partial_t^l J^{-2} \rho_0 \partial_t^7 v^i,_k\, \mathrm{d}x \Big|_{t=T} \\
&\quad = \sum_{l=1}^{7} c_l \int_\Omega \rho_0 \partial_t^{8-l} a_i^k(0) \partial_t^l J^{-2}(0) \rho_0 \partial_t^7 v^i,_k(T)\mathrm{d}x \\
&\qquad + \sum_{l=1}^{7} c_l \int_\Omega \rho_0 \int_0^T (\partial_t^{8-l} a_i^k \partial_t^l J^{-2})_t \mathrm{d}t'\, \rho_0 \partial_t^7 v^i,_k(T)\mathrm{d}x.
\end{aligned}$$

The first integral on the right-hand side is estimated using Young's inequality, and is bounded by $\tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t)$, while the second integral can be estimated in the identical fashion as the corresponding spacetime integral. As such, we have shown that $\mathcal{R}_1$ has the claimed bound (9.35).

Analysis of $\int_0^T \mathcal{R}_2(t)\mathrm{d}t$. Using (5.5), we integrate by parts, to find that

$$\begin{aligned}\int_0^T \mathcal{R}_2(t)\mathrm{d}t &= -\sum_{l=1}^{7} c_l \int_0^T \int_\Omega \kappa \partial_t^{8-l} a_i^k \, \rho_0^2 \partial_t^l J^{-2} \, \partial_t^8 v^i{,_k} \, \mathrm{d}x\mathrm{d}t \\ &\leqq \sum_{l=1}^{7} c_l \sqrt{\kappa T} \sup_{[0,T]} \|\partial_t^{8-l} a_i^k \rho_0 \partial_t^l J^{-2}\|_0 \|\sqrt{\kappa}\rho_0 \partial_t^8 v,_k^i\|_{L^2(0,T;L^2(\Omega))} \\ &\leqq C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t)\right),\end{aligned}$$

the last inequality following from the fact that $\|\sqrt{\kappa}\rho_0\partial_t^8 Dv\|^2_{L^2(0,T;L^2(\Omega))}$ and $\|\rho_0\partial_t^7 Dv\|^2_{L^\infty(0,T;L^2(\Omega))}$ are both contained in the energy function.

Analysis of $\int_0^T \mathcal{R}_3(t)\mathrm{d}t$. This remainder integral is estimated in the same way as $\int_0^T \mathcal{R}_2(t)\mathrm{d}t$.

Analysis of the integral $\int_0^T \mathcal{I}_0(t)\mathrm{d}t$. Integrating $\mathcal{I}_0$ from 0 to T, we see that

$$\int_0^T \mathcal{I}_0(t)\mathrm{d}t = \frac{1}{2}\int_\Omega \rho_0 |\partial_t^8 v(T)|^2 \mathrm{d}x - \tilde{M}_0.$$

Analysis of the integral $\int_0^T \mathcal{I}_1(t)\mathrm{d}t$**.** To estimate $\mathcal{I}_1$, we again integrate by parts using (5.5), to obtain

$$\mathcal{I}_1 = -\int_\Omega \partial_t^8 a_i^k \rho_0^2 J^{-2}\, \partial_t^8 v^i{,_k}\, \mathrm{d}x.$$

Using the differentiation identity (5.4), the same anti-symmetric commutation that we used for the $\bar{\partial}^4$-differentiated problem can be employed once again to yield

$$\begin{aligned}\rho_0^2(\partial_t^7 v^r{,_s} A_i^s)\,(\partial_t^8 v^i{,_k} A_r^k) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\rho_0 D_\eta \partial_t^7 v(t)|^2 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\rho_0 \mathrm{curl}_\eta \partial_t^7 v(t)|^2 \\ &\quad + \frac{1}{2}\rho_0{}^2 \partial_t^7 v^k{,_r}\, \partial_t^7 v^b{,_s}\, (A_j^r A_m^s)_t [\delta_m^j \delta_b^k - \delta_b^j \delta_m^k],\end{aligned}$$

and

$$-\rho_0^2(\partial_t^7 v,_s^r\, A_r^s)\,(\partial_t^8 v,_k^i\, A_i^k) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\rho_0 \mathrm{div}_\eta \partial_t^7 v|^2 + \frac{1}{2}\rho_0^2 \partial_t^7 v,_s^r\, \partial_t^7 v,_k^i\, (A_r^s A_i^k)_t.$$

Hence,

$$\mathcal{I}_1 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho_0{}^2 \left(|D\partial_t^7 v|^2 - J^{-1}|\,\mathrm{curl}_\eta\, \partial_t^7 v|^2 - J^{-1}|\,\mathrm{div}_\eta\, \partial_t^7 v|^2\right)\mathrm{d}x + \mathcal{R},$$

and

$$\int_0^T \mathcal{I}_1 \mathrm{d}t = \int_\Omega \frac{{\rho_0}^2}{2}\left(|D\partial_t^7 v(T)|^2 - \frac{|\operatorname{curl}_\eta \partial_t^7 v(T)|^2}{J} - \frac{|\operatorname{div}_\eta \partial_t^7 v(T)|^2}{J}\right)\mathrm{d}x$$
$$-\tilde{M}_0 + \int_0^T \mathcal{R}(t)\mathrm{d}t, \tag{9.36}$$

where the remainder integral $\mathcal{R}$ satisfies (9.35).

Analysis of the integral $\int_0^T \mathcal{I}_2(t)\mathrm{d}t$. $\partial_t^8 J^{-2} = -2J^{-3}\partial_t^8 J$ plus lower-order terms, which have at most seven time derivatives acting on J. For such lower-order terms, we integrate by parts with respect to ∂_t, and estimate the resulting integrals in the same manner as we estimated the remainder term $\mathcal{R}_1$, and obtain the same bound.

Thus,

$$\begin{aligned}\mathcal{I}_2 &= 2\int_\Omega \rho_0^2 J^{-3} a_s^r \partial_t^8 \eta^s,_r \ a_i^k \partial_t^8 v^i,_k \,\mathrm{d}x + \mathcal{R} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\rho_0^2}{J^3} a_s^r \partial_t^7 v,_r^s \ a_i^k \partial_t^7 v,_k^i \,\mathrm{d}x - \int_\Omega \rho_0^2 \left(\frac{a_s^r a_i^k}{J^3}\right)_t \partial_t^7 v,_r^s \partial_t^7 v,_k^i \,\mathrm{d}x + \mathcal{R} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho_0^2 J^{-1} |\operatorname{div}_\eta \partial_t^7 v|^2 \mathrm{d}x + \mathcal{R},\end{aligned}$$

so that

$$\int_0^T \mathcal{I}_2(t)\mathrm{d}t = \int_\Omega \rho_0^2 J^{-1} |\operatorname{div}_\eta \partial_t^7 v(T)|^2 \mathrm{d}x - \tilde{M}_0 + \int_0^T \mathcal{R}(t)\mathrm{d}t.$$

Analysis of the integral $\int_0^T \mathcal{I}_3(t)\mathrm{d}t$. This closely follows our analysis of the integral $\mathcal{I}_1$. We first integrate by parts, using (5.5), to obtain

$$\mathcal{I}_3 = -\kappa \int_\Omega \partial_t^9 a_i^k \rho_0^2 J^{-2} \partial_t^8 v^i,_k \,\mathrm{d}x.$$

We then use the formula (5.4) for horizontally differentiating the cofactor matrix:

$$\mathcal{I}_3 = \kappa \int_\Omega {\rho_0}^2 J^{-3} \partial_t^8 v^r,_s \ [a_i^s a_r^k - a_r^s a_i^k] \, \partial_t^8 v^i,_k \ \mathrm{d}x + \mathcal{R},$$

where the remainder $\mathcal{R}$ satisfies (9.25). We decompose the highest-order term in $\mathcal{I}_3$ as the sum of the following two integrals:

$$\begin{aligned}\mathcal{I}_{3a} &= \kappa \int_\Omega \rho_0^2 J^{-3} \, (\partial_t^8 v^r,_s a_i^s)(\partial_t^8 v^i,_k a_r^k)\mathrm{d}x, \\ \mathcal{I}_{3b} &= -\kappa \int_\Omega {\rho_0}^2 J^{-1} \, |\operatorname{div}_\eta \partial_t^8 v|^2 \mathrm{d}x.\end{aligned}$$

Since

$$\begin{aligned}\mathcal{I}_{3a} &= \kappa \int_\Omega \frac{\rho_0^2}{J^3}\left[\partial_t^8 v,_s^i a_r^s \partial_t^8 v,_k^i a_r^k + (\partial_t^8 v,_s^r a_i^s - \partial_t^8 v,_s^i a_r^s)\partial_t^8 v,_k^i a_r^k\right]\mathrm{d}x \\ &= \kappa \int_\Omega \rho_0 J^{-1} \partial_t^8 v^i,_s A_r^s \partial_t^8 v^i,_k A_r^k \mathrm{d}x - \kappa \int_\Omega \rho_0^2 J^{-1} |\operatorname{curl}_\eta \partial_t^8 v|^2 \mathrm{d}x,\end{aligned}$$

and letting $D_\eta \partial_t^8 v = \partial_t^8 Dv\ A$ (matrix multiplication of $\partial_t^8 Dv$ with A), we thus have that

$$\int_0^T \mathcal{I}_3(t)\mathrm{d}t = \kappa \int_0^T \int_\Omega \frac{\rho_0}{J} |D_\eta \partial_t^8 v|^2 \,\mathrm{d}x\mathrm{d}t - \kappa \int_0^T \int_\Omega \frac{\rho_0^2}{J} |\operatorname{curl}_\eta \partial_t^8 v|^2 \mathrm{d}x\mathrm{d}t \\ -\kappa \int_0^T \int_\Omega {\rho_0}^2 J^{-1} \,|\operatorname{div}_\eta \partial_t^8 v|^2 \mathrm{d}x\mathrm{d}t + \int_0^T \mathcal{R}(t)\mathrm{d}t.$$

Analysis of the integral $\int_0^T \mathcal{I}_4(t)\mathrm{d}t$. Integrating by parts, and using (5.2),

$$\int_0^T \mathcal{I}_4(t)\mathrm{d}t = 2\kappa \int_0^T \int_\Omega \rho_0^2 J^{-3} a_s^r \partial_t^8 v^s,_r \ a_i^k \partial_t^8 v^i,_k \,\mathrm{d}x\mathrm{d}t + \int_0^T \mathcal{R}(t)\mathrm{d}t \\ = 2\kappa \int_0^T \int_\Omega {\rho_0}^2 J^{-1} \,|\operatorname{div}_\eta \partial_t^8 v|^2 \mathrm{d}x\mathrm{d}t + \int_0^T \mathcal{R}(t)\mathrm{d}t.$$

Analysis of the integral $\int_0^T \mathcal{I}_5(t)\mathrm{d}t$. Integrating by parts, and using the Cauchy–Schwarz inequality, we see that

$$\int_0^T \mathcal{I}_5 \mathrm{d}t = -\kappa \int_0^T \int_\Omega \rho_0^2 \partial_t J^{-2}\, \partial_t^8 a_i^k\, \partial_t^8 v^i,_k \,\mathrm{d}x\mathrm{d}t \\ \leqq \sqrt{\kappa T} \sup_{[0,T]} \left(\|\partial_t J^{-2}\|_{L^\infty(\Omega)} \|\rho_0 \partial_t^8 a_i^k\|_0 \right) \|\sqrt{\kappa} \rho_0 \partial_t^8 v^i,_k \|_{L^2(0,T;L^2(\Omega))} \\ \leqq C\sqrt{T}\, P(\sup_{t\in[0,T]} \tilde{E}(t)),$$

the last inequality following from the Sobolev embedding theorem and the $L^\infty(0, T)$ control of $\|\rho_0 \partial_t^8 a_i^k(t)\|_0$.

Analysis of the integral $\int_0^T \mathcal{I}_6(t)\mathrm{d}t$. Estimating in the same fashion as for $\mathcal{I}_5$ shows that $\int_0^T \mathcal{I}_6(t)\mathrm{d}t \leqq C\sqrt{T}\, P(\sup_{t\in[0,T]} \tilde{E}(t))$.

The sum $\sum_{a=0}^6 \int_0^T \mathcal{I}_a(t)\mathrm{d}t$. By considering the sum of all the integrals $\int_0^T \mathcal{I}_a(t)\mathrm{d}t$ for $a = 0, \ldots, 6$, we obtain the inequality

$$\sup_{[0,T]} \frac{1}{2} \left(\int_\Omega \rho_0 |\partial_t^8 v|^2 \mathrm{d}x + \int_\Omega \frac{\rho_0^2}{J} |\partial_t^7 Dv(t)|^2 \mathrm{d}x \right) + \kappa \int_0^T \int_\Omega \frac{\rho_0^2}{J} |D_\eta \partial_t^8 v|^2 \mathrm{d}x\mathrm{d}t \\ \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t) \right) \\ + \sup_{[0,T]} \int_\Omega \frac{\rho_0^2}{J} |\partial_t^7 \operatorname{curl} v|^2 \mathrm{d}x + \kappa \int_0^T \int_\Omega \frac{\rho_0^2}{J} |\operatorname{curl}_\eta \partial_t^8 v|^2 \mathrm{d}x\mathrm{d}t.$$

We next have

$$\kappa \int_0^T \int_\Omega \frac{\rho_0^2}{J} |D_\eta \partial_t^8 v|^2 \mathrm{d}x\mathrm{d}t = \kappa \int_0^T \int_\Omega \rho_0^2 J^{-1} \partial_t^8 v^i,_r A_k^r \partial_t^8 v^i,_s A_k^s \mathrm{d}x\mathrm{d}t$$

$$= \kappa \int_0^T \int_\Omega \rho_0^2 \partial_t^8 v,_k^i \partial_t^8 v,_k^i \mathrm{d}x\mathrm{d}t$$

$$+\kappa \int_0^T \int_\Omega \rho_0^2 \left[\frac{A_k^r A_k^s}{J} - \delta_k^r \delta_k^s \right] \partial_t^8 v,_r^i \partial_t^8 v,_s^i \mathrm{d}x\mathrm{d}t.$$

It thus follows from (9.3) that

$$\sup_{[0,T]} \frac{1}{2} \left(\int_\Omega \rho_0 |\partial_t^8 v|^2 \mathrm{d}x + \int_\Omega \frac{\rho_0^2}{J} |\partial_t^7 Dv|^2 \mathrm{d}x \right) + \frac{\kappa}{2} \int_0^T \int_\Omega \rho_0^2 |D\partial_t^8 v|^2 \mathrm{d}x\mathrm{d}t$$

$$\leqq \tilde{M}_0 + \delta \sup_{[0,T]} \tilde{E} + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t) \right)$$

$$+ \sup_{[0,T]} \int_\Omega \frac{\rho_0^2}{J} |\partial_t^7 \operatorname{curl} v|^2 \mathrm{d}x + \kappa \int_0^T \int_\Omega \frac{\rho_0^2}{J} |\operatorname{curl}_\eta \partial_t^8 v|^2 \mathrm{d}x\mathrm{d}t.$$

The curl estimates (9.5) provide the bound for the last two integrals from which the desired result is obtained and the proof of the proposition is completed. □

Corollary 2. (Estimates for $\partial_t^7 v(t)$)

$$\sup_{t\in[0,T]} \|\partial_t^7 v(t)\|_0^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E} + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t) \right).$$

Proof. The weighted embedding estimate (3.2) shows that

$$\|\partial_t^7 v(t)\|_0^2 \leqq C \int_\Omega \rho_0^2 \left(|\partial^7 v|^2 + |D\partial_t^7 v|^2 \right) \mathrm{d}x.$$

Now

$$\int_\Omega \rho_0^2 |\partial_t^7 v|^2 \mathrm{d}x \leqq M_0 + \int_\Omega \rho_0^2 \left| \int_0^t \partial_t^8 v \mathrm{d}t' \right|^2 \mathrm{d}x \leqq M_0 + T^2 \sup_{t\in[0,T]} \|\sqrt{\rho_0}\partial_t^8 v\|_0^2.$$

Thus, Proposition 5 shows that

$$\sup_{t\in[0,T]} \|\partial_t^7 v(t)\|_0^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E} + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t) \right).$$

□

9.6. The $\partial_t^2\bar\partial^3$, $\partial_t^4\bar\partial^2$, and $\partial_t^6\bar\partial$ Problems

Since we have provided detailed proofs of the energy estimates for the two end-point cases of all space derivatives, the $\bar\partial^4$ problem, and all time derivatives, the ∂_t^8 problem, we have covered all of the estimation strategies for all possible error terms in the three remaining intermediate problems. Meanwhile, the energy contributions for the three intermediate are found in the identical fashion as for the $\bar\partial^4$ and ∂_t^8 problems. As such we have the additional estimate

Proposition 6. *For $\delta > 0$ and letting the constant M_0 depend on $1/\delta$, for $\alpha = 1, 2$,*

$$\sup_{t\in[0,T]}\sum_{a=1}^{3}\Big[|\partial_t^{2a}\eta^\alpha(t)|^2_{3.5-a} + \|\sqrt{\rho_0}\bar\partial^{4-a}\ \partial_t^{2a}v(t)\|_0^2 + \|\rho_0\bar\partial^{4-a}\ \partial_t^{2a}D\eta(t)\|_0^2$$
$$+\kappa\int_0^t \|\rho_0\bar\partial^{4-a}\ \partial_t^{2a}Dv(s)\|_0^2 ds\Big] \leqq \tilde M_0 + \delta\sup_{[0,T]}\tilde E + C\sqrt{T}\,P(\sup_{[0,T]}\tilde E).$$

9.7. Additional Elliptic-Type Estimates for Normal Derivatives

Our energy estimates provide a priori control of horizontal and time derivatives of η; it remains to gain a priori control of the normal (or vertical) derivatives of η. This is accomplished via a bootstrapping procedure relying on having $\partial_t^7 v(t)$ bounded in $L^2(\Omega)$.

Proposition 7. *For $t\in[0,T]$, $\partial_t^5 v(t)\in H^1(\Omega)$, $\rho_0\partial_t^6 J^{-2}(t)\in H^1(\Omega)$ and*

$$\sup_{[0,T]}\Big(\|\partial_t^5 v\|_1^2 + \|\rho_0\partial_t^6 J^{-2}\|_1^2\Big) \leqq \tilde M_0 + \delta\sup_{[0,T]}\tilde E + C\sqrt{T}\,P(\sup_{[0,T]}\tilde E).$$

Proof. We begin by taking six time-derivatives of (7.2a') to obtain

$$2\kappa\partial_t^7\big[A_i^k(\rho_0 J^{-1}),_k\big] + 2\partial_t^6\big[A_i^k(\rho_0 J^{-1}),_k\big] = -\partial_t^7 v^i.$$

According to Lemma 2, and the bound on $\|\partial_t^7 v(t)\|_0^2$ given by Corollary 2,

$$\sup_{[0,T]}\Big\|\partial_t^6\Big[2A_i^k(\frac{\rho_0}{J}),_k\Big]\Big\|_0^2 \leqq \tilde M_0 + \delta\sup_{t\in[0,T]}\tilde E(t) + C\sqrt{T}\,P\bigg(\sup_{t\in[0,T]}\tilde E(t)\bigg). \tag{9.37}$$

For $\beta = 1, 2$,

$$\begin{aligned}2A_i^k(\rho_0 J^{-1}),_k &= \rho_0 a_i^k J^{-2},_k + 2\rho_0,_k a_i^k J^{-2}\\ &= \rho_0 a_i^3 J^{-2},_3 + 2\rho_0,_3 a_i^3 J^{-2} + \rho_0 a_i^\beta J^{-2},_\beta + 2\rho_0,_\beta a_i^\beta J^{-2}.\end{aligned} \tag{9.38}$$

Letting ∂_t^6 act on equation (9.38), we have that

$$\rho_0 a_i^3 \partial_t^6 J^{-2},_3 + 2\rho_0,_3 a_i^3 \partial_t^6 J^{-2} = \underbrace{2\partial_t^6 \big[A_i^k (\rho_0 J^{-1}),_k\big]}_{\mathcal{J}_1} - \underbrace{\rho_0 \partial_t^6 (a_i^\beta J^{-2},_\beta)}_{\mathcal{J}_2} - \underbrace{2\rho_0,_\beta \partial_t^6 (a_i^\beta J^{-2})}_{\mathcal{J}_3}$$
$$- \underbrace{(\partial_t^6 a_i^3)[\rho_0 J^{-2},_3 + 2\rho_0,_3 J^{-2}]}_{\mathcal{J}_4} + \underbrace{\sum_{a=1}^{5} c_a \partial_t^a a_i^3 \partial_t^{6-a} [\rho_0 J^{-2},_3 + 2\rho_0,_3 J^{-2}]}_{\mathcal{J}_5}.$$

Bounds for $\mathcal{J}_1(t)$. The inequality (9.37) establishes the $L^2(\Omega)$ bound for $\mathcal{J}_1(t)$.
Bounds for $\mathcal{J}_2(t)$. According to Proposition 6,

$$\sup_{[0,T]} \Big(\|\sqrt{\rho_0} \partial_t^6 v\|_0^2 + \|\rho_0 \bar\partial D \partial_t^5 v\|_0^2 \Big) \leqq \tilde M_0 + \delta \sup_{[0,T]} \tilde E + C\sqrt{T}\, P(\sup_{[0,T]} \tilde E), \tag{9.39}$$

so that with (9.4), we see that the differentiation identity (5.2) shows that

$$\sup_{t\in[0,T]} \|\mathcal{J}_2(t)\|_0^2 \leqq \tilde M_0 + \delta \sup_{t\in[0,T]} \tilde E(t) + C\sqrt{T}\, P\bigg(\sup_{t\in[0,T]} \tilde E(t) \bigg).$$

Bounds for $\mathcal{J}_3(t)$. The estimate for $\mathcal{J}_3(t)$ follows from the inequality for $\beta = 1, 2$

$$\left\| \frac{\rho_0,_\beta}{\rho_0} \right\|_{L^\infty(\Omega)} \leqq C \left\| \frac{\rho_0,_\beta}{\rho_0} \right\|_2 \leqq C \|\rho_0,_\beta\|_3,$$

where the first inequality follows from the Sobolev embedding theorem, and the second from the higher-order Hardy inequality Lemma 1 since $\rho_0,_\beta \in H^3(\Omega) \cap \dot H_0^1(\Omega)$ for $\beta = 1, 2$.

Thus,

$$\begin{aligned} \|2\rho_0,_\beta \partial_t^6 (a_i^\beta J^{-2})\|_0^2 &= \|2\rho_0 \partial_t^6 (a_i^\beta J^{-2}) \frac{\rho_0,_\beta}{\rho_0} \|_0^2 \\ &\leqq \|2\rho_0 \partial_t^6 (a_i^\beta J^{-2})\|_0^2 \left\| \frac{\rho_0,_\beta}{\rho_0} \right\|_{L^\infty(\Omega)}^2 \\ &\leqq \tilde M_0 + \delta \sup_{t\in[0,T]} \tilde E(t) + C\sqrt{T}\, P\bigg(\sup_{t\in[0,T]} \tilde E(t) \bigg), \end{aligned} \tag{9.40}$$

thanks to (9.39) and (9.4), and the fact that $\|\rho_0\|_4$ is bounded by assumption, from which it follows that

$$\sup_{t\in[0,T]} \|\mathcal{J}_3(t)\|_0^2 \leqq \tilde M_0 + \delta \sup_{t\in[0,T]} \tilde E(t) + C\sqrt{T}\, P\bigg(\sup_{t\in[0,T]} \tilde E(t) \bigg).$$

Bounds for $\mathcal{J}_4(t)$. Identity (5.6) shows that a^3_i is quadratic in $\bar\partial\eta$, and in particular, depends only on horizontal derivatives. From the estimate (9.39) and the weighted embedding (3.2), we may infer that

$$\sup_{t\in[0,T]} \|\bar\partial\partial_t^5 v(t)\|_0^2 \leqq \tilde M_0 + \delta \sup_{t\in[0,T]} \tilde E(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde E(t)\right).$$

Thus $\sup_{t\in[0,T]} \|\mathcal{J}_4(t)\|_0^2 \leqq \tilde M_0 + \delta \sup_{t\in[0,T]} \tilde E(t) + C\sqrt{T}\, P(\sup_{t\in[0,T]} \tilde E(t))$.
Bounds for $\mathcal{J}_5(t)$. Each summand in $\mathcal{J}_5(t)$ is a lower-order term, such that the time-derivative of each summand is controlled by the energy function $\tilde E(t)$; as such, the fundamental theorem of calculus shows that

$$\sup_{t\in[0,T]} \|\mathcal{J}_5(t)\|_0^2 \leqq \tilde M_0 + \delta \sup_{t\in[0,T]} \tilde E(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde E(t)\right).$$

We have therefore shown that for all $t \in [0, T]$,

$$\left\| \rho_0 a^3_i \partial_t^6 J^{-2},_3 + 2\rho_0,_3\, a^3_i \partial_t^6 J^{-2} \right\|_0^2 \leqq \tilde M_0 + \delta \sup_{t\in[0,T]} \tilde E(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde E(t)\right),$$

and our objective is to infer that the $L^2(\Omega)$-norm of each summand on the right-hand side is uniformly bounded on $[0, T]$.

To this end, we expand the $L^2(\Omega)$-norm to obtain the inequality

$$\|\rho_0 |a^3_\cdot| \partial_t^6 J^{-2},_3(t)\|_0^2 + 4\| |a^3_\cdot \rho_0,_3|\, \partial_t^6 J^{-2}(t)\|_0^2 + 4\int_\Omega \rho_0\rho_0,_3 |a^3_\cdot|^2 \partial_t^6 J^{-2} \partial_t^6 J^{-2},_3 \, dx \leqq \tilde M_0 + \delta \sup_{[0,T]} \tilde E + C\sqrt{T}\, P(\sup_{[0,T]} \tilde E). \tag{9.41}$$

For each $\kappa > 0$, solutions to our degenerate parabolic approximation (7.2) have sufficient regularity to ensure that $\rho_0|\partial_t^6 J^{-2}|^2,_3$ is integrable. As such, we integrate-by-parts with respect to x_3 to find that

$$4\int_\Omega \rho_0\rho_0,_3 |a^3_\cdot|^2 \partial_t^6 J^{-2} \partial_t^6 J^{-2},_3 \, dx = -2 \left\| |a^3_\cdot \rho_0,_3|\, \partial_t^6 J^{-2}(t) \right\|_0^2 - 2\int_\Omega \rho_0 (\rho_0,_3 |a^3_\cdot|^2),_3 (\partial_t^6 J^{-2})^2 \, dx. \tag{9.42}$$

Substitution of (9.42) into (9.41) yields

$$\|\rho_0 |a^3_\cdot| \partial_t^6 J^{-2},_3(t)\|_0^2 + 2\| |a^3_\cdot \rho_0,_3|\, \partial_t^6 J^{-2}(t)\|_0^2 \leqq \tilde M_0 + \delta \sup_{[0,T]} \tilde E + C\sqrt{T}\, P(\sup_{[0,T]} \tilde E) + C\int_\Omega \rho_0 |\partial_t^6 J^{-2}|^2 \, dx. \tag{9.43}$$

Using (9.3), we see that $|a^3_\cdot|^2$ has a strictly positive lower-bound. By the physical vacuum condition (1.5), for $\epsilon > 0$ taken sufficiently small, there are constants $\theta_1, \theta_2 > 0$ such that $|\rho_0,_3(x)| \geqq \theta_1$ whenever $1-\epsilon \leqq x_3 \leqq 1$ and $0 \leqq x \leqq \epsilon$, and $\rho_0(x) > \theta_2$ whenever $\epsilon \leqq x \leqq 1-\epsilon$; hence, by readjusting the constants on the right-hand side of (9.43), we find that

$$\|\rho_0 \partial_t^6 J^{-2},_3(t)\|_0^2 + 2\|\partial_t^6 J^{-2}(t)\|_0^2 \leqq \tilde{M}_0 + \delta \sup_{[0,T]} \tilde{E}(t) + C\sqrt{T}\, P(\sup_{[0,T]} \tilde{E}(t)) + C\int_\Omega \rho_0 |\partial_t^6 J^{-2}|^2 \,\mathrm{d}x. \tag{9.44}$$

By Proposition 6, for $\beta = 1, 2$,

$$\sup_{t\in[0,T]} \|\rho_0 \partial_t^6 J^{-2},_\beta(t)\| \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t)\right),$$

and by the fundamental theorem of calculus and Proposition 5,

$$\sup_{[0,T]} \|\rho_0 \partial_t^6 J^{-2}\| \leqq \tilde{M}_0 + \delta \sup_{[0,T]} \tilde{E} + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t)\right).$$

These two inequalities, combined with (9.44), show that

$$\|\rho_0 \partial_t^6 J^{-2}(t)\|_1^2 + \|\partial_t^6 J^{-2}(t)\|_0^2 \leqq \tilde{M}_0 + \delta \sup_{[0,T]} \tilde{E} + C\sqrt{T}\, P(\sup_{[0,T]} \tilde{E}) + C\int_\Omega \rho_0 |\partial_t^6 J^{-2}|^2 \,\mathrm{d}x.$$

We use Young's inequality and the fundamental theorem of calculus (with respect to t) for the last integral to find that for $\theta > 0$

$$\begin{aligned}
\int_\Omega \rho_0 \partial_t^6 J^{-2}\, \partial_t^6 J^{-2} \,\mathrm{d}x &\leqq \theta \left\|\partial_t^6 J^{-2}(t)\right\|_0^2 + C_\theta \left\|\rho_0 \partial_t^6 J^{-2}(t)\right\|_0^2 \\
&\leqq \theta \left\|\partial_t^6 J^{-2}(t)\right\|_0^2 + C_\theta \left\|\rho_0 \partial_t^5 Dv(t)\right\|_0^2 \\
&\leqq \theta \left\|\partial_t^6 J^{-2}(t)\right\|_0^2 + \tilde{M}_0 + \delta \sup_{[0,T]} \tilde{E} + C\sqrt{T}\, P(\sup_{[0,T]} \tilde{E}),
\end{aligned}$$

where we have used the fact that $\|\rho_0 \partial_t^7 Dv(t)\|_0^2$ is contained in the energy function $\tilde{E}(t)$. We choose $\theta \ll 1$ and once again readjust the constants; as a result, we see that on $[0, T]$

$$\|\rho_0 \partial_t^6 J^{-2}(t)\|_1^2 + \left\|\partial_t^6 J^{-2}(t)\right\|_0^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t)\right). \tag{9.45}$$

With $J_t = a_i^j v^i{,}_j$, we see that

$$a_i^j \partial_t^5 v^i{,}_j = \partial_t^6 J - v^i{,}_j \partial_t^5 a_i^j - \sum_{a=1}^{4} c_a \partial_t^a a_i^j \partial_t^{5-a} v^i{,}_j , \tag{9.46}$$

so that using (9.45) together with the fundamental theorem of calculus for the last two terms on the right-hand side of (9.46), we see that

$$\left\| a_i^j \partial_t^5 v^i{,}_j (t) \right\|_0^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t) \right),$$

from which it follows that

$$\left\| \operatorname{div} \partial_t^5 v(t) \right\|_0^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t) \right).$$

Proposition 3 provides the estimate

$$\| \operatorname{curl} \partial_t^5 v(t) \|_0^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t) \right),$$

and Proposition 6 shows that for $\alpha = 1, 2$,

$$|\partial_t^5 v^\alpha(t)|_{0.5}^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t) \right).$$

We thus conclude from Proposition 1 that

$$\sup_{t\in[0,T]} \left\| \partial_t^5 v(t) \right\|_1^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t) \right).$$

□

Having a good bound for $\partial_t^5 v(t)$ in $H^1(\Omega)$ we proceed with our bootstrapping.

Proposition 8. *For $t \in [0, T]$, $v_{ttt}(t) \in H^2(\Omega)$, $\rho_0 \partial_t^4 J^{-2}(t) \in H^2(\Omega)$ and*

$$\sup_{t\in[0,T]} \left(\|v_{ttt}(t)\|_2^2 + \|\rho_0 \partial_t^4 J^{-2}(t)\|_2^2 \right) \leqq M_0 + \delta \sup_{[0,T]} E + C\sqrt{T}\, P(\sup_{[0,T]} E).$$

Proof. We take four time-derivatives of (7.2a') to obtain

$$\kappa \partial_t^5 \left[A_i^k (\rho_0 J^{-1}),_k \right] + 2\partial_t^4 \left[A_i^k (\rho_0 J^{-1}),_k \right] = -\partial_t^5 v^i .$$

According to Lemma 2, and the bound on $\|\partial_t^5 v(t)\|_1^2$ given by Proposition 7,

$$\sup_{t\in[0,T]} \left\| \partial_t^4 \left[2A_i^k (\rho_0 J^{-1}),_k \right] \right\|_1^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t) \right). \tag{9.47}$$

For $\beta = 1, 2$,

$$
\begin{aligned}
2A_i^k(\rho_0 J^{-1}),_k &= \rho_0 a_i^k J^{-2},_k + 2\rho_0,_k a_i^k J^{-2} \\
&= \rho_0 a_i^3 J^{-2},_3 + 2\rho_0,_3 a_i^3 J^{-2} + \rho_0 a_i^\beta J^{-2},_\beta + 2\rho_0,_\beta a_i^\beta J^{-2},
\end{aligned}
\tag{9.48}
$$

Letting ∂_t^4 act on equation (9.48), we have that

$$
\begin{aligned}
&\rho_0 a_i^3 \partial_t^4 J^{-2},_3 + 2\rho_0,_3 a_i^3 \partial_t^4 J^{-2} \\
&\quad = \underbrace{2\partial_t^4\big[A_i^k(\rho_0 J^{-1}),_k\big]}_{\mathcal{J}_1} - \underbrace{\rho_0 \partial_t^4(a_i^\beta J^{-2},_\beta)}_{\mathcal{J}_2} - \underbrace{2\rho_0,_\beta \partial_t^4(a_i^\beta J^{-2})}_{\mathcal{J}_3} \\
&\qquad - \underbrace{(\partial_t^4 a_i^3)[\rho_0 J^{-2},_3 + 2\rho_0,_3 J^{-2}]}_{\mathcal{J}_4} + \underbrace{\sum_{a=1}^{3} c_a \partial_t^a a_i^3 \partial_t^{4-a}[\rho_0 J^{-2},_3 + 2\rho_0,_3 J^{-2}]}_{\mathcal{J}_5}.
\end{aligned}
\tag{9.49}
$$

In order to estimate $\partial_t^4 J^{-2}(t)$ in $H^1(\Omega)$, we first estimate horizontal derivatives of $\partial_t^4 J^{-2}(t)$ in $L^2(\Omega)$. As such, we consider for $\alpha = 1, 2$,

$$
\begin{aligned}
&\rho_0 a_i^3 \partial_t^4 J^{-2},_{3\alpha} + 2\rho_0,_3 a_i^3 \partial_t^4 J^{-2},_\alpha \\
&\qquad = \sum_{l=1}^{5} \mathcal{J}_l,_\alpha - (\rho_0 a_i^3),_\alpha \partial_t^4 J^{-2},_3 - 2(\rho_0,_3 a_i^3),_\alpha \partial_t^4 J^{-2}.
\end{aligned}
\tag{9.50}
$$

Bounds for $\mathcal{J}_1,_\alpha$. The estimate (9.47) shows that

$$
\|\mathcal{J}_1,_\alpha(t)\|_0^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\bigg(\sup_{t\in[0,T]} \tilde{E}(t)\bigg).
$$

Bounds for $\mathcal{J}_2,_\alpha$. Proposition 6 provides the estimate

$$
\sup_{t\in[0,T]} \Big(\|\bar\partial^2 v_{ttt}(t)\|_0^2 + \|\rho_0 \bar\partial^2 D v_{ttt}(t)\|_0^2\Big) \leqq \tilde{M}_0 + \delta \sup_{[0,T]} \tilde{E} + C\sqrt{T}\, P(\sup_{[0,T]} \tilde{E}).
\tag{9.51}
$$

We write

$$
\mathcal{J}_2,_\alpha = \rho_0 \partial_t^4(a_i^\beta,_\alpha J^{-2},_\beta + a_i^\beta J^{-2},_{\beta\alpha}) + \rho_0,_\alpha \partial_t^4(a_i^\beta J^{-2},_\beta).
$$

Using (9.4), for $\alpha = 1, 2$, the highest-order term in $\rho_0 \partial_t^4(a_i^\beta J^{-2},_{\beta\alpha})$ satisfies the inequality

$$
\|\rho_0 a_i^\beta \partial_t^4 J^{-2},_{\beta\alpha}\|_0^2 \leqq C\|\rho_0 \bar\partial^2 D v_{ttt}\|_0^2,
$$

which has the bound (9.51), and the lower-order terms have the same bound using the fundamental theorem of calculus; for example

$$\begin{aligned}\|\rho_0 J^{-2},_{\beta\alpha}\, \partial_t^4 a_i^\beta\, J^{-2},_{\beta\alpha}\|_0^2 &\leqq \|\rho_0 J^{-2},_{\alpha\beta}\|_{L^6(\Omega)} \|\partial_t^4 a_i^\beta\|_{L^3(\Omega)} \\ &\leqq C\|\rho_0 J^{-2},_{\alpha\beta}\|_1 \|\partial_t^4 a_i^\beta\|_{0.5} \leqq \tilde{M}_0,\end{aligned}$$

where we have used Hölder's inequality, the Sobolev embedding theorem, and (9.4) for the final inequality. On the other hand, $\rho_0,_\alpha\, \partial_t^4 (a_i^\beta J^{-2},_\beta)$ is estimated in the same manner as (9.40), which shows that

$$\|\mathcal{J}_2,_\alpha(t)\|_0^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t)\right).$$

Bounds for $\mathcal{J}_3,_\alpha$. Using the fact that $\|\partial_t \mathcal{J}_3,_\alpha\|_0^2$ can be bounded by the energy function, the fundamental theorem of calculus shows that

$$\|\mathcal{J}_3,_\alpha(t)\|_0^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t)\right).$$

Bounds for $\mathcal{J}_4,_\alpha$. Again, using the fact that the vector a_i^3 only contains horizontal derivatives of η^i, (9.4) shows that for $\alpha = 1, 2$,

$$\begin{aligned}\|\left(\partial_t^4 a_i^3\, [\rho_0 J^{-2},_3 + 2\rho_0,_3\, J^{-2}]\right),_\alpha\|_0^2 &\leqq C\|\bar{\partial}^2 v_{ttt}\|_0^2 + \tilde{M}_0 \\ &\leqq \tilde{M}_0 + \delta \sup_{[0,T]} \tilde{E} + C\sqrt{T}\, P(\sup_{[0,T]} \tilde{E}),\end{aligned}$$

the last inequality following from (9.51), and thus

$$\|\mathcal{J}_4,_\alpha(t)\|_0^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t)\right).$$

Bounds for $\mathcal{J}_5,_\alpha$. These are lower-order terms, estimated with the fundamental theorem of calculus and (9.4), yielding

$$\|\mathcal{J}_5,_\alpha(t)\|_0^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t)\right).$$

Bounds for $-(\rho_0 a_i^3),_\alpha\, \partial_t^4 J^{-2},_3 - 2(\rho_0,_3\, a_i^3),_\alpha\, \partial_t^4 J^{-2}$. The bounds for these terms follows in the same fashion as for $\mathcal{J}_2,_\alpha$ and show that

$$\|(\rho_0 a_i^3),_\alpha\, \partial_t^4 J^{-2},_3 + 2(\rho_0,_3\, a_i^3),_\alpha\, \partial_t^4 J^{-2}\|_0^2 \leqq \tilde{M}_0 + \delta \sup_{[0,T]} \tilde{E} + C\sqrt{T}\, P(\sup_{[0,T]} \tilde{E}).$$

We have hence bounded the $L^2(\Omega)$-norm of the right-hand side of (9.50) by $\tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\, T\, P(\sup_{t\in[0,T]} \tilde{E}(t))$. Using the same integration-by-parts argument just given above in the proof of Proposition 7, we conclude that for $\alpha = 1, 2$,

$$\sup_{[0,T]} \left(\|\partial_t^4 J^{-2},_\alpha\|_0^2 + \|\rho_0 \partial_t^4 J^{-2},_\alpha\|_1^2\right) \leqq \tilde{M}_0 + \delta \sup_{[0,T]} \tilde{E} + C\sqrt{T}\, P(\sup_{[0,T]} \tilde{E}). \tag{9.52}$$

From the inequality (9.52), we may infer that for $\alpha = 1, 2$,

$$\sup_{t\in[0,T]} \| \operatorname{div} v_{ttt},_\alpha (t)\|_0^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t)\right), \quad (9.53)$$

and according to Proposition 3, for $\alpha = 1, 2$,

$$\sup_{t\in[0,T]} \| \operatorname{curl} v_{ttt},_\alpha (t)\|_0^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t)\right). \quad (9.54)$$

The boundary regularity of $v_{ttt},_\alpha$, $\alpha = 1, 2$, follows from Proposition 6:

$$\sup_{t\in[0,T]} |v_{ttt},_\alpha (t)|_{0.5}^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t)\right). \quad (9.55)$$

Thus, the inequalities (9.53), (9.54), and (9.55) together with (6.4) and (9.52) show that

$$\sup_{t\in[0,T]} \left(\|v_{ttt},_\alpha (t)\|_1^2 + \|\rho_0 \partial_t^4 J^{-2},_\alpha (t)\|_1^2\right) \leqq \tilde{M}_0 + \delta \sup_{[0,T]} \tilde{E} + C\sqrt{T}\, P(\sup_{[0,T]} \tilde{E}). \quad (9.56)$$

In order to estimate $\|\partial_t^4 J^{-2},_3 (t)\|_0^2$, we next differentiate (9.49) in the vertical direction x_3 to obtain

$$\rho_0 a_i^3 \partial_t^4 J^{-2},_{33} + 3\rho_0,_3 a_i^3 \partial_t^4 J^{-2},_3 = \sum_{l=1}^5 \mathcal{J}_l,_3 - \rho_0 a_i^3,_3 \partial_t^4 J^{-2},_3 - 2(\rho_0,_3 a_i^3),_3 \partial_t^4 J^{-2}. \quad (9.57)$$

Following our estimates for the horizontal derivatives, inequality (9.56) together with Propositions 6 and 7 show that the right-hand side of (9.57) is bounded in $L^2(\Omega)$ by $\tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P(\sup_{t\in[0,T]} \tilde{E}(t))$.

It follows that for $k = 1, 2, 3$,

$$\|\rho_0 a_i^3 \partial_t^4 J^{-2},_{k3} + 3\rho_0,_3 a_i^3 \partial_t^4 J^{-2},_k \|_0^2 \leqq \tilde{M}_0 + \delta \sup_{[0,T]} \tilde{E} + C\sqrt{T}\, P(\sup_{[0,T]} \tilde{E}).$$

Note that the coefficient in front of $\rho_0,_3 a_i^3 \partial_t^4 J^{-2},_k$ has changed from 2 to 3, but the identical integration-by-parts argument that we used in the proof of Proposition 7 is employed, once again, and shows that

$$\|\rho_0 \partial_t^4 J^{-2}(t)\|_2^2 + \|\partial_t^4 J^{-2}(t)\|_1^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t)\right).$$

Thus $\| \operatorname{div} v_{ttt}(t)\|_1^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P(\sup_{t\in[0,T]} \tilde{E}(t))$. From Proposition 3, $\| \operatorname{curl} v_{ttt}(t)\|_1^2 \leqq \tilde{M}_0 + \delta \sup_{[0,T]} \tilde{E} + C\, T\, P(\sup_{[0,T]} \tilde{E})$ and with the bound on v_{ttt}^α given by Proposition 6, Proposition 1 provides the estimate

$$\|v_{ttt}(t)\|_2^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t)\right).$$

□

Proposition 9. *For* $t \in [0,T]$, $v_t(t) \in H^3(\Omega)$, $\rho_0 \partial_t^2 J^{-2}(t) \in H^3(\Omega)$ *and*

$$\sup_{t\in[0,T]} \left(\|v_t(t)\|_3^2 + \|\rho_0 \partial_t^2 J^{-2}(t)\|_3^2 \right) \leqq \tilde{M}_0 + \delta \sup_{[0,T]} \tilde{E} + C\sqrt{T}\, P(\sup_{[0,T]} \tilde{E}).$$

Proof. We take two time-derivatives of (7.2a') to obtain

$$\kappa \partial_t^3 \big[A_i^k (\rho_0 J^{-1}),_k\big] + 2\partial_t^2 \big[A_i^k (\rho_0 J^{-1}),_k\big] = -\partial_t^3 v^i.$$

According to Lemma 2, and the bound on $\|\partial_t^3 v(t)\|_2^2$ given by Proposition 8,

$$\sup_{t\in[0,T]} \left\| \partial_t^2 \big[2A_i^k (\rho_0 J^{-1}),_k\big] \right\|_2^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t) \right). \tag{9.58}$$

Letting ∂_t^2 act on equation (9.48), we have that

$$\begin{aligned} &\rho_0 a_i^3 \partial_t^2 J^{-2},_3 + 2\rho_0,_3 a_i^3 \partial_t^2 J^{-2} \\ &\quad = 2\partial_t^2 \big[A_i^k (\rho_0 J^{-1}),_k\big] - \rho_0 \partial_t^2 (a_i^\beta J^{-2},_\beta) - 2\rho_0,_\beta \partial_t^2 (a_i^\beta J^{-2}) \\ &\qquad - (\partial_t^2 a_i^3)[\rho_0 J^{-2},_3 + 2\rho_0,_3 J^{-2}] + c_a \partial_t a_i^3 \partial_t [\rho_0 J^{-2},_3 + 2\rho_0,_3 J^{-2}]. \end{aligned}$$

The bound (9.58) allows us to proceed by using the same argument that we used in the proof of Proposition 8, and this leads to the desired inequality. □

Proposition 10. *For* $t \in [0,T]$, $\eta(t) \in H^4(\Omega)$, $\rho_0 J^{-2}(t) \in H^4(\Omega)$ *and*

$$\sup_{t\in[0,T]} \left(\|\eta(t)\|_4^2 + \|\rho_0 J^{-2}(t)\|_4^2 \right) \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t) \right).$$

Proof. From (7.2a'), we see that

$$\kappa \partial_t \big[A_i^k (\rho_0 J^{-1}),_k\big] + 2\big[A_i^k (\rho_0 J^{-1}),_k\big] = -v_t^i.$$

According to Lemma 2, and the bound on $\|v_t(t)\|_3^2$ given by Proposition 9,

$$\sup_{t\in[0,T]} \left\| 2A_i^k (\rho_0 J^{-1}),_k \right\|_2^2 \leqq \tilde{M}_0 + \delta \sup_{t\in[0,T]} \tilde{E}(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde{E}(t) \right). \tag{9.59}$$

Since

$$\rho_0 a_i^3 J^{-2},_3 + 2\rho_0,_3 a_i^3 J^{-2} = 2\big[A_i^k (\rho_0 J^{-1}),_k\big],$$

we can use the bound (9.59) and proceed by using the same argument that we used in the proof of Proposition 8 to conclude the proof. □

We now just have to estimate the two last terms of $\tilde{E}(t)$.

Proposition 11.

$$\sup_{[0,T]} \left(\| \operatorname{curl}_\eta v\|_3^2 + \|\rho_0 \bar\partial^4 \operatorname{curl}_\eta v\|_0^2\right) \leqq M_0 + \delta \sup_{[0,T]} E + C\sqrt{T}\, P(\sup_{[0,T]} E).$$

Proof. Letting D^3 act on the identity (9.7) for $\operatorname{curl}_\eta v$, we see that the highest-order term scales like

$$D^3 \operatorname{curl} u_0 + \int_0^t D^4 v\, Dv\, A\, A \mathrm{d}t'.$$

Integration by parts shows that the highest-order contribution to the term $D^3 \operatorname{curl}_\eta v(t)$ can be written as

$$D^3 \operatorname{curl} u_0 - \int_0^t D^4 \eta\, [Dv\, A\, A]_t \mathrm{d}t' + D^4\eta(t)\, Dv(t)\, A(t)\, A(t),$$

which, according to Proposition 10, has $L^2(\Omega)$-norm bounded by

$$M_0 + \delta \sup_{t\in[0,T]} E(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} E(t)\right),$$

after readjusting the constants; thus, the inequality for the $H^3(\Omega)$-norm of $\operatorname{curl}_\eta v(t)$ is proved

The same type of analysis works for the weighted estimate. After integration by parts in time, the highest-order term in the expression for $\rho_0 \bar\partial^4 \operatorname{curl}_\eta v(t)$ scales like

$$\rho_0 \bar\partial^4 \operatorname{curl} u_0 - \int_0^t \rho_0 \bar\partial^4 D\eta\, [Dv\, A\, A]_t \mathrm{d}t' + \rho_0 \bar\partial^4 D\eta(t)\, Dv(t)\, A(t)\, A(t).$$

Hence, the inequality (9.23) shows that the weighted estimate holds as well. □

10. Proof of Theorem 1 (The Main Result)

10.1. Time of Existence and Bounds Independent of κ and Existence of Solutions to (1.9)

Combining the estimates from Propositions 3, 4, 5, 6, 7, 8, 9, 10, 11 and Corollary 2, we obtain the following inequality on $(0, T_\kappa)$:

$$\sup_{t\in[0,T]} \tilde E(t) \leqq \tilde M_0 + \delta \sup_{t\in[0,T]} \tilde E(t) + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde E(t)\right).$$

By choosing δ sufficiently small, we have that

$$\sup_{t\in[0,T]} \tilde E(t) \leqq \tilde M_0 + C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \tilde E(t)\right).$$

Using our continuation argument, presented in Section 9 of [8], this provides us with a time of existence T_1 independent of κ and an estimate on $(0, T_1)$ independent of κ of the type:

$$\sup_{t\in[0,T_1]} \tilde{E}(t) \leqq 2\tilde{M}_0,$$

as long as conditions (9.3) and (9.4) of Subsection 9.2 hold. These conditions can now be verified by using the fundamental theorem of calculus and further shrinking the time-interval, if necessary. For example, since

$$\|\eta(t)\|_{3.5} \leqq 2\|e\|_{3.5} + 2\int_0^t \|v(t')\|_{3.5}\mathrm{d}t',$$

we see that for t taken sufficiently small, $\|\eta(t)\|_{3.5}^2 \leqq 2|\Omega| + 1$. The other conditions in Subsection 9.2 are satisfied with similar arguments. This leads us to a time of existence $T_2 > 0$ independent of κ for which we have the estimate on $(0, T_2)$

$$\sup_{t\in[0,T_2]} \tilde{E}(t) \leqq 2\tilde{M}_0. \tag{10.1}$$

In particular, our sequence of solutions $\{\eta^\kappa\}_{\kappa>0}$ to our approximate κ-problem (7.2) satisfy the κ-independent bound (10.1) on the κ-independent time-interval $(0, T_2)$.

10.2. The Limit as $\kappa \to 0$

By the κ-independent estimate (10.1), standard compactness arguments provide the existence of a strongly convergent subsequences for $\epsilon > 0$

$$\eta^{\kappa'} \to \eta \text{ in } L^2((0, T_2); H^3(\Omega))$$
$$v_t^{\kappa'} \to v_t \text{ in } L^2((0, T_2); H^2(\Omega)).$$

Consider the variational form of (7.2a): for all $\varphi \in L^2(0, T_2; H^1(\Omega))$,

$$\int_0^{T_2}\Big[\int_\Omega \rho_0 (v^{\kappa'})_t^i\,\varphi^i\,\mathrm{d}x - \int_\Omega \rho_0^2 (J^{\kappa'})^{-2}(a^{\kappa'})_i^k\,\varphi^i{,_k}\,\mathrm{d}x$$
$$-\kappa\int_\Omega \rho_0^2\partial_t[(J^{\kappa'})^{-2}(a^{\kappa'})_i^k]\,\varphi^i{,_k}\,\mathrm{d}x\Big]\mathrm{d}t = 0.$$

The strong convergence of the sequences $(\eta^{\kappa'}, v_t^{\kappa'})$ show that the limit (η, v_t) satisfies

$$\int_0^{T_2}\Big[\int_\Omega \rho_0 v_t^i\,\varphi^i\,\mathrm{d}x - \int_\Omega \rho_0^2 J^{-2} a_i^k\,\varphi^i{,_k}\Big]\mathrm{d}x\mathrm{d}t = 0,$$

which shows that η is a solution to (1.9) on the κ-independent time interval $(0, T_2)$. A standard argument shows that $v(0) = u_0$ and $\eta(0) = e$.

10.3. Uniqueness of Solutions to the Compressible Euler Equations (1.9)

Suppose that (η, v) and $(\bar{\eta}, \bar{v})$ are both solutions of (1.9) with the same initial data that satisfy the estimate (1.13). Let

$$\delta v = v - \bar{v}, \quad \delta\eta = \eta - \bar{\eta} \quad \delta a = a - \bar{a}, \quad \delta J^{-2} = J^{-2} - \bar{J}^{-2}, \quad \text{etc.}$$

Then δv satisfies

$$\rho_0 \delta v_t^i + \delta a_i^k (\rho_0^2 J^{-2}),_k + \bar{a}_i^k (\rho^2 \delta J^{-2}),_k = 0 \text{ in } (0,T] \times \Omega,$$
$$\delta v = 0 \text{ on } \{t=0\} \times \Omega.$$

Consider the energy function

$$\mathcal{E}(t) = \sum_{a=0}^{4} \|\partial_t^{2a} \delta\eta(t)\|_{4-a}^2 + \sum_{a=0}^{3} \Big[\|\rho_0 \bar{\partial}^{4-a} \partial_t^{2a} D\delta\eta(t)\|_0^2 + \|\sqrt{\rho_0} \bar{\partial}^{4-a} \partial_t^{2a} \delta v(t)\|_0^2 \Big] + \sum_{a=0}^{3} \|\rho_0 \partial_t^{2a} \delta J^{-2}(t)\|_{4-a}^2 + \|\rho_0 \partial_t^7 D\delta v(t)\|_0^2 + \|\rho_0 \partial_t^8 \delta v(t)\|_0^2.$$

Given the transport-type structure for the curl of $\delta\eta$ and its space and time derivatives, together with the assumed smoothness of η and $\bar{\eta}$, we can proceed in the same fashion as our estimates in Section 9, and using that $\delta v(0) = 0$, we obtain

$$\sup_{t\in[0,T]} \mathcal{E}(t) \leqq C\sqrt{T}\, P\left(\sup_{t\in[0,T]} \mathcal{E}(t) \right),$$

which shows that $\delta v(t) = 0$ on $[0,T]$. The extra regularity assumption on the initial data is being used in a way similar as in our proof for uniqueness, on page 887 in [9], in order to estimate forcing terms which do not form an exact in time derivative and would otherwise be of order too high.

10.4. Optimal Regularity for Initial Data

For the purposes of constructing solutions to our degenerate parabolic κ-problem (7.2), in Section 7.1, we smoothed our initial data so that both our initial velocity field u_0^ϑ is smooth, and and our initial density function ρ_0^ϑ is smooth, positive in the interior, and vanishing on the boundary Γ with the physical vacuum condition (1.5).

Our a priori estimates then allow us to pass to the limit $\lim_{\vartheta\to 0} u_0^\vartheta = u_0$ and $\lim_{\vartheta\to 0} \rho_0^\vartheta = \rho_0$. By construction, $\rho_0 \in H^4(\Omega)$, satisfies $\rho_0 > 0$ in Ω, and the physical vacuum condition (1.5) near the boundary Γ. Similarly, the initial velocity field need only satisfy $E(0) < \infty$.

11. The Case of General $\gamma > 1$

We denote by a_0 the integer satisfying the inequality

$$1 < 1 + \frac{1}{\gamma - 1} - a_0 \leqq 2.$$

The general higher-order energy function is given by

$$E_\gamma(t) = \sum_{a=0}^{4} \|\partial_t^{2a}\eta(t)\|_{4-a}^2 + \sum_{a=0}^{4} [\|\rho_0 \bar\partial^{4-a}\partial_t^{2a} D\eta(t)\|_0^2 + \|\sqrt{\rho_0}\bar\partial^{4-a}\partial_t^{2a} v(t)\|_0^2]$$
$$+ \sum_{a=0}^{3} \|\rho_0 \partial_t^{2a} J^{-2}(t)\|_{4-a}^2 + \|\operatorname{curl}_\eta v(t)\|_3^2 + \|\rho_0 \bar\partial^4 \operatorname{curl}_\eta v(t)\|_0^2$$
$$+ \sum_{a=0}^{a_0} \|\sqrt{\rho_0}^{\,1+\frac{1}{\gamma-1}-a}\partial_t^{7+a_0-a} Dv(t)\|_0^2,$$

and we set $M_0^\gamma = P(E_\gamma(0))$.

Notice the last sum in E_γ appears whenever $\gamma < 2$, and the number of time-differentiated problems increases as γ approaches 1. We explain this last summation of norms in E_γ with a particular example. Consider the case in which $\gamma = \frac{3}{2}$. Then, $\rho_0 \sim d^2$ near Γ, $a_0 = 1$, and the last summation is written as

$$\sum_{a=0}^{a_0} \|\sqrt{d}^{\,1+\frac{1}{\gamma-1}-a}\partial_t^{7+a_0-a} Dv(t,\cdot)\|_0^2 = \|d^{\frac{3}{2}}\partial_t^8 Dv(t)\|_0^2 + \|d^{\frac{1}{2}}\partial_t^7 Dv(t)\|_0^2,$$

which is equivalent to

$$\int_\Omega \rho_0^{\frac{3}{2}} |\partial_t^8 Dv(t)|^2 \mathrm{d}x + \int_\Omega |\rho_0^{\frac{1}{2}} |\partial_t^7 Dv(t)|^2 \mathrm{d}x. \tag{11.1}$$

The Euler equations with $\gamma = \frac{3}{2}$ are written as

$$\rho_0 v_t^i + a_i^k (\rho_0^{\frac{3}{2}} J^{-\frac{3}{2}})_{,k} = 0. \tag{11.2}$$

Energy estimates on the ninth time-differentiated problem produce the first integral in (11.1), while the second integral is obtained using our elliptic-type estimates on the seventh time-differentiated version of (11.2). (Notice that the value of γ does not play a role in our elliptic-type estimates.) Having control on the two integrals in (11.1) then shows that we are back in the situation for the case in which $\gamma \geqq 2$, that is, we see that $\partial_t^7 v(t)$ is even better than $L^2(\Omega)$, which allows us to proceed as before. In particular, for $\gamma < 2$ the power on ρ_0 in the first integral in (11.1) is greater than one, and by weighted embedding estimates, this means that the embedding occurs into a less regular Sobolev space; this accounts for the need to have more time-differentiated problems when $\gamma < 2$.

Using this energy function, the same methodology as we used for the case $\gamma = 2$, shows that $\sup_{t\in[0,T]} E_\gamma(t)$ remains bounded for $T > 0$ taken sufficiently small.

Theorem 4. (Existence and uniqueness for the case $\gamma > 1$) *Suppose that* $\rho_0 \in H^4(\Omega)$, $\rho_0(x) > 0$ *for* $x \in \Omega$, $\rho_0 = 0$ *on* Γ, *and* ρ_0 *satisfies* (1.5). *Furthermore, suppose that* u_0 *is given such that* $M_0 < \infty$. *Then there exists a solution to* (1.9) (*and hence* (1.1)) *on* $[0, T]$ *for* $T > 0$ *taken sufficiently small, such that*

$$\sup_{t\in[0,T]} E_\gamma(t) \leqq 2M_0.$$

Moreover if the initial data are such that

$$\sum_{a=0}^{5}\Big[\|\partial_t^{2a}\eta(0)\|_{5-a}^2 + \|\rho_0\partial_t^{2a}\bar\partial^{5-a}D\eta(0)\|_0^2 + \|\sqrt{\rho_0}\bar\partial^{5-a}\partial_t^{2a}v(0)\|_0^2\Big]$$

$$+\|\operatorname{curl}_\eta v(0)\|_4^2 + \|\rho_0\bar\partial^5\operatorname{curl}_\eta v(0)\|_0^2 + \sum_{a=0}^{a_0}\|\sqrt{\rho_0}^{\,1+\frac{1}{\gamma-1}-a}\partial_t^{9+a_0-a}Dv(t)\|_0^2$$

is finite, then the solution is unique.

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